Advanced Studies in Pure Mathematics 51, 2008
Surveys on Geometry and Integrable Systems
pp. 55–111

Generalized Weierstraß representations of surfaces

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§1. Introduction

The classical Weierstraß representation

$$\phi(z,\bar{z}) = \mathrm{Re}\left\{\int_{z_0}^{z}(\frac{1}{2}f(1-g^2), \frac{i}{2}f(1+g^2), fg)dz\right\}$$

has been a very useful tool for the construction and the investigation of minimal surfaces in $\mathbb{R}^3$. While the differential equation describing these surfaces is highly nonlinear, the Weierstraß data, a pair consisting of a holomorphic function f and a meromorphic function g, is completely unconstrained. Moreover, the relation between the Weierstraß data (f,g) and ϕ is sufficiently direct that it is possible to relate geometric properties of the surface to the properties of the Weierstraß data.

In recent years a generalized Weierstraß representation was found, which applies to the construction of all surfaces of constant mean curvature in $\mathbb{R}^3$. If the mean curvature H vanishes, i.e. if the surface actually is a minimal surface, then the new procedure leads to the classical Weierstraß representation in a straightforward fashion. If $H \neq 0$, then the generalized Weierstraß representation is a new tool for the construction of conformal immersions of these surfaces. The Weierstraß data consists again of a pair of functions (Q,f), where Q is holomorphic and f is meromorphic. More precisely, Qdz^2 is the Hopf differential of the surface to be constructed and f is closely related with the conformal factor of the induced metric. In spite of this, the relation between the Weierstraß data and the geometry of the conformally immersed surface is much less direct than in the classical case. However, it turns out that at least some

Received July 11, 2001.
Revised October 5, 2001; December 13, 2007 (updated).
2000 *Mathematics Subject Classification.* Primary 58E20.

features of surfaces, such as the number of umbilical points and also the number of embedded ends, are well reflected in the Weierstraß data.

In this paper we will attempt to give a survey of the use of the generalized Weierstraß representation for the construction of constant mean curvature surfaces in $\mathbb{R}^3$. Since the original paper [26] introducing this method actually discussed harmonic maps into arbitrary compact symmetric spaces and since the notion of harmonic maps also exists for pseudo-Riemannian spaces, it is not surprising that there are generalized Weierstraß representations also for other classes of surfaces, such as the surfaces in $\mathbb{R}^3$ of constant Gauß curvature $K = -1$, timelike surfaces of constant mean curvature in three dimensional Minkowski space, and affine spheres in $\mathbb{R}^3$.

While the number of papers using methods similar to the one used in [26] is still reasonably small, we have still been unable to present all results in sufficient detail. We would like to apologize sincerely to everyone whose work we failed to represent adequately.

Acknowledgements: Over the years I have profited greatly from communication with many people. In particular I would like to mention Franz Pedit, Fran Burstall, Ivan Sterling, Martin Guest, and Junichi Inoguchi. I would also like to thank Martin Guest, Junichi Inoguchi, Magdalena Toda, and the referee for thoughtful and pertinent questions and comments on a preliminary version of this article. Finally, I would like to thank Professors M. Guest, R. Miyaoka and Y. Ohnita for their generous invitation to participate in the 9th MSJ-IRI in Tokyo and for the opportunity to publish this survey in these conference proceedings.

§2. Constructing potentials from surfaces

2.1 Let M be a connected, smooth, orientable surface and $\phi : M \longrightarrow \mathbb{R}^3$ an immersion of constant mean curvature. We can assume that $\phi \circ \pi$ is conformal, where $\pi : \mathbb{D} \longrightarrow M$ realizes the universal cover of M. In this article we will always assume that $\mathbb{D}$ is noncompact. For the case $\mathbb{D} \cong S^2$ we refer to [56],[9],[36].

By a result of Ruh and Vilms [48] we know that ϕ has constant mean curvature if and only if the Gauß map $N : M \longrightarrow S^2$ is harmonic. Kenmotsu [41] has shown how one can construct ϕ from N. In the method discussed below we will construct both N and ϕ from an "extended moving frame". To this end we write $S^2 = SU(2)/U(1)$, where $U(1)$ is represented as the subgroup $U(1) \cong \{\, \mathrm{diag}(\alpha, \alpha^{-1}) \mid |\alpha| = 1\}$ of diagonal matrices in $SU(2)$. Then there exists some frame $F : \mathbb{D} \longrightarrow SU(2)$, such that the diagram

$$
\begin{tikzcd}
 & SU(2) \arrow[d] \\
\mathbb{D} \arrow[ur] \arrow[dr] & \\
 & S^2
\end{tikzcd}
\tag{2.1.1}
$$

commutes, i.e. $N \equiv F \bmod U(1)$.

One can characterize the harmonicity of N by differential equations for the Maurer-Cartan form of F. More precisely, let $\mathfrak{g} = \mathfrak{p} + \mathfrak{k}$ denote the Cartan decomposition of $\mathfrak{g} = su(2, \mathbb{C}) = \mathrm{Lie}(SU(2))$ associated with the symmetric space $S^2 \cong SU(2)/U(1)$. In particular, $\mathfrak{k} = \mathrm{Lie}(U(1))$. Then the Maurer-Cartan form

$$\alpha = F^{-1}dF \tag{2.1.2}$$

of F decomposes as $\alpha = \alpha_{\mathfrak{p}} + \alpha_{\mathfrak{k}}$. Decomposing $\alpha_{\mathfrak{p}}$ further into the (1,0)-part $\alpha'_{\mathfrak{p}}$ and the (0,1)-part $\alpha''_{\mathfrak{p}}$, where $\alpha_{\mathfrak{p}} = \alpha'_{\mathfrak{p}} + \alpha''_{\mathfrak{p}}$, one introduces a "loop" parameter $\lambda \in S^1$ as in [56] by

$$\alpha_\lambda = \lambda^{-1}\alpha'_{\mathfrak{p}} + \alpha_{\mathfrak{k}} + \lambda\alpha''_{\mathfrak{p}}. \tag{2.1.3}$$

For later reference we state the explicit form of α_λ as it is used in this article (see e.g. the Appendix of [17]): Let z be a conformal coordinate and denote by $ds^2 = e^u dz d\bar{z}$ the metric, by H the mean curvature and by $Q(z)dz^2$ the Hopf differential, so that α_λ is given by $\alpha_\lambda =$

$$\begin{pmatrix} \frac{1}{4}u_z & -\frac{1}{2}\lambda^{-1}He^{\frac{u}{2}} \\ \lambda^{-1}Qe^{-\frac{u}{2}} & -\frac{1}{4}u_z \end{pmatrix} dz + \begin{pmatrix} -\frac{1}{4}u_{\bar{z}} & -\lambda\bar{Q}e^{-\frac{u}{2}} \\ \frac{1}{2}\lambda He^{\frac{u}{2}} & \frac{1}{4}u_{\bar{z}} \end{pmatrix} d\bar{z}. \tag{2.1.4}$$

Then one has the easy but crucial

Theorem 2.1.1. *([26]) The map $N : \mathbb{D} \longrightarrow S^2$ is harmonic if and only if α is integrable for all $\lambda \in S^1$, i.e.*

$$d\alpha_\lambda + \frac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = 0 \quad \textit{for all } \lambda \in S^1. \tag{2.1.5}$$

It is well known that the integrability of α is equivalent to the existence of some map $\hat{F} : \mathbb{D} \times S^1 \longrightarrow SU(2)$ such that $\hat{F}^{-1}d\hat{F} = \alpha_\lambda$

and $\hat{F}(z,1) = F(z)$. *Such a map $\hat{F}$ is called an extended framing for N. From now on we will simply write F for the extended framing.*

As an immediate consequence of the theorem above we obtain

Corollary 2.1.2. *Let F be the extended framing associated with some harmonic map $N : \mathbb{D} \longrightarrow S^2$. Then the map, denoted by N again, $N : \mathbb{D} \times S^1 \longrightarrow S^2$, defined by $N \equiv F \operatorname{mod} U(1)$, forms an S^1-family of harmonic maps.*

2.2 At this point one needs to observe that by the introduction of the loop parameter λ the extended framing $F(\cdot, \lambda)$ is, for fixed λ, actually a framing for the harmonic map $N(\cdot, \lambda)$. From a more technical point of view, F can be considered as a map from $\mathbb{D}$ into the group of loops in $Sl(2,\mathbb{C})$, i.e. the "loop group" $\Lambda Sl(2,\mathbb{C})$. Actually, from the definition of α_λ we see that odd powers of λ have coefficient matrices in $\mathfrak{p}$ and even powers of λ have coefficient matrices in $\mathfrak{k}$ (for the Lie algebra generated by $\alpha_\lambda(z,\bar{z}), z \in \mathbb{D}$). Therefore, α_λ is an element of the *twisted loop algebra*

$$(2.2.1) \qquad \Lambda sl(2,\mathbb{C})_\sigma = \{\beta(\lambda) \in sl(2,\mathbb{C}) \mid \beta(-\lambda) = \sigma_3 \beta(\lambda) \sigma_3\},$$

where $\sigma_3 = \operatorname{diag}(1,-1)$ denotes the third Pauli matrix. And F is in the twisted loop group

$$(2.2.2) \qquad \Lambda Sl(2,\mathbb{C})_\sigma = \{g(\lambda) \in Sl(2,\mathbb{C}) \mid g(-\lambda) = \sigma_3 g(\lambda) \sigma_3\}.$$

In this case the twisting amounts to having even functions of λ on the diagonal and odd functions of λ off the diagonal. Since topological questions are not completely irrelevant for the splitting theorems mentioned below, we will tacitly assume that all the functions of λ are elements of the *Wiener algebra*

$$(2.2.3) \qquad \mathcal{A} = \{f(\lambda) = \sum_{n \in \mathbb{Z}} f_n \lambda^n \mid \sum_{n \in \mathbb{Z}} | f_n | < \infty\}.$$

To get the loop group technicalities out of the way we note that for our purposes there are three subgroups of special importance:

$$(2.2.4) \qquad \Lambda^+ Sl(2,\mathbb{C})_\sigma = \{f(\lambda) = \sum_{0 \le n < \infty} f_n \lambda^n \in \Lambda Sl(2,\mathbb{C})_\sigma\}$$

$$(2.2.5) \qquad \Lambda^- Sl(2,\mathbb{C})_\sigma = \{f(\lambda) = \sum_{-\infty < n \le 0} f_n \lambda^n \in \Lambda Sl(2,\mathbb{C})_\sigma\}$$

$$(2.2.6) \quad \Lambda SU(2)_\sigma = \{f \in \Lambda Sl(2,\mathbb{C})_\sigma \mid f(\lambda) \in SU(2) \ \forall \lambda \in S^1\}.$$

Since we have assumed that all coefficient functions are in the Wiener algebra, all the groups mentioned so far are Banach Lie groups (and Banach subgroups, where applicable). The corresponding Banach Lie algebras will be denoted by lower case letters, as in $\Lambda sl(2,\mathbb{C})_\sigma$. Crucial ingredients for the generalized Weierstraß representation are the following two group splitting results.

To ensure uniqueness in the first of these results we need to use the group $\Lambda_*^- Sl(2,\mathbb{C})_\sigma$, which is defined by the requirement that the coefficient of λ^0 is the identity matrix I.

Theorem 2.2.1. *(Birkhoff Splitting)*

$$\Lambda Sl(2,\mathbb{C})_\sigma = \bigcup_{n=0}^{\infty} \Lambda^- Sl(2,\mathbb{C})_\sigma \cdot \mathrm{diag}(\lambda^n, \lambda^{-n}) \cdot \Lambda^+ Sl(2,\mathbb{C})_\sigma \tag{2.2.7}$$

Moreover, the group multiplication map

$$\Lambda_*^- Sl(2,\mathbb{C})_\sigma \times \Lambda^+ Sl(2,\mathbb{C})_\sigma \longrightarrow \Lambda_*^- Sl(2,\mathbb{C})_\sigma \cdot \Lambda^+ Sl(2,\mathbb{C})_\sigma \tag{2.2.8}$$

is a complex analytic diffeomorphism onto an open and dense subset of $\Lambda Sl(2,\mathbb{C})_\sigma$. *For + and − interchanged the analogous statements hold.*

For a proof of this theorem see [47],[26],[16]. Note that if one ignores the subscript $*$ above, then, in the case $n = 0$, the two representations $a_- \cdot b_+ = \hat{a}_- \cdot \hat{b}_+$ are equivalent to $a_- = \hat{a}_- \cdot c, b_+ = c^{-1} \cdot \hat{b}_+$, where c does not depend on λ, i.e. c is a λ-independent diagonal matrix.

For the second basic splitting result we need the group $\Lambda_P^+ Sl(2,\mathbb{C})_\sigma$, which is defined by the requirement that the (diagonal) coefficient matrix of λ^0 has only real, non-negative entries.

Theorem 2.2.2. *(Iwasawa Splitting)*

$$\Lambda Sl(2,\mathbb{C})_\sigma = \Lambda SU(2)_\sigma \cdot \Lambda^+ Sl(2,\mathbb{C})_\sigma \tag{2.2.9}$$

Moreover, the group multiplication map

$$\Lambda SU(2)_\sigma \times \Lambda_P^+ Sl(2,\mathbb{C})_\sigma \longrightarrow \Lambda Sl(2,\mathbb{C})_\sigma \tag{2.2.10}$$

is a real analytic diffeomorphism. For + replaced by − the analogous statement holds.

For a proof of this theorem see [47],[26],[16]. If one ignores the restriction "P" above, then two representations $a \cdot b_+ = \hat{a} \cdot \hat{b}_+$ are equivalent to $a = \hat{a} \cdot k$ and $\hat{b}_+ = k^{-1} \cdot b_+$, where k does not depend on λ and is unitary, i.e. $k = \mathrm{diag}(e^{it}, e^{-it})$, for $t \in \mathbb{R}$.

Remark 2.2.3. 1. Usually we call any decomposition of the form $g = u \cdot v_+$ with $u \in \Lambda SU(2)_\sigma$ and $v_+ \in \Lambda^+ Sl(2, \mathbb{C})_\sigma$ an Iwasawa splitting of g. The freedom just explained will then be referred to as the *freedom in the Iwasawa splitting.*

2. We would like to point out that the prototype of most splitting theorems used at present in geometric contexts seems to be Theorem 4.1 of [3]. Notwithstanding, of course, Birkhoff's original contribution.

3. The two theorems above work so well in the generalized Weierstraß representation because they are in some sense "complementary", namely, a relation of the form $V_- = F \cdot V_+$ can be interpreted as an (almost) 1-1 correspondence between V_- and F :

a) Given V_-, one obtains F (almost uniquely) via the Iwasawa splitting.

b) Given F, one obtains V_- (almost uniquely) via the Birkhoff splitting.

4. The splitting theorems have been stated for the full (twisted) loop group over the Wiener algebra. It will be important for applications, like the ones involving dressing (see section 5), to have these theorems in this generality. However, the actually geometrically relevant quantities are all holomorphic in $\lambda \in \mathbb{C}^*$ and are thus contained in a much smaller loop group. The extended frames, for example, are defined and holomorphic for $\lambda \in \mathbb{C}^*$, since α_λ is holomorphic in $\lambda \in \mathbb{C}^*$

2.3 Returning to the geometry under consideration we want to replace the extended frame F, whose Maurer-Cartan form α_λ satisfies some nonlinear differential equation, by some holomorphic extended frame $C = C(z, \lambda)$, which contains the same information as F. However, while the integrability condition for α_λ is nonlinear and nontrivial, the integrability condition for the Maurer-Cartan form of C will be trivial !

To achieve this, we will introduce a gauge $V_+ : \mathbb{D} \longrightarrow \Lambda^+ Sl(2, \mathbb{C})_\sigma$ such that

$$C = F \cdot V_+ \tag{2.3.1}$$

is holomorphic in z and in λ. For our setting it is also useful to require

$$C(z_0, \lambda) = I, \tag{2.3.2}$$

where z_0 is a base point chosen once and for all.

