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Eigenfunctions for substitution tiling systems

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Abstract.

We prove that for the uniquely ergodic $\mathbb{R}^d$-action associated with a primitive substitution tiling of finite local complexity, every measurable eigenfunction coincides with a continuous function almost everywhere. Thus, topological weak-mixing is equivalent to measure-theoretic weak-mixing for such actions. If the expansion map for the substitution is a pure dilation by $\theta > 1$ and the substitution has a fixed point, then failure of weak-mixing is equivalent to θ being a Pisot number.

§1. Introduction

In this article we consider self-affine (substitution) tilings of $\mathbb{R}^d$ and associated dynamical systems. These tilings are of translationally finite local complexity, that is, they have a finite number of tiles and patches of a given size, up to translation. A self-affine tiling has the property that if we inflate it by a certain expanding linear map, then the original tiling may be obtained by subdividing the inflated tiles according to a prescribed rule. Self-affine tilings have been extensively studied; they arise, in particular, in connection with Markov partitions of toral automorphisms and as models for quasicrystals. See the next section for some historical comments.

We focus on the dynamical system, which may be defined (under appropriate assumptions) as the translation action by $\mathbb{R}^d$ on the orbit closure of a given tiling. Assuming primitivity of the tiling substitution, we obtain a uniquely ergodic action. The spectrum (more precisely,

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the spectral measure of the associated unitary group) is a fundamental invariant of the dynamical system, and it is closely related to the diffraction spectrum studied by physicists (see [15, 32]). In particular, the point spectrum (the set of eigenvalues) corresponds to the Bragg peaks ("sharp bright spots") in the diffraction picture associated with the tiling.

The new result (announced in [59]) proved here is that every measurable eigenfunction can be chosen to be continuous. Thus, the topological and ergodic-theoretic point spectra are the same (such systems are sometimes called homogeneous [51]). This extends a result of Host [23] on symbolic substitution $\mathbb{Z}$-actions. The condition that all eigenfunctions are continuous has been used recently in the work on mathematical quasicrystals, see [3, 31].

The characterization and existence of eigenvalues have a link with Number Theory, namely with Pisot (or PV) numbers. We present a proof of the following statement: if the expanding map is a pure dilation by $\theta > 1$, then the dynamical system has non-trivial eigenvalues (that is, it is not weak-mixing) if and only if θ is Pisot. A similar result was obtained in [18] in the context of diffraction spectrum, by different methods.

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§2. Historical Remarks

Here we make some historical remarks, in order to put our work in a larger context. They are necessarily incomplete, and we apologize for any inadvertent omissions. We refer to the survey by E. A. Robinson, Jr. [52] for more details.

The story begins in mathematical logic [64, 9] with the discovery of *aperiodic prototile sets*, i.e. sets of prototiles which can tile the plane, but only in a such a way that the resulting tiling does not have any translational symmetries. One of the most interesting aperiodic prototile sets was constructed in the early 1970's by R. Penrose [39]. Penrose tilings have an additional feature that they can be generated using a tiling substitution. See [20] for a detailed description of Penrose tilings and many other aperiodic sets.

In 1984 physicists discovered what came to be known as *quasicrystals* [55]. These are metallic alloys, which, like a crystal, have a sharp X-ray diffraction pattern, but unlike a crystal, have an aperiodic atomic structure. Aperiodicity was inferred from the "forbidden" 5-fold symmetry of the diffraction pattern. It turned out that Penrose tilings have similar features, so they became a focus of many investigations, both by physicists and by mathematicians. Other tilings have been studied from this point of view as well. See [57, 2] for an introduction to the mathematics of quasicrystals and further references.

The third source of our subject is symbolic dynamics (and closely related word combinatorics) and ergodic theory. Prototiles may be considered as a kind of geometric symbols and entire tilings as infinite (multi-dimensional for $d > 1$) words. It is useful to consider not just an individual tiling, but a space of tilings, together with the translation action. A new feature is that this is an action by $\mathbb{R}^d$, rather than $\mathbb{Z}^d$, and topologically, the tiling space is not a Cantor set, but a solenoid. Tiling dynamical systems were first introduced by D. Rudolph [53], and further investigated by C. Radin and M. Wolff [48] and E. A. Robinson, Jr. [50], see also the book [47]. As already mentioned in the Introduction, the dynamical spectrum is closely related to the diffraction spectrum (see [15, 32]), and this connection has been used to obtain new results about diffraction, see e.g. [56].

Tilings with an inflation symmetry, of which the Penrose tiling is an example, may be viewed as generalizations of substitution sequences. Substitutions (also called morphisms) have been studied in many fields, including ergodic theory and dynamical systems, see e.g. [44] and references therein. The spectral theory of substitution dynamical systems was developed by M. Queffelec [45], B. Host [23] and other authors. Independently, *adic transformations* were introduced by Vershik [61] as a general model for measure-preserving systems. Their theory was developed by A. M. Vershik, A. Livshits, and others (see [35, 62] and references therein), which, in the stationary case, was linked to the theory of substitutions [36, 22]. The importance of Pisot numbers for the spectral properties of substitution and adic systems was suggested by A. Livshits in the mid 1980's [personal communication]. B. Host discussed it in [23, (6.2)]. A complete algebraic characterization of eigenvalues for substitution systems, using the notion of Pisot families, was obtained by S. Ferenczi, C. Mauduit and A. Nogueira [16]. A geometric realization of a certain Pisot substitution as a "domain exchange" on a torus, which also yielded a self-similar tiling of the plane with fractal boundaries, was discovered by G. Rauzy [49], which inspired a lot of new research. An axiomatic framework was introduced by W. Thurston [60], who studied self-similar tilings of the plane and their expansion constants. This theory was further developed by R. Kenyon [25, 28]. Self-affine tilings have been used for constructions of Markov partitions, see e.g. [6, 25, 40, 24, 29]. There are also strong links with numeration systems and beta-expansions, see [10] and references therein.

Although it had precursors, such as [48, 8], it seems that [58] was the first systematic attempt to extend substitution dynamics to the tiling setting. It was continued in [59] which extended B. Mossé's results [38] on recognizability. The present paper is a further study in this direction.

Finally, we should add that there are many other interesting developments in the subject which go far beyond the scope of this paper, among them the study of C^*-algebras arising from substitution tilings [1], deformations of tiling spaces [12], conjugacies for tiling dynamical systems [21], tiling dynamical systems as $\mathbb{G}$-solenoids and laminated spaces [7], etc.

