Advanced Studies in Pure Mathematics 51, 2008
Surveys on Geometry and Integrable Systems
pp. 143–161

Darboux transformations and generalized self-dual Yang-Mills flows

Gu Chaohao

§1. Introduction

The self-dual Yang-Mills equations are important partial differential equations in mathematical physics and have significant applications in mathematics [2, 3]. Besides the outstanding results related to 4-dimensional topology [5], it was known long ago that the self-dual Yang-Mills equations are an integrable system in the sense that they are the integrability condition of an overdetermined system of partial differential equations with a spectral parameter λ [12, 16]. Consequently, the rapidly developing theory of integrable systems can be applied to the self-dual Yang-Mills equations. Moreover, it has been found that many known soliton equations are reductions of the self-dual Yang-Mills equations [1, 17].

The self-dual Yang-Mills equations are equations in 4 dimensional space. It is interesting to generalize the equations to higher dimensions [13, 14, 18]. In [14], the generalization to $\mathbf{R}^{4n}$ is a simple case. In [8], more general integrable systems called generalized self-dual Yang-Mills flows (GSYMF) in $\mathbf{R}^{2n,1}$ were introduced. These are dynamical systems on the moduli space of the generalized self-dual Yang-Mills equations. The systems are very general in the sense that almost all known soliton equations of all dimensions are the reduction of these systems. It has been found that the Darboux transformation method is applicable to the GSYMF, giving new solutions explicitly. For the multi-soliton solutions, the phenomena of separation of solitons and confinement of solitons occurs.

Received February 17, 2001.
2000 *Mathematics Subject Classification.* Primary 35Q51; Secondary 35Q58, 70S15.

In the present paper we give an overview of the generalized self-dual Yang-Mills flows and the Darboux transformation method for obtaining explicit solutions.

In §2, we give a very brief sketch of some basic facts on Yang-Mills fields and self-duality. §3 is devoted to the derivation of the GSYMF from the Lax equations. In §4, the Darboux transformations (DT) for the GSYMF, as a method to obtain explicit GSYMF from a known one, are described. The algorithm is purely algebraic, depending on the choice of several arbitrary holomorphic functions. The most important cases of $U(N)$ and $SU(N)$ are emphasized. §5 is devoted to the explicit construction of solitons. The asymptotic behavior of solitons, which is a generalization of the usual KdV solitons, is much more complicated. In particular the multi-solitons may exhibit separation or confinement of single solitons as $|t| \to \infty$. In §6 we point out that the AKNS system in $\mathbf{R}^{n,1}$ can be a reduction of the GSYMF. Hence we can say the GSYMF may be the most general integrable system known to date, since almost all soliton equations for spaces of lower dimensions (≤ 4) are the reduction of the self-dual Yang-Mills equations.

§2. Yang-Mills fields, self-duality

First we recall some well-known basic facts on self-dual Yang-Mills fields. Let G be a matrix Lie group, g its Lie algebra and $\mathbf{R}^N$ an N-dimensional flat space $\{(x_1, \cdots, x_N)\}$ with a metric of the form $\mathrm{d}s^2 = \sum_{\alpha,\beta}^N \eta_{\alpha\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta$, where $\eta_{\alpha\alpha} = \pm 1$, $\eta_{\alpha\beta} = 0$ $(\alpha \neq \beta)$. The Yang-Mills potential is a 1-form valued in the Lie algebra g

$$\mathcal{A} = A_\alpha \mathrm{d}x^\alpha \tag{1}$$

and the field strength is

$$\mathcal{F} = \mathrm{d}\mathcal{A} + \mathcal{A} \wedge \mathcal{A} = \frac{1}{2} F_{\lambda\mu} \mathrm{d}x^\lambda \wedge \mathrm{d}x^\mu \tag{2}$$

where

$$F_{\lambda\mu} = \partial_\lambda A_\mu - \partial_\mu A_\lambda - [A_\mu, A_\lambda]. \tag{3}$$

Differentiating $\mathcal{F}$, we obtain the Bianchi identity

$$F_{(\lambda\mu|\nu)} = 0. \tag{4}$$

Here $F_{\lambda\mu|\nu}$ is the gauge derivative of $F_{\lambda\mu}$ defined by

$$F_{\lambda\mu|\nu} = \partial_\nu F_{\lambda\mu} + [A_\nu, F_{\lambda\mu}] \tag{5}$$

and $(\lambda\mu|\nu)$ is the symbol for the sum of cyclic permutations, i.e.

$$F_{(\lambda\mu|\nu)} = F_{\lambda\mu|\nu} + F_{\mu\nu|\lambda} + F_{\nu\lambda|\mu}.$$

The Yang-Mills equations are

$$\eta^{\mu\nu} F_{\lambda\mu|\nu} = 0. \tag{6}$$

A transformation from a potential $\mathcal{A}$ to $\mathcal{A}'$ defined by

$$\mathcal{A}' = \phi\mathcal{A}\phi^{-1} - (\mathrm{d}\phi)\phi^{-1} \qquad (\phi : \mathbf{R}^N \to G) \tag{7}$$

is called a gauge transformation; $\mathcal{A}'$ and $\mathcal{A}$ are called gauge equivalent. Gauge equivalent potentials are regarded as the same physical entity. The Yang-Mills theory is the most important theory in modern physics and has had great influence on mathematics. In mathematical terms, $\mathcal{A}$ is the connection form and $\mathcal{F}$ is the curvature of a G-bundle over $\mathbf{R}^N$. The case $N = 4$ is the most important, but the study of the higher dimensional case is also interesting [9, 13, 14, 18].

