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On the Castelnuovo-Weil lattices, I

Viet Nguyen-Khac and Tetsuji Shioda

Abstract.

The Castelnuovo-Weil lattice of a curve refers to the ring of correspondence of the curve or the endomorphism ring of its Jacobian variety, viewed as a lattice with respect to the natural trace form. The main subject is the minimal norm $\mu(C)$ of this lattice, and we discuss the question as to whether $\mu(C) = 2g$ holds true.

§1. Introduction

In the development of algebraic geometry, curves have always played an important role as the basic objects of dimension one. Yet some higher dimensional varieties have been indispensable for studying curves themselves, such as the Jacobian varieties, introduced by Riemann, and the correspondence theory of curves, started by Hurwitz and developped by the Italian geometers, especially by Castelnuovo and Severi.

Given two curves C and C', a correspondence between them is simply a divisor D on the product surface $C \times C'$. A correspondence is said to be equivalent to 0 (denoted $D \equiv 0$) if it is linearly equivalent to a divisor of the form $\mathfrak{a} \times C' + C \times \mathfrak{b}$ where $\mathfrak{a}$ or $\mathfrak{b}$ is a divisor on C or C'. Castelnuovo's theory gives a numerical criterion for equivalence; namely it attaches to every correspondence an integer $\gamma(D) \geq 0$, called the "equivalence defect", such that

$$\gamma(D) = 0 \Longleftrightarrow D \equiv 0.$$

Let D' be the correspondence between C' and C obtained by switching the two factors of the product $C \times C'$. The composition of correspondences can be defined in a natural way, and then $D' \circ D$ is a correspondence between C and itself. If Z is any such correspondence, the number

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of its fixed points is given by the intersection number $(Z.\Delta)$ of Z with the diagonal Δ on $C \times C$. Put

$$S(Z) = d(Z) + d'(Z) - (Z.\Delta) \tag{1.1}$$

where $d(Z), d'(Z)$ are the degrees, i.e. the intersection numbers of Z with $x \times C, C \times x'$. As is easily seen, $S(Z)$ depends only on the equivalence class ξ of Z so that one can write $S(Z) = \sigma(\xi)$. Then σ has the properties of a trace on the ring of correspondence classes. The equivalence defect can then be expressed as $\gamma(D) = S(D' \circ D)$, and the Castelnuovo's inequality asserts that

$$S(D' \circ D) \geq 0, \quad S(D' \circ D) = 0 \Leftrightarrow D \equiv 0. \tag{1.2}$$

In other words, we have

$$\sigma(\xi'\xi) \geq 0, \quad \sigma(\xi'\xi) = 0 \Leftrightarrow \xi = 0. \tag{1.3}$$

In this form, it is the fundamental theorem on correspondences, and Weil has given two methods of proof valid in any characteristic: the first one in terms of correspondence theory ([13]), and the second one in terms of endomorphisms of Jacobian variety ([14], cf. [15]). We recall some of the arguments in §2. [There is a third proof based on the Hodge's index theorem (cf. [4], [5]).]

The most remarkable application of this positivity theorem is Weil's proof of the Riemann hypothesis for curves over a finite field.

Now the ring $\mathfrak{C} = \mathfrak{C}(C)$ of correspondence classes on C can be regarded as a positive-definite even integral lattice for which the norm of an element $\xi \in \mathfrak{C}$ is given by $\sigma(\xi' \cdot \xi)$. We propose to call this lattice the *Castelnuovo-Weil lattice* of a curve C (CWL in short).

In this paper, we discuss what we can say about the minimal norm $\mu(C)$ of this lattice:

$$\mu(C) := \mathrm{Min}\ \sigma(\xi' \cdot \xi) \quad (\xi \in \mathfrak{C}, \xi \neq 0). \tag{1.4}$$

§2. Problem setting

Let k be a field. By a curve, we mean a smooth, geometrically irreducible, projective curve defined over k. Let C be a curve of genus $g \geq 1$ and let J_C denote the Jacobian variety of C.

Given any correspondence D on $C \times C$, the map $cl(P - Q) \mapsto cl(D(P) - D(Q))$ defines an endomorphism α_D of J_C. Here $cl(\mathfrak{a})$ denotes the linear equivalence class of a divisor $\mathfrak{a}$ on C, and for any $P \in C$, $D(P)$ is the image of P under the correspondence D, i.e. a divisor on C such that $D.(P \times C) = P \times D(P)$. Then $D \mapsto \alpha_D$ induces a ring homomorphism from $\mathfrak{C}(C)$ to $\mathrm{End}(J_C)$, and this gives rise to an isomorphism:

$$\mathfrak{C}(C) \cong \mathrm{End}(J_C), \quad \xi = [D] \mapsto \alpha = \alpha_D \tag{2.1}$$

(see [14, VI, §43, Cor. 2]).