It is straightforward to see that C is holomorphic in z if and only if the $(0,1)$-part of the Maurer-Cartan form of $F \cdot V_+$ vanishes. This is equivalent to

$$\partial_{\bar{z}} V_+ = -(\alpha_{\mathfrak{k}}^{''} + \lambda \cdot \alpha_{\mathfrak{p}}^{''}) \cdot V_+, \tag{2.3.3}$$

where $\alpha_{\mathfrak{k}}^{''}$ and $\alpha_{\mathfrak{p}}^{''}$ denote the $(0,1)-$ parts of $\alpha_{\mathfrak{k}}$ and $\alpha_{\mathfrak{p}}$ respectively. As shown in the Appendix of [26], this equation admits a global solution $V_+ : \mathbb{D} \longrightarrow \Lambda^+ Sl(2,\mathbb{C})_\sigma$. Moreover, we can assume that V_+ is holomorphic in $\lambda \in \mathbb{C}^*$ and also that $C(z_0, \lambda) = I$. This result uses the local solvability of the $\bar{\partial}-$problem and the fact that every holomorphic cocycle on $\mathbb{D}$ is a boundary cycle [7]. Note that the holomorphic extended frame C in 2.3.1 is not uniquely determined. Any holomorphic gauge $W_+ : \mathbb{D} \longrightarrow \Lambda^+ Sl(2,\mathbb{C})_\sigma$ satisfying $W_+(z_0,\lambda) = I$ produces another holomorphic extended frame

$$\tilde{C} = C \cdot W_+ = F \cdot (V_+ \cdot W_+). \tag{2.3.4}$$

We will see later (e.g. section 2.4) that for certain purposes one or the other holomorphic extended frame is preferable.

2.4 It certainly seems to be useful to look for some holomorphic extended frame C which is uniquely determined by F and which, in turn, determines F essentially uniquely. Such a curve would be F_-, if F splits as $F = F_- \cdot F_+$ for all $z \in \mathbb{D}$. In general, this is not possible. However, one can prove:

Theorem 2.4.1. *([26]) If F is an extended framing of a harmonic map $N : \mathbb{D} \longrightarrow S^2$, then there exists some discrete set $S \subset \mathbb{D}$ such that the extended framing F can be split in the form*

$$F = F_- \cdot F_+ \quad for \ all \ \ z \in \mathbb{D} \setminus S \tag{2.4.1}$$

with $F_- \in \Lambda_*^- Sl(2,\mathbb{C})_\sigma$. Then F_- is uniquely determined by F and, considered as a function on all of $\mathbb{D}$, meromorphic. Moreover, we have $F_-(z_0,\lambda) = I$.

Remark 2.4.2. 1. Since $F(z_0,\lambda) = I$, we know that F_- is not singular at $z = z_0$.

2. To verify that F_- is meromorphic, one observes first that the Maurer-Cartan form ξ of F_- is integrable and that by 2.4.1 it is obtained from the Maurer-Cartan form of F by some gauge transformation which does not contain any negative powers of λ. It follows from this that ξ

is a $(1,0)-$form. More precisely, it is obtained from $\alpha'_{\mathfrak{p}}$ by conjugation with some matrix independent of λ. Therefore the $\bar{z}-$derivative of F_- vanishes and F_- is holomorphic where defined.

3. It is easy to see that the argument just given cannot be applied to F_+.

The Maurer-Cartan form

$$\xi = F_-^{-1} dF_- \tag{2.4.2}$$

is called the *normalized potential* for N (or for F). This ξ is uniquely determined by the harmonic map N. In contrast, the Maurer-Cartan forms associated with the holomorphic extended frames C associated with N are called *holomorphic potentials* for N. Holomorphic potentials are not uniquely determined by N.

Theorem 2.4.3. *([26]) a) The normalized potential ξ for some harmonic map $N : \mathbb{D} \longrightarrow S^2$ is of the form*

$$\xi = \lambda^{-1} \xi_{-1} dz, \tag{2.4.3}$$

where $\xi_{-1} : \mathbb{D} \longrightarrow sl(2,\mathbb{C})$ is meromorphic.

b) The holomorphic potentials

$$\eta = C^{-1} dC \tag{2.4.4}$$

are of the form

$$\eta = \lambda^{-1}\eta_{-1} + \lambda^0 \eta_0 + \lambda^1 \eta_1 + \dots \tag{2.4.5}$$

where all $\eta_j : \mathbb{D} \longrightarrow sl(2,\mathbb{C})$, $j = -1, 0, 1, \dots$ are holomorphic.

c) If η is a holomorphic potential for the harmonic map $N : \mathbb{D} \longrightarrow G/K$ and if $W_+ : \mathbb{D} \longrightarrow \Lambda^+ Sl(2,\mathbb{C})_\sigma$ is a holomorphic map satisfying $W_+(z_0, \lambda) = I$, then the gauged potential

$$\tilde{\eta} = W_+{}^{-1} \eta W_+ + W_+{}^{-1} dW_+ \tag{2.4.6}$$

is also a holomorphic potential for N.

Recall that the coefficient matrices at even powers of λ are diagonal and the coefficient matrices at odd powers of λ are off-diagonal.

Remark 2.4.4. 1. The coefficient matrix of normalized potentials and the coefficient matrix at λ^{-1} of holomorphic potentials is in all cases of the form $\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$. A comparison with 2.1.4 shows that the corresponding immersion of constant mean curvature is minimal if and only if $a = 0$, and it is totally umbilical if and only if $b = 0$. For the first case see section 3.5. The second case we have excluded from our discussion.

2. By the theorem above, every normalized potential has a meromorphic coefficient function. More generally, potentials for which the coefficient functions at all powers of λ are meromorphic will be called *meromorphic potentials* in this article. In particular, normalized potentials are meromorphic potentials. In many articles (including previous articles of the author) the notion of a meromorphic potential is used synonymously with what we call normalized potential. In this article the notion of a meromorphic potential will not be so restricted (see section 6.6).

§3. Constructing surfaces from potentials

3.1 In this section we will reverse the steps carried out in the previous section and construct constant mean curvature surfaces from potentials.

We start from some holomorphic potential η, i.e. some holomorphic $(1,0)$–form on $\mathbb{D}$ of the following type

$$\eta = \lambda^{-1}\eta_{-1} + \lambda^0\eta_0 + \lambda^1\eta_1 + \dots \tag{3.1.1}$$

where the coefficient matrices at even powers of λ are diagonal and the coefficient matrices at odd powers of λ are off-diagonal.

In view of 2.4.4, to pass from the level of potentials to the level of "holomorphic extended frames" we solve the *complex ordinary differential equation*

$$dC = C \cdot \eta, \quad C(z_0, \lambda) = I. \tag{3.1.2}$$

In view of 2.3.1, the transition from the holomorphic extended frame to the extended frame is established by the Iwasawa splitting

$$C = F \cdot V_+ \tag{3.1.3}$$

where we can assume $F(z_0, \lambda) = I$. Using Lemma 4.5 of [26] and Theorem 3.1 of [22] we have

Theorem 3.1.1. *Let η be a holomorphic potential and let F be constructed as above. If b_{-1} denotes the $(1,2)-$entry of η_{-1}, and if $b_{-1} \neq 0$, then $N \equiv F \operatorname{mod} U(1)$ is a harmonic map from $\mathbb{D} \setminus \hat{S}$ to S^2 with extended framing F, where $\hat{S} = \{z \in \mathbb{D} \mid b_{-1}(z) = 0\}$.*

The case $b_{-1} = 0$ will be treated in section 3.5.

If the mean curvature H of the associated immersion is different from 0, then the final step, constructing the surface of constant mean curvature associated with N, can be carried out via Kenmotsu's work [41]. But using the associated family, one can construct the immersion associated with N directly using the extended framing ([4], Theorem 1.2):

Theorem 3.1.2. *(Sym-Bobenko Formula) If F is the extended framing of some harmonic map $N : \mathbb{D} \longrightarrow S^2$, associated with an immersion ϕ of constant mean curvature $H \neq 0$, then*

$$\phi = -\frac{1}{2H}\{\partial_t F \cdot F^{-1} + F \cdot \frac{i}{2} \operatorname{diag}(1,-1) \cdot F^{-1}\}, \tag{3.1.4}$$

where $\lambda = \exp(it)$.

Here we have realized $\mathbb{R}^3$ as $su(2)$. A natural isomorphism of $\mathbb{R}^3$ with $su(2)$ is given by the spin representation (see e.g. the Appendix of [17]).

Remark 3.1.3. It is important to note that this formula yields the same immersion ϕ for the extended frames F and $F \cdot k$, where $k \in SU(2)$ is diagonal and independent of λ. In particular, the freedom in the Iwasawa splitting (see Remark 2.2.3) has *no* effect on the final immersion, if $H \neq 0$.

3.2 Starting, conversely, from some normalized potential ξ, one can proceed as above. However, due to possible monodromy in the solutions of the ODE involved, one may have to introduce "cuts" into $\mathbb{D}$. On the other hand, if one wants to construct immersions without singularities on $\mathbb{D}$, then some conditions need to be satisfied at the poles of ξ. First of all we note that smooth immersions on $\mathbb{D}$ have holomorphic Hopf differentials on $\mathbb{D}$. In addition, for the smoothness on all of $\mathbb{D}$ of the immersion produced from ξ, two more conditions need to be satisfied:

(3.2.1) The solution F_- to $dF_- = F_- \cdot \xi$, $F_-(z_0, \lambda) = I$ is meromorphic on $\mathbb{D}$

(3.2.2) The extended framing F obtained from F_- via $F_- = F \cdot V_+$ is smooth on $\mathbb{D}$.

Condition 3.2.1 is necessary by Theorem 2.4.1 and condition 3.2.2 is obviously necessary. For the case $H = 0$ see section 3.5. Assume now $H \neq 0$. Converting the matrix differential equation $dF_- = F_- \cdot \xi$, with

$$\xi = \lambda^{-1} \begin{pmatrix} 0 & f \\ Q/f & 0 \end{pmatrix} dz,$$

into a scalar second order ordinary differential equation, one obtains

$$y'' - \frac{f'}{f} y' - \lambda^{-2} Q y = 0. \tag{3.2.3}$$

One observes that only regular singular points occur. Thus for every point of $\mathbb{D}$ one of the two solutions, say y_1, can already be assumed to be meromorphic. For the second solution, y_2, to be meromorphic a simple residue condition needs to be satisfied

Theorem 3.2.1. *([17], Theorem 2.8.2) The equation*

$$dF_- = F_- \cdot \xi, \quad F_-(z_0, \lambda) = I$$

has a global meromorphic solution in $\Lambda^- Sl(2, \mathbb{C})_\sigma$ *if and only if*

$$\oint \frac{f}{{y_1}^2} dz = \mathrm{res}_p \frac{f}{{y_1}^2} = 0 \tag{3.2.4}$$

for any zero or pole p of f.

Condition 3.2.4 can be expressed in terms of the orders of the roots and poles of the coefficient functions of ξ.

Theorem 3.2.2. *([17]) Let*

$$\xi = \lambda^{-1} \begin{pmatrix} 0 & f \\ Q/f & 0 \end{pmatrix} dz, \tag{3.2.5}$$

be a normalized potential for which the solution to the differential equation $dC = C \cdot \xi$, $C(0, \lambda) = I$, is meromorphic. Then for ξ to yield a *constant mean curvature surface without branch points under the generalized Weierstraß representation it is necessary and sufficient that for every $z_0 \in \mathbb{D}$, where f has a pole or zero, the following conditions are satisfied: Let m denote the order of z_0 as a zero of the Hopf differential Q. If f has a pole of order n at z_0, then $n = 2$ or for some integer $r \geq 1$*

$$(3.2.6) \qquad n = r(2m+4) \qquad \text{or} \qquad n = r(2m+4)+2.$$

If f has a zero of order n at z_0, then for some integer $r \geq 1$

$$(3.2.7) \qquad n = r(2m+4) \qquad \text{or} \qquad n = r(2m+4)-2.$$

Remark 3.2.3. 1. The conditions for the smoothness of the immersion are expressed for holomorphic potentials and for normalized potentials in quite different ways. If a normalized potential ξ as above is actually holomorphic, then for the vanishing orders n and m of f and Q at some point $z_0 \in \mathbb{D}$ we have $n \neq m$. But then 3.2.7 does not make sense for positive n. Therefore only $n = 0$ is possible, if the associated immersion is supposed to have no branch points at z_0. In other words, if ξ is holomorphic, then the immersion does not have any branch points if and only if f never vanishes.

2. When considering non-umbilical points, one deals with the case $m = 0$. In this case the theorem simply states that n is even.

3. Constructing a constant mean curvature immersion ϕ from a holomorphic potential η or a normalized potential ξ is called a *generalized Weierstraß representation* of ϕ. The potentials η or ξ are called the Weierstraß data for ϕ.

4. The first, still fairly slow, computer implementation of the generalized Weierstraß representation was carried out by Lerner and Sterling [44]. Inspired by their work, Pinkall and Gunn improved the algorithm considerably, leading to a fast visualization method for surfaces of constant mean curvature. Recently, Schmitt has produced further improvements [43].

3.3 Examples 1. Consider the holomorphic potential

$$(3.3.1) \qquad \eta = \lambda^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} dz.$$

The associated family of surfaces corresponding to η contains (for $\lambda = 1$) the round cylinder. In this case the Weierstraß representation can be computed explicitly by hand.

2. Consider the holomorphic potential

$$\eta = \lambda^{-1} \begin{pmatrix} 0 & X \\ \bar{X} & 0 \end{pmatrix} dz, \tag{3.3.2}$$

where $X = a\lambda^{-1} + \lambda b, \quad a, b \in \mathbb{R}$. If $a = b = 0$, then the map ϕ defined in 3.1.4 is constant, and thus degenerate. If $a = 0, b \neq 0$, then the map ϕ describes a minimal surface [24]. If $a \neq 0, b = 0$, then the map ϕ describes (part of) a sphere. This follows from the fact that in this case the Hopf differential vanishes identically and thus every point of the surface is an umbilical point. Assume now $ab \neq 0$. Since the associated family does not change, if we replace λ by $-\lambda$, or by λ^{-1}, we can assume $a > 0$ and $a \geq b$. If $ab > 0$ and $a + b = \frac{1}{2}$, then ϕ describes the associated family of an unduloid (the unduloid is the surface obtained for $\lambda = 1$). In particular, this is a *Delaunay surface*, i.e. a surface of revolution of constant mean curvature. For more information about Delaunay surfaces we refer to [42],[29],[30].

3. Consider the potential

$$\eta = \lambda^{-1} \begin{pmatrix} 0 & 1 \\ z^n & 0 \end{pmatrix} dz \tag{3.3.3}$$

where n is a non-negative integer. The associated family of surfaces contains (for $\lambda = 1$) a *Smyth surface* [51]. Its metric is invariant under a one-parameter group of self-isometries. All such surfaces of constant mean curvature have been determined by Smyth [51]. The associated family of such a surface contains either a Delaunay surface or a Smyth surface.

4. Consider the potential

$$\xi = \begin{pmatrix} 0 & \lambda^{-1}\alpha + \lambda\beta \\ \lambda^{-1}\beta + \lambda\alpha & 0 \end{pmatrix} \tag{3.3.4}$$

where α and β are holomorphic 1-forms on $\mathbb{C}^*$ of the form

$$\alpha = -h\frac{dz}{z}, \tag{3.3.5}$$

and

$$\beta = (h + \sum_{n=1}^{\infty} b_n(z^{-n} + z^n))\frac{dz}{z}, \tag{3.3.6}$$

where $h, b_n \in \mathbb{R}$ and $h \neq 0$.

Theorem 3.3.1. *([42], Theorem 7.1.1) Let α and β be as above and assume that the coefficients of β satisfy*

$$\int_0^{2\pi} \cos\left(4\sum_{n=1}^{\infty} \frac{b_n}{n}\sin(nt)\right) dt = 0. \tag{3.3.7}$$

Then the potential ξ generates, for $\lambda = 1$, an immersion of a cylinder which has constant mean curvature and two planar symmetries. By choosing α and β appropriately one can obtain any number of umbilical points.

By different methods more examples, such as immersions of cylinders and tori, will be given in the following sections.