If $N = 4$ and $\mathbf{R}^4$ is Euclidean, a duality operator $*$ (Hodge operator) can be applied to $\mathcal{F}$:

$$(*\mathcal{F})_{\alpha\beta} = \frac{1}{2}\epsilon_{\alpha\beta\gamma\delta}\eta^{\gamma\lambda}\eta^{\delta\mu}\mathcal{F}_{\lambda\mu} \tag{8}$$

where

$$\epsilon_{\alpha\beta\gamma\delta} = \begin{cases} 1 & (\alpha,\beta,\gamma,\delta) \text{ is an even permutation of } (1,2,3,4), \\ -1 & (\alpha,\beta,\gamma,\delta) \text{ is an odd permutation of } (1,2,3,4), \end{cases} \tag{9}$$

A Yang-Mills potential is called self-dual (respectively, anti-self-dual) if

$$*\mathcal{F} = \mathcal{F} \qquad (\text{respectively, } *\mathcal{F} = -\mathcal{F}) \tag{10}$$

or

$$F_{12} = \pm F_{34}, \qquad F_{23} = \pm F_{14}, \qquad F_{31} = \pm F_{24}, \tag{10$'$}$$

where "+" (respectively, "−") is for the self-dual (respectively, anti-self-dual) case.

It is well-known that a self-dual (respectively, anti-self-dual) Yang-Mills potential satisfies the Yang-Mills equation. This can be seen from the definitions if the Bianchi identity is used.

The concept of self-duality can be extended to $\mathbf{R}^{2,2}$ where the metric can be written as

$$\mathrm{d}s^2 = 2(\mathrm{d}x_1\,\mathrm{d}x_4 - \mathrm{d}x_2\,\mathrm{d}x_3). \tag{11}$$

In this case the self-dual condition is

$$F_{12} = 0, \qquad F_{34} = 0, \qquad F_{14} - F_{23} = 0. \tag{12}$$

Here we have changed the sign of $\epsilon_{\alpha\beta\gamma\delta}$, e.g. $\epsilon_{1234} = -1$ (or we consider the anti-self-dual case).

From $F_{12} = 0$ we see that the equations

$$\partial_1\phi = \phi A_1, \qquad \partial_2\phi = \phi A_2 \tag{13}$$

are completely integrable on any plane $x_3 =$ constant, $x_4 =$ constant. Hence we can find $\phi(x_1, x_2, x_3, x_4) \in G$ such that after the gauge transformation defined by ϕ,

$$A_1' = \phi A_1 \phi^{-1} - (\partial_1\phi)\phi^{-1} = 0, \qquad A_2' = \phi A_2 \phi^{-1} - (\partial_2\phi)\phi^{-1} = 0. \tag{14}$$

This special gauge is called the R-gauge [19].

Let $p_2 = x_4$, $p_1 = x_3$. Under an R-gauge, the equations (12) become

$$\frac{\partial A_{p_1}}{\partial p_2} - \frac{\partial A_{p_2}}{\partial p_1} - [A_{p_1}, A_{p_2}] = 0, \qquad \frac{\partial A_{p_1}}{\partial x_2} - \frac{\partial A_{p_2}}{\partial x_1} = 0 \tag{15}$$

and $A_{x_1} = A_{x_2} = 0$. Hence the self-dual Yang-Mills equations in $\mathbf{R}^{2,2}$ can be simplified to (15) in which A_{x_1} and A_{x_2} disappear.

§3. Generalized self-dual Yang-Mills fields and their flows

We consider the generalization of the self-dual Yang-Mills fields in the higher dimensional flat space $\mathbf{R}^{n,n}$.

Let $(x, p) = (x_1, \cdots, x_n; p_1, \cdots, p_n)$ be the coordinates of points in $\mathbf{R}^{n,n}$ $(n \geq 2)$ $(i = 1, 2, \cdots, n)$. We start with the Lax equations

$$L\Psi = \left(\frac{\partial}{\partial p_i} - \lambda\frac{\partial}{\partial x_i}\right)\Psi = -A_i\Psi. \tag{16}$$

Here Ψ is an $N \times N$ matrix in G, $A_i = A_i(x, p)$ take values in the Lie algebra g of G, and λ is the spectral parameter.

By calculation, we see that the integrability condition for (16) is

$$\frac{\partial A_i}{\partial p_j} - \frac{\partial A_j}{\partial p_i} - [A_i, A_j] = 0 \tag{17}$$

and

$$\frac{\partial A_i}{\partial x_j} - \frac{\partial A_j}{\partial x_i} = 0. \tag{18}$$

If $n = 2$, these are just the self-dual Yang-Mills equations in the R-gauge. Thus (17) and (18) may be considered as the generalized self-dual Yang-Mills equations with $A_{p_i} = A_i$, $A_{x_i} = 0$. The generalization was initiated by Takasaki [13] for even n. However, as an integrable system, it makes sense for odd n too [7].

From (17) we see that there is a matrix-valued function J such that

$$A_i = -\frac{\partial J}{\partial p_i} J^{-1}, \tag{19}$$

and (18) becomes

$$\frac{\partial}{\partial x_i}\left(\frac{\partial J}{\partial p_j} J^{-1}\right) - \frac{\partial}{\partial x_j}\left(\frac{\partial J}{\partial p_i} J^{-1}\right) = 0. \tag{20}$$

From (18) we see that there is a matrix-valued function K such that

$$A_i = \frac{\partial K}{\partial x_i} \tag{21}$$

and (17) becomes

$$\frac{\partial^2 K}{\partial p_j \partial x_i} - \frac{\partial^2 K}{\partial p_i \partial x_j} - \left[\frac{\partial K}{\partial x_i}, \frac{\partial K}{\partial x_j}\right] = 0. \tag{22}$$

Consequently (20) or (22) is an equivalent form of the generalized self-dual Yang-Mills equations.

The metric of $\mathbf{R}^{n,n}$ is defined as

$$\mathrm{d}s^2 = \sum_{k=0}^{\frac{n}{2}-1} (\mathrm{d}x_{2k+1}\,\mathrm{d}p_{2k+2} - \mathrm{d}x_{2k+2}\,\mathrm{d}p_{2k+1}) \ (n \text{ even}) \tag{23}$$

and

$$\mathrm{d}s^2 = \sum_{k=0}^{[\frac{n}{2}]-1} (\mathrm{d}x_{2k+1}\,\mathrm{d}p_{2k+2} - \mathrm{d}x_{2k+2}\,\mathrm{d}p_{2k+1}) + \mathrm{d}x_n\,\mathrm{d}p_n \ (n \text{ odd}). \tag{24}$$

As in the 4-dimensional case, it can be shown that:

Theorem 1. *If n is even, a solution of generalized self-dual Yang-Mills equations satisfies the Yang-Mills equations.*

If Ψ is a solution to (16) and $\det \Psi \neq 0$, it is called a Lax representation of the generalized self-dual Yang-Mills field.