§3. Preliminaries and Statement of Results

We begin with tiling preliminaries, following [33, 52]. We emphasize that our tilings are *translationally finite*, thus excluding the pinwheel tiling [46] and its relatives.

3.1. Tilings.

Fix a set of types (or colors) labeled by $\{1, \ldots, m\}$. A *tile* in $\mathbb{R}^d$ is defined as a pair $T = (A, i)$ where $A = \text{supp}(T)$ (the support of T) is a compact set in $\mathbb{R}^d$ which is the closure of its interior, and $i = \ell(T) \in \{1, \ldots, m\}$ is the type of T. (The tiles are not assumed to be homeomorphic to the ball or even connected.) A *tiling* of $\mathbb{R}^d$ is a set $\mathcal{T}$ of tiles such that $\mathbb{R}^d = \bigcup\{\text{supp}(T) : T \in \mathcal{T}\}$ and distinct tiles (or rather, their supports) have disjoint interiors.

A patch P is a finite set of tiles with disjoint interiors. The *support of a patch* P is defined by $\text{supp}(P) = \bigcup\{\text{supp}(T) : T \in P\}$. The *diameter of a patch* P is $\text{diam}(P) = \text{diam}(\text{supp}(P))$. The *translate* of a tile $T = (A, i)$ by a vector $g \in \mathbb{R}^d$ is $T + g = (A + g, i)$. The translate of a patch P is $P + g = \{T + g : T \in P\}$. We say that two patches P_1, P_2 are *translationally equivalent* if $P_2 = P_1 + g$ for some $g \in \mathbb{R}^d$. Finite subsets of $\mathcal{T}$ are called $\mathcal{T}$-patches.

Definition 3.1. A tiling $\mathcal{T}$ has (translational) *finite local complexity* (FLC) if for any $R > 0$ there are finitely many $\mathcal{T}$-patches of diameter less than R up to translation equivalence.

Definition 3.2. A tiling $\mathcal{T}$ is called *repetitive* if for any patch $P \subset \mathcal{T}$ there is $R > 0$ such that for any $x \in \mathbb{R}^d$ there is a $\mathcal{T}$-patch P' such that $\text{supp}(P') \subset B_R(x)$ and P' is a translate of P.

3.2. Tile-substitutions, self-affine tilings.

We study *perfect* (geometric) substitutions, in which a tile is "blown up" by an expanding linear map and then subdivided. Other possibilities, where the substitution is combinatorial, and/or there is no perfect geometric subdivision, have also been considered, see e.g. [41, 42, 43, 12].

A linear map $\phi : \mathbb{R}^d \to \mathbb{R}^d$ is *expansive* if all its eigenvalues lie outside the unit circle.

Definition 3.3. Let $\mathcal{A} = \{T_1, \ldots, T_m\}$ be a finite set of tiles in $\mathbb{R}^d$ such that $T_i = (A_i, i)$; we will call them *prototiles*. Denote by $\mathcal{P}_{\mathcal{A}}$ the set of patches made of tiles each of which is a translate of one of T_i's. A map $\omega : \mathcal{A} \to \mathcal{P}_{\mathcal{A}}$ is called a *tile-substitution* with expansion ϕ if

$$\operatorname{supp}(\omega(T_j)) = \phi A_j \quad \text{for } j \le m. \tag{3.1}$$

In plain language, every expanded prototile ϕT_j can be decomposed into a union of tiles (which are all translates of the prototiles) with disjoint interiors.

The substitution ω is extended to all translates of prototiles by $\omega(x + T_j) = \phi x + \omega(T_j)$, and to patches by $\omega(P) = \cup\{\omega(T) : T \in P\}$. This is well-defined due to (3.1). The substitution ω also acts on the space of tilings whose tiles are translates of those in $\mathcal{A}$.

To the substitution ω we associate its $m \times m$ substitution matrix S, with S_{ij} being the number of tiles of type i in the patch $\omega(T_j)$. The substitution ω is called *primitive* if the substitution matrix is primitive, that is, if there exists $k \in \mathbb{N}$ such that S^k has only positive entries. We say that $\mathcal{T}$ is a fixed point of a substitution if $\omega(\mathcal{T}) = \mathcal{T}$.

Definition 3.4. Given a primitive tile-substitution ω, let X_ω be the set of all tilings whose every patch is a translate of a subpatch of $\omega^n(T_j)$ for some $j \le m$ and $n \in \mathbb{N}$. (Of course, one can use a specific j by primitivity.) The set X_ω is called the *tiling space* corresponding to the substitution.

Definition 3.5. A repetitive FLC fixed point of a primitive tile-substitution is called a *self-affine tiling*. It is called *self-similar* if the expansion map is a similitude, that is, $|\phi(x)| = \theta|x|$ for all $x \in \mathbb{R}^d$, with some $\theta > 1$.

It is often convenient to work with self-affine tilings; doing this is not a serious restriction, since every primitive tile-substitution has a periodic point in the tiling space and replacing ω by ω^n does not change the tiling space.

We say that a tile-substitution ω has FLC if for any $R > 0$ there are finitely many subpatches of $\omega^n(T_j)$ for all $j \le m$, $n \in \mathbb{N}$, of diameter less than R, up to translation. This obviously implies that all tilings in X_ω have FLC, and is equivalent to it if the tile-substitution is primitive.

Remark. A primitive substitution tiling space is not necessarily of finite local complexity, see [26, 14, 17]. Thus we have to assume FLC explicitly.

Lemma 3.6. [40, Prop. 1.2] *Let ω be a primitive tile-substitution of finite local complexity. Then every tiling $\mathcal{S} \in X_\omega$ is repetitive.*

3.3. Tiling topology and tiling dynamical system

We use a tiling metric on X_ω, which is based on a simple idea: two tilings are close if after a small translation they agree on a large ball around the origin. There is more than one way to make this precise. We say that two tilings $\mathcal{T}_1, \mathcal{T}_2$ *agree* on a set $K \subset \mathbb{R}^d$ if

$$\mathrm{supp}(\mathcal{T}_1 \cap \mathcal{T}_2) \supset K.$$

For $\mathcal{T}_1, \mathcal{T}_2 \in X_\omega$ let

$$\widetilde{d}(\mathcal{T}_1, \mathcal{T}_2) := \inf\{r \in (0, 2^{-1/2}) : \ \exists\, g,\ \|g\| \le r \text{ such that}$$
$$\mathcal{T}_1 - g \ \text{ agrees with } \mathcal{T}_2 \text{ on } B_{1/r}(0)\}.$$

Then

$$d(\mathcal{T}_1, \mathcal{T}_2) = \min\{2^{-1/2}, \widetilde{d}(\mathcal{T}_1, \mathcal{T}_2)\}.$$

Theorem 3.7. [53] (see also [52]). *(X_ω, d) is a complete metric space. It is compact, whenever the space has finite local complexity. The action of $\mathbb{R}^d$ by translations on X_ω, given by $g(\mathcal{S}) = \mathcal{S} - g$, is continuous.*

This continuous translation action $(X_\omega, \mathbb{R}^d)$ is called the (topological) tiling dynamical system associated with the tile-substitution.