Further, by considering the representation on the Tate module $T_\ell(J_C)$, ℓ being a prime number $\neq \mathrm{char}(k)$, the trace $\sigma : \mathfrak{C}(C) \to \mathbb{Z}$ is transformed to the trace Tr on $\mathrm{End}(T_\ell(J_C))$ in the sense of linear algebra. So we can work equally in either $\mathrm{End}(J_C)$ or $\mathfrak{C}(C)$.

Now the class of the diagonal on $C \times C$, $\delta = [\Delta]$, is the identity of the ring $\mathfrak{C}(C)$, and we have

$$\sigma(\delta) = \mathrm{Tr}(1) = 2g. \tag{2.2}$$

This follows from (1.1) since we have $d = d' = 1$ for $Z = \Delta$ and $(\Delta^2) = 2 - 2g$ (the Euler number of C) in terms of correspondences, while it is obvious in the linear algebra setting.

Hence the minimal norm is always smaller or at most equal to $2g$:

$$\mu(C) \leq 2g. \tag{2.3}$$

More than fifteen years ago one of the authors has posed a naive question:

Question 2.1. *Does the equality $\mu(C) = 2g$ always hold?*

In the positive direction, this is true for small genus and also for the "generic" curve of any genus g. Namely we have:

Proposition 2.2. *Question 2.1 is true for curves of genus $g \leq 2$ and hyperelliptic curves of genus 3.*

The proof will be postponed till the end of this section.

Proposition 2.3. *For the generic curve of any genus g over an algebraically closed field k, we have $\mu(C) = 2g$.*

Proof This follows from Mori's result that one has $\mathrm{End}(J_C) \simeq \mathbb{Z}$ for a generic curve; see [7]. If we denote by $h \geq 1$ the rank of $\mathrm{End}(J_C)$, then the Picard number of the product surface $X = C \times C$ is equal to $\rho = h + 2$ by (2.1). We claim that $h = 1$, i.e. $\rho = 3$ for C generic. Indeed, considering the moduli space of genus g curves, if C specializes to C' and $X = C \times C$ to $X' = C' \times C'$, then we have $\rho \leq \rho'$ by the intersection theory. By Mori, we have $h' = 1$ and $\rho' = 3$ for some C'. This shows that $\rho = 3$ and hence $h = 1$ for C generic. *q.e.d.*

For the proof of Proposition 2.2, let us recall Weil's first proof ([13]) for the positivity (1.3):

Theorem 2.4 (Castelnuovo-Weil). *We have $\sigma(\alpha.\alpha') > 0$ for any non-zero α.*

For proving this, Weil ([13, II]) argues as follows. Assume that D is an effective p_1-flat correspondence (i.e. it has no vertical components). Letting $d = d(D)$, $e = d'(D)$, we write the value of D at P as $D(P) = \sum_{i=1}^{d} Q_i$. Because of p_1-flatness assumption, one sees that the 1-cycle $U := (C \times D).(D' \times C)$ in $C \times C \times C$ is well defined, so by the definition of the composition product, we have $D \circ D' = p_{13*}(U)$.

Let V be the cycle in $C \times C \times C$ obtained from U by interchanging two first factors. Then V is p_1-flat w.r.t. the first factor in $C \times C \times C$ and $D \circ D' = p_{23*}(V)$. Also $V(P) = D(P) \times D(P) = \sum_{i=1}^{d} Q_i \times Q_i + \sum_{1 \le i \ne j \le d} Q_i \times Q_j$. So V can be written in the form $V = V_0 + V_1$, where $V_0(P) = \sum_{i=1}^{d} Q_i \times Q_i$ and $V_1(P) = \sum_{1 \le i \ne j \le d} Q_i \times Q_j$. Furthermore it can be verified that $p_{23*}(V_0) = e\Delta$. Thus one obtains $D \circ D' = e\Delta + T$, where $T := p_{23*}(V_1)$. Obviously $d(T) = d'(T) = e(d-1)$.