Remark 1. *If we replace (16) by*

$$L_i \Psi = (-A_i + \lambda B_i)\Psi, \tag{25}$$

then the integrability condition is still given by the equations

$$F_{p_i p_j} = 0, \qquad F_{p_i x_j} - F_{p_j x_i} = 0, \qquad F_{x_i x_j} = 0. \tag{26}$$

Here $A_{x_i} = B_i$, $A_{p_i} = A_i$ are the gauge potentials in a general gauge.

In the general case, let ϕ be a function valued in G and $\Psi' = \phi\Psi$, then

$$\begin{aligned} L_i \Psi' &= (L_i \phi)\Psi + \phi(-A_i + \lambda B_i)\Psi \\ &= (L_i \phi)\phi^{-1}\Psi' + \phi(-A_i + \lambda B_i)\phi^{-1}\Psi' = -A_i'\Psi' + \lambda B_i'\Psi' \end{aligned}$$

with

$$\begin{aligned} A_i' &= \phi A_i \phi^{-1} - (\partial_{p_i}\phi)\phi^{-1}, \\ B_i' &= \phi B_i \phi^{-1} - (\partial_{x_i}\phi)\phi^{-1}. \end{aligned} \tag{27}$$

This is exactly a gauge transformation.

We turn to the flows of the generalized self-dual Yang-Mills equations under the R-gauge in $\mathbf{R}^{n,n}$.

Suppose the functions Ψ and A_i depend on (x,p) and a "time" t and an evolution equation for Ψ

$$\frac{\partial \Psi}{\partial t} = V\Psi = \sum_{a=0}^{m+q} V_a \lambda^{m-a}\Psi, \quad (q \geq 0), \tag{28}$$

is satisfied. Besides, (16) and (28) constitute an integrable system in $\mathbf{R}^{2n,1}$. Here the V_a's are independent of the spectral parameter λ and can be determined in the following way.

Considering the condition of integrability for (16) and (28) we obtain a system of equations for the V_a's:

$$\frac{\partial V_0}{\partial x_i} = 0, \tag{29}$$

$$\frac{\partial V_a}{\partial x_i} = \frac{\partial V_{a-1}}{\partial p_i} + [A_i, V_{a-1}], \quad (a = 1, 2, \cdots, m),$$

and

$$\frac{\partial V_{m+q}}{\partial p_i} = [V_{m+q}, A_i], \tag{30}$$

$$\frac{\partial V_{a-1}}{\partial p_i} = \frac{\partial V_a}{\partial x_i} + [V_{a-1}, A_i], \quad (a = m + q, \cdots, m + 2)$$

and a system of evolution equations for the A_i's:

$$\frac{\partial A_i}{\partial t} + \frac{\partial V_m}{\partial p_i} - \frac{\partial V_{m+1}}{\partial x_i} + [A_i, V_m] = 0. \tag{31}$$

Theorem 2. *If the A_i's satisfy the generalized self-dual Yang-Mills equations (17), (18), then (29) and (30) are completely integrable and V_a ($a = 0, 1, \cdots, m, m+1, \cdots, m+q$) can be expressed as integro-differential expressions of the A_i's.*

Proof. From the first equation of (30) we have

$$V_0 = V_0(p, t). \tag{32}$$

Assume that $V_1, V_2, \cdots, V_{a-1}$ ($a < m$) are constructed already. By differentiating (29) with respect to x_j, we have

$$\frac{\partial^2 V_a}{\partial x_i \partial x_j} = \frac{\partial^2 V_a}{\partial x_j \partial x_i}$$

if (17) and (18) hold. Hence $V_a(x, p)$ can be obtained by integrating the right hand sides of (29) along any paths connecting $(0, p)$ and (x, p) in the plane p_i = constant, i.e.

$$V_a = \sum_i \int_{(0,p)}^{(x,p)} \left(\frac{\partial V_{a-1}}{\partial p_i} + [A_i, V_{a-1}] \right) \mathrm{d}x_i + V_a^0(p, t). \tag{33}$$

The integral is independent of the choice of paths and $V_0(p, t)$, $V_a^0(p, t)$ are arbitrary functions.

Let

$$V_a = JW_aJ^{-1}, \qquad a = m+q, \cdots, m+1. \tag{34}$$

Equation (30) becomes

$$W_{m+q} = W_{m+q}(x,t), \tag{35}$$

$$\frac{\partial W_{a-1}}{\partial p_i} = \frac{\partial W_a}{\partial x_i} + \left[J^{-1}\frac{\partial J}{\partial x_i}, W_a\right], \quad (a = m+q, \cdots, m+2). \tag{36}$$

When (17) and (18) are satisfied, we have

$$W_{a-1} = \sum_i \int_{(x,0)}^{(x,p)} \left(\frac{\partial W_a}{\partial x_i} + \left[J^{-1}\frac{\partial J}{\partial x_i}, W_a\right]\right) \mathrm{d}p_i + W_{a-1}^0(x,t). \tag{37}$$

Here, the path of integration is any curve on the plane x_i = constant connecting $(x,0)$ and (x,p) and $W_{m+q}(x,t)$, $W_a^0(x,t)$ $(m < a < m+q)$ are arbitrary functions of integration. The theorem is proved.

Equations (31) can be considered as dynamical equations on the moduli space of generalized self-dual Yang-Mills equations.