Theorem 3.8. *If ω is a primitive tiling substitution with FLC, then the dynamical system $(X_\omega, \mathbb{R}^d)$ is minimal, that is, for every $\mathcal{S} \in X_\omega$, the orbit $\{\mathcal{S} - g : \ g \in \mathbb{R}^d\}$ is dense in X_ω.*

This follows from Lemma 3.6 and Gottschalk's Theorem [19], see [52, Sec. 5] for details.

Definition 3.9. A vector $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d$ is said to be an eigenvalue for the continuous $\mathbb{R}^d$-action if there exists an eigenfunction $f \in C(X_\omega)$, that is, $f \not\equiv 0$ and for all $g \in \mathbb{R}^d$ and all $\mathcal{S} \in X_\omega$,

$$f(\mathcal{S} - g) = e^{2\pi i \langle g, \alpha \rangle} f(\mathcal{S}). \tag{3.2}$$

Here $\langle \cdot, \cdot \rangle$ denotes the standard scalar product in $\mathbb{R}^d$.

Note that this "eigenvalue" is actually a vector. In physics it might be called a "wave vector." More generally, for an action of a locally compact Abelian group G, the eigenvalues are elements of the dual group $\widehat{G}$. For a single transformation (translation by a vector g), the eigenvalue is the more familiar $e^{2\pi i\langle g,\alpha\rangle}$, a point on the unit circle.

3.4. Measurable dynamics.

A topological dynamical system is said to be uniquely ergodic if it has a unique invariant Borel probability measure.

Theorem 3.10. *If ω is a primitive tiling substitution with FLC, then the dynamical system $(X_\omega, \mathbb{R}^d)$ is uniquely ergodic.*

This result has appeared in the literature in several slightly different versions. We refer to [33, 52] for the proof.

Let μ be the unique invariant measure from Theorem 3.10. The measure-preserving tiling dynamical system is denoted by $(X_\omega, \mathbb{R}^d, \mu)$.

Definition 3.11. A vector $\alpha \in \mathbb{R}^d$ is an eigenvalue for the measure-preserving system $(X_\omega, \mathbb{R}^d, \mu)$ if there exists an eigenfunction $f \in L^2(X_\omega, \mu)$, that is, f is not the zero function in L^2 and for all $g \in \mathbb{R}^d$, the equation (3.2) holds for μ-a.e. $\mathcal{S} \in X_\omega$.

By ergodicity, all the eigenvalues are simple and the eigenfunctions have a constant modulus a.e., see [63]. To distinguish between the measure-theoretic and topological settings, we can speak about measurable and continuous eigenfunctions.

Theorem 3.12. *If ω is a primitive tiling substitution with FLC, then every measurable eigenfunction for the system $(X_\omega, \mathbb{R}^d, \mu)$ coincides with a continuous function μ-a.e.*

This extends the result of Host [23] on $\mathbb{Z}$-actions associated to primitive one-dimensional symbolic substitutions.

Remark. Continuous and measurable eigenfunctions for linearly recurrent Cantor systems were recently investigated in [13, 11]. In the latter paper necessary and sufficient conditions for being an eigenvalue are established, and it is proved that not every measurable eigenfunction is a.e. continuous for such systems.

A dynamical system is said to be *weak-mixing* if it has no nonconstant eigenfunctions. This notion is considered both in the topological and the measure-theoretic category. As a consequence of Theorem 3.12, for our systems measure-theoretic weak-mixing is equivalent to topological weak-mixing.

Theorem 3.12 was announced in [59]. In the case of aperiodic substitution spaces, the result is immediate from [58, Th. 5.1] and [59, Th. 1.1], so that here we only need to deal with the case when some periods are present.

3.5. Characterization of eigenvalues.

Let ω be a primitive tiling substitution of finite local complexity and let $\mathcal{S} \in X_\omega$. Consider the set of translation vectors between tiles of the same type:

$$\Xi := \{x \in \mathbb{R}^d : \exists T, T' \in \mathcal{S},\ T' = T + x\}. \tag{3.3}$$

It is clear that Ξ does not depend on the tiling $\mathcal{S}$. The vectors $x \in \Xi$ are sometimes called *return vectors*; they are tiling analogs of return words in word combinatorics. We also need the group of translation symmetries

$$\mathcal{K} = \{x \in \mathbb{R}^d : \mathcal{S} - x = \mathcal{S}\},$$

which does not depend on $\mathcal{S}$ either. The tiling space is said to be *aperiodic* if $\mathcal{K} = \{0\}$, *sub-periodic* if $0 < \operatorname{rank}(\mathcal{K}) < d$, and *periodic* if $\operatorname{rank}(\mathcal{K}) = d$.

Theorem 3.13. *Let ω be a primitive tiling substitution of finite local complexity with expansion map ϕ, which has a fixed point (a self-affine tiling). Then the following are equivalent for $\alpha \in \mathbb{R}^d$:*

(i) *α is an eigenvalue for the topological dynamical system $(X_\omega, \mathbb{R}^d)$;*

(ii) *α is an eigenvalue for the measure-preserving system $(X_\omega, \mathbb{R}^d)$;*

(iii) *α satisfies the following two conditions:*

$$\lim_{n\to\infty} e^{2\pi i \langle \phi^n z, \alpha\rangle} = 1 \quad \textit{for all } z \in \Xi, \tag{3.4}$$

and

$$e^{2\pi i\langle g, \alpha\rangle} = 1 \quad \textit{for all } g \in \mathcal{K}. \tag{3.5}$$

This theorem is also a generalization of the corresponding result from [23]. Theorem 3.12 is immediate from Theorem 3.13, since eigenvalues are simple for an ergodic system and hence normalized measurable and continuous eigenfunctions for the same eigenvalue must coincide a.e.