In particular, if $d = 1$, then the part V_1 vanishes. Hence one concludes $D \circ D' = e\Delta$. It follows that, if $d = 1$,

$$\sigma(\alpha.\alpha') = e\sigma(\delta) = 2ge. \tag{2.4}$$

[This fact can be also shown by a direct argument as follows (cf. [13, II, §9, Prop.4]). Let $D'(Q) = \sum_{i=1}^{e} P_i$. Since $P_i \times Q$, $1 \le i \le e$ are lying in different fibre components, so $D(P_i) = Q$, $1 \le i \le e$. It follows that $D(D'(Q)) = eQ$. Therefore $D \circ D' = e\Delta$, because $D \circ D'$ is symmetric and by the definition of the composition product.]

Let $\mathfrak{d}$ be a divisor on C such that there are d linearly independent functions $\varphi_1, \ldots, \varphi_d \in \mathfrak{L}(\mathfrak{d}) := \{f \in \overline{k}(C) : \quad (f) + \mathfrak{d} \ge 0\} \cup \{0\}$. We assume also $Q_1, \dots, Q_d \notin \mathrm{Supp}\,(\mathfrak{d})$ and

$$(*) \qquad \begin{array}{l} \varphi_i(P_j) \neq \infty, \text{ for all } i, j = 1, \dots, d, \\ \det\big(\varphi_i(Q_j)\big)_{1 \leq i,j \leq d} \neq 0. \end{array}$$

A computation on the d-fold product $C \times \cdots \times C$ (provided $d \geq 2$) shows that $\deg(T.\Delta) \leq 2e \deg(\mathfrak{d})$. Consequently for $\alpha := [D]$ one obtains

$$(2.5) \qquad \sigma(\alpha.\alpha') \geq 2e\big[d + g - 1 - \deg(\mathfrak{d})\big],$$

Letting $\mathfrak{d} = \mathfrak{k}$ be the canonical divisor of C and taking $d = g$ ([13, II,§14]), we get

$$(2.6) \qquad \sigma(\alpha.\alpha') \geq 2e,$$

provided $g \geq 2$ (the case $g = 1$ is obvious). Since $e > 0$ for non-zero α, this proves Theorem 2.4.

Now we turn to the proof of Proposition 2.2. Keeping the notation above, note first that if $e \geq g$, then we have $\sigma(\alpha.\alpha') \geq 2g$. So it remains to treat the case (i) $e = 1, g \leq 3$ and (ii) $g = 3, e = 2$. Suppose that $e = 1$; then, after changing the symmetric role between d, e and by (2.4), we obtain $\sigma(\alpha.\alpha') = 2gd \geq 2g$. In general, the problem is reduced to the existence of special linear systems on C satisfying condition $(*)$. Next, in the case when C is a hyperelliptic curve of genus 3, we proceed as follows. Assuming $e = d = 2$ (otherwise inequality (2.5) proves Prop. 2.2), we take $|\mathfrak{d}| = g_2^1$ and Q_1, Q_2 not lying in the same fiber of the hyperelliptic morphism. It is easy to make the choice so that $(*)$ is satisfied, hence the proof of Proposition 2.2 follows.

As we see in §4, Question 2.1 will have a counterexample. Yet we do not know any counterexample in the hyperelliptic case:

Question 2.5. *Does $\mu(C) = 2g$ hold for any hyperelliptic curve C of genus g?*

§3. μ-problem for polarized abelian varieties

Let us consider the corresponding problem for polarized abelian varieties, leaving for a moment the case of Jacobian varieties.

Let (A, L) be a polarized abelian variety of dimension d over k. Recall that L is an ample line bundle. It corresponds to an isogeny $\varphi_L \colon A \to A^\vee = Pic^0(A), \quad x \mapsto \tau_x^* L \otimes L^{-1}$ (τ_x is the translate to x). So for $\xi \in \mathrm{End}(A)$, $\xi^\vee : A^\vee \to A^\vee$ we know that the Rosati conjugate $\xi' \in \mathrm{End}^0(A)$ is defined as $\xi' = \varphi_L^{-1}.\xi^\vee.\varphi_L : A \to A$. In particular, if L is a principal polarization, i.e. $\deg(\varphi_L) = 1$, then one sees that $\xi' \in \mathrm{End}(A)$ (in the case $A = \mathrm{Jac}(C)$ this fact is clearly true via correspondence language above). For instance by using ℓ-adic representation on Tate's modules one can see in the form of matrices: $\xi \longleftrightarrow \xi_T$, $\varphi_L \longleftrightarrow m$, $\xi' \longleftrightarrow m^{-1}.{}^t\xi_T.m$ which is the conjugate of the transpose matrix.