§4. Darboux transformations

The Darboux transformation of GSYMF is actually a modified gauge transformation

$$\widetilde{\Psi} = S\Psi \tag{38}$$

of the solutions of Lax equations with

$$S = \lambda I + \alpha \tag{39}$$

such that $(L_i\widetilde{\Psi})\widetilde{\Psi}^{-1}$ is independent of λ. S is called a Darboux matrix. Here I is the unit matrix and α is to be determined.

We can write

$$L_i\widetilde{\Psi} = \left(\frac{\partial}{\partial p_i} - \lambda\frac{\partial}{\partial x_i}\right)\widetilde{\Psi} = -\widetilde{A}_i\widetilde{\Psi}, \tag{40}$$

$$\frac{\partial\widetilde{\Psi}}{\partial t} = \sum V_a'\lambda^{m-a}\widetilde{\Psi}. \tag{41}$$

Substituting (38) and (39) in (40), we obtain

$$L_i\widetilde{\Psi} = \left(\left(\frac{\partial}{\partial p_i} - \lambda\frac{\partial}{\partial x_i}\right)\alpha - (\lambda I + \alpha)A_i\right)\Psi = -\widetilde{A}_i(\lambda I + \alpha)\Psi. \tag{42}$$

Hence

$$\widetilde{A}_i = A_i + \frac{\partial \alpha}{\partial x_i}, \tag{43}$$

and

$$\frac{\partial \alpha}{\partial p_i} = \alpha A_i - \widetilde{A}_i\alpha = [\alpha, A_i] - \frac{\partial \alpha}{\partial x_i}\alpha. \tag{44}$$

Similarly, from (41) we obtain

$$\begin{aligned}
&\widetilde{V}_0 = V_0,\\
&\widetilde{V}_a = V_a + \alpha V_{a-1} - \widetilde{V}_{a-1}\alpha \ (a = 1, 2, \cdots, m),\\
&\widetilde{V}_{m+q} = \alpha V_{m+q}\alpha^{-1},\\
&\widetilde{V}_{m+k} = \alpha V_{m+k}\alpha^{-1} + (V_{m+k+1} - \widetilde{V}_{m+k+1})\alpha^{-1} \ (k = q-1, \cdots, 1)
\end{aligned} \tag{45}$$

and

$$\frac{\partial \alpha}{\partial t} + \alpha V_m - \widetilde{V}_m\alpha + V_{m+1} - \widetilde{V}_{m+1} = 0. \tag{46}$$

Since the generalized self-dual Yang-Mills equations (17), (18) and the evolution equation (31) are consequences of the Lax equations (16) and (28) (provided that $\det \Psi \not\equiv 0$), the $\widetilde{A}_i$'s are solutions to (17), (18) and (31) as well. We have

Theorem 3. *Let A_i be a solution to the generalized self-dual Yang-Mills equations (17), (18) and the evolution equations (31), and let Ψ be the Lax representation. If α is a solution to (44) and (46), then the*

$$\widetilde{A}_i = A_i + \frac{\partial \alpha}{\partial x_i} \tag{47}$$

satisfy (17), (18) and (31), in which the V_a are replaced by the $\widetilde{V}_a$ in (45). Moreover,

$$\widetilde{\Psi} = (\lambda I + \alpha)\Psi$$

is the Lax representation of $\widetilde{A}_i$.

The transformation $(\Psi, A_i) \to (\widetilde{\Psi}, \widetilde{A}_i)$ defined by (38), (39) and (47) is called a Darboux transformation.

System (44) and (46) is nonlinear and very complicated. However, we are able to construct explicit solutions as follows:

Let λ_α $(\alpha = 1, 2, \cdots, N)$ be N numbers (at least two of them not equal, $\lambda_\alpha \neq 0$) and $h_\alpha(\lambda_\alpha, x, p)$ be N column vectors, satisfying (16) and (28) for $\lambda = \lambda_\alpha$ $(\alpha = 1, 2, \cdots, N)$, i.e.

$$\frac{\partial h_\alpha}{\partial p_i} = \lambda_\alpha \frac{\partial h_\alpha}{\partial x_i} - A_i h_\alpha, \quad \frac{\partial h_\alpha}{\partial t} = \sum_a V_a \lambda_\alpha^{m-a} h_\alpha. \tag{48}$$

Define

$$H = [h_1, h_2, \cdots, h_N]. \tag{49}$$

Here we require that $\det H \neq 0$.

Theorem 4. *The matrix α defined by*

$$\alpha = -H\Lambda H^{-1} \text{ with } \Lambda = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_N \end{pmatrix} \tag{50}$$

satisfies equations (44) and (46).

Proof. From (48) it is easily seen that

$$\frac{\partial H}{\partial p_i} = \frac{\partial H}{\partial x_i}\Lambda - A_i H. \tag{51}$$

Hence

$$\begin{aligned} \frac{\partial \alpha}{\partial p_i} &= -\frac{\partial H}{\partial x_i}\Lambda^2 H^{-1} + H\Lambda H^{-1}\frac{\partial H}{\partial x_i}\Lambda H^{-1} - A_i\alpha + \alpha A_i, \\ \frac{\partial \alpha}{\partial x_i} &= -\frac{\partial H}{\partial x_i}\Lambda H^{-1} + H\Lambda H^{-1}\frac{\partial H}{\partial x_i} H^{-1}. \end{aligned} \tag{52}$$

By using (50), the definition of α, it is easily seen that (44) is satisfied.

Similarly, we have

$$\frac{\partial H}{\partial t} = \sum_{a=0}^{m+q} V_a H \Lambda^{m-a}. \tag{53}$$

Hence

$$
\begin{aligned}
\frac{\partial \alpha}{\partial t} &= -\sum_{a=0}^{m+q} V_a H \Lambda^{m-a+1} H^{-1} + \sum_{a=0}^{m+q} H\Lambda H^{-1} V_a H \Lambda^{m-a} H^{-1} \\
&= \sum_{a=0}^{m+q} V_a (-\alpha)^{m-a} \alpha - \alpha \sum_{a=0}^{m+q} V_a (-\alpha)^{m-a}. \qquad (54)
\end{aligned}
$$

From (45) it can be seen that the right hand side of (54) is just $\widetilde{V}_m \alpha - \alpha V_m + \widetilde{V}_{m+1} - V_{m+1}$. Thus (46) is satisfied.