Theorem 3.13 does not address the question when the eigenvalues are present, so additional analysis is needed. This turns out to be closely related to Number Theory, more precisely, to *Pisot numbers* (also called PV-numbers) and their generalizations. Recall that an algebraic integer $\theta > 1$ is a Pisot number if all its Galois conjugates θ' (other roots of the minimal polynomial) satisfy $|\theta'| < 1$.

Theorem 3.14. *Let $\theta > 1$, and let $\phi(x) = \theta x$ on $\mathbb{R}^d$ be the expansion map. Let ω be a primitive tile-substitution of finite local complexity with expansion ϕ, admitting a fixed point. Then the associated measure-preserving system is not weak-mixing if and only if θ is a Pisot number.*

A similar result was obtained by Gähler and Klitzing [18], where the diffraction spectrum was considered, from a different point of view. We present a proof using our methods. Both [18] and our proof rely on a result of R. Kenyon, which says that under the assumptions of the theorem, $\Xi \subset \mathbb{Z}[\theta]b_1 + \cdots + \mathbb{Z}[\theta]b_d$, for some basis $\{b_1, \ldots, b_d\}$ of $\mathbb{R}^d$. We include a proof of the latter as well, since it is not easy to extract from the literature.

Remarks. 1. We do not have a general theorem like Theorem 3.14, for an arbitrary expansion map ϕ, but some partial results which involve complex Pisot numbers and Pisot families, may be found in [58, 52].

2. It is interesting to compare the simple criterion for weak mixing in Theorem 3.14 with the case of substitution $\mathbb{Z}$-actions, for which a more complicated characterization was obtained by S. Ferenczi, C. Mauduit and A. Nogueira [16]. The reason, roughly, is that here we only consider "geometric" tiling substitutions, so the algebraic conjugates of the expansion constant do not enter into the picture.

3. In this paper we do not address the difficult question whether the spectrum is *pure discrete*, see [58, 4, 33], as well as the more recent [5, 31] and references therein.

§4. Continuous eigenfunctions

In this section we prove Theorem 3.13. To deduce Theorem 3.12, we note that for a primitive tile-substitution ω there exist $\mathcal{T} \in X_\omega$ and $k \in \mathbb{N}$ such that $\omega^k(\mathcal{T}) = \mathcal{T}$ (see [52, Th. 5.10]). Since replacing ω by ω^k does not change the tiling space (ϕ should be replaced by ϕ^k), the result would follow.

The implication (i) $\Rightarrow$ (ii) in Theorem 3.13 is obvious.

Proof of (ii) $\Rightarrow$ (iii). The necessity of (3.4) is proved in [58, Th. 4.3]. We should note that in [58] it was assumed that the expansion map ϕ is diagonalizable, but the proof of [58, Th. 4.3] works for any expansion map. The straightforward details are left to the interested reader.

Next we prove the necessity of (3.5). Let $g \in \mathcal{K}$. Then $\mathcal{S} - g = \mathcal{S}$ for every $\mathcal{S} \in X_\omega$. If f_α is a measurable eigenfunction corresponding to $\alpha \in \mathbb{R}^d$, then

$$f_\alpha(\mathcal{S}) = f_\alpha(\mathcal{S} - g) = e^{2\pi i \langle g, \alpha \rangle} f(\mathcal{S})$$

for μ-a.e. $\mathcal{S} \in X_\omega$. It follows that $e^{2\pi i \langle g, \alpha \rangle} = 1$, as desired.

Proof of (iii) $\Rightarrow$ (i). Let $\mathcal{T}$ be the self-affine tiling whose existence we assumed. Suppose that (3.4), (3.5) hold and define

$$f_\alpha(\mathcal{T} - x) = e^{2\pi i \langle x, \alpha \rangle} \quad \text{for } x \in \mathbb{R}^d. \tag{4.1}$$

The orbit $\{\mathcal{T} - x : \ x \in \mathbb{R}^d\}$ is dense in X_ω by minimality. If we show that f_α is uniformly continuous on this orbit, then we can extend f_α to X_ω, and this extension will satisfy the eigenvalue equation (3.2) by continuity. We will need several lemmas; the first one will be useful in the next section as well.

Lemma 4.1. [25, 60] *Suppose that there is a primitive tiling substitution of finite local complexity with expansion ϕ. Then all the eigenvalues of ϕ are algebraic integers.*

Proof. We provide a short proof for completeness. Consider $\langle \Xi \rangle$, the subgroup of $\mathbb{R}^d$ generated by Ξ. It is a free finitely generated Abelian group by FLC. Thus, we can find a set of free generators $v_1, \dots, v_\ell$ for $\langle \Xi \rangle$. Consider the $d \times \ell$ matrix $V = [v_1 \dots v_\ell]$. By the definition of substitution tilings, $\phi \Xi \subset \Xi$, hence ϕ acts on $\langle \Xi \rangle$, and so there is an integer $\ell \times \ell$ matrix M such that

$$\phi V = VM. \tag{4.2}$$

Note that $\ell \geq d$ and $\operatorname{rank}(V) = d$ since Ξ spans $\mathbb{R}^d$. If λ is an eigenvalue of ϕ^T with the eigenvector x, then λ is also an eigenvalue of M^T with the eigenvector $V^T x$. (The superscript T denotes the transpose of a matrix.) Since M is an integer matrix, it follows that λ is an algebraic integer. Q.E.D.

Lemma 4.2. *If (3.4) holds for some $z \in \Xi$, then the convergence in (3.4) is exponential; in fact, there exist $\rho \in (0,1)$, depending only on the expansion ϕ, and $C_z > 0$ such that*

$$\left| e^{2\pi i \langle \phi^n z, \alpha \rangle} - 1 \right| < C_z \rho^n \quad \textit{for} \quad n \in \mathbb{N}. \tag{4.3}$$

Below we will show that (4.3) holds with a constant C independent of z, but this lemma is the first step.

Proof. This can be deduced from [37, L. 2], but we sketch a (well-known) direct proof for the reader's convenience.