3.4 Wu's Formula In general, interest in the loop group method is concentrated on the construction of CMC surfaces with specific preassigned properties. Thus the goal is to choose potentials in order to obtain surfaces with such properties. For this purpose it is useful to compute the potentials for some known examples, and this can be achieved by a remarkably simple formula. The crucial observation behind this formula—as explained in more detail in section 7.7.6—is that the theory presented so far is in fact a "real form" of a complex theory, in which z and $\bar{z}$ become independent variables. A thorough investigation of this complex theory has not yet been carried out.

Theorem 3.4.1. *([58]) Let $\phi : \mathbb{D} \longrightarrow \mathbb{R}^3$ be an associated family of immersions of constant mean curvature $H \neq 0$. If $ds^2 = \zeta(z,\bar{z})dzd\bar{z}$ denotes the metric and $\lambda^{-2}Q(z)dz^2$ the Hopf differential, then the conformal factor ζ is the restriction of a meromorphic map $\hat{\zeta} : \mathbb{D} \times \bar{\mathbb{D}} \longrightarrow \mathbb{C}$ to the subset $\{(z,\bar{z}) \mid z \in \mathbb{D}\}$ of $\mathbb{D} \times \bar{\mathbb{D}}$, and the normalized potential, uniquely associated with ϕ, is of the form*

$$\xi = \lambda^{-1}\begin{pmatrix} 0 & f \\ Q/f & 0 \end{pmatrix} dz, \tag{3.4.1}$$

where

$$f(z) = \frac{\zeta(z,0)}{\sqrt{\zeta(0,0)}}. \tag{3.4.2}$$

3.5 Minimal Surfaces The concrete expression 2.1.4 for the Maurer-Cartan form of the extended frames shows:

a) A holomorphic potential $\eta = \lambda^{-1}\eta_{-1} + \lambda^0\eta_0 + \lambda^1\eta_1 + \dots$ corresponds to a minimal surface iff $\eta_{-1} = \begin{pmatrix} 0 & 0 \\ p & 0 \end{pmatrix} dz$,

b) A normalized potential $\xi = \lambda^{-1}\xi_{-1}$ corresponds to a minimal surface iff $\xi_{-1} = \begin{pmatrix} 0 & 0 \\ p & 0 \end{pmatrix} dz$.

Let ξ be the normalized potential of some minimal surface. Then

$$F_- = \begin{pmatrix} 1 & 0 \\ \lambda^{-1}\int_{z_0}^z p(w)dw & 1 \end{pmatrix}. \tag{3.5.1}$$

We put

$$q(z) = \int_{z_0}^z p(w)dw. \tag{3.5.2}$$

Because of the very simple form of F_- one can find an Iwasawa splitting explicitly. We have $F_- = F \cdot V_+$, where

$$F = (1+|q|^2)^{-1/2} \begin{pmatrix} 1 & -\lambda\bar{q} \\ \lambda^{-1}q & 1 \end{pmatrix}, \tag{3.5.3}$$

and

$$V_+ = (1+|q|^2)^{-1/2} \begin{pmatrix} 1+|q|^2 & \lambda\bar{q} \\ 0 & 1 \end{pmatrix}. \tag{3.5.4}$$

Interpreting F as the coordinate frame of some minimal surface we obtain (e.g. by comparing the Maurer-Cartan forms)

$$ds^2 = 4(1+|q|^2)^2 dz d\bar{z}, \tag{3.5.5}$$

$$Q = q_z = p. \tag{3.5.6}$$

In contrast with the case $H \neq 0$, the freedom in the Iwasawa splitting (see Remark 2.2.3) does affect the associated minimal immersions. Let F and $\tilde{F} = F \cdot k$ be coordinate frames obtained via Iwasawa splitting:

$$F_- = \tilde{F} \cdot \tilde{V}_+ = F \cdot V_+. \tag{3.5.7}$$

Then, setting $k = \mathrm{diag}(e^{i\alpha}, e^{-i\alpha})$, we obtain for the associated conformal factor and the Hopf differential

$$d\tilde{s}^2 = (1+|q|^2)^2 \cdot e^{Re(h)} \cdot dz d\bar{z} \tag{3.5.8}$$

(3.5.9) $$\tilde{Q} = Qe^{\bar{h}/2},$$

where h is an antiholomorphic map satisfying

(3.5.10) $$\mathrm{Im}(h) = -4\alpha.$$

Since F and $\tilde{F}$ yield the same Gauß map, and since the relation between Gauß maps and meromorphic potentials is one-to-one, the freedom in the Iwasawa splitting describes the family of minimal surfaces possessing the same Gauß map. The splitting 3.5.3 and 3.5.4 given above represents one natural choice of minimal surface associated with a fixed Gauß map.

Before making a precise connection with the classical Weierstraß representation we note that the normalized potential for minimal surfaces contains exactly one meromorphic function, which defines — via the freedom in the Iwasawa splitting — many minimal surfaces and one Gauß map, which is the same for all these minimal surfaces. The second function that appears in the classical Weierstraß representation enters as the function h above, representing the freedom in the Iwasawa splitting.

Let F be the extended coordinate framing of some minimal surface and denote the entries of F by a, b and $-\bar{b}, \bar{a}$ as usual. Then from 2.1.4 we derive, since $H = 0$,

(3.5.11) $$\bar{a}_z = -\frac{1}{4}u_z\bar{a},$$

(3.5.12) $$b_z = -\frac{1}{4}u_z b,$$

whence, since u is real,

(3.5.13) $$a = e^{-\frac{1}{4}u}s,$$

(3.5.14) $$b = \lambda e^{-\frac{1}{4}u}\bar{r},$$

where s and r are holomorphic functions. In view of 3.5.3 and the relation $F = \tilde{F} \cdot k$ for the general frame, we see that r and s are independent of λ. If $\phi : \mathbb{D} \longrightarrow \mathbb{R}^3$ is the minimal immersion which corresponds to F, then (A.5.6) and (A.6.3) of [17] show that

(3.5.15) $$G^{-1} \cdot J(\phi_z) \cdot G = -\frac{i}{2}\lambda^{-1}e^{u/2}F\sigma_+F^{-1},$$

where

$$G = i \begin{pmatrix} 0 & \lambda^{-1/2} \\ \lambda^{1/2} & 0 \end{pmatrix} \text{ and } \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{3.5.16}$$

while

$$J(x,y,z) = -\frac{i}{2} \begin{pmatrix} z & x-iy \\ x+iy & -z \end{pmatrix}. \tag{3.5.17}$$

As a consequence

$$\phi_z = (\frac{1}{2}(s^2 - \lambda^{-4}r^2), -\frac{i}{2}(s^2 + \lambda^{-4}r^2), -sr\lambda^{-2}). \tag{3.5.18}$$

If we give up the requirement that F is I at z_0, we can consider the frame

$$\hat{F} = DF, \quad \text{where } D = \operatorname{diag}(\sqrt{i}\lambda^{-1}, \sqrt{i}^{-1}\lambda). \tag{3.5.19}$$

Then for the corresponding minimal immersion $\hat{\phi}$ we obtain

$$\hat{\phi}_z = \lambda^{-2}(\frac{1}{2}(r^2 - s^2), \frac{i}{2}(s^2 + r^2), sr), \tag{3.5.20}$$

which is a version of the classical Weierstraß representation. One obtains the usual formula for $\mu = r^2$ and $\nu = s/r$. Note that the special splitting 3.5.3 corresponds to $s = 1$ and $r = -q$, equivalently $\mu = q^2$ and $\nu = -q^{-1}$. Note that ν corresponds to the stereographically projected Gauß map. For $q = -z^{-1}$ one obtains $\nu = z$ and $\mu = z^{-2}$, which describes the catenoid. More examples are contained in [25].

From now on we will assume, unless stated otherwise, that "constant mean curvature" means that H is constant and nonzero.

§4. Surfaces of finite type

4.1 The generalized Weierstraß representation constructs all constant mean curvature surfaces. However, as it is not easy to trace properties from the input for the Iwasawa splitting to its output, the procedure is difficult to use in most concrete cases. We will see in section 5 that geometric features can be controlled in some cases via dressing. Still, the construction of the associated surfaces is not very explicit, at least from a non-numerical point of view. It is therefore very fortunate that for a large and important class of constant mean curvature surfaces an independent geometric construction procedure is known, which, in addition, is relatively explicit. We follow in essence the exposition of [10],[12] and [8].

For $d \in 2\mathbb{N}+1$ we set

$$\xi_* = \Lambda_d = \{ \sum_{n=-d}^{d} \xi_{-n}\lambda^n \in \Lambda su(2)_\sigma, \xi_d \neq 0\}. \tag{4.1.1}$$

Then the basic results are Theorem 3.2 and Theorem 2.5 of [10], which we shall summarize. Note that if $\xi \in \Lambda_d$, then ξ_{d-1} is diagonal, since d is odd. Therefore $\xi_{d-1} \in \mathfrak{k}^{\mathbb{C}}$. Specializing the general procedure (see (2.5) in [10]) we set for any $\tau \in \mathfrak{k}^{\mathbb{C}}$

$$r(\tau) = \frac{1}{2}\tau \tag{4.1.2}$$

Using this notation we can state

Theorem 4.1.1. *([10]) For each $d \in 2\mathbb{N}+1$ and $\xi_* \in \Lambda_d$, there is a unique solution $\xi : \mathbb{R}^2 \longrightarrow \Lambda_d$, to the differential equation*

$$\frac{\partial \xi}{\partial z} = [\xi, \lambda^{-1}\xi_d + r(\xi_{d-1})], \quad \xi(z_0) = \xi_*. \tag{4.1.3}$$

Moreover, in this case, the $\Lambda su(2)_\sigma$-valued 1-form α given by

$$\alpha = (\lambda^{-1}\xi_d + r(\xi_{d-1}))dz + (\lambda\xi_{-d} + \overline{r(\xi_{d-1})})d\bar{z} \tag{4.1.4}$$

satisfies the Maurer-Cartan equations. In addition, the extended framing F defined by

$$dF = F \cdot \alpha, \qquad F(z_0, \lambda) = I \tag{4.1.5}$$

induces a harmonic map $N : \mathbb{D} \longrightarrow S^2$ and a surface of constant mean curvature $\phi : \mathbb{D} \longrightarrow \mathbb{R}^3$ via the Sym-Bobenko formula.

Constant mean curvature surfaces defined this way are said to be of *finite type*. A crucial observation is that from the theorem above one can construct naturally a very special holomorphic potential for surfaces of finite type (see Theorem 4.2.1 below).

4.2 Consider the potential

$$\eta = \lambda^{d-1}\xi_* dz, \tag{4.2.1}$$

where $\xi_* \in \Lambda_d$ is, as in the previous section, independent of z and $\bar{z}$. Then the holomorphic extended frame C defined by the differential equation 3.1.2 is of the form

$$C(z, \lambda) = e^{z\lambda^{d-1}\xi_*}. \tag{4.2.2}$$

Consider the Iwasawa splitting

$$C = \hat{F} \cdot \hat{V}_+. \tag{4.2.3}$$

The trivial, but important, consequence of this is that

$$\hat{F}^{-1} \cdot \xi_* \cdot \hat{F} \in \Lambda_d, \tag{4.2.4}$$

which is easy to verify, since C commutes with ξ_* and $\hat{\xi} = \hat{F}^{-1} \cdot \xi_* \cdot \hat{F}$ is skew-hermitian.

It is straightforward to verify that

$$\frac{\partial \hat{\xi}}{\partial z} = [\hat{\xi}, \hat{F}^{-1}\frac{\partial \hat{F}}{\partial z}], \quad where \;\; \hat{\xi}(0, \lambda) = \xi_*. \tag{4.2.5}$$

Since $\hat{\xi} \in \Lambda_d$, one can derive from 4.2.3 and 4.2.4 that $\hat{F}^{-1}\frac{\partial \hat{F}}{\partial z} = (\lambda^{-1}\hat{\xi}_d + r(\hat{\xi}_{d-1}))$ and thus has exactly the form of the $(1,0)$-part of α in 4.1.4. Since $\hat{F}^{-1}\frac{\partial \hat{F}}{\partial z}$ is skew-hermitian, $\hat{F}^{-1}d\hat{F}$ has the form 4.1.4. Therefore, the surface derived from a potential of the form 4.2.1 is of finite type. Conversely, if we start from a surface of finite type, then with the notation of Theorem 4.1.1, ξ satisfies the differential equation 4.1.3 with initial condition ξ_*. If the potential η in 4.2.1 is formed with the same ξ_*, then the argument above shows that $\hat{\xi}$ satisfies the same differential equation with the same initial condition as ξ. Therefore $\xi = \hat{\xi}$. From this it is easy to verify that the Maurer-Cartan forms of F and $\hat{F}$ also coincide, whence $F = \hat{F}$. Thus we obtain

Theorem 4.2.1. *([10]) If ξ satisfies 4.1.3 and ϕ is the constant mean curvature immersion of finite type defined from F, which is given by 4.1.4 and 4.1.5, then ϕ coincides with the constant mean curvature immersion induced from the constant potential $\eta = \lambda^{d-1}\xi_* dz$ via the generalized Weierstraß representation. Conversely, if $\xi_* \in \Lambda_d$ and $\eta = \lambda^{d-1}\xi_* dz$, then the associated constant mean curvature immersion is of finite type.*

Remark 4.2.2. 1. We would like to emphasize that equation 4.2.4 implies that ξ only involves finitely many powers of λ.

2. Surfaces of finite type have been investigated by many authors. We have followed closely the work of Burstall and Pedit [10],[12]. A starting point for many investigations has been the work of Pinkall and Sterling [46].

3. A beautiful and also very explicit description of all constant mean curvature surfaces of finite type has been given by algebro-geometric methods. This includes a discussion of periodicity conditions. The connection with ξ_* is given by the algebraic curve $\det(\xi_*(\lambda) - \mu I) = 0$. For details see [4],[5].

Example 4.2.3. (1) $d = 1$. *In this case the potentials have the form* $\eta = \xi_* = (\lambda^{-1}A_{-1} + A_0 + \lambda A_1)dz$. *From 3.3.2 one sees that the potentials of the Delaunay surfaces are of this type.*

(2) $d = 3$. *In this case a special choice of* ξ_* *leads to the Wente torus.*

4.3 In the last section we have obtained a very explicit and simple definition of surfaces of finite type in terms of certain holomorphic potentials. Since the Hopf differential can be obtained as the determinant of the coefficient matrix at λ^{-1} in any potential of the surface, it is clear that surfaces of finite type do not have any umbilics. In particular, compact constant mean curvature surfaces of genus > 1 are not of finite type. On the other hand, not every surface without umbilics is of finite type. As an example we mention the potentials

$$\eta = \lambda^{-1}\begin{pmatrix} 0 & 1 \\ c & 0 \end{pmatrix}$$

which do not yield surfaces of finite type, if $c \in \mathbb{R}$, $c \neq 0, \pm 1$ [44]. Despite the somewhat obscure definition of "finite type", we know

Theorem 4.3.1. *([46]) Every constant mean curvature surface with doubly periodic metric is of finite type. In particular, every constant mean curvature torus is of finite type.*

For immersions of cylinders the situation is quite different: some constant mean curvature cylinders are of finite type, while others have umbilics [22],[42] and are therefore not of finite type.

Remark 4.3.2. The notion of "finite type" as spelled out in Theorem 4.1.1 is quite roundabout. One of its consequences is that $F \cdot \xi_* \cdot F^{-1}$ is polynomial in λ (see the remark above). It would seem to be more natural to define "finite type" by this condition, namely that $F \cdot \xi_* \cdot F^{-1}$

is polynomial in λ, without assuming the reality of ξ_* [36]. It would be interesting to understand when this condition implies "finite type" in the sense of this article.

§5. Dressing

5.1 Dressing is a deformation technique which, applied to constant mean curvature surfaces, produces new surfaces from old ones. Since its invention by Zakharov and Shabat [59], dressing methods have been applied in many situations. For example, constructing solutions to the KdV equation via flows on infinite dimensional Grassmannians is in essence dressing, producing nontrivial solutions from the trivial one. For the situation discussed in this paper the basic result is

Theorem 5.1.1. *Let F be the extended frame of some constant mean curvature surface. Let $h_+ \in \Lambda^+ Sl(2,\mathbb{C})_\sigma$ and consider the Iwasawa splitting*

$$h_+ \cdot F = \hat{F} \cdot \hat{V}_+. \tag{5.1.1}$$

Then $\hat{F}$ is the extended frame of some constant mean curvature surface.