Remark 2. *The column solutions h_α in (48) can be obtained from $\Psi(\lambda)$ by setting $h_\alpha = \Psi(\lambda_\alpha) l_\alpha$, where l_α satisfies*

$$L_i l_\alpha = \frac{\partial l_a}{\partial p_i} - \lambda \frac{\partial l_\alpha}{\partial x_i} = 0,$$

i.e. $l_\alpha = l_\alpha(\lambda_\alpha p + x)$. It is easily seen that the l_α's are holomorphic functions of $\lambda_\alpha p + x$.

Remark 3. *As in many other cases, we can apply Darboux transformations successively to obtain an infinite sequence of solutions by the purely algebraic algorithm indicated in Theorem 4, provided that a seed solution (Ψ, A_i) is known.*

Remark 4. *It can be seen that $L_i(\Psi(\bar{\lambda})^* \Psi(\lambda)) = 0$, $\partial_t(\Psi(\bar{\lambda})^* \Psi(\lambda)) = 0$. If we take a suitable initial condition for $\Psi(\lambda)$, the $\Psi(\lambda)$ can be normalized so that $\Psi(\bar{\lambda})^* \Psi(\lambda) = I$.*

For the case of $U(N)$, we consider (x, p) as real coordinates of $\mathbf{R}^{n,n}$. The potential A_i's should take values in the Lie algebra $u(N)$, i.e.

$$A_i^* + A_i = 0. \qquad (55)$$

Moreover, we can choose the functions of integration $V_0(p)$, $V_a^0(p)$ $(a = 1, 2, \cdots, m)$ such that V_0, V_1, $\cdots$, V_m take values in $u(N)$ too. In this case we can apply the Darboux transformation as before. However, we should consider the additional requirement

$$\widetilde{A}_i^* + \widetilde{A}_i = 0 \qquad (56)$$

which is equivalent to

$$\frac{\partial \alpha^*}{\partial x_i} + \frac{\partial \alpha}{\partial x_i} = 0 \qquad (57)$$

since (55) and (43) hold. For this purpose we choose the parameters λ_α's and l_α's as follows. Let μ be a complex number ($\mu \neq \bar{\mu}$) and

$$\lambda_\alpha = \mu \text{ or } \bar{\mu} \qquad (\alpha = 1, 2, \cdots, N). \tag{58}$$

We assume that Ψ is normalized and choose the l_α such that

$$l_\alpha^* l_\beta = 0 \text{ if } \lambda_\alpha \neq \lambda_\beta. \tag{59}$$

Then we have

$$h_\alpha^* h_\beta = 0 \text{ if } \lambda_\alpha \neq \lambda_\beta. \tag{60}$$

From (50), the definition of α, we have

$$\alpha h_\beta = -\lambda_\beta h_\beta, \qquad h_\beta^* \alpha^* = -\lambda_\beta^* h_\beta^*. \tag{61}$$

Hence

$$h_\beta^*(\alpha^* + \alpha) h_\gamma = -h_\beta^*(\mu^* + \mu) h_\gamma \tag{62}$$

holds for all $\beta, \gamma = 1, 2, \cdots, N$. Since the columns $\{h_\alpha\}$ are linearly independent, we have

$$\alpha^* + \alpha = -(\mu + \bar{\mu}) I \tag{63}$$

and (57) holds. Thus we obtain:

Theorem 5. *If $A_i \in u(N)$, $V_a \in u(N)$ and parameters $\lambda_1, \cdots, \lambda_N$; $l_1, \cdots, l_N$ for constructing the Darboux matrix satisfy (58) and (59) and the condition of linear independence, then $\widetilde{A}_i \in u(N)$.*

Remark 5. *The matrices α and S can be expressed in a geometric way. Without loss of generality, let $\lambda_a = \mu$ ($a = 1, 2, \cdots, k$), $\lambda_b = \bar{\mu}$ ($b = k+1, \cdots, N$). At each point of $\mathbf{R}^{n,n}$, the l_a's span a k-dimensional subspace L_1 of $\mathbf{C}_n$ and the l_b's span the orthogonal complement L_2. They are holomorphic functions of $\mu p + x$ and $\bar{\mu} p + x$, respectively. Define Σ_1 to be the bundle of k-dimensional spaces on $\mathbf{R}^{n,n} \times \mathbf{R}^1$ spanned by the $\Psi(\mu) l_a$, and let π be the Hermitian projection from $\mathbf{C}^n \times (\mathbf{R}^{n,n} \times \mathbf{R}^1)$ on Σ_1 and $\pi^\perp$ the complement of π. Then from (61) we see that*

$$\alpha = -(\mu \pi + \bar{\mu} \pi^\perp) \tag{64}$$

and

$$S = (\lambda - \mu)\pi + (\lambda - \bar{\mu})\pi^\perp = (\lambda - \mu)\left(I + \frac{\mu - \bar{\mu}}{\lambda - \mu}\pi^\perp\right). \tag{65}$$

Hence the Darboux transformation of Ψ can be written as

$$\Psi_1 = \left(I + \frac{\mu - \bar{\mu}}{\lambda - \mu} \pi^{\perp} \right) \Psi. \tag{66}$$

Here both Ψ_1 and Ψ are normalized. Consequently, a Darboux transformation is defined by a complex number μ and a holomorphic map ρ from $\mathbf{C}^n$ to the Grassmannian $G_{N,k}$ such that the subspace L_1 is the image of $\rho(\mu p + x)$.

Remark 6. *The above statements are valid for the case of $SU(N)$, since we have* $\mathrm{tr}\alpha = k\mu + (N-k)\bar{\mu}$ = *constant and* $\mathrm{tr}\widetilde{A}_i = 0$, *from* $\mathrm{tr}A_i = 0$.