We continue the argument in the proof of the previous lemma. Let $z \in \Xi$. By the definition of free generators, there is a unique vector $a(z) \in \mathbb{Z}^\ell$ such that $z = Va(z)$. Then we have from (4.2):

$$\langle \phi^n z, \alpha \rangle = \langle \phi^n V a(z), \alpha \rangle = \langle V M^n a(z), \alpha \rangle = \langle M^n a(z), V^T \alpha \rangle := \zeta_n.$$

It follows from the Caley-Hamilton Theorem that the sequence ζ_n satisfies a recurrence relation with integer coefficients:

$$\zeta_{n+\ell} + a_1\zeta_{n+\ell-1} + \cdots + a_n\zeta_n = 0, \tag{4.4}$$

where $t^\ell + a_1 t^{\ell-1} + \cdots + a_\ell$ is the characteristic polynomial p of the integer matrix M. Let

$$\zeta_n = \langle \phi^n z, \alpha \rangle := K_n + \epsilon_n,$$

where K_n is the nearest integer to ζ_n. By (3.4), we have $\epsilon_n \to 0$. Since the sequence ζ_n satisfies (4.4) and K_n are integers, we conclude that K_n satisfy the same recurrence relation for n sufficiently large. Therefore, also ϵ_n satisfy (4.4), hence they can be expressed in terms of the zeros of p, for n sufficiently large. More precisely, there exist $\alpha_j \in \mathbb{C}$ and polynomials $q_j \in \mathbb{Z}[x]$ such that

$$\epsilon_n = \sum_{j=1}^{\ell_1} \alpha_j q_j(n) \theta_j^n \quad \text{for } n \geq n_0, \tag{4.5}$$

where θ_j, $j = 1, \ldots, \ell_1$, are the distinct zeros of p. Since $\epsilon_n \to 0$, it is not hard to see that all θ_j, which occur in (4.5) with nonzero coefficients, must satisfy $|\theta_j| < 1$. Then

$$|\epsilon_n| \leq \text{const} \cdot \rho^n,$$

where $\max\{|\theta_j| : |\theta_j| < 1\} < \rho < 1$, and we conclude that $\left|e^{2\pi i \langle \phi^n z, \alpha \rangle} - 1\right| = \left|e^{2\pi i \epsilon_n} - 1\right|$ satisfies (4.3) for appropriate C_z. Q.E.D.

Next we need a lemma which is analogous to [40, Th. 1.5] and [58, Lem. 6.5]. We do not include a complete proof, but deduce it from [34].

Lemma 4.3. *Let ω be a primitive tile-substitution of finite local complexity with expansion ϕ, which has a fixed point $\mathcal{T}$, and let Ξ be the set of translation vectors between tiles of the same type in X_ω. Then there exist $k \in \mathbb{N}$ and a finite set U in $\mathbb{R}^d$, with $\phi^k U \subset \Xi$, such that for any $z \in \Xi$ there exist $N \in \mathbb{N}$ and $u(j), w(j) \in U$, $0 \leq j \leq N$, so that*

$$z = \sum_{j=0}^{N} \phi^{kj}(u(j) + w(j)).$$

Proof. We can find $k \in \mathbb{N}$ so that S^k is strictly positive, where S is the substitution matrix of ω. Then $\mathcal{T}$ is also a fixed point of ω^k, a tile-substitution with expansion ϕ^k and a strictly positive substitution matrix. Now we can apply [34, Lem. 4.5] to obtain the desired result. Q.E.D.

Corollary 4.4. *If (3.4) holds, then the convergence is uniform in $z \in \Xi$, that is,*

$$\lim_{n\to\infty} \sup_{z\in\Xi} \left| e^{2\pi i\langle \phi^n z, \alpha\rangle} - 1 \right| = 0.$$

Proof. Let

$$C := \max\{C_u : \ u \in U\},$$

where C_u is from Lemma 4.2 and U is from Lemma 4.3. Let $z \in \Xi$ and consider the expansion from Lemma 4.3. Then we have

$$\begin{aligned}
\left| e^{2\pi i\langle \phi^n z, \alpha\rangle} - 1 \right| &= \left| \exp\Big(2\pi i \Big\langle \sum_{j=0}^{N} \phi^{n+kj}(u(j) + w(j)), \alpha \Big\rangle\Big) - 1 \right| \\
&\le \sum_{j=0}^{N} \left| \exp(2\pi i\langle \phi^{n+k(j-1)} \phi^k u(j), \alpha\rangle) - 1 \right| \\
&+ \sum_{j=0}^{N} \left| \exp(2\pi i\langle \phi^{n+k(j-1)} \phi^k w(j), \alpha\rangle) - 1 \right| \\
&\le 2C \sum_{j=0}^{N} \rho^{n+k(j-1)} < 2C\rho^{-k}(1-\rho^k)^{-1}\rho^n,
\end{aligned}$$

which implies the desired statement. Here we used that $|e^{i(a+b)} - 1| \le |e^{ia} - 1| + |e^{ib} - 1|$ for real a, b and (4.3). Q.E.D.

Extending the work of Mossé [38] on primitive aperiodic substitution $\mathbb{Z}$-actions, we proved in [59] that if X_ω is aperiodic, then the substitution map ω is bijective on X_ω. In the general case, we proved in [59, Th. 1.2] that if $\omega(\mathcal{S}_1) = \omega(\mathcal{S}_2)$, then $\mathcal{S}_2 = \mathcal{S}_1 - \phi^{-1}g$ for some $g \in \mathcal{K}$. The next lemma is a quantitative version of this statement, which follows from it easily.

Lemma 4.5. *Let ω be a primitive tile-substitution of finite local complexity. Then for every $\epsilon > 0$ there is $\delta > 0$ such that for any $\mathcal{S}, \mathcal{S}' \in X_\omega$ with $d(\mathcal{S}, \mathcal{S}') < \delta$ there exist $\mathcal{U}, \mathcal{U}' \in X_\omega$ such that $\omega(\mathcal{U}) = \mathcal{S}$, $\omega(\mathcal{U}') = \mathcal{S}'$, and $d(\mathcal{U}, \mathcal{U}') < \epsilon$.*

Proof. Recall that $\mathcal{K}$ is the set of periods for X_ω, so that $\mathcal{S} - h = \mathcal{S}$ for all $h \in \mathcal{K}$. Consider the following equivalence relation on X_ω:

$$\mathcal{S}_1 \sim \mathcal{S}_2 \iff \mathcal{S}_2 = \mathcal{S}_1 - \phi^{-1}g \ \text{ for some } g \in \mathcal{K}.$$