The conclusion of the Castelnuovo-Weil theorem for (A, L) can be deduced from the ampleness of L and the following well-known formula ([6], V, §3, [8], §21, [1], V, 17.3) for $\mathrm{Tr}(\xi.\xi')$:

$$\mathrm{Tr}(\xi.\xi') = \frac{2d}{L^d}(L^{d-1}.\xi^*(L)).$$

In particular, if the curve C is equipped with a k-rational point, say P_0, then for the canonically polarized Jacobian (J_C, Θ) one gets

$$\mathrm{Tr}(\xi.\xi') = 2(\psi_{P_0}(C).\xi^*(\Theta)), \tag{3.1}$$

where $\psi_{P_0} : C \to J_C, \quad P \mapsto cl(P - P_0)$ is the canonical embedding.

By analogy with J_C one defines

$$\mu(A) := \min \mathrm{Tr}(\xi.\xi'),$$

where ξ runs over all non-zero endomorphism of A.

The first thing to observe in this case is that $\mu(A) < 2d$ can happen even for principally polarized abelian varieties of dimension $d = 2$. Indeed, we have:

Proposition 3.1. *Let $A = E \times E$ be the selfproduct of an elliptic curve E with the decomposable principal polarization $\Theta = o \times E + E \times o$. Then we have $\mu(A) = 2 < 2d = 4$.*

Proof Immediate. N.B. This does not contradict the genus 2 case of Prop.2.2, because (A, Θ) cannot be isomorphic to a canonically polarized Jacobian of genus 2 curve by [16].

Nevertheless, for isogenies, we have the following proposition.

Proposition 3.2. *Let (A, L) be a polarized abelian variety of dimension d. Let $\xi \in \mathrm{End}(A)$ be an isogeny. Then $\mathrm{Tr}(\xi.\xi') \geq 2d$, where $\xi' \in \mathrm{End}^0(A)$ denotes the Rosati conjugate w.r.t. the polarization L.*

Proof. Since $\xi.\xi'$ is symmetric and positive, so by [6], V, §3, we know that $2d$ eigenvalues $\lambda_1, \dots, \lambda_{2d}$ of $\xi.\xi'$ are real positive numbers. On the other hand ξ and ξ' (as in $\mathrm{End}^0(A)$) have the same characteristic polynomial. Hence

$$\deg(\xi) = \deg(\xi') = \det(\xi_T) = \det({}^t\xi_T),$$

so $\deg(\xi.\xi') = (\det(\xi_T))^2$ is a non-zero integer, as ξ is an isogeny. It remains to use the (Cauchy) arithmetic-geometric mean inequality

$$\frac{1}{2d}\sum_{i=1}^{2d} \lambda_i \geq \sqrt[2d]{\prod_{i=1}^{2d} \lambda_i} = \sqrt[2d]{\deg(\xi.\xi')} \geq 1$$

as required. *q.e.d.*

Corollary 3.3. *In the notations above,* $\mathrm{Tr}(\xi.\xi') = 2d$ *characterizes the automorphisms of* (A, L).

Corollary 3.4. *Let* (A, L) *be a simple polarized abelian variety of dimension* d. *Then* $\mu(A) = 2d$ *holds.*

§4. Refined Question

As is well-known, the existence of a covering from C to a curve of genus ≥ 1 gives rise to a decomposition of J_C, so it may also affect the existence of special linear systems on C. The aim of this section is to clarify this situation, which will lead to a counterexample to Question 2.1 and to the refined conjecture below. Let $f\colon C \to C_1$ be a finite covering of curves of genera g, g_1 respectively, and let $d = \deg(f)$. Then f induces two homomorphisms (the norm and conorm maps):

$$f_*\colon J_C \to J_{C_1},\ f_*(\mathcal{O}_C(\sum n_i P_i)) := \mathcal{O}_{C_1}(\sum n_i f(P_i)),$$

$$f^*\colon J_{C_1} \to J_C,\ f^*(\mathcal{O}_{C_1}(\sum m_i Q_i)) := \mathcal{O}_C(\sum m_i f^*(Q_i))$$

(sometimes f_* is denoted as N_f). We have the following two simple properties:

1) $f_* = f^{*\vee}$ (this justifies the term "norm" and "conorm").

2) $f_*.f^* = d_{J_{C_1}}$ the multiplication by d on J_{C_1}.