In 5.1.1 we can assume that the coefficient matrix at λ^0 of h_+ has only non-negative entries. Therefore we can assume that h_+ is in the identity component of $\Lambda^+ Sl(2,\mathbb{C})_\sigma$. Hence we can connect h_+ with I by a continuous curve h_+^s, $0 \le s \le 1$, $h_+^0 = I$, $h_+^1 = h_+$. Then the family of immersions ϕ^s associated with $\hat{F}^s$, ${h_+}^s \cdot F = \hat{F}^s \cdot \hat{V}^s$, represents a continuous deformation within the class of constant mean curvature immersions, starting from $\phi = \phi^0$ and ending at $\hat{\phi} = \phi^1$.

Remark 5.1.2. This already shows that dressing as defined above is an important procedure. It turns out, however, that for the most interesting applications the theorem above needs to be generalized.

For this one observes that the Maurer-Cartan form of F depends on λ in a holomorphic fashion, with λ varying in $\mathbb{C}^*$. Therefore, the solution F to the equation $dF = F \cdot \alpha$, $F(0,\lambda) = I$, also depends holomorphically on $\lambda \in \mathbb{C}^*$. This leads to the introduction of the group $\Lambda_r Sl(2,\mathbb{C})_\sigma$ of matrices defined on the circle $|\lambda| = r$, for $0 < r \le 1$. We also say $h_+ \in \Lambda_r^+ Sl(2,\mathbb{C})_\sigma$ if $h_+ \in \Lambda_r Sl(2,\mathbb{C})_\sigma$ is holomorphic in the interior of the disk of radius r. Note that for $h_+ \in \Lambda_r^+ Sl(2,\mathbb{C})_\sigma$ the expression $h_+ \cdot F$ does make sense on the circle $|\lambda| = r$. In this situation we have the following

Theorem 5.1.3. *(r-Iwasawa Splitting) Let $h \in \Lambda_r Sl(2,\mathbb{C})_\sigma$. Then there exist $U, V_+ \in \Lambda_r Sl(2,\mathbb{C})_\sigma$ such that $h = U \cdot V_+$ and*

(5.1.2) $V_+ \in \Lambda_r^+ Sl(2,\mathbb{C})_\sigma$

(5.1.3) *U extends holomorphically to $A_r = \{\lambda \in \mathbb{C} \mid r < |\lambda| < 1/r\}$*

(5.1.4) *U is unitary when restricted to the unit circle.*

For a proof of this theorem, see [45]. The result follows easily from [3]. The group of matrices in $\Lambda_r Sl(2,\mathbb{C})_\sigma$ satisfying the last two conditions will be denoted by $\Lambda_r SU(2)_\sigma$.

Corollary 5.1.4. *Theorem 5.1.1 remains true if h is chosen in $\Lambda_r^+ Sl(2,\mathbb{C})_\sigma$.*

If $\hat{F}$ is defined as in Theorem 5.1.1 by some $h \in \Lambda_r^+ Sl(2,\mathbb{C})_\sigma$ we will say that *the frame $\hat{F}$ and the corresponding immersion $\hat{\phi}$ is obtained by r-dressing from F and ϕ respectively.* If $r = 1$, then we will generally use "dressing" instead of "1-dressing".

Since a group splitting is involved in the r-dressing procedure, computations by hand are possible only in very special cases. For example one can prove

Theorem 5.1.5. *([17], section 3.3) a) If $\xi = \lambda^{-1} \begin{pmatrix} 0 & f \\ Q/f & 0 \end{pmatrix} dz$ is the normalized potential of some constant mean curvature immersion ϕ, then dressing with $h_+ \in \Lambda^+ Sl(2,\mathbb{C})_\sigma$ yields a constant mean curvature immersion $\hat{\phi}$ with normalized potential $\hat{\xi} = \lambda^{-1} \begin{pmatrix} 0 & \hat{f} \\ Q/\hat{f} & 0 \end{pmatrix} dz$.*

b) If $h_+ = \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix} dz$, then $\hat{f} = s^2 \cdot f$.

c) If $h_+ = \begin{pmatrix} 1 & 0 \\ t\lambda & 1 \end{pmatrix} dz$, then $\hat{f} = (1 + tb_1)^{-2} \cdot f$.

d) If $h_+ = \begin{pmatrix} 1 & t\lambda \\ 0 & 1 \end{pmatrix} dz$, then $\hat{f} = (1 + tc_1)^2 \cdot f$.

Here $b_1(z) = \int_{z_0}^{z} f(w)dw$ and $c_1(z) = \int_{z_0}^{z} \frac{Q(w)}{f(w)}dw$.

Clearly, Q is invariant under the dressing operation. The transformation $f \longmapsto \hat{f}$ is quite explicit for the three simple dressing matrices listed above. Moreover, these three matrices generate $\Lambda_r^+ Sl(2,\mathbb{C})_\sigma$ as a

Banach Lie group. Nevertheless, since $f \longmapsto \hat{f}$ is non-linear and involves integration, it is difficult to find an explicit formula for more general h_+. It would be helpful to understand the dressing transformation $f \longrightarrow \hat{f}$ analytically in a way which reflects the group structure of dressing.

From a geometric point of view, work of Inoguchi-Toda and also Mahler seems to show that the Bäcklund transformations defined by Bianchi (see e.g. [52]) can be obtained by special dressing matrices (different from the ones listed above). One hopes that eventually this will lead to a better understanding of more general dressing matrices.

Another geometric result related to dressing is

Theorem 5.1.6. *([22], Theorem 5.3) If $\phi : \mathbb{D} \longrightarrow \mathbb{R}^3$ is a constant mean curvature surface, which induces a complete metric on $\mathbb{D}$, then all surfaces in the dressing orbit of ϕ are complete.*

5.2 In general, the dressing orbits of the group $\Lambda_r^+ Sl(2,\mathbb{C})_\sigma$ are quite large:

Theorem 5.2.1. *a) ([21]) If the surface ϕ_0 has umbilics, then the isotropy group under the dressing action is trivial.*

b) If the surface ϕ_0 is generated by the constant holomorphic potential η, then the isotropy group $\mathrm{Iso}(\phi_0)$ *under the dressing action is*

$$\mathrm{Iso}(\phi_0) = \{g \in \Lambda_r^+ Sl(2,\mathbb{C})_\sigma \mid [g, \eta_{-1}] = 0\}. \tag{5.2.1}$$

Proof. Only b) needs some argument. Assume $h_+ \cdot F_0 = F_0 \cdot V_+$, then for the holomorphic extended frame C_0 associated with η we have $h_+ \cdot C_0 = C_0 \cdot W_+$. Since $C_0 = \exp(z\eta)$, this is equivalent to the formula $\exp(z \cdot ad(\eta)) \cdot h_+ = W_+$, which, in turn, is equivalent to the condition

$$(ad(\eta))^m h_+ \text{ only contains } \lambda^k \text{ for } k \geq 0.$$

Writing $\eta = \lambda^{-1}\eta_{-1} + \eta_+$, it is easy to see that this statement is equivalent to the condition

$$(ad(\lambda^{-1} \cdot \eta_{-1}))^m h_+ \text{ only contains } \lambda^k \text{ for } k \geq 0.$$

Recall that both coefficients of the off-diagonal matrix η_{-1} do not vanish, as $H \neq 0$, since we do not consider totally umbilical surfaces. Now the claim follows by a straightforward induction. □

Parts a) and b) of the theorem above do not cover the case of a general, never vanishing Hopf differential. It would be interesting to investigate this case.

5.3 Dressing and r-dressing can be defined not only on the level of frames, but also on the level of holomorphic extended frames:

$$h_+ \cdot C = \hat{C} \cdot \hat{W}_+. \tag{5.3.1}$$

Here we require that C and $\hat{C}$ are defined for $|\lambda| = 1$.

There is considerable freedom in the choice of h_+. Given $\hat{C}$ as above, one can also choose $\tilde{C} = \hat{C} \cdot U_+$, where U_+ is holomorphic in z and holomorphic in λ in the disk of radius 1. On the other hand, one can consider $\hat{C} = h_+ \cdot C \cdot {h_+}^{-1}$. If η is the Maurer-Cartan form of C, then the Maurer-Cartan form of $\hat{C}$ is $h_+ \cdot \eta \cdot {h_+}^{-1}$. If this is defined on the unit circle, then we would call this latter one-form a potential dressed from η.

Let C, $\hat{C}$ and η, $\hat{\eta}$ be as above. Then one obtains for the Maurer-Cartan forms η and $\hat{\eta}$ and W_+ the equation

$$d\hat{W}_+ = \hat{W}_+ \cdot \eta - \hat{\eta} \cdot \hat{W}_+ \tag{5.3.2}$$

Thus, in order to relate η and $\hat{\eta}$, one needs only to solve a linear ODE. However, we are interested in solutions involving only non-negative powers of λ, while η and $\hat{\eta}$ also involve some negative powers of λ. Thus the existence of such a solution $\hat{W}_+$ is a delicate and difficult problem.

The existence of a solution to 5.3.2 implies that C and $\hat{C} \cdot \hat{W}_+$ solve the same linear ODE, whence $A \cdot C = \hat{C} \cdot \hat{W}_+$ with some $A(\lambda)$ independent of z. An evaluation at $z = z_0$ yields $A(\lambda) = \hat{W}_+(z_0, \lambda)$, and hence a dressing equation 5.3.1. Thus dressing with some h_+ is equivalent to solving the *gauge equation* 5.3.2 with some $\hat{W}_+$ (not necessarily normalized as to satisfy $W_+(z_0, \lambda) = I$).

One could perhaps hope that all potentials with the same Hopf differential $Q(z)dz^2$ can be dressed onto each other. This is not true. It turns out that it is not even possible in general to solve the gauge equation 5.3.2 with a formal power series $\hat{W}_+ = \sum_{n=0}^{\infty} \hat{W}_n \lambda^n$. The formal solvability has been characterized by Wu [57]. (There are two misprints in [57]: in 5.12 there should be a + sign between p'/p and q'/q and b in 5.14 should be a q.) It would be desirable to find an analytic connection between "convergent solutions" and "formal solutions".

5.4 There have been several applications of the dressing transformation. One of the first results in the context of constant mean curvature surfaces related round cylinders to constant mean curvature tori:

Theorem 5.4.1. *([27], 5.3) Let ϕ be an associated family of constant mean curvature surfaces, which contains a torus. Then there exists some $0 < r < 1$ such that ϕ can be obtained by r-dressing from the associated family ϕ_0 of the round cylinder.*

This result was generalized by Burstall and Pedit [12]. Note that for the description of tori $r = 1$ does not work. This stems from the fact that the points defining the hyperelliptic curve, which is used to describe constant mean curvature tori in the algebro-geometric approach, lie between the r-circle and the $1/r$-circle. To make this more precise we quote two results from [19], which refer to 1-dressing.

Theorem 5.4.2. *Other than round cylinders there are no surfaces with translational symmetry in the dressing orbit of the round cylinder.*

Corollary 5.4.3. *The 1-dressing action does not act transitively on the set of all constant mean curvature surfaces without umbilics.*

The following result is in the same spirit and context as the results above, but admits r-dressing.

Theorem 5.4.4. *([20], Theorem 4.9) Every constant mean curvature surface, for which the metric is invariant under a discrete group of translations, and which is in the r-dressing orbit of the round cylinder, is of finite type.*

5.5 Recently, dressing was used to construct surfaces which are not of finite type and have nontrivial topology:

Theorem 5.5.1. *([22], Theorem 5.4 and Theorem 5.5) Consider the holomorphic potential*

$$\eta = \frac{1}{2}(\lambda + \lambda^{-1}) \begin{pmatrix} 0 & A \\ B\cos(2\pi z) & 0 \end{pmatrix} dz, \tag{5.5.1}$$

with associated constant mean curvature immersion ϕ_0. Then for all sufficiently small $A, B \neq 0$ there exists some $h_+ \in \Lambda^+ Sl(2,\mathbb{C})_\sigma$ such that the constant mean curvature immersions ϕ obtained from ϕ_0 by dressing with h_+ are complete and for $\lambda = \pm i$ one obtains constant mean curvature immersions of a cylinder with umbilics at $z = 1/4$ and $z = 3/4$.

Starting from [22], Kilian [42] has found many more cylinders with umbilics. See Example 4 in section 3.3.

5.6 We have seen in section 3.5 that also for minimal surfaces there is a 1-1-relation between normalized potentials and Gauß maps. However, there are many minimal surfaces with the same Gauß map. And these surfaces are all generated from the same potential via the freedom in the Iwasawa splitting (which becomes irrelevant in the case of mean curvature $H \neq 0$). Thus the primary interest is to understand how dressing affects the Gauß map. This is equivalent to asking how dressing affects the normalized potential of a minimal surface. Let $\xi = \lambda^{-1} \begin{pmatrix} 0 & 0 \\ p & 0 \end{pmatrix} dz$ be the normalized potential of a minimal surface, and

$$C_- = \begin{pmatrix} 1 & 0 \\ \lambda^{-1} q & 1 \end{pmatrix}$$

its associated holomorphic extended frame, where $q = \int_{z_0}^{z} p(w)dw$. Then the dressing equation $h_+ \cdot C_- = \hat{C}_- \cdot \hat{V}_+$ yields after a straightforward computation that $\hat{C}_-$ is of the form

$$\hat{C}_- = \begin{pmatrix} 1 & 0 \\ \lambda^{-1} \hat{q} & 1 \end{pmatrix},$$

where $\hat{q}$ is given by the explicit linear fractional formula $\hat{q} = q/(a + bq)$, for certain complex numbers a and b. In particular, dressing preserves the class of minimal surfaces and the dressing orbit has complex dimension 2. Recall from section 3.5 that q describes the Gauß map of the associated surfaces. Thus $\hat{q}$ belongs to a surface with a Gauß map which is different from the one of the undressed surfaces. It was, of course, classically known that this fractional linear formula transforms Gauß maps of minimal surfaces to Gauß maps of minimal surfaces [25],[56],[3].

§6. Symmetries

6.1 In the previous sections we have discussed the main features of the generalized Weierstraß representation of surfaces. For the construction of specific examples fundamental groups need to be incorporated as well. First we present the general framework [18],[19].

Let $\phi : \mathbb{D} \longrightarrow \mathbb{R}^3$ be an immersion of constant mean curvature, which induces a complete metric on $\mathbb{D}$. As always in this article $\mathbb{D}$

denotes an open and simply connected subset of $\mathbb{C}$. Assume now that R is an orientation-preserving rigid motion of $\mathbb{R}^3$ such that

$$R\phi(\mathbb{D}) = \phi(\mathbb{D}). \tag{6.1.1}$$

Then it is well known that there exists a holomorphic map $\gamma : \mathbb{D} \longrightarrow \mathbb{D}$ such that for all $z \in \mathbb{D}$

$$R\phi(z) = \phi(\gamma z) \tag{6.1.2}$$

Therefore we define *symmetries* for ϕ to be pairs (γ, R), where γ is biholomorphic on $\mathbb{D}$ and R is a rigid motion as above, such that 6.1.2 holds. Each symmetry induces a transformation formula for the frame F associated with ϕ,

$$F(\gamma z) = R_0 \cdot F(z) \cdot k(z), \tag{6.1.3}$$

where R_0 is the (orientation preserving, orthogonal) linear part of the affine transformation R and k is a unitary diagonal matrix. As a consequence one obtains

$$\gamma^* \alpha = k^{-1} \cdot \alpha \cdot k + k^{-1} \cdot dk. \tag{6.1.4}$$

From this it is easy to see that, after introducing the parameter $\lambda \in S^1$,

$$\gamma^* \alpha_\lambda = k^{-1} \cdot \alpha_\lambda \cdot k + k^{-1} \cdot dk. \tag{6.1.5}$$

This implies that for the extended frame, which by abuse of notation we will also denote by F, there exists some $\chi \in \Lambda SU(2)$ such that $\chi|_{\lambda=1} = R_0$ and

$$F(\gamma z, \lambda) = \chi(\lambda) \cdot F(z, \lambda) \cdot k(z), \tag{6.1.6}$$

where k is the unitary diagonal matrix of 6.1.3, hence independent of λ.