Observe that $\mathcal{K}$ is a discrete subgroup of $\mathbb{R}^d$ and $\phi\mathcal{K}$ is a subgroup of finite index in $\mathcal{K}$. Thus, all the equivalence classes are finite. Denote the equivalence class of $\mathcal{S}$ by $[\mathcal{S}]$, and the set of equivalence classes by $\widehat{X}_\omega$. Consider the induced metric on $\widehat{X}_\omega$:

$$\widehat{d}([\mathcal{S}_1], [\mathcal{S}_2]) := \min\{d(\mathcal{S}_1', \mathcal{S}_2') : \ \mathcal{S}_1' \sim \mathcal{S}_1, \ \mathcal{S}_2' \sim \mathcal{S}_2\}.$$

It is readily seen that $(\widehat{X}_\omega, \widehat{d})$ is compact. Consider the map $\widehat{\omega} : \widehat{X}_\omega \to X_\omega$ given by

$$\widehat{\omega}([\mathcal{V}]) = \omega(\mathcal{V}) \ \text{ for } \mathcal{V} \in X_\omega.$$

This is well-defined, since for $\mathcal{V}' \sim \mathcal{V}$ we have $\mathcal{V}' = \mathcal{V} - g$ for some $g \in \phi^{-1}\mathcal{K}$, hence $\omega(\mathcal{V}') = \omega(\mathcal{V}) - \phi g = \omega(\mathcal{V})$. Since ω is a continuous surjection onto X_ω, we have that $\widehat{\omega}$ is a continuous surjection from $\widehat{X}_\omega$ onto X_ω. We claim that $\widehat{\omega}$ is 1-to-1. Indeed, if $\widehat{\omega}([\mathcal{S}_1]) = \widehat{\omega}([\mathcal{S}_2])$, then $\omega(\mathcal{S}_1) = \omega(\mathcal{S}_2)$, and $\mathcal{S}_1 \sim \mathcal{S}_2$ by [58, Th. 1.2]. It follows that $\widehat{\omega}^{-1}$ is uniformly continuous, which is precisely the desired statement. Q.E.D.

Conclusion of the proof of (iii) $\Rightarrow$ (i) *in Theorem 3.13.* Recall that we need to show the uniform continuity of f_α, given by (4.1), on the orbit $\{\mathcal{T} - x : \ x \in \mathbb{R}^d\}$. We have $\omega(\mathcal{T}) = \mathcal{T}$, so for any $h \in \mathbb{R}^d$, by [58, Th. 1.2],

$$\omega^{-1}(\mathcal{T} - h) = \{\mathcal{T} - \phi^{-1}h - \phi^{-1}g : \ g \in \mathcal{K}\}.$$

Applying Lemma 4.5, we obtain that for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$\text{if } \ d(\mathcal{T} - x, \mathcal{T} - y) < \delta \ \text{ then}$$

$$\exists\, g, g' \in \mathcal{K}, \ d(\mathcal{T} - \phi^{-1}x - \phi^{-1}g, \mathcal{T} - \phi^{-1}y - \phi^{-1}g') < \epsilon. \tag{4.6}$$

Fix $\eta \in (0,1)$. By Corollary 4.4, we can choose $n \in \mathbb{N}$ such that $|e^{2\pi i \langle \phi^n z, \alpha\rangle} - 1| < \eta/2$ for all $z \in \Xi$. Applying (4.6) n times, we can find $\delta > 0$ such that if $d(\mathcal{T} - x, \mathcal{T} - y) < \delta$, then there exist $g_j, g_j' \in \mathcal{K}$, for $1 \le j \le n$, with

$$d\Big(\mathcal{T} - \phi^{-n}x - \sum_{j=1}^n \phi^{-n+j-1}g_j, \mathcal{T} - \phi^{-n}y - \sum_{j=1}^n \phi^{-n+j-1}g_j'\Big) < \epsilon$$

where

$$\epsilon := \frac{\eta}{4\pi\|\phi^n\|\|\alpha\|}.$$

By the definition of the metric d, this means that there exists $h \in \mathbb{R}^d$, with $\|h\| \le \epsilon$, such that $\mathcal{T} - \phi^{-n}x - \sum_{j=1}^n \phi^{-n+j-1}g_j - h$ agrees with $\mathcal{T} - \phi^{-n}y - \sum_{j=1}^n \phi^{-n+j-1}g_j'$ on $B_{1/\epsilon}(0)$. Agreement on any neighborhood of the origin implies that the tilings share the tiles containing the origin. Thus, by the definition of the set Ξ,

$$\phi^{-n}(x-y) + \sum_{j=1}^{n} \phi^{-n+j-1}(g_j - g'_j) - h \in \Xi.$$

Therefore, there exists $z \in \Xi$ such that

$$x - y = \phi^n z + \phi^n h + w, \quad \text{where } w := \sum_{j=1}^{n} \phi^{j-1}(g'_j - g_j) \in \mathcal{K}$$

(here we use that $\phi\mathcal{K} \subset \mathcal{K}$). Finally,

$$\begin{aligned} |f_\alpha(x) - f_\alpha(y)| &= \left|e^{2\pi i\langle x-y,\alpha\rangle} - 1\right| \\ &= \left|e^{2\pi i(\langle \phi^n z,\alpha\rangle + \langle \phi^n h,\alpha\rangle + \langle w,\alpha\rangle)} - 1\right| \\ &\le \left|e^{2\pi i\langle \phi^n z,\alpha\rangle} - 1\right| + \left|e^{2\pi i\langle \phi^n h,\alpha\rangle} - 1\right| \\ &\le \eta/2 + 2\pi|\langle \phi^n h, \alpha\rangle| \\ &\le \eta/2 + 2\pi\|\phi^n\|\|\alpha\|\epsilon = \eta, \end{aligned}$$

and we are done. We used the condition (3.5) to get rid of the term with w. Q.E.D.

§5. Pisot substitutions

Proof of necessity in Theorem 3.14. We need to prove that if the dynamical system $(X_\omega, \mathbb{R}^d)$ has a non-constant eigenfunction, then θ is a Pisot number. Recall that here we assume the expansion map to be a pure dilation: $\phi(x) = \theta x$.

Let $\alpha \neq 0$ be an eigenvalue. The set of translation vectors Ξ between tiles of the same type spans $\mathbb{R}^d$, hence we can find $z \in \Xi$ such that $\langle z, \alpha\rangle \neq 0$. By Theorem 3.13, the distance from $\theta^n\langle z, \alpha\rangle$ to the nearest integer tends to zero, as $n \to \infty$. We know that θ is algebraic (see Lemma 4.1), hence θ is a Pisot number by the classical result of Pisot (see e.g. [54]). Q.E.D.