§5. Solitons and interactions

We start with the trivial solution $A_i = 0$. In order to obtain the solution $\Psi_0(\lambda)$ to the Lax equations, we have to determine $V_0, V_1, \cdots, V_m$. For $A_i = 0$, (29), (30) become

$$V_0 = V_0(p),$$

$$\frac{\partial V_a}{\partial x_i} = \frac{\partial V_{a-1}}{\partial p_i} \quad (a = 1, 2, \cdots, m+q), \tag{67}$$

$$\frac{\partial V_{m+q}}{\partial p_i} = 0.$$

We assume that V_0, V_a^0, W_{m+k}^0, W_{m+q} are independent of t. It is easily seen that

$$V(\lambda) = U_1(\lambda p + x) + U_2\left(p + \frac{x}{\lambda}\right), \tag{68}$$

where U_1, U_2 are matrix-valued polynomials of degrees m and q, and $U_1, U_2 \in u(N)$ (or $su(N)$) for real λ. Thus

$$\Psi_0 = \exp[V(\lambda)t]. \tag{69}$$

We can find explicit expressions for H and α. We assume $N = 2$ in order to simplify the expressions, and let

$$V(\lambda) = \begin{pmatrix} \mathrm{i}v(\lambda) & 0 \\ 0 & -\mathrm{i}v(\lambda) \end{pmatrix} \tag{70}$$

with

$$v(\lambda) = u_1(\lambda p + x) + u_2\left(p + \frac{x}{\lambda}\right). \tag{71}$$

Here u_1, u_2 are polynomials of degrees m and q with real coefficients.

The matrix H takes the form

$$H = \begin{pmatrix} e^{\mathrm{i}v(\mu)t} & -\overline{g(\mu)}e^{\mathrm{i}v(\bar{\mu})t} \\ g(\mu)e^{-\mathrm{i}v(\mu)t} & e^{-\mathrm{i}v(\bar{\mu})t} \end{pmatrix}. \tag{72}$$

Here $g(\mu)$ is a holomorphic function of $\mu p + x$ and $\mu = \sigma + \mathrm{i}\tau$ $(\tau \neq 0)$. The condition $h_2^* h_1 = 0$ holds. By calculation we have

$$\begin{aligned} \Delta &= \det H = e^{wt} + |g(\mu)|^2 e^{-wt}, \\ \alpha &= -\frac{1}{\Delta}\begin{pmatrix} \mu e^{wt} + \bar{\mu}|g|^2 e^{-wt} & (\mu - \bar{\mu})\bar{g}e^{\mathrm{i}ut} \\ (\mu - \bar{\mu})ge^{-\mathrm{i}ut} & \mu|g|^2 e^{-wt} + \bar{\mu}e^{wt} \end{pmatrix}. \end{aligned} \tag{73}$$

Here

$$w = \mathrm{i}(v(\mu) - v(\bar{\mu})), \qquad u = v(\mu) + v(\bar{\mu}) \tag{74}$$

are real functions of p and x. Consequently,

$$A_i = \frac{\partial \alpha}{\partial x_i} = \begin{pmatrix} a_{i11} & a_{i12} \\ -\bar{a}_{i12} & -a_{i11} \end{pmatrix} \tag{75}$$

with

$$\begin{aligned} a_{i11} &= 2\tau\mathrm{i}\frac{1}{(e^{wt} + |g(\mu)|^2 e^{-wt})^2}\left(\frac{\partial |g|^2}{\partial x_i} - 2|g|^2 \frac{\partial w}{\partial x_i}t\right), \\ a_{i12} &= -2\tau\mathrm{i}\frac{\left(\frac{\partial \bar{g}}{\partial x_i} + \bar{g}\frac{\partial u}{\partial x_i}t\mathrm{i}\right)e^{\mathrm{i}ut}}{e^{wt} + |g(\mu)|^2 e^{-wt}} \\ &\quad + 2\tau\mathrm{i}\frac{(e^{wt} - |g(\mu)|^2 e^{-wt})\frac{\partial w}{\partial x_i} + e^{-wt}\frac{\partial |g|^2}{\partial x_i}}{(e^{wt} + |g(\mu)|^2 e^{-wt})^2}\bar{g}e^{\mathrm{i}ut}. \end{aligned} \tag{76}$$

These are the explicit formulae for single solitons. The asymptotic behavior of the single solitons as $t \to \pm\infty$ depends on μ, g and v. For the general case it is quite complicated.

We assume

$$g(\mu) = P(\mu)\exp\{Q(\mu)\} \tag{77}$$

where P and Q are polynomials of $\mu p + x$. Imagine an observer who moves in the space $\mathbf{R}^{n,n}$ with constant velocity. His world line l in $\mathbf{R}^{n,n,1}$ is

$$p = \pi t + \pi_0, \quad x = \xi t + \xi_0. \tag{78}$$

Here (π, ξ) is his velocity and (π_0, ξ_0) is his starting point. In his observation, v, P, Q behave as polynomials of t. Write $Q = q + \mathrm{i}s$, where q and s are real. Then $|g|^2 = |P|^2 e^{2q}$. Along l,

$$q - wt = C_0 t^m + C_1 t^{m-1} + \cdots + C_{m-1}t + C_m \tag{79}$$

where the coefficients $C_0, C_1, \cdots, C_m$ are polynomials in π, ξ and π_0, ξ_0. If $|A_i| \to 0$ as $t \to \pm\infty$ along l, then l is called a vanishing line of the soliton. If at least one of $C_0, C_1, \cdots, C_{m-1}$ is not zero and $|P| \neq 0$ along l, then the world line l is a vanishing line. If along l, $P = 0$ and e^{-q+wt} approaches infinity, l is also a vanishing line. The asymptotic behavior of the single soliton is seen by the observer, who is going along the non-vanishing world line.