6.2 Equation 6.1.6 can be evaluated in two "directions". On the one hand, since $H \neq 0$, we can apply the Sym-Bobenko formula, and obtain

$$\phi(\gamma z, \lambda) = \chi(\lambda) \cdot \phi(z, \lambda) \cdot \chi^{-1}(\lambda) + i\lambda\partial_\lambda \chi(\lambda) \cdot \chi^{-1}(\lambda). \tag{6.2.1}$$

On the other hand, from 6.1.6, one obtains for all holomorphic extended frames associated with F the transformation formula

$$C(\gamma z, \lambda) = \chi(\lambda) \cdot C(z, \lambda) \cdot W_+(z, \lambda). \tag{6.2.2}$$

The holomorphic potential $\eta = C^{-1} \cdot dC$ then satisfies

$$\gamma^*\eta = W_+{}^{-1} \cdot \eta \cdot W_+ + W_+{}^{-1} \cdot dW_+. \tag{6.2.3}$$

6.3 Since we are interested in constructing constant mean curvature immersions from potentials, it is natural to start with potentials η satisfying 6.2.3. Then the corresponding holomorphic extended frame C satisfies

$$C(\gamma z, \lambda) = \rho(\lambda) \cdot C(z, \lambda) \cdot W_+(z, \lambda) \tag{6.3.1}$$

with some $\rho \in \Lambda Sl(2, \mathbb{C})_\sigma$. If $\rho \in \Lambda SU(2)_\sigma$, then the extended frame F obtained from C via Iwasawa splitting satisfies 6.1.6 for $\chi = \rho$ and the Sym-Bobenko formula produces a constant mean curvature immersion satisfying 6.2.1 with $\chi = \rho$. Unfortunately, in general, ρ will *not* be unitary for all $\lambda \in S^1$. As a consequence, the transition from C to F does *not* result in a formula like 6.1.1. And therefore we do *not* know anything about the transformation behaviour of the immersion constructed from C via F and the Sym-Bobenko formula.

There seem to be at least three avenues to overcome this complication:

1. One needs to find some criterion for η which ensures that ρ will be unitary for all $\lambda \in S^1$. While this seems to be the most desirable solution, no general criterion seems to be known. However, in some special cases such criteria have been found [42].

2. Assume now that η satisfies 6.2.3, and therefore

$$C(\gamma z, \lambda) = \rho(\lambda) \cdot C(z, \lambda) \cdot W_+(z, \lambda) \tag{6.3.2}$$

for some $\rho \in \Lambda Sl(2, \mathbb{C})_\sigma$. We carry out the Iwasawa splitting $\rho = \rho_u \cdot \rho_+$ for ρ and dress the holomorphic extended frame C with ρ_+: $\rho_+ \cdot C = \hat{C} \cdot \hat{V}_+$. Then we obtain

$$C(\gamma z, \lambda) = \rho_u(\lambda) \cdot \hat{C} \cdot S_+. \tag{6.3.3}$$

This leads to

$$F(\gamma z, \lambda) = \rho_u \cdot \hat{F} \cdot \hat{k}, \tag{6.3.4}$$

with $\hat{k}$ independent of λ. Applying the Sym-Bobenko formula to this equation one finally obtains

$$\phi(\gamma z, \lambda) = \rho_u \cdot \hat{\phi}(z, \lambda) \cdot \rho_u{}^{-1} + i\lambda\partial_\lambda\rho_u(\lambda) \cdot \rho_u(\lambda)^{-1}. \tag{6.3.5}$$

Since ϕ and $\hat{\phi}$ are different surfaces, this approach does not seem to be useful for the construction of surfaces admitting specific fundamental groups.

3. Assume again that η satisfies 6.2.3, whence C satisfies 6.3.2. This third approach assumes that ρ is of the form $\rho = \rho_+ \cdot \rho_u \cdot \rho_+^{-1}$ with $\rho_+ \in \Lambda_r^+ Sl(2,\mathbb{C})_\sigma$ and $\rho_u \in \Lambda SU_r(2)_\sigma$. Then dressing C with ${\rho_+}^{-1}$ yields ${\rho_+}^{-1} \cdot C = \hat{C} \cdot \hat{V_+}$ and 6.3.2 translates into

$$\hat{C}(\gamma z, \lambda) = \rho_u \cdot \hat{C} \cdot \hat{S_+}. \tag{6.3.6}$$

Note that this equation is, in contrast to 6.3.3, an equation involving only one surface. Similarly, for the new frame $\hat{F}$ one obtains the transformation formula

$$\hat{F}(\gamma z, \lambda) = \rho_u \cdot \hat{F}(z, \lambda) \cdot \hat{k}(z) \tag{6.3.7}$$

and for the associated immersion

$$\hat{\phi}(\gamma z, \lambda) = \rho_u(\lambda) \cdot \hat{\phi}(z, \lambda) \cdot \rho_u(\lambda)^{-1} + i\lambda\partial_\lambda \rho_u(\lambda) \cdot \rho_u(\lambda)^{-1}. \tag{6.3.8}$$

Clearly, for the new immersion $\hat{\phi}$ the pair (γ, ρ_u) is a symmetry.

6.4 From 6.3.8 one can show that the invariance of $\hat{\phi}$ under the operation of γ, $\hat{\phi}(\gamma z, \lambda_0) = \hat{\phi}(z, \lambda_0)$, is equivalent to the so called *closing conditions*

$$\rho_u(\lambda_0) = \pm I, \tag{6.4.1}$$

$$\partial_\lambda \mid_{\lambda=\lambda_0} \rho_u = 0. \tag{6.4.2}$$

As a consequence, if for fixed $\lambda_0 \in S^1$ the immersion $\hat{\phi}$ factors through a Riemann surface M, then the closing conditions need to be satisfied for all $\gamma \in \pi_1(M)$.

More precisely, if $\operatorname{Ker}\hat{\phi}$ denotes the biholomorphic maps γ of $\mathbb{D}$ which fix $\hat{\phi}$ for $\lambda = \lambda_0$, then $\gamma \in \operatorname{Ker}\hat{\phi}$ if and only if the associated ρ_u satisfies the closing conditions. In particular

$$\pi_1(M) \subset \operatorname{Ker}\hat{\phi} \quad \text{if} \quad \hat{\phi} \text{ factors through M at } \lambda = \lambda_0. \tag{6.4.3}$$

In this case, one can factor $\hat{\phi}$ through the Riemann surface $M' = \mathbb{D}/\operatorname{Ker}\hat{\phi}$ [19]. In most cases, factoring through a given Riemann surface is only possible for a few values of λ. In general, $\operatorname{Ker}\hat{\phi} = \{\mathrm{id}\}$ for all but finitely many λ's. It would be interesting to determine the "degenerate cases", i.e. those where $\rho_u = I$ for all $\lambda \in S^1$. In particular, it is an open question whether there exist complete immersions with this property.

6.5 If one starts with some biholomorphic map γ of $\mathbb{D}$ and a rigid motion R satisfying $\phi(\gamma z) = R\phi(z)$, then γ is uniquely determined modulo $\operatorname{Ker}\phi$. Conversely, given some $\gamma \in \operatorname{Aut}\mathbb{D}$ and a symmetry (γ, R), then R is uniquely determined by γ as a rigid motion of $\mathbb{R}^3 \cong su(2)$ [18]. Thus the map $\gamma \longrightarrow \chi(\gamma, \lambda)$ is a homomorphism up to sign:

$$\chi(\gamma\mu, \lambda) = \pm\chi(\gamma, \lambda) \cdot \chi(\mu, \lambda). \tag{6.5.1}$$

The map $\gamma \longrightarrow W_+(\gamma, z, \lambda)$ of 6.3.1 is a cocycle up to sign:

$$W_+(\gamma\mu, z, \lambda) = \pm W_+(\mu, z, \lambda) W_+(\gamma, \mu z, \lambda). \tag{6.5.2}$$

6.6 From 6.4.1 it is clear that for the construction of constant mean curvature surfaces one needs to find good potentials to start with. In addition one needs to find the cocyles W_+. However, these can be chosen to be trivial if M is non-compact.

Theorem 6.6.1. *([22]) If M is a non-compact Riemann surface and $\phi : M \longrightarrow \mathbb{R}^3$ a constant mean curvature immersion with universal cover $\pi : \mathbb{D} \longrightarrow M$, then the associated family of ϕ can be generated from a holomorphic potential $\eta = \sum_{j=-1}^{\infty} \eta_j \lambda^j$ on $\mathbb{D}$, which is invariant under $\pi_1(M) \subset \operatorname{Aut}\mathbb{D}$:*

(6.6.1) *η is a holomorphic $(1,0)$-form on $\mathbb{D}$ and $\gamma^*\eta = \eta \; \forall \gamma \in \pi_1(M)$.*

For compact M one can show

Theorem 6.6.2. *If M is a compact Riemann surface and $\phi : M \longrightarrow \mathbb{R}^3$ a constant mean curvature immersion with universal cover $\pi : \mathbb{D} \longrightarrow M$, then the associated family of ϕ can be generated from a meromorphic potential $\eta = \sum_{j=-1}^{\infty} \eta_j \lambda^j$ on $\mathbb{D}$, which is invariant under $\pi_1(M) \subset \operatorname{Aut}\mathbb{D}$:*

(6.6.2) *η is a meromorphic (1,0)-form on $\mathbb{D}$ and $\gamma^*\eta = \eta \; \forall \gamma \in \pi_1(M)$.*

We emphasize (see also section 2.4) that the notion of a "normalized potential" involves only one power of λ, namely λ^{-1}, and a meromorphic coefficient function, while the meromorphic potentials occurring in the theorem above involve more than one power of λ by the results of section 2.3 of [21]. The proofs of the two theorems above are almost identical.

The first one uses the fact that holomorphic cocycles on a non-compact Riemann surface with values in a Banach Lie group are holomorphic boundaries. The second theorem uses the fact that meromorphic cocycles on compact Riemann surfaces are meromorphic boundaries (see [7] for both cases).

Remark 6.6.3. As mentioned in the previous section, an important goal is to construct surfaces with a given fundamental group Γ. For the non-compact case one can start, due to Theorem 6.6.1, from a holomorphic potential η satisfying 6.6.1. This implies for the holomorphic extended frame C the equation $C(\gamma.z,\lambda) = \rho(\gamma,\lambda)\cdot C(z,\lambda)$ for $\gamma \in \Gamma$, $z \in \mathbb{D}$, $\lambda \in S^1$. If ρ is unitary for all $\gamma \in \Gamma$ and all $\lambda \in S^1$, then one obtains for the extended frame F associated to C, $C = F\cdot V_+$, the equation $F(\gamma.z,\lambda) = \rho(\gamma,\lambda)\cdot F(z,\lambda)\cdot k(z)$ and the Sym-Bobenko formula implies formula 6.3.8 for $\phi = \hat{\phi}$ derived from $F = \hat{F}$. So it only remains to satisfy the closing conditions 6.4.1 and 6.4.2, which is not too hard. The crucial problem is that it is quite difficult to choose η so that ρ is unitary for all $\gamma \in \Gamma$. An instance where this task has been carried out successfully is the work of Kilian [42]. A more general situation occurs when ρ is not unitary, but can be represented in the form $\rho = h_+ \cdot \rho_u \cdot h_+{+}^{-1}$ with $h_+ \in \Lambda_r^+ Sl(2,\mathbb{C})_\sigma$ and $\rho_u \in \Lambda_r SU(2)_\sigma$. In this case the third approach outlined in section 6.3 can be applied, producing an immersion $\hat{\phi}$ of constant mean curvature. If in addition the closing conditions 6.4.1 and 6.4.2 are satisfied for $\hat{\phi}$, then $\hat{\phi}$ factors through $\Gamma\backslash\mathbb{D} = M$. This way one can produce surfaces with fundamental group Γ from $\Gamma-$invariant potentials after some dressing. This has been carried out successfully (see Theorem 5.5.1) in [22] for the construction of cylinders of constant mean curvature, initiating the study [42]. In [28] the same approach is used for the construction of trinoids and more generally N-noids. Based on the dressing idea [22] outlined above and a special representation of the holomorphic extended frame [28], Schmitt has developed a numerical algorithm which produces pictures of cylinders, trinoids and N-noids [43],[49], available at *www.gang.umass.edu.* The fact that the pictures of trinoids and N-noids actually do represent surfaces of the indicated topological type is not proved in [43], or [49]. In some cases, however, this follows from [42] or [22]. We expect that the other cases will follow from [28].

It should be noted also that by completely different methods (almost) embedded trinoids and planar N-noids have been investigated (see e.g. [34],[35], and the references listed there). Some pictures have been produced following this approach.

6.7 There are various groups associated with a constant mean curvature immersion $\phi : M \longrightarrow \mathbb{R}^3$, its lift $\psi : \mathbb{D} \longrightarrow \mathbb{R}$ and the canonical projection $\pi : \mathbb{D} \longrightarrow M$. Let

$$(6.7.1) \qquad \operatorname{Aut}\phi(M) = \{\text{orientation preserving rigid motions } R \text{ of } \mathbb{R}^3 \text{ such that } R\phi(M) = \phi(M)\}.$$

Recall that in this article $\mathbb{D}$ always denotes a simply connected open subset of $\mathbb{C}$.

Theorem 6.7.1. *([19]) Let $\phi : M \longrightarrow \mathbb{R}^3$ be a complete immersion of constant mean curvature with simply connected cover $\pi : \mathbb{D} \longrightarrow M$. Then* $\operatorname{Aut}\phi(M)$ *contains a one-parameter group if and only if the map $\psi : \mathbb{D} \longrightarrow \mathbb{R}^3$ is in the associated family of a Delaunay surface.*

As mentioned earlier, for complete surfaces every $R \in \operatorname{Aut}\phi(M)$ induces an automorphism $\gamma = \gamma_R$ of $\mathbb{D}$ satisfying $\phi(\gamma z) = R\phi(z)$. Thus the theorem above has the

Corollary 6.7.2. *A complete constant mean curvature surface different from the sphere, possesses a one-parameter group of symmetries if and only if $\mathbb{D} = \mathbb{C}$ and the surface $\psi : \mathbb{D} \longrightarrow \mathbb{R}^3$ is in the associated family of a Delaunay surface.*

A very similar result holds if one only considers self-isometries:

Theorem 6.7.3. *([51]) Let $\phi : M \longrightarrow \mathbb{R}^3$ be a complete immersion of constant mean curvature with simply connected cover $\pi : \mathbb{D} \longrightarrow M$. Then $\mathbb{D}$ admits a one-parameter group of self-isometries for the induced metric if and only if $\mathbb{D} = \mathbb{C}$ and, with the induced metric, $\mathbb{D}$ is isometrically isomorphic to the simply connected cover of a Delaunay surface or of a Smyth surface. More precisely,*

a) if $\mathbb{D}$ admits a one-parameter group of translations, then $\mathbb{D}$ is isometric to the simply connected cover of a Delaunay surface,

b) if $\mathbb{D}$ admits a one-parameter group of rotations around a fixed point, then $\mathbb{D}$ is isometric to the simply connected cover of a Smyth surface.

Note that the existence of a one-parameter group of self-isometries implies that the corresponding metric only depends on one variable or is constant. Very little is known about surfaces with large but discrete groups of symmetries or self-isometries. Constant mean curvature cylinders have been investigated by Kilian [42] and constant mean curvature surfaces with doubly periodic frames, and in particular tori, have been investigated in [46],[20],[4],[5]. Recently, work on trinoids and more generally N-noids of genus zero has been started [28],[43],[49] (see also Remark 6.6.3).

§7. Weierstraß representations for other classes of surfaces

7.1 For the discussion of constant mean curvature surfaces we have used the loop group $\Lambda Sl(2,\mathbb{C})_\sigma$ and some of its subgroups. For other classes one needs to choose different groups $\mathcal{H}$ and subgroups $\mathcal{H}_+, \mathcal{H}_-$ and $\mathcal{U}$. But certain features will remain the same. In particular, for the surfaces in a given class under consideration:

1. For every surface there is a "frame" F in $\mathcal{U}$
2. There is a "Birkhoff splitting" $F = F_- \cdot F_+$ with $F_\pm \in \mathcal{H}_\pm$.

Using these features one can — in the language of the previous sections — construct immersions and normalized potentials $F_-^{-1} \cdot dF_-$. It is, of course, crucial that one can also construct frames from potentials.