Proof of sufficiency in Theorem 3.14. We need to show that if θ is Pisot, then there are non-zero eigenvalues for the dynamical system. The proof relies on the following result.

Theorem 5.1 (Kenyon). *Let $\mathcal{T}$ be a self-similar tiling with expansion map $\phi(x) = \theta x$ for some $\theta > 1$ and let Ξ be the set of return vectors* *of the tiling $\mathcal{T}$, defined in (3.3). Then there exists a basis $\{b_1, \dots, b_d\}$ of $\mathbb{R}^d$ such that*

$$\Xi \subset \mathbb{Z}[\theta]b_1 + \cdots + \mathbb{Z}[\theta]b_d. \tag{5.1}$$

First we finish the proof of sufficiency. Suppose that there are non-trivial periods, that is, $\mathcal{K} \neq \{0\}$. Since $\theta\mathcal{K} \subset \mathcal{K}$ and $\mathcal{K}$ is a discrete subgroup of $\mathbb{R}^d$, we obtain that $\theta \in \mathbb{N}$. Then it follows from (5.1) that the group generated by Ξ, which we denoted by $\langle\Xi\rangle$, is discrete. It is a lattice in $\mathbb{R}^d$, since it spans $\mathbb{R}^d$. It is clear that the all points of the dual lattice $\langle\Xi\rangle'$ satisfy both (3.4) and (3.5) (using that $\mathcal{K} \subset \Xi$), hence there are non-trivial eigenvalues.

Now suppose that the tiling is aperiodic, that is, $\mathcal{K} = \{0\}$. Let $\{b_1^*, \dots, b_d^*\}$ be the dual basis for $\{b_1, \dots, b_d\}$, that is, $\langle b_i, b_j^*\rangle = \delta_{ij}$. We claim that the set

$$\mathbb{Z}[\theta^{-1}]b_1^* + \cdots + \mathbb{Z}[\theta^{-1}]b_d^*$$

is contained in the group of eigenvalues. Indeed, suppose $\alpha = \sum_{j=1}^d b_j^* p_j(\theta^{-1})$ for some polynomials $p_j \in \mathbb{Z}[x]$. Let $z \in \Xi$. By (5.1), we can write $z = \sum_{j=1}^d b_j q_j(\theta)$ for some polynomials $q_j \in \mathbb{Z}[x]$. Then

$$\langle \phi^n z, \alpha\rangle = \theta^n \sum_{j=1}^d q_j(\theta)p_j(\theta^{-1}) = \theta^{n-k}P(\theta)$$

for some $k \in \mathbb{N}$ and $P \in \mathbb{Z}[x]$. We have $\mathrm{dist}(\theta^{n-k}P(\theta), \mathbb{Z}) \to 0$, as $n \to \infty$ (see [54]), so (3.4) is satisfied and α is an eigenvalue by Theorem 3.13. Q.E.D.

Proof of Theorem 5.1. Our proof is based on [60, 27] and a personal communication from Rick Kenyon; however, we do not need quasiconformal maps as in [27] which is concerned with more general tilings.

Instead of the set Ξ, it is more convenient to work with *control points*, see [60, 25, 40].

Definition 5.2. [40] Let $\mathcal{T}$ be a fixed point of a primitive substitution with expansive map ϕ. For each $\mathcal{T}$-tile T, fix a tile γT in the patch $\omega(T)$; choose γT with the same relative position for all tiles of the same type. This defines a map $\gamma : \mathcal{T} \to \mathcal{T}$ called the *tile map.* Then define the *control point* for a tile $T \in \mathcal{T}$ by

$$c(T) = \bigcap_{n=0}^{\infty} \phi^{-n}(\gamma^n T).$$

Note that the control points are not uniquely defined; they depend on the choice of γ.

Let $\mathcal{C} = \mathcal{C}(\mathcal{T}) = \{c(T) : T \in \mathcal{T}\}$ be the set of control points for all tiles. The control points have the following properties:

(a) $T' = T + c(T') - c(T)$, for any tiles T, T' of the same type;
(b) $\phi(c(T)) = c(\gamma T)$, for $T \in \mathcal{T}$.

Therefore,

(c) $\Xi \subset \mathcal{C} - \mathcal{C}$.
(d) $\phi(\mathcal{C}) \subset \mathcal{C}$.

The definition above works for a general expansion ϕ, but now we again assume that $\phi(x) = \theta x$. Observe that it is enough to prove the inclusion

$$\mathcal{C} \subset \mathbb{Q}(\theta)e_1 + \cdots + \mathbb{Q}(\theta)e_d \tag{5.2}$$

for some basis $\{e_1, \ldots, e_d\}$. Indeed, the Abelian group $\langle \mathcal{C} \rangle$ is finitely generated. Let $\{w_1, \ldots, w_N\}$ be a set of free generators. By (5.2), $w_j = \sum_{j=1}^d e_j \frac{p_j^{(i)}(\theta)}{q_j^{(i)}(\theta)}$, for $i \le N$, for some polynomials $p_j^{(i)}, q_j^{(i)} \in \mathbb{Z}[x]$. Then we obtain (5.1), with $b_j = e_j \left(\prod_{j=1}^d \prod_{i=1}^N q_j^{(i)}(\theta)\right)^{-1}$, in view of the property (c) above.

Now pick any set $\{e_1, \ldots, e_d\} \subset \mathcal{C}$ which spans $\mathbb{R}^d$. Consider the vector space

$$\widetilde{\mathcal{C}} := \mathrm{Span}_{\mathbb{Q}(\theta)} \mathcal{C}$$

over the field $\mathbb{Q}(\theta)$. We want to show that $\{e_1, \ldots, e_d\}$ is a basis for $\widetilde{\mathcal{C}}$. Let π be any linear projection from $\widetilde{\mathcal{C}}$ onto $\mathrm{Span}_{\mathbb{Q}(\theta)}\{e_1, \ldots, e_d\}$. Since the vector space is over $\mathbb{Q}(\theta)$, we obviously have

$$\pi(\theta^n x) = \theta^n \pi(x) \quad \text{for } x \in \widetilde{\mathcal{C}},\ n \in \mathbb{Z}. \tag{5.3}$$

Thus we have a map

$$f(x) = \pi(x) \quad \text{for } x \in \mathcal{C}_\infty, \quad \text{where } \mathcal{C}_\infty := \bigcup_{n=0}^{\infty} \theta^{-n}\mathcal{C}. \tag{5.4}$$

This is consistent in view of (5.3). Now f is defined on a dense subset of $\mathbb{R}^d$; let us show that it is uniformly Lipschitz on this set, then f can be extended to $\mathbb{R}^d$ by continuity. Fix any norm $\|\cdot\|$ on $\mathbb{R}^d$.