Example 1. Let

$$v = a = \text{real constant}, \quad g = P(\mu p + x).$$

Then $w = 0$, $q = 0$. From (76) we have

$$\alpha_{i11} = 2\tau\mathrm{i}\frac{1}{(1+|P|^2)^2}\frac{\partial |P|^2}{\partial x_i},$$

$$\alpha_{i12} = -2\tau\mathrm{i}\left(\frac{1}{1+|P|^2}\frac{\partial \bar{P}}{\partial x_i} - \frac{\bar{P}}{(1+|P|^2)^2}\frac{\partial |P|^2}{\partial x_i}\right)e^{2ait}.$$

The soliton is periodic with time t and $x_i = x_i^0$, $p_i = p_i^0$, $t = t$ are non-vanishing world lines if $|P| \neq 0$ or $\dfrac{\partial \bar{P}}{\partial x_i} \neq 0$.

Example 2. Let

$$v = v(\mu), \quad g = \exp\{Q(x)\}.$$

Then $w = -2\mathrm{i}\,\mathrm{Im}\,v$, $u = 2\,\mathrm{Re}\,v$, $P = 1$, $q = \mathrm{Re}\,Q$, $s = \mathrm{Im}\,Q$. From (76), we have

$$a_{i11} = \frac{-4\tau\mathrm{i}}{(e^{-q+wt} + e^{q-wt})^2} \frac{\partial w}{\partial x_i} t,$$

$$a_{i12} = -2\tau\mathrm{i}\frac{\frac{\partial}{\partial x_i}(q - \mathrm{i}s)}{e^{-q+wt} + e^{q-wt}} e^{\mathrm{i}(u-s)}$$

$$+2\tau\mathrm{i}\frac{(e^{-q+wt} - e^{q-wt})\frac{\partial w}{\partial x_i}t - 2e^{q-wt}\frac{\partial q}{\partial x_i}}{(e^{-q+wt} + e^{q-wt})^2} e^{\mathrm{i}(u-s)t}.$$

The vanishing world lines are defined by the expansion of $q - wt$, since $|P| = 1$.

§6. Multi-solitons and the interaction of solitons

Let (Ψ, A) be a single soliton obtained from the trivial solution with "parameters" (μ_1, g_1), with v given. It will be denoted by $\Sigma_{(\mu_1,g_1)}$. Applying the Darboux transformation with parameters (μ_2, g_2) to (Ψ, A) we obtain a solution $\Sigma_{(\mu_1,g_1,\mu_2,g_2)}$ which is called a double soliton. The Lax representation of the double soliton is

$$\Psi_2 = (\lambda I + \alpha_2')(\lambda I + \alpha_1)\Psi. \tag{80}$$

Here Ψ is the Lax representation of the trivial solution, $\lambda I + \alpha_1$ is the Darboux matrix based on Ψ and defined by (μ_1, g_1), $\lambda I + \alpha_2'$ is the Darboux matrix based on $\Psi_1 = (\lambda I + \alpha_1)\Psi$ and defined by (μ_2, g_2).

Let $\mathcal{O}_{(\mu_1,g_1)}$, $\mathcal{O}_{(\mu_2,g_2)}$ be the sets of vanishing world lines of $\Sigma_{(\mu_1,g_1)}$ and $\Sigma_{(\mu_2,g_2)}$, respectively. Consider the behavior of $\Sigma_{(\mu_1,g_1,\mu_2,g_2)}$ along a line $l_1 \in \mathcal{O}_{(\mu_1,g_1)}$. From the definition it is seen that as $t \to \infty$, α_1 behaves as a constant matrix α_1^0. Then

$$\Psi_2 \sim (\lambda I + \alpha_2'')(\lambda I + \alpha_1^0)\Psi. \tag{81}$$

Hence Ψ_2 behaves asymptotically as a Darboux transformation with parameters (μ_2, g_2), based on $\Psi_1' = (\lambda I + \alpha_1^0)$ which is a Lax representation of the trivial solution. Consequently, $\Sigma_{(\mu_1,g_1,\mu_2,g_2)}$ behaves asymptotically as a single soliton $\Sigma'_{(\mu_2,g_2)}$ along l_1. By the theorem of permutability, $\Psi_2 = (\lambda I + \alpha_1')(\lambda I + \alpha_2)\Psi$. Similarly, it can be seen that, along a line $l_2 \in \mathcal{O}_{(\mu_2,g_2)}$, $\Sigma_{(\mu_1,g_1,\mu_2,g_2)}$ behaves as a single soliton $\Sigma'_{(\mu_1,g_1)}$. Thus we have:

Theorem 6. *If $l \in \mathcal{O}_{(\mu_1,g_1)}$ (respectively, $\mathcal{O}_{(\mu_2,g_2)}$), then along l the double soliton $\Sigma_{(\mu_1,g_1,\mu_2,g_2)}$ behaves asymptotically as a single soliton $\Sigma'_{(\mu_2,g_2)}$ (respectively, $\Sigma'_{(\mu_1,g_1)}$).*

In particular, if $l \in \mathcal{O}_{(\mu_1,g_1)} \cap \mathcal{O}_{(\mu_2,g_2)}$, then $\Sigma_{(\mu_1,g_1,\mu_2,g_2)}$ approaches the trivial solution. Let $S_{(\mu_1,g_1)}$ (respectively, $S_{(\mu_2,g_2)}$) be the set of non-vanishing world lines of $\Sigma_{(\mu_1,g_1)}$ (respectively, $\Sigma_{(\mu_2,g_2)}$). If $S_{(\mu_1,g_1)} \cap S_{(\mu_2,g_2)} = \emptyset$, then $\Sigma_{(\mu_1,g_1,\mu_2,g_2)}$ splits into two single solitons $\Sigma'_{(\mu_1,g_1)}$ and $\Sigma'_{(\mu_2,g_2)}$ as $t \to \infty$. If $S_{(\mu_1,g_1)} \cap S_{(\mu_2,g_2)} \neq \emptyset$ and $l \in S_{(\mu_1,g_1)} \cap S_{(\mu_2,g_2)}$, then along l, the behavior of $\Sigma_{(\mu_1,g_1,\mu_2,g_2)}$ can be regarded as confinement of solitons.