Let us mention also the following fact. We identify $J_C \simeq J_C^\vee$ and $J_{C_1} \simeq J_{C_1}^\vee$ via canonical polarizations φ_Θ and φ_{Θ_1} respectively. If $\imath :=$

$f^*\colon J_{C_1} \to J_C$ is injective, then one can consider the pull-back of Θ via $\imath$ which defines a polarization on J_{C_1}. We have the following standard commutative diagram

$$\begin{array}{ccc} J_C & \xrightarrow{\varphi_\Theta} & J_{C_1}^\vee \\ {\scriptstyle \imath}\big\uparrow & & \big\downarrow{\scriptstyle \imath^\vee} \\ J_{C_1} & \xrightarrow{\varphi_{\imath^*\Theta}} & J_{C_1}^\vee \end{array}$$

so $\varphi_{\imath^*\Theta} = f_*.f^* = d_{J_{C_1}}$. It means that $\imath^*\Theta = d\Theta_1$. In other words, if f^* is injective, then $\varphi_{\imath^*\Theta}$ (the polarization on J_{C_1} induced by f^*) is d times canonical one (i.e. φ_{Θ_1}).

Let $\xi_1 \in \mathrm{End}(J_{C_1})$. Then $\xi := f^*.\xi_1.f_* \in \mathrm{End}(J_C)$, or equivalently $d^2\xi_1 = f_*.\xi.f^*$. If C has a k-rational point, say P_0, $P_0' := f(P_0)$, then by using (3.1) and the standard commutative diagram above one has

$$\begin{aligned} \mathrm{Tr}(\xi.\xi') &= 2(\psi_{P_0}(C).\xi^*(\Theta)) \\ &= 2(f_*\psi_{P_0}(C).\xi_1^*.\imath^*(\Theta)) \\ &= 2(\psi_{P_0'}(f_*(C)).d\xi_1^*(\Theta_1)) = d^2\mathrm{Tr}(\xi_1.\xi_1'). \end{aligned}$$

Proposition 4.1. *Taking* $\xi_1 = \delta_{C_1}$ *one gets* $\mathrm{Tr}(\xi.\xi') = 2d^2 g_1$.

Remark 4.2. It can be seen easily that in Proposition 4.1

$$\xi \longleftrightarrow \Gamma_f^* := (id \times f)^*(\Gamma_f) = (f \times f)^*(\Delta_{C_1}),$$

where $\Gamma_f \subset C \times C_1$ denotes the graph of f. So using bilinear form $\sigma(\cdot,\cdot)$ we obtain another proof of this statement.

Counterexample to Question 2.1 Take a bi-elliptic curve C of genus $g \geq 5$ (which clearly exists). Namely assume that C admits a double covering to an elliptic curve C_1. Then, with the notation above, we have $d = 2$ and

$$\mathrm{Tr}(\xi.\xi') = 2d^2 = 8 < 2g.$$

Thus the naive question 2.1 is false for any $g \geq 5$.

We refine this situation as follows.

Question 4.3. *In the notations of* §1 *we have* $\mu(C) = \mu_0$, *where* $\mu_0 := \mathrm{Min}\{2d^2 g_1\}$, *the* Min *is taken over all coverings* $f\colon C \to C_1$ *with* $d = \deg f$ *and* $g_1 := g(C_1) \geq 1$.

Precisely Question 4.3 consists of two parts. Let L be the sublattice generated by all endomorphisms corresponding to the coverings $f\colon C \to C_1$ with $g_1 \geq 1$. Then the problem is, in fact, divided into two statements: $\mu(C) = \mu(L)$ and $\mu(L) = \mu_0$. As a heuristic consideration for the second part one might involve the following descending argument. Assume we have two endomorphisms γ_1, γ_2 corresponding to the coverings $f\colon C \to C_i$ of degree d_i and $g_i := g(C_i) \geq 1$, $i = 1, 2$. Let us denote by C_{12} the (normalized) image of the natural morphism $f_1 \times f_2\colon C \to C_1 \times C_2$ and consider the induced morphism $f_{12}\colon C \to C_{12}$, say of degree d_{12} and $g_{12} := g(C_{12})$. Then by using the Castelnuovo-Weil inequality (Theorem 2.4) one can see easily that either $d_{12}^2 g_{12} \leq \mathrm{Min}\{d_1^2 g_1, d_2^2 g_2\}$, or the angle between γ_1 and γ_2 is big (the angle between two endomorphisms can be defined in a natural way via the positive definite bilinear form $\sigma(\cdot,\cdot)$ defined in §2).