Lemma 5.3. *There exists $L_1 > 0$ such that*

$$\|\pi(\xi) - \pi(\xi')\| \le L_1 \|\xi - \xi'\|, \quad \forall\, \xi, \xi' \in \mathcal{C}. \tag{5.5}$$

Proof. This argument is from [60]. One can move "quasi-efficiently" between control points by moving from "neighbor to neighbor." More precisely, let W be the set of vectors in $\mathbb{R}^d$ obtained as the difference vectors between control points of neighboring $\mathcal{T}$-tiles. Note that W is finite by FLC. There is a constant $C_1 = C_1(\mathcal{T})$ such that $\forall\ \xi, \xi' \in \mathcal{C}$, there exist $p \in \mathbb{N}$ and $\xi_1 := \xi,\ \xi_2, \ldots, \xi_{p-1} \in \mathcal{C},\ \xi_p := \xi'$ such that $\xi_{i+1} - \xi_i \in W$ for $i = 1, \ldots, p-1$, and

$$\sum_{i=1}^{p-1} \|\xi_{i+1} - \xi_i\| \le C_1 \cdot \|\xi - \xi'\|.$$

(This is an exercise; see [30] for a detailed proof.) Let

$$C_2 := \max\{\|\pi(w)\|/\|w\| :\ w \in W\}.$$

Now we can estimate:

$$\begin{aligned}
\|\pi(\xi) - \pi(\xi')\| = \|\pi(\xi - \xi')\| &= \left\|\sum_{i=1}^{p-1} \pi(\xi_{i+1} - \xi_i)\right\| \\
&\le \sum_{i=1}^{p-1} \|\pi(\xi_{i+1} - \xi_i)\| \\
&\le C_2 \sum_{i=1}^{p-1} \|\xi_{i+1} - \xi_i\| \\
&\le C_1 C_2 \|\xi - \xi'\|.
\end{aligned}$$

Q.E.D.

In view of (5.3), the last lemma implies that f is Lipschitz on $\mathcal{C}_\infty$ with the uniform Lipschitz constant L_1. We extend f to a Lipschitz function on $\mathbb{R}^d$ by continuity; it satisfies

$$f(\theta x) = \theta f(x) \quad \text{for all } x \in \mathbb{R}^d. \tag{5.6}$$

Note that the extension f need not coincide with π on $\widetilde{\mathcal{C}}$; we started with π not on the entire $\mathbb{Q}(\theta)$-vector space, but only on the control points and their preimages under the expansion.

Lemma 5.4. *The function f depends only on the tile type in $\mathcal{T}$, up to an additive constant: if $T, T + x \in \mathcal{T}$ and $\xi \in \mathrm{supp}(T)$, then*

$$f(\xi + x) = f(\xi) + \pi(x). \tag{5.7}$$

Proof. Observe that $x \in \Xi \subset \mathcal{C} - \mathcal{C} \subset \widetilde{\mathcal{C}}$, so $\pi(x)$ is defined. It is enough to check (5.7) on a dense set. Suppose $\xi = \theta^{-k}c(S) \in \mathrm{supp}(T)$ for some $S \in \omega^k(T)$. Then $S + \theta^k x \in \omega^k(T + x) \subset \mathcal{T}$ and we have

$$\begin{aligned} f(\xi + x) &= f(\theta^{-k}c(S) + x) \\ &= f(\theta^{-k}c(S + \theta^k x)) \\ &= \theta^{-k}\pi(c(S + \theta^k x)) \\ &= \theta^{-k}\pi(c(S)) + \theta^{-k}\pi(\theta^k x) \\ &= f(\xi) + \pi(x), \end{aligned}$$

as desired. Q.E.D.

Conclusion of the proof of Theorem 3.14. We mimic the argument of Thurston [60] but provide more details.

The function $f : \mathbb{R}^d \to \mathbb{R}^d$ is Lipschitz, hence it is differentiable almost everywhere. Let x be a point where the total derivative $H = Df(x)$ exists. Then

$$f(x + u) = f(x) + Hu + \Psi(u) \quad \text{for all} \quad u \in \mathbb{R}^d,$$

where

$$\|\Psi(u)\|/\|u\| \to 0, \quad \text{as } u \to 0. \tag{5.8}$$

Multiplying by θ^k, using (5.6) and substituting $v = \theta^k u$, we obtain

$$f(\theta^k x + v) = f(\theta^k x) + Hv + \theta^k \Psi(\theta^{-k} v) \quad \text{for all} \quad v \in \mathbb{R}^d.$$

For a set $A \subset \mathbb{R}^d$ denote by $[A]^{\mathcal{T}}$ the $\mathcal{T}$-patch of $\mathcal{T}$-tiles whose supports intersect A. By repetitivity, there exists $R > 0$ such that $B_R(0)$ contains a translate of the patch $[B_1(\theta^k x)]^{\mathcal{T}}$ for all $k \in \mathbb{N}$. This implies, in view of Lemma 5.4, that there exist $x_k \in B_R(0)$, for $k \geq 1$, such that

$$f(x_k + v) = f(x_k) + Hv + \theta^k \Psi(\theta^{-k} v) \quad \text{for all} \quad v \in B_1(0).$$

There exists a limit point x' of the sequence $\{x_k\}$; then by continuity of f and (5.8),

$$f(x' + v) = f(x') + Hv \quad \text{for all} \quad v \in B_1(0).$$

Thus, f is flat on some neighborhood. Applying (5.6) again, we obtain that f is flat on an arbitrarily large neighborhood. By repetitivity, a translate of $[B_1(0)]^{\mathcal{T}}$ occurs in every sufficiently large neighborhood, therefore, by Lemma 5.4, the function f is flat on $B_1(0)$. Using (5.6) for the last time, we conclude that f is flat everywhere, and since $f(0) = 0$

(see (5.3) and (5.4)), f is linear. But $f(e_j) = \pi(e_j) = e_j$ for the set $\{e_j\}_{j\le d}$ which we chose in the beginning of the proof, hence f is the identity map, $\pi(\xi) = \xi$ for all $\xi \in \mathcal{C}$, and we conclude that (5.2) holds, as desired. Q.E.D.

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