3. There is an "Iwasawa splitting" $F_- = F \cdot F_+$ such that

a) in this equation the Birkhoff splitting and the Iwasawa splitting are essentially inverse to each other: starting from F the Birkhoff splitting produces F_-, starting from F_- the Iwasawa splitting produces F.

b) the frames obtained via Iwasawa splitting from normalized potentials are frames of surfaces in the class considered.

In most cases there is a "Sym-Bobenko formula" which produces the immersion from the extended frame F. However, there are some cases (e.g. see section 7.4) where such a formula does not seem to be known. We would like to point out that in the examples listed in the next few sections, the relevant Iwasawa splitting is no longer globally defined. Also, the "loops" are no longer maps from S^1, but from other curves, like the real line. This requires generalized splitting theorems. The basic results in this direction can be found in [1] and [40]. The splittings actually used in the examples below can easily be derived from these results.

7.2 Pseudospherical Surfaces In this section we discuss the class of surfaces of constant Gauß curvature $K = -1$ following [55]. We will always use asymptotic line coordinates. Frequently we will even assume that these coordinates form a Chebyshev net. The starting point for this approach is the fact that a pseudospherical surface M, endowed with the second fundamental form, is a Lorentz manifold, and the Gauß map $N : M \longrightarrow S^2$ of a Lorentz manifold M is (Lorentz) harmonic if and only if M is pseudospherical. One needs to note that "harmonic" is defined here by the vanishing of the tension field, where the "metric coefficients" involved are for M the ones of the pseudo-Riemannian structure and for S^2 the ones of the usual Riemannian structure. Viewing S^2 as a quotient of $SO(3)$, $S^2 = SO(3)/SO(2)$, we can lift the Gauß map N to a map into $SO(3)$. Among these lifts there is a "normalized" one, for which the Maurer-Cartan form looks particularly simple. Expressing the coordinate frame in terms of the angle ϕ between the asymptotic lines, one obtains the normalized frame U by rotating the coordinate frame by the angle $\Theta = \phi/2$ and then taking its transpose. (The latter step is only to obtain the form of the moving frame equations as in the constant mean curvature case.)

With these conventions the moving frame equations then read (for the coefficient matrices in asymptotic line coordinates):

$$\partial_x U = U \cdot \begin{pmatrix} 0 & -\phi_x & 0 \\ \phi_x & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \tag{7.2.1}$$

$$\partial_y U = U \cdot \begin{pmatrix} 0 & 0 & -\sin\phi \\ 0 & 0 & -\cos\phi \\ \sin\phi & \cos\phi & 0 \end{pmatrix}. \tag{7.2.2}$$

Introducing the *real, positive* parameter λ one obtains the extended frame equations

$$\partial_x U = U \cdot \begin{pmatrix} 0 & -\phi_x & 0 \\ \phi_x & 0 & \lambda \\ 0 & -\lambda & 0 \end{pmatrix}, \tag{7.2.3}$$

$$\partial_y U = U \cdot \lambda^{-1} \begin{pmatrix} 0 & 0 & -\sin\phi \\ 0 & 0 & -\cos\phi \\ \sin\phi & \cos\phi & 0 \end{pmatrix}. \tag{7.2.4}$$

Proposition 7.2.1. *The extended frame equations can be solved simultaneously for all λ if and only if the corresponding surface is pseudospherical.*

The family of surfaces, parametrized by λ, is called the associated family. Since λ is (for geometric applications) a real parameter, we need to be careful when defining the loop groups for pseudospherical surfaces. However, the crucial observation is that all geometrically relevant quantities actually are analytic in $\lambda \in \mathbb{C}^*$. Therefore we can carry out all loop group operations in a standard loop group, remembering that for geometric evaluations λ needs to be chosen real and positive. With this in mind we consider the group $\Lambda SO(3)_P$ whose elements $g(\lambda)$ are defined by the following four conditions:

$$g \in \Lambda Sl(3, \mathbb{C}) \tag{7.2.5}$$

$$g(\lambda)^T = g(\lambda)^{-1} \tag{7.2.6}$$

$$\overline{g(\overline{\lambda})} = g(\lambda) \tag{7.2.7}$$

$$g(-\lambda) = P \cdot g(\lambda) \cdot P^{-1} \tag{7.2.8}$$

where $P = \mathrm{diag}(1, 1, -1)$ and the subscript P will denote in this section "twisting by P", which differs from the use of the subscript P in section 2.2. Thus the elements of $\Lambda SO(3)_P$ are loops in $\Lambda Sl(3, \mathbb{C})$, which have real coefficients at all powers of λ, are in $SO(3)$ for every real λ, and are twisted by P. We set $\Lambda^+ SO(3)_P = \Lambda^+ Sl(3, \mathbb{C}) \bigcap \Lambda SO(3)_P$, and define $\Lambda^- SO(3)_P$ analogously.

Now we are ready to define the "loop group" used for pseudospherical surfaces:

$$\mathcal{H} = \Lambda SO(3)_P \times \Lambda SO(3)_P, \tag{7.2.9}$$

and the relevant subgroups

$$\mathcal{U} = \{(g, g) \mid g \in \Lambda SO(3)_P\} \tag{7.2.10}$$

$$\mathcal{H}^+ = \Lambda^+ SO(3)_P \times \Lambda^- SO(3)_P \tag{7.2.11}$$

$$\mathcal{H}^- = \Lambda^- SO(3)_P \times \Lambda^+ SO(3)_P. \tag{7.2.12}$$

As with previous conventions we write for example $(g, h) \in \mathcal{H}_*^+$, if g and h have I as the coefficient of λ^0. We will write $(g, h) \in \mathcal{H}_\sharp^+$, if we require only I for the coefficient of g of λ^0. Then the "Birkhoff splitting" means to write an element of $\mathcal{H}$ as a product of elements in $\mathcal{H}^+$ and $\mathcal{H}_*^-$. It is not difficult to see that $\mathcal{H}_*^- \cdot \mathcal{H}^+$ is open and dense in $\mathcal{H}$, and the group splitting is an analytic operation on this set [55]. Similarly, $\mathcal{U} \cdot \mathcal{H}_\sharp^+$ is open and dense and the group splitting is an analytic operation on this set. (To verify this one considers $(g, h) \in \mathcal{H}$ and forms the element $h^{-1} g \in \Lambda SO(3)_P$. By the previous result, in any neighbourhood of g

there is some $g' \in \Lambda SO(3)_P$ such that $h^{-1}g' = v_- \cdot v_+^{-1}$ with $v_+ \in \Lambda_*^+ SO(3)_P$ and $v_- \in \Lambda^- SO(3)_P$. As a consequence, setting $u = hv_- = g'v_+$ we obtain $(g', h) = (u, u)(v_+, v_-)$ and the first part of the claim follows. The second part is a consequence of the analyticity of the map $\Lambda^- SO(3)_P \cdot \Lambda_*^+ SO(3)_P \longrightarrow \Lambda^- SO(3)_P \times \Lambda_*^+ SO(3)_P$.) The "Birkhoff splitting" here is actually a pair of two standard Birkhoff splittings, while the "Iwasawa splitting" here for a pair (g, h) means writing $g = u \cdot v_+$ and $h = u \cdot v_-$, which is equivalent to $h^{-1} \cdot g = {v_-}^{-1} \cdot v_+$.

Following the outline at the beginning of this section we want to produce "normalized potentials" by splitting the "frame". For a pseudospherical surface we consider its normalized frame U and form the element $F = (U, U) \in \mathcal{U}$. This is the "frame", which will work for our purposes. We perform the Birkhoff splitting in $\mathcal{H}$:

$$F = F_- \cdot F_+, \quad \textit{where} \quad F_\pm \in \mathcal{H}^\pm. \tag{7.2.13}$$

In this case it turns out that

$$F_- = (U_-, U_+) \tag{7.2.14}$$

is such that U_- only depends on the variable y, while U_+ only depends on x. The normalized potential is

$$\xi = (\xi_-, \xi_+) = F_-^{-1} dF_-, \tag{7.2.15}$$

where

$$\xi_- = U_-^{-1} dU_- = \lambda^{-1} \xi_{-1}(y) dy, \tag{7.2.16}$$

$$\xi_+ = U_+^{-1} dU_+ = \lambda \xi_1(x) dx, \tag{7.2.17}$$

and

$$\xi_{-1}(y) = \begin{pmatrix} 0 & 0 & \sin\phi(0,y) \\ 0 & 0 & \cos\phi(0,y) \\ -\sin\phi(0,y) & -\cos\phi(0,y) & 0 \end{pmatrix} dy, \tag{7.2.18}$$

$$\xi_1(x) = \begin{pmatrix} 0 & 0 & -\sin\hat{\phi}(x) \\ 0 & 0 & -\cos\hat{\phi}(x) \\ \sin\hat{\phi}(x) & \cos\hat{\phi}(x) & 0 \end{pmatrix} dx \tag{7.2.19}$$

where $\hat{\phi}(x) = \phi(0,0) - \phi(x,0)$.

Remark 7.2.2. We have constructed a "normalized potential" for each pseudospherical surface. Since the group splittings used in this case are not globally defined, the normalized potentials may acquire singularities. At this point it has not yet been investigated what type of singularities can occur. At any rate, locally around a given base point, the potential can be assumed to be smooth. It would be interesting to investigate whether there are analogues of the "holomorphic potentials" of the constant mean curvature case. These would need to be defined on simply connected Lorentz surfaces and would perhaps be globally smooth.

For the converse construction, producing surfaces from potentials we start from two matrices of the form 7.2.18 and 7.2.19, where we replace the functions $\phi(x,0)$ and $\phi(0,y)$ by some arbitrary smooth functions $a(x)$ and $b(y)$. Next we solve the two ODE's 7.2.15 for U_- and U_+ with trivial initial condition. Next we perform an Iwasawa splitting of (U_-,U_+). As pointed out above, this is equivalent to performing the classical Birkhoff splitting $U_-^{-1}\cdot U_+ = W_+\cdot W_-^{-1}$. Then one sets $U = U_-\cdot W_+ = U_+\cdot W_-$ and shows that U is a frame associated with a pseudospherical surface, which is obtained by the *Sym formula*

$$\psi(x,y,\lambda) = \frac{\partial U}{\partial t}\cdot U^{-1}, \quad \text{where } \lambda = e^t. \tag{7.2.20}$$

For details, examples, and proofs we refer to [55].

7.3 Timelike Surfaces in Minkowski Space $\mathbb{E}_1^3$ of Constant Mean Curvature This case has similarities with both the theory of surfaces of constant mean curvature and the theory of surfaces of constant negative Gauß curvature. For details see [23]. Consider the Minkowski space $\mathbb{E}_1^3$ defined by the metric $\langle .,. \rangle = -du_1^2 + du_2^2 + du_3^2$. The starting point for this approach is the fact that the Gauß map $N : M \longrightarrow S_1^2 = Sl(2,\mathbb{R})/K$, where $K = \{\mathrm{diag}(a,a^{-1}) \mid a \neq 0\}$, of a timelike surface in $\mathbb{E}_1^3$, is harmonic if and only if the (Lorentzian) mean curvature is constant. As in the previous cases, "harmonic" is defined by the vanishing of the tension field. In the case under consideration both the timelike surface M and the Lorentz sphere S_1^2 carry a Lorentz metric. Lifting the Lorentzian frame, which defines an element of the Lorentz group, to the double cover $Sl(2,\mathbb{R})$, one obtains the moving frame equations (we will always use null coordinates):

$$\partial_y U = U \cdot \begin{pmatrix} \frac{1}{4}\omega_y & -\frac{1}{2}H\exp(\frac{1}{2}\omega) \\ R\exp(-\frac{1}{2}\omega) & -\frac{1}{4}\omega_y \end{pmatrix}, \tag{7.3.1}$$

$$\partial_x U = U \cdot \begin{pmatrix} -\frac{1}{4}\omega_x & -Q\exp(-\frac{1}{2}\omega) \\ \frac{1}{2}H\exp(\frac{1}{2}\omega) & \frac{1}{4}\omega_x \end{pmatrix}, \tag{7.3.2}$$

where $Q = \langle \phi_{xx}, N\rangle$, $R = \langle \phi_{yy}, N\rangle$, $H = 2\exp(-\omega)\langle \phi_{xy}, N\rangle$ and the metric is $ds^2 = e^{\omega}dxdy$. Note that we have two Hopf differentials this time.

Introducing the *real, positive* parameter λ one obtains the extended frame equations

$$\partial_y U = U \cdot \begin{pmatrix} \frac{1}{4}\omega_y & -\lambda^{-1}\frac{1}{2}H\exp(\frac{1}{2}\omega) \\ \lambda^{-1}R\exp(-\frac{1}{2}\omega) & -\frac{1}{4}\omega_y \end{pmatrix}, \tag{7.3.3}$$

$$\partial_x U = U \cdot \begin{pmatrix} -\frac{1}{4}\omega_x & -\lambda Q\exp(-\frac{1}{2}\omega) \\ \lambda\frac{1}{2}H\exp(\frac{1}{2}\omega) & \frac{1}{4}\omega_x \end{pmatrix}. \tag{7.3.4}$$

For our loop group approach the following result is basic:

Proposition 7.3.1. *The extended frame equations can be solved simultaneously for all λ if and only if the corresponding timelike surface in $\mathbb{E}^3_1$ has constant mean curvature.*

The family of surfaces parametrized by λ, is called the *associated family.*

As in the case of pseudospherical surfaces the parameter λ is real, but all geometrically relevant quantities are actually analytic for $\lambda \in \mathbb{C}^*$. Therefore we can carry out all loop group operations in a standard loop group, remembering that for geometric evaluations λ needs to be chosen real and positive.

With this in mind we consider the group $\Lambda Sl(2,\mathbb{R})_\sigma$, whose elements $g(\lambda)$ are defined by the following three conditions:

$$g \in \Lambda Sl(2,\mathbb{C}) \tag{7.3.5}$$

$$\overline{g(\overline{\lambda})} = g(\lambda) \tag{7.3.6}$$

$$g(-\lambda) = \sigma_3 \cdot g(\lambda) \cdot \sigma_3^{-1}, \tag{7.3.7}$$

where $\sigma_3 = \mathrm{diag}(1,-1)$. Thus the elements of $\Lambda Sl(2,\mathbb{R})_\sigma$ are loops in $Sl(2,\mathbb{C})$, which have real coefficients at all powers of λ, are in $Sl(2,\mathbb{R})$ for real λ, and are twisted by σ_3. We set $\Lambda^+ Sl(2,\mathbb{R})_\sigma = \Lambda^+ Sl(2,\mathbb{C})_\sigma \bigcap \Lambda Sl(2,\mathbb{R})$ and define $\Lambda^- Sl(2,\mathbb{R})_\sigma$ similarly.

Now we define the "loop group" used for timelike surfaces of constant mean curvature in $\mathbb{E}^3_1$:

$$\mathcal{H} = \Lambda Sl(2,\mathbb{R})_\sigma \times \Lambda Sl(2,\mathbb{R})_\sigma \tag{7.3.8}$$

and the relevant subgroups

$$\mathcal{U} = \{(g,g) \mid g \in \Lambda Sl(2,\mathbb{R})_\sigma\} \tag{7.3.9}$$

$$\mathcal{H}^- = \Lambda^+ Sl(2,\mathbb{R})_\sigma \times \Lambda^- Sl(2,\mathbb{R})_\sigma \tag{7.3.10}$$

$$\mathcal{H}^+ = \Lambda^- Sl(2,\mathbb{R})_\sigma \times \Lambda^+ Sl(2,\mathbb{R})_\sigma. \tag{7.3.11}$$

As usual we add a subscript $*$ to indicate that the coefficient of λ^0 is the identity matrix. The comments concerning the Birkhoff and Iwasawa splittings in the previous section also apply here.