Proposition 4.4. *Question 4.3 implies Question 2.5.*

Proof. This is another application of Theorem 2.4. Indeed applying what we said above to the natural morphism $f_1 \times \iota\colon C \to C_1 \times \mathbb{P}^1$ where $f_1\colon C \to C_1$ is a morphism of degree d_1 and $\iota\colon C \to \mathbb{P}^1$ is the canonical hyperelliptic morphism, in view of Theorem 2.4, we have

$$g \leq (d_1 - 1) + d_1 g_1.$$

The right-hand side is clearly $< d_1^2 g_1$, provided $d_1 \geq 2$. *q.e.d.*

It should be noted that Question 4.3 has a close connection with the following important question in the theory of abelian varieties: which curves can appear in a polarized abelian variety? We shall treat this feature in a future publication.

§5. Some consequences

Let C be a curve of genus g over a finite field $\mathbb{F}_q$. The Hasse-Weil theorem says that the number $N_q(C)$ of $\mathbb{F}_q$-rational points on C can be at most $q + 1 + 2g\sqrt{q}$. If this bound attained such a curve C is called maximal.

Theorem 5.1. *If C is not maximal, and $\mu(C) = 2g$, then*

$$N_q(C) < q + 1 + 2g\sqrt{q - 3/4}.$$

Proof. Let us consider 2-dimensional lattice L_2 associated to

$$f(m,n) := 2gqm^2 + 2(q+1-N_q(C))mn + 2gn^2$$

whose discriminant $D(L_2)$ is $(q+1-N_q(C))^2 - 4g^2q$. Note that $f(m,n)$ is nothing but $\sigma(m\Gamma_\pi + n\Delta, m\Gamma_\pi + n\Delta)$ in the notation of §§2, 4, where π denotes the Frobenius morphism. So the positive definiteness of $\sigma(\cdot,\cdot)$ gives us the mentioned Hasse-Weil bound. Moreover if C is not maximal, then $\operatorname*{Min}_{(m,n)\neq(0,0)} f(m,n) \geq \mu(C)$. In the dimension 2 the Hermite constant

$$\gamma(L_2) := \operatorname*{Min}_{(m,n)\neq(0,0)} f(m,n)/\sqrt{-D(L_2)}$$

is known to be $\leq \sqrt{4/3}$ ([2], I). Moreover the equality holds for the form $2x^2 + 2xy + 2y^2$. Hence

$$N_q(C) < q + 1 + 2g\sqrt{q - \frac{3}{4}\left(\frac{\mu(C)}{2g}\right)^2}$$

and one obtains the statement of the theorem. *q.e.d.*

If $q = 2, 3$ one may take simply $m = n = 1$ in the argument above to get

$$N_q(C) \leq q + 1 + gq$$

that is Serre's bound ([9]) for this case. It should be noted that Serre's bound

$$N_q(C) \leq q + 1 + gm, \quad m := [2\sqrt{q}]$$

can be obtained in a similar vein as of §3 by dealing with $\xi_r := \pi + r\delta \in \mathrm{End}(J_C) \otimes \mathbb{R}$, where π denotes the Frobenius endomorphism and r is a (real) root of $x^2 - (m+1)x + q = 0$. To this end it suffices to use arguments of §3 taking into account

$$\xi_r\xi_r' = r\big(\pi + \pi' + (m+1)\delta\big)$$

and

$$\sqrt[g]{\det\big((\xi_r)_T\big)} \geq r.$$

In conclusion we make a remark concerning a further generalization. Taking $X = C_1 \times C_2$ in the Weil formalism of correspondences, with two different curves C_1, C_2 as in §1, one can define more generally the CWL of type $\mathfrak{C}(C_1, C_2)$ (cf. [12]). The case $C_2 = E-$ an elliptic curve leads to the well-known Mordell-Weil lattices treated in [3], [10] or [11].

For arbitrary C_1, C_2 the question about μ seems very difficult and still unknown.

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Viet Nguyen-Khac
Institute of Mathematics
18 Hoang Quoc Viet
10307 Hanoi VIETNAM

Current address
Vietnam Information Security Lab.
17 Hoang Sam str., Nghia Do - Cau Giay
Hanoi VIETNAM
E-mail address: `nkviet@math.ac.vn`

Tetsuji Shioda
Department of Mathematics
Rikkyo University
3-34-1 Nishi-Ikebukuro Toshima-ku
Tokyo 171-8501 JAPAN
E-mail address: `shioda@rkmath.rikkyo.ac.jp`