The situation for p-solitons is similar. The most interesting phenomena of the interaction of KdV solitons and many other solitons are thus extended to the case of GSYMF. Furthermore, confinement of solitons seems to be an interesting phenomenon in soliton theory.

§7. Reduction to AKNS systems in $\mathbf{R}^{n,1}$

As has been pointed out by many authors, almost all known soliton equations can be obtained as reductions of self-dual Yang-Mills equations. Of course, each of these equations at most has four independent variables. In recent years, integrable systems with more independent variables have beeen considered by many authors [4, 10, 15]. Almost all of these integrable systems with many independent variables belong to the framework of the AKNS system [11]. However, we have:

Theorem 7. *The AKNS system in the space $\mathbf{R}^{n,1}$ is a reduction of the generalized self-dual Yang-Mills flows.*

Proof. Let $\Psi = \Psi(x,p,t)$ satisfy the Lax equation (16), (28), and let J_i $(i = 1, 2, \cdots, n)$ be constant diagonal matrices. Define

$$
\begin{aligned}
\Phi(p,t) &= \exp(-\textstyle\sum x_j J_j)\Psi, \\
P_i(p,t) &= -\exp(-\textstyle\sum x_j J_j) A_i \exp(\textstyle\sum x_j J_j), \\
U_a(p,t) &= \exp(-\textstyle\sum x_j J_j) V_a \exp(\textstyle\sum x_j J_j).
\end{aligned}
\tag{82}
$$

Assume that the right-hand sides are independent of x. Substituting them into the Lax equations (16), (28), we obtain

$$\begin{aligned} \frac{\partial \Phi}{\partial p_i} &= (\lambda J_i + P_i)\Phi, \\ \frac{\partial \Phi}{\partial t} &= \sum_{a=0}^{m+q} U_a \lambda^{m-a}\Phi. \end{aligned} \tag{83}$$

This is just the AKNS system in $\mathbf{R}^{n,1}$.

From this theorem, one is lead to believe that the GSYMF is the most general known integrable system.

§ Acknowledgments

The work is supported by the Chinese National Research Project "Nonlinear Science", the Scientific Foundation of the National Education Department of China for Doctor Program and the Research Foundation of the Education Commission of Shanghai.

References

[1] M. J. Ablowitz, S. Chakrayarty and L. A. Takhtajan, A self-dual Yang-Mills hierarchy and its reduction to integrable systems in 1+1 and 2+1 dimensions, Commun. Math. Phys., **159** (1993), 289–314.

[2] M. F. Atiyah, Instantons in two and four dimensions, Commun. Math. Phys., **93** (1984), 437–451.

[3] M. F. Atiyah, Geometry of Yang-Mills Fields, Lezioni Fermiane, Accademia Nazionale dei Lincei Scuola Normaie Superione, Pisa, 1979.

[4] R. Beals and K. Tenenblat, An intrinsic generalization for the wave and sine-Gordon equation, In: Differential Geometry, Pitman Monograph, **52**, Longman, Harlow, 1991, pp. 25–46.

[5] D. S. Freed and K. K. Uhlenbeck, Instantons and Four-Manifolds, Math. Sciences Res. Inst. Publ. I, Springer-Verlag, 1984.

[6] C. H. Gu, Bäcklund transformations and Darboux transformations, In: Soliton theory and its applications, (ed. C. H. Gu), Springer-Verlag and Zhejiang Science and Technology Publishing House, 1995, pp. 122–151.

[7] C. H. Gu, Integrable evolution systems based on generalized self-dual Yang-Mills equations and their soliton-like solutions, Lett. Math. Phys., **35** (1995), 61–74.

[8] C. H. Gu, Generalized self-dual Yang-Mills flows: explicit solutions and reductions, Proc. of KdV'95, Acta Appl. Math., **39** (1995), 349–360.

[9] C. H. Gu, On the interaction of solitons for a class of integrable systems in the space-time $\mathbf{R}^{n+1}$, Lett. Math. Phys., **26** (1992), 199–209.

[10] C. H. Gu and H. S. Hu, Explicit solutions to the intrinsic generalization for the wave and sine-Gordon equations, Lett. Math. Phys., **29** (1993), 1–11.

[11] C. H. Gu and Z. X. Zhou, On Darboux transformations for soliton equations in high-dimensional space-time, Lett. Math. Phys., **32** (1994), 1–10.

[12] L. J. Mason, Integrable and solvable systems, In: Further advances in twistor theory, (eds. L. J. Mason, L. P. Hughston and P. Z. Kobak), Longman Scientific Technical, 1995, pp. 1–73.

[13] K. Takasaki, A new approach to the self-dual Yang-Mills equations, Commun. Math. Phys., **94** (1984), 35–59.

[14] K. Takasaki, Integrable systems in gauge theory, Kähler geometry and super KP hierarchy — symmetries and algebraic points of view, In: Proc. International Congr. Math., Kyoto, 1990, Vol. II, pp. 1205–1214.

[15] C. L. Terng and K. Uhlenbeck, Bäcklund transformations and loop group actions, Commun. Pure Appl. Math., **53** (2000), 1–75.

[16] R. S. Ward, Integrable systems and twistors, In: Integrable systems, twistors, loop groups, and Riemann surfaces, (eds. N. J. Hitchin, G. B. Segal and R. S. Ward), Oxford, Clarendon Press, 1999, pp. 121–134.

[17] R. S. Ward, Integrable and solvable systems and relations among them, Phil. Trans. Roy. Soc. London Ser. A, **315** (1985), 451–457.

[18] R. S. Ward, Completely solvable gauge-field equations in dimension greater than four, Nuclear Phys. B, **236** (1984), 381–396.

[19] C. N. Yang, Condition of self-duality for SU_2 gauge fields on Euclidean 4-dimensional spaces, Phys. Rev. Lett., **38** (1977), 1377–1379.

Institute of Mathematics
Fudan University
Shanghai 200433
P. R. CHINA

E-mail address:
`guch@fudan.ac.cn`