To produce a normalized potential from a surface we consider its extended frame $U \in \Lambda Sl(2,\mathbb{C})_\sigma$ and form the element $F = (U, U)$. Again, this is the "frame" which will work for our purposes. We perform the Birkhoff splitting in $\mathcal{H}$:

$$F = F_- \cdot F_+, \quad \text{where } F_\pm \in \mathcal{H}^\pm. \tag{7.3.12}$$

In this case it turns out that

$$F_- = (U_+, U_-) \tag{7.3.13}$$

is such that U_- only depends on the variable y, while U_+ only depends on the variable x. Thus the normalized potential is

$$\xi = (\xi', \xi'') = F_-^{-1} dF_-, \tag{7.3.14}$$

where

$$\xi' = U_+^{-1} dU_+ = \lambda \begin{pmatrix} 0 & \frac{Q(x)}{f(x)} \\ -\frac{1}{2} H f(x) & 0 \end{pmatrix} dx, \tag{7.3.15}$$

$$\xi'' = U_-^{-1} dU_- = \lambda^{-1} \begin{pmatrix} 0 & \frac{1}{2} H g(y) \\ \frac{R(x)}{g(y)} & 0 \end{pmatrix} dy, \tag{7.3.16}$$

where $f(x) = \exp(\omega(x,0) - \frac{1}{2}\omega(0,0))$ and $g(y) = \exp(\omega(0,y) - \frac{1}{2}\omega(0,0))$. These are the "normalized potentials" for the case under consideration. Remark 7.2.2 applies here as well.

For the converse construction, producing surfaces from potentials, we start from two matrices of the form 7.3.15 and 7.3.16, where we replace the functions $\omega(x,0)$ and $\omega(0,y)$ by some arbitrary smooth functions $a(x)$ and $b(y)$. Next we solve the two ODE's $\partial_x U_+ = U_+ \cdot \xi'$ and $\partial_y U_- = U_- \cdot \xi''$ for U_- and U_+ with trivial initial condition. Then we perform an Iwasawa splitting of (U_-, U_+). As pointed out before, this is equivalent to performing the classical Birkhoff splitting $U_-^{-1} \cdot U_+ = W_+ \cdot W_-^{-1}$. Finally we set $U = U_- \cdot W_+ = U_+ \cdot W_-$ and show that U is a frame associated with a timelike surface in $\mathbb{E}_1^3$ of constant mean curvature, which is obtained by the *Generalized Sym formula*

$$(7.3.17) \qquad \phi(x,y,\lambda) = -\frac{1}{H}\left\{\frac{\partial U}{\partial t} \cdot U^{-1} - \frac{1}{2} U \cdot \sigma_3 \cdot U^{-1}\right\}, \quad \lambda = e^t.$$

For details, examples, and proofs we refer to [23].

7.4 Affine Spheres By definition, affine spheres are Blaschke surfaces and therefore the Gauß curvature K never vanishes. According to whether $K > 0$ or $K < 0$ different coordinates are natural and different loop groups are needed. The case $K > 0$ is very similar to the case of constant mean curvature surfaces in $\mathbb{R}^3$. Therefore we will restrict here to the case $K < 0$ and use asymptotic line coordinates u and v. Since we consider proper affine spheres, the affine mean curvature H is constant and different from 0. We will assume without loss of generality that $H = -1$. Then the moving frame equations for the frame $\hat{U} = (f_u, f_v, f)$ of an affine sphere are

$$(7.4.1) \qquad \partial_u \hat{U} = \hat{U} \begin{pmatrix} \omega_u & 0 & 1 \\ Ae^{-\omega} & 0 & 0 \\ 0 & e^{\omega} & 0 \end{pmatrix},$$

$$(7.4.2) \qquad \partial_v \hat{U} = \hat{U} \begin{pmatrix} 0 & Be^{-\omega} & 0 \\ 0 & \omega_v & 0 \\ e^{\omega} & 0 & 1 \end{pmatrix}.$$

Gauging the moving frame $\hat{U}$ by $D = \operatorname{diag}(\lambda^{-1}e^{-\frac{1}{2}\omega}, \lambda e^{-\frac{1}{2}\omega}, 1)$, with λ *real and positive*, we obtain for the *modified* (extended) frame $U = \hat{U}D$ the equations

$$(7.4.3) \quad U^{-1} \cdot U_u = \frac{\omega_u}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 & 0 & e^{\omega/2} \\ Ae^{-\omega} & 0 & 0 \\ 0 & e^{\omega/2} & 0 \end{pmatrix},$$

$$(7.4.4) \quad U^{-1} \cdot U_v = \frac{\omega_v}{2} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \lambda^{-1} \begin{pmatrix} 0 & Be^{-\omega} & 0 \\ 0 & 0 & e^{\omega/2} \\ e^{\omega/2} & 0 & 0 \end{pmatrix}.$$

For the loop group approach the following result is basic:

Proposition 7.4.1. *The modified frame equations 7.4.3 and 7.4.4 can be solved simultaneously for all λ if and only if the corresponding affine surface is a proper affine sphere.*

The family parametrized by λ is called the *associated family.*

As in the previous two sections the parameter λ is real, but all geometrically relevant quantities are actually analytic for $\lambda \in \mathbb{C}^*$. Therefore, in this case also, we can carry out all loop group operations in a standard loop group, remembering that for geometric evaluations λ needs to be chosen real and positive. With this in mind we consider the group $G[\lambda]$, whose elements are defined by the following four conditions

$$g \in \Lambda Sl(3,\mathbb{C}) \tag{7.4.5}$$

$$\overline{g(\overline{\lambda})} = g(\lambda) \tag{7.4.6}$$

$$Q \cdot g(\epsilon\lambda) \cdot Q^{-1} = g(\lambda) \tag{7.4.7}$$

$$T \cdot [g(-\lambda)^{-1}]^t \cdot T = g(\lambda) \tag{7.4.8}$$

(7.4.9)

where $\epsilon = \exp(2\pi i/3)$, $Q = \mathrm{diag}(\epsilon, \epsilon^2, 1)$, and

$$T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus the elements of $G[\lambda]$ are loops in $Sl(3,\mathbb{C})$, which have real coefficients at all powers of λ and satisfy two twisting conditions. It turns out that the Lie algebra $\mathfrak{g}[\lambda]$ of $G[\lambda]$ is the loop part of a Kac-Moody algebra of type $A_2^{(2)}$. Keeping with the notation of this article we set $G^{\pm}[\lambda] = G[\lambda] \bigcap \Lambda^{\pm} Sl(2,\mathbb{C})$ and we will add a subscript $*$ if we want to indicate that the coefficient at λ^0 is I. (Note that [14] uses the opposite convention.)

Next we introduce the "loop group" used for affine spheres

$$\mathcal{H} = G[\lambda] \times G[\lambda], \tag{7.4.10}$$

and the relevant subgroups

$$\mathcal{U} = \{(g,g) \mid g \in G[\lambda]\} \tag{7.4.11}$$

$$\mathcal{H}^- = G^-[\lambda] \times G^+[\lambda] \tag{7.4.12}$$

$$\mathcal{H}^+ = G^+[\lambda] \times G^-[\lambda]. \tag{7.4.13}$$

The comments made in section 7.2 about the Birkhoff and Iwasawa splittings also apply here.

To produce a normalized potential from a surface we consider its modified frame $U \in G[\lambda]$ and form the element $F = (U, U) \in \mathcal{U}$. Again, this is the "frame", which will work for our purposes. We perform the Birkhoff splitting in $\mathcal{H}$:

$$F = F_- \cdot F_+, \quad \text{where } F_\pm \in \mathcal{H}^\pm. \tag{7.4.14}$$

It turns out that

$$F_- = (V_-, V_+) \tag{7.4.15}$$

is such that V_- only depends on the variable u, while V_+ only depends on the variable v. Thus the normalized potential is

$$\xi = (\xi_-, \xi_+) = F_-^{-1} dF_-, \tag{7.4.16}$$

where
(7.4.17)

$$\xi_- = V_-^{-1} dV_- = \lambda^{-1} D_-^{-1} \begin{pmatrix} 0 & Be^{-\omega} & 0 \\ 0 & 0 & e^{\omega/2} \\ e^{\omega/2} & 0 & 0 \end{pmatrix} D_- dv = \lambda^{-1} T_-,$$

(7.4.18)

$$\xi_+ = V_+^{-1} dV_+ = \lambda D_+^{-1} \begin{pmatrix} 0 & 0 & e^{\omega/2} \\ Ae^{-\omega} & 0 & 0 \\ 0 & e^{\omega/2} & 0 \end{pmatrix} D_+ du = \lambda T_+,$$

and where D_- and D_+ are independent of λ and of the form $\mathrm{diag}(a, a^{-1}, 1)$.

This way we have constructed a "normalized potential" for each proper affine sphere. The comments made in section 7.2 concerning singularities and "holomorphic potentials" apply here as well.

For the converse construction we start from an arbitrary normalized potential $\xi = (\lambda^{-1} T_-, \lambda T_+)$, where

$$T_-(v) = \begin{pmatrix} 0 & \beta_-(v) & 0 \\ 0 & 0 & \alpha_-(v) \\ \alpha_-(v) & 0 & 0 \end{pmatrix} dv \tag{7.4.19}$$

$$T_+(u) = \begin{pmatrix} 0 & 0 & \alpha_+(u) \\ \beta_+(u) & 0 & 0 \\ 0 & \alpha_+(u) & 0 \end{pmatrix} du \tag{7.4.20}$$

and α_+ and α_- never vanish. (One can show that after changing u to $-u$ and/or v to $-v$, if necessary, one can assume $\alpha_+(0) = \alpha_-(0) > 0$. Such potentials are called "normalized" in [14].)

Next we solve the ODE's 7.4.17 and 7.4.18 with initial condition I, producing some V_+ and V_-. Then we perform an Iwasawa splitting

$$(V_-, V_+) = (\tilde{U}, \tilde{U}) \cdot (L_+, L_-) \tag{7.4.21}$$

In the previous examples a Sym-type formula was applied at this point, producing the required immersion. Such a formula does not seem to be available for affine spheres. However, one can show that (after changing u into $-u$ and/or v into $-v$, if necessary) there is a λ-independent gauge $C_0 \in G[\lambda]$ such that $U = C_0(0,0)^{-1} \cdot \tilde{U} \cdot C$ is the frame of a proper affine sphere. For details, examples, and proofs we refer to [14].

7.5 Willmore Surfaces in $\mathbb{R}^3$. Part I. In the next two sections we will discuss Willmore surfaces. For details we refer to [37]. There are actually two approaches to describe Willmore surfaces via loop groups. The first one is parallel to the previous sections.

An immersion

$$\phi : M \longrightarrow \mathbb{R}^3 \tag{7.5.1}$$

is called a *Willmore surface* if it is a critical point of the Willmore functional

$$\mathcal{W}(M) = \int_M H^2 dA. \tag{7.5.2}$$

Equivalently, an immersion is a Willmore surface if the mean curvature H satisfies the non-linear elliptic equation

$$\Delta^M H + 2H(H^2 - K) = 0 \tag{7.5.3}$$

In this part we assume that M has no umbilical points and that ϕ is conformal. Under these assumptions Bryant [6] has defined a *conformal Gauß map*

$$\check{N}_\phi : M \longrightarrow S^{3,1} \subset \mathbb{E}^{4,1}, \tag{7.5.4}$$

where $\mathbb{E}^{4,1}$ is the five-dimensional Minkowski space with metric

$$\langle x, y \rangle = -x_0 y_4 - x_4 y_0 + \sum_{j=1}^{3} x_j y_j \tag{7.5.5}$$

and

$$S^{3,1} = \{y \in \mathbb{E}^{4,1} \mid \langle y, y \rangle = 1\}. \tag{7.5.6}$$

We note that $S^{3,1}$ carries naturally a pseudo-Riemannian metric induced from $\mathbb{E}^{4,1}$.

As in the previous sections one has

Theorem 7.5.1. *([6]) The immersion $\phi : M \longrightarrow \mathbb{R}^3$ is a Willmore immersion if and only if the conformal Gauß map $\check{N}_\phi : M \longrightarrow S^{3,1}$ is harmonic. Moreover, ϕ can be retrieved from $\check{N}_\phi$ by a projection and every harmonic map $\check{N} : M \longrightarrow S^{3,1}$ induces naturally a Willmore immersion.*

In this theorem harmonicity is defined as usual by the vanishing of the tension field. The metrics entering into this stress tensor are the Riemannian metric on M induced by the immersion ϕ and the pseudo-Riemannian metric on $S^{3,1}$ induced from $\mathbb{E}^{4,1}$. As a consequence, the discussion of Willmore immersions is equivalent to the discussion of harmonic maps from M to $S^{3,1}$. At this point the theory becomes parallel to [26]. We observe that

$$S^{3,1} = SO_o(4,1)/SO_o(3,1), \tag{7.5.7}$$

where

$$SO(4,1) = \{A \in \mathrm{Mat}(5, \mathbb{R}) \mid \langle Au, Av \rangle = \langle u, v \rangle \ \forall\ u, v \in \mathbb{E}^{4,1}\}. \tag{7.5.8}$$

It is easy to see that the isotropy subgroup of $e_3 = (0,0,1,0)^t$ is naturally isomorphic with $SO(3,1)$. The subscript o denotes the identity component. Moreover, $SO_o(3,1)$ is the fixed point set of the involutive automorphism $g \mapsto \tau g \tau, \tau = \mathrm{diag}(1,1,1,-1,1)$ in $SO_o(3,1)$. In particular, $S^{3,1}$ is the *semisimple symmetric space* defined by τ. Thus we can follow the general procedure of [26], which for such symmetric spaces can be found in [10].

Let $M = \mathbb{D}$ be a simply connected subset of $\mathbb{C}$. We consider the (generalized) Cartan decomposition

$$so(4,1) = so(3,1) \oplus \mathfrak{m} = \mathfrak{h} \oplus \mathfrak{m}, \tag{7.5.9}$$

where $so(4,1)$ and $so(3,1) = \mathfrak{h}$ are the Lie algebras of the corresponding Lie groups defined above, and $\mathfrak{m}$ is the (-1)-eigenspace of (the differential of) τ. We consider the *extended lift*

$$F : \mathbb{D} \longrightarrow SO_0(4,1) \tag{7.5.10}$$

and its Maurer-Cartan form

$$\alpha = F^{-1}dF. \tag{7.5.11}$$

We decompose α in conformal coordinates z and $\bar{z}$ as

$$\alpha = \alpha'_{\mathfrak{m}} + \alpha_{\mathfrak{h}} + \alpha''_{\mathfrak{m}}, \tag{7.5.12}$$

where $\alpha = \alpha_{\mathfrak{m}} + \alpha_{\mathfrak{h}}$ is the decomposition of α relative to 7.5.12 and $\alpha'_{\mathfrak{m}}$ is the (1,0)-part of α, while $\alpha''_{\mathfrak{m}}$ is the (0,1)-part of α.

Introducing the *loop parameter* $\lambda \in S^1$ we set

$$\alpha_\lambda = \lambda^{-1}\alpha'_{\mathfrak{m}} + \alpha_{\mathfrak{h}} + \lambda\alpha''_{\mathfrak{m}}. \tag{7.5.13}$$

We then have

Proposition 7.5.2. *([10]) The extended Maurer-Cartan form α_λ is integrable for all $\lambda \in S^1$, i.e.*

$$d\alpha_\lambda + \frac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = 0 \quad \textit{for all} \quad \lambda \in S^1 \tag{7.5.14}$$

if and only if $\check{N}_\phi$ is harmonic, i.e. if and only if ϕ is a Willmore immersion.

Next we introduce the loop groups

$$\Lambda SO_0(4,1)^{\mathbb{C}} = \{S^1 \longrightarrow SO_0(4,1)^{\mathbb{C}}\} \tag{7.5.15}$$

$$\Lambda^+ SO_0(4,1)^{\mathbb{C}} = \Lambda SO_0(4,1)^{\mathbb{C}} \bigcap \Lambda^+ Sl(5,\mathbb{C}) \tag{7.5.16}$$

$$\Lambda^- SO_0(4,1)^{\mathbb{C}} = \Lambda SO_0(4,1)^{\mathbb{C}} \bigcap \Lambda^- Sl(5,\mathbb{C}) \tag{7.5.17}$$

$$\Lambda SO_0(4,1) = \{S^1 \longrightarrow SO_0(4,1)\} \tag{7.5.18}$$

and the corresponding twisted loop groups twisted by τ. From 7.5.14 we know that

$$F^{-1}dF = \alpha_\lambda, \quad F(0,0,\lambda) \tag{7.5.19}$$

is solvable and

$$F \in \Lambda SO_0(4,1). \tag{7.5.20}$$

Applying the classical Birkhoff splitting we obtain

$$F = F_- \cdot F_+, \tag{7.5.21}$$

$$\xi = F_-^{-1} \cdot dF_-. \tag{7.5.22}$$

As in the case of surfaces of constant mean curvature in $\mathbb{R}^3$ one verifies that F_- only depends on z and that ξ is of the form

(7.5.23) $$\xi = \lambda^{-1}\xi_{-1}dz, \quad \xi_{-1} \in \mathfrak{m}.$$

This way we have obtained normalized potentials for Willmore surfaces without umbilics.

Conversely, starting from some ξ of the form 7.5.23 we solve the ODE 7.5.22 for some F_- with initial condition $F_-(0,\lambda) = I$. Then we apply a generalized Iwasawa splitting

(7.5.24) $$F_- = F \cdot F_+^{-1},$$

where $F \in \Lambda SO_0(4,1)^{\mathbb{C}}_\tau$ and $F_+ \in \Lambda^+ SO_0(4,1)^{\mathbb{C}}$.

We note that this is possible near $z = 0$, since there $F_- = I$ and the splitting of F_- near $z = 0$ is a consequence of the Lie algebra decomposition

(7.5.25) $$\Lambda so(4,1)^{\mathbb{C}}_\tau = \Lambda so(4,1)_\tau + \Lambda^+ so(4,1)_\tau.$$

For a more general statement we refer to [40],[1].

It is easy to verify that $F^{-1}dF$ is of the form

(7.5.26) $$F^{-1}dF = \lambda^{-1}\gamma'_{\mathfrak{m}} + \gamma_{\mathfrak{h}} + \lambda\gamma''_{\mathfrak{m}}.$$

But this implies (see e.g. [10]) that F is the frame of a harmonic map $\check{N} : \mathbb{D} \longrightarrow S^{3,1}$. Thus we obtain a Willmore immersion from $\mathbb{D}$ to $\mathbb{R}^3$ by the theorem above.

7.6 Willmore Surfaces in $\mathbb{R}^3$. Part II. We have seen in the last section that Willmore surfaces in $\mathbb{R}^3$ without umbilics have a Weierstrass type representation in the spirit of [26]. We have pointed out that in the normalized potentials, and even in the frames and the immersions, singularities may occur due to the fact that the group splittings are not global. In the presentation of the last section, in addition umbilics show up as singularities of the harmonic maps involved. Interestingly, it turns out that in the case of Willmore surfaces there is a (somewhat) different loop group procedure, which manages to handle the umbilical points as non-singular points. Of course, the singularities which are caused by the non-global group splittings remain.

Let $\phi : \mathbb{D} \longrightarrow \mathbb{R}^3$ be a Willmore immersion and $F : \mathbb{D} \longrightarrow \Lambda SO_0(4,1)^{\mathbb{C}}$ an extended lift. Then decomposing the Maurer-Cartan form $\alpha =$

$F^{-1}dF$ according to the involution induced by τ yields the description of section 7.5. Part I which is associated with the symmetric space $S^{3,1} = SO_o(4,1)/SO_o(3,1)$. However, there is another natural involution on $\Lambda SO_0(4,1)^{\mathbb{C}}$, induced by $\sigma = \mathrm{diag}(-1,1,1,1-1)$. This involution has the fixed point group $SO(3) \times SO(1,1)$ and yields the symmetric space

$$Gr^5_{3,0} = SO_0(4,1)/(SO(3) \times SO(1,1)), \tag{7.6.1}$$

the Grassmannian of three-dimensional spacelike subspaces of $\mathbb{E}^{4,1}$.

Decomposing α relative to the involution σ yields

$$\alpha = \delta_{\mathfrak{k}} + \delta_{\mathfrak{p}}, \tag{7.6.2}$$

where $\mathfrak{k}$ and $\mathfrak{p}$ are the eigenspaces of the involution σ for the eigenvalues 1 and -1 respectively. Decomposing further (in conformal coordinates) we obtain

$$\alpha = \delta'_{\mathfrak{p}} + \delta_{\mathfrak{k}} + \delta''_{\mathfrak{p}}, \tag{7.6.3}$$

where $\delta'_{\mathfrak{p}}$ is a (1,0)-form and $\delta''_{\mathfrak{p}}$ is a (0,1)-form. Now one can introduce a parameter $\mu \in S^1$ producing

$$\alpha^{(\mu)} = \mu^{-1}\delta'_{\mathfrak{p}} + \delta_{\mathfrak{k}} + \delta''_{\mathfrak{p}}. \tag{7.6.4}$$

It is crucial for this part of the discussion that the Willmore property can be rephrased not only relative to the extended Maurer-Cartan frame α_λ defined in 7.5.13, but also relative to $\alpha^{(\mu)}$.

Theorem 7.6.1. *([37], Theorem 2) The map $\phi : \mathbb{D} \longrightarrow \mathbb{R}^3$ is a Willmore immersion if and only if $\alpha^{(\mu)}$ is integrable for all $\mu \in S^1$:*

$$d\alpha^{(\mu)} + \frac{1}{2}[\alpha^{(\mu)} \wedge \alpha^{(\mu)}] = 0 \quad \textit{for all} \quad \mu \in S^1. \tag{7.6.5}$$

So far in this article the parameter λ was always used in relation with harmonic Gauß maps. Therefore *the roles of λ and μ are interchanged compared to* [37].

Just as in the previous section, the integrability condition 7.6.5 allows us to solve the differential equation $F^{-1}dF = \alpha^{(\mu)}, F(0,\mu) = I$. The classical Birkhoff splitting then yields

$$F = F_- \cdot F_+, \tag{7.6.6}$$

where F, F_- and F_+ are in the twisted loop groups which are defined relative to σ. To indicate our use of the parameter μ in constrast to λ we will use M in place of Λ, as in $MSO_o(4,1)^{\mathbb{C}}_\sigma$. Then the Maurer-Cartan form of F_- is of the form ([37], section 4.1):

$$F_-^{-1}dF_- = \mu^{-1}\begin{pmatrix} 0 & a^t & 0 \\ b & 0 & a \\ 0 & b^t & 0 \end{pmatrix}. \tag{7.6.7}$$

Here a and b are 1-forms with values in $\mathbb{R}^3$ satisfying

$$b^t \cdot b = b^t \cdot a = 0 \quad \text{and} \quad b \neq 0. \tag{7.6.8}$$

Moreover, a and b are closed 1-forms of the type

$$b = \beta dz \quad \text{and} \quad a = \gamma dz + \zeta d\bar{z}. \tag{7.6.9}$$

These are the normalized potentials $\xi^{(\mu)}$ used in this part.

So far things are "as usual". The converse construction, however, will involve some "unusual" steps. Starting from some normalized potential $\xi^{(\mu)}$, which is of the form 7.6.7 and satisfies 7.6.8 and 7.6.9, the right side of 7.6.7 defines an integrable 1-form. Solving the ODE $F^{-1}dF = \xi^{(\mu)}$ with trivial initial condition we obtain some matrix function $F_-^{(\mu)} : \mathbb{D} \longrightarrow M^-SO_o(4,1)_\sigma$. The generalized Iwasawa splitting [40] yields

$$F_-^{(\mu)} = F^{(\mu)} \cdot F_+^{(\mu)}, \tag{7.6.10}$$

with $F^{(\mu)} \in MSO_0(4,1)_\sigma$ and $F_+^{(\mu)} \in MSO_0(4,1)^{\mathbb{C}}_\sigma$.

The crucial point is whether one is able to construct a Willmore immersion from the map $F^{(\mu)}$. It is shown in [37] that the freedom in the splitting 7.6.10 can used so that the resulting extended lift produces a harmonic map via projection onto $S^{3,1}$.

Remark 7.6.2. The case of Willmore immersions has some interesting aspects:

1. The property of being a Willmore immersion is rephrased equivalently by the harmonicity of a map which is *not* the usual Gauß map. A similar situation was encountered by Inoguchi [39] in his investigation of surfaces of constant mean curvature in hyperbolic space $\mathbb{H}^3$, and most recently in [11]. It would be interesting to look for surface classes which are defined by the harmonicity of some "natural map" and to investigate

to what extent such classes can be treated by some loop group approach akin to the ones presented in this article.

2. The transition from the frame to the harmonic map or the immersion is not given by a "Sym-type formula". However, as in the case of affine spheres the freedom in the generalized Iwasawa splitting permits one to choose the splitting so that the transition can be made. It would be interesting to understand for which surface classes a Sym-type formula does exist.

7.7 Remarks 7.7.1. *Other surface classes*: There are several other classes of surfaces which are natural candidates for a "loop group approach". Among these are surfaces of constant mean curvature in the sphere S^3 and the hyperbolic space $\mathbb{H}^3$, as well as surfaces of positive constant Gauß curvature in $\mathbb{R}^3$ and surfaces of constant Gauß curvature in $\mathbb{E}^3_1$. Spacelike surfaces of constant mean curvature in $\mathbb{E}^3_1$ fall into this class as well [54].

There are several types of "loop group involvement". On the one hand, surfaces like the ones of constant positive Gauß curvature in $\mathbb{R}^3$ are parallel to surfaces of constant mean curvature, and thus admit a "loop group approach". Next, surfaces like the surfaces of constant mean curvature in hyperbolic space $\mathbb{H}^3$ have been investigated quite successfully by introducing "loop parameters" and "extended frames" [33],[32],[54]. From the point of view of this article, however, it would be most interesting to see whether the surfaces of the classes listed above (and possibly others) can be characterized directly by the harmonicity of some "generalized Gauß map" into some symmetric (?) space and whether it is possible to construct all these Gauß maps (and thus all the surfaces in the class considered) from unconstrained "potentials". Work in this direction is presently being conducted by several mathematicians (Inoguchi, Inoguchi-Toda [39], Rossman and others).

7.7.2. *More powers of* λ: Hamiltonian stationary Lagrangian surfaces in $\mathbb{C}^2$ have been investigated via a loop group approach in [38]. An interesting feature of this paper is the use of potentials in which not only λ^{-1} occurs, but also λ^{-2}. On the other hand, the integration of the relevant differential equations can be carried out by hand. In this sense the loop group approach is only a guide and not essential for the theory. In contrast, in the investigation of harmonic maps into k-symmetric spaces [10], potentials involving negative powers $\lambda^{-k}, \ldots, \lambda^{-1}$ occur. In this case, however, only special harmonic maps ("primitive harmonic maps") can be treated.

7.7.3. *More features* of the theory of constant mean curvature in $\mathbb{R}^3$ can be carried over to the surface classes discussed in the last few sections. This certainly applies to dressing. *Wu's Formula* seems to work in all cases. *Bäcklund transformations* also appear. These transformations usually represent very specific geometric constructions of new surfaces from known ones (such as using line congruences). In all cases known to the author they can be interpreted as dressing transformations, and are usually associated with very simple dressing matrices. This is well known for the pseudospherical case [53], but seems to apply also to the Tzitzeica transformation of affine spheres and to "Date's direct method" [13]. It would be interesting to see this made explicit.

7.7.4. *A general theory* providing a framework for all known loop group approaches to the description of surfaces (and other geometric quantities) is still missing. At this point the method seems to work best for maps $\phi : M \longrightarrow M'$ for which there exists a "Gauß map" $N : M \longrightarrow S$ from M into some (not necessarily Riemannian) symmetric space S, such that N is " harmonic" if and only if the map ϕ is in the class considered. The paper [11] may be of particular influence in this context.

7.7.5. *Higher Dimensions*: The presentation so far has addressed exclusively maps from surfaces. In view of the fact that the theories presented so far all relate to "harmonic" maps, one should expect that one can extend the theory to include "pluriharmonic" maps. This seems to be the case [15]. There does exist, however, one paper which uses a loop group approach for maps defined on higher dimensional manifolds [31] in a substantially different way. In this case isometric immersions of space forms are discussed. The paper concentrates on immersions of "finite type". It seems that also "potentials" could have been considered. It would be interesting to clarify this and to relate these potentials to the parameter space of the Cartan-Kähler theory. In general it would be very interesting to find loop group approaches to higher dimensional "completely integrable soliton" equations in a direct geometric context.

7.7.6 *Complex Theory*: While a general theory of classes of surfaces which can be treated by some unified loop group approach seems to be out of reach at this point, it seems to be at least feasible to expect that some "complexified" theory should be able to provide a "higher point of view", at least for some cases. Most of the surface classes discussed in the previous sections are treated in a double-loop-group setting and their potentials consist of pairs of Lie algebra valued differential forms. The only exception is the class of surfaces of constant mean curvature in $\mathbb{R}^3$. It is easy to see that also this case has naturally a double-loop-group setting. For $g \in \Lambda Sl(2,\mathbb{C})_\sigma$ we put $\hat{g} = (g, (\bar{g}^t)^{-1})$. Thus $\Lambda Sl(2,\mathbb{C})_\sigma$ is identified with a subgroup of the double loop group $\mathcal{H} = \Lambda Sl(2,\mathbb{C})_\sigma \times \Lambda Sl(2,\mathbb{C})_\sigma$. If F is the frame of a constant mean curvature surface, then $\hat{F} = (F,F) \in \mathcal{U}$, the diagonal subgroup in $\mathcal{H}$. As in the previous cases we set $\mathcal{H}^- = \Lambda^- Sl(2,\mathbb{C})_\sigma \times \Lambda^+ Sl(2,\mathbb{C})_\sigma$ and $\mathcal{H}^+ = \Lambda^+ Sl(2,\mathbb{C})_\sigma \times \Lambda^- Sl(2,\mathbb{C})_\sigma$. Then $\mathcal{H}^- \cdot \mathcal{H}^+$ is open and dense in $\mathcal{H}$. The Birkhoff splitting $\hat{F} = \hat{F}_- \cdot \hat{F}_+$, which is $F = F_- \cdot F_+$ and $F = (\bar{F_-}^t)^{-1} \cdot (\bar{F_+}^t)^{-1}$, yields the pair of potentials $\hat{\xi} = (\xi, -\bar{\xi}^t) \in \operatorname{Lie}\mathcal{H}_-$, the Lie algebra of $\mathcal{H}_-$ [24]. Here the first factor depends only on z, while the second factor depends only on $\bar{z}$. Conversely, starting from $\hat{\xi}$, one solves the pair of ODE's $d\hat{F} = \hat{F} \cdot \hat{\xi}$ with initial condition $\hat{F}(z_0) = (I,I)$, and splits $\hat{F} = (F,F) \cdot (V_+, V_-)$ in $\mathcal{U} \cdot \mathcal{H}_+$. It is easy to see that, because of the special form of $\hat{\xi}$, the matrix F is unitary. Clearly, the latter splitting is a double-loop-group version of the classical Iwasawa splitting. It is natural to generalize the procedure above. Let $\Omega = (\xi(z), \eta(w))$ be a differential form in two independent variables z and w which takes values in the Lie algebra of $\mathcal{H}_-$. Then, solving the pair of ODE's $d\hat{F} = \hat{F} \cdot \Omega$ with initial condition I, one obtains a pair of functions $\hat{F}_- = (U_-(z), U_+(w))$. The generalized Iwasawa splitting $\mathcal{H} \approx \mathcal{U} \cdot \mathcal{H}_+$ then produces some $U(z,w)$, such that $(U_-(z), U_+(w)) = (U,U) \cdot (V_+, V_-)$, where U, V_+, V_- depend on z and on w. Thus we obtain the familiar formula

$$U = U_- \cdot V_+^{-1} = U_+ \cdot V_-^{-1}. \tag{7.7.1}$$

It is tempting to predict that U plays the role of some (perhaps modified) frame of some complex surface. This surface should have many interesting features. In particular, it should contain the surfaces of constant mean curvature in $\mathbb{R}^3$, the pseudospherical surfaces in $\mathbb{R}^3$, and the timelike surfaces of constant mean curvature in $\mathbb{E}^3_1$ as "real forms".

Update added, December 2007: Since this article was written, progress has been made on several fronts. A list of additional references ([60]-[108]) is provided below concerning matters closely related to the questions discussed in this article and in particular to the remarks above. This list is by no means complete and we apologize to everyone whose work is not mentioned.

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