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Some highlights from the history of probabilistic number theory

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Abstract.

In this survey lecture it is intended to sketch some parts [chosen according to the author's interests] of the [early] history of Probabilistic Number Theory, beginning with PAUL TURÁNs proof (1934) of the HARDY–RAMANUJAN result on the "normal order" of the additive function $\omega(n)$, the ERDŐS–WINTNER Theorem, and the ERDŐS–KAC Theorem. Next, mean–value theorems for arithmetical functions, and the KUBILIUS model and its application to limit laws for additive functions will be described in short.

Subsuming applications of the theory of almost–periodic functions under the concept of "Probabilistic Number Theory", the problem of *"uniformly–almost–even functions with prescribed values"* will be sketched, and the KNOPFMACHER – SCHWARZ – SPILKER theory of integration of arithmetical functions will be sketched. Next, K.–H. INDLEKOFERs elegant theory of integration of functions $\mathbb{N} \to \mathbb{C}$ of will be described.

Finally, it is tried to scratch the surface of the topic "universality", where important contributions came from the university of Vilnius.

About fifteen years ago the author got interested in the History of the Frankfurt Mathematical Seminary, and in the history of number theory. Here it is intended to sketch some highlights from the history of Probabilistic Number Theory. And this task is not difficult, using, for example, the monographs of P. D. T. A. ELLIOTT ([37], [38]) and G. TENENBAUM ([228]), a paper of mine from 1994 on the Development of Probabilistic Number Theory,[1], a paper of J.–L. MAUCLAIRE [180], and

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[1]The present paper is a partly updated version of this paper [210], which was based on a lecture in Vilnius, 1993.

a paper of K.–H. INDLEKOFER ([106], from 2002). A survey paper by J. KUBILIUS [134], unfortunately in Russian, gives the stage of the theory of value–distribution for additive and multiplicative functions until 1972 (with more than 200 references). Great progress in this theory was made possible by the mean–value theorems of E. WIRSING and G. HALÁSZ (see section 4).

§1. Introduction

1.1. Number Theory without Probability Theory

Number Theory is an old mathematical discipline; important contributions to number theory in the 19^{th} century were given by C.–F. GAUSS, A.–M. LEGENDRE, P. L. TCHEBYCHEFF, B. RIEMANN, LEJEUNE G. DIRICHLET, J. HADAMARD, CH. DE LA VALLÉE–POUSSIN, and in the early 20^{th} century by E. LANDAU, G. H. HARDY, S. RAMANUJAN, J. E. LITTLEWOOD, and by many others.[2]

At present, Number Theory uses many methods from other parts of mathematics, for example:

- Elementary Calculations (partial summation, comparison with integrals, inequalities, elementary algebra and combinatorics).
- Generating functions $\sum_1^\infty \frac{f(n)}{n^s}$, where $f : \mathbb{N} \to \mathbb{C}$. For example, the **mean–value** $M(f)$, *if it exists*, equals

$$(1.1)\quad M(f) \left[\overset{def}{=} \lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} f(n) \right] = \lim_{\sigma\to 1+} \zeta^{-1}(\sigma) \cdot \sum_{n=1}^\infty f(n) \cdot n^{-\sigma}.$$

- Sieve Methods (see [78], [79], [92], [201]).
- Complex Analysis (Cauchy's integral theorem, theorem of residues, theory of entire functions, results on zeros of meromorphic functions, Weierstraß factorization, ...).
- Asymptotic Analysis (LAPLACEs method, saddle point method, Tauberian theorems, ...), see [81], [245], [11].
- Estimates of Exponential Sums $\sum_{N<n\le 2N} \exp(2\pi i\cdot h(n) \ll \dots$ for real–valued functions h ([240], [243], [93]).

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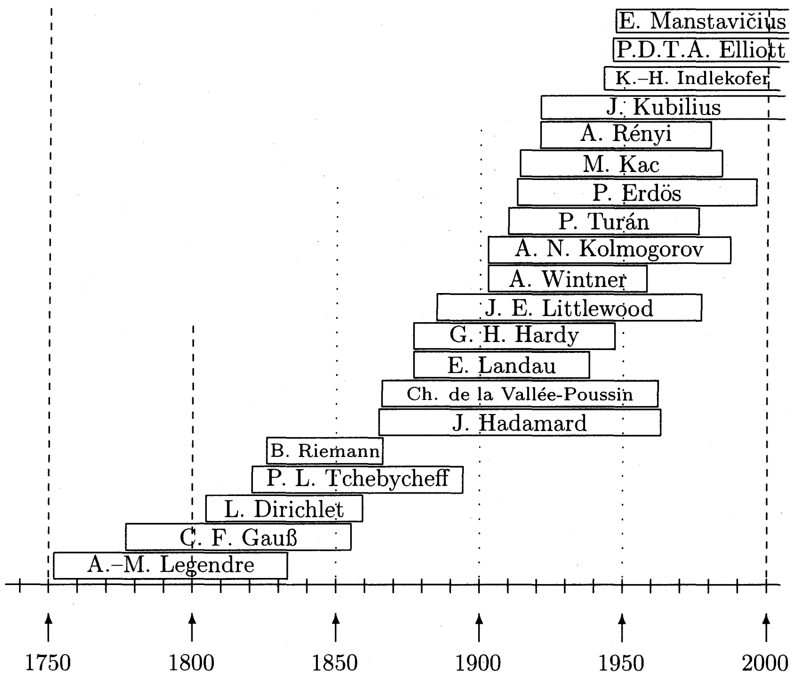
[2]We leave aside contributions to algebraic number theory, for example by L. DIRICHLET, E. E. KUMMER, R. DEDEKIND, ...



Chronological Table

- Special functions (Gamma–function, Beta–function, some integrals, Theta–functions, ...) and Zeta–functions (see, for example, [112], [117], [153], [125]).
- Modular functions, modular forms, elliptic curves (see, for example, [15]).
- Ideas from Geometry (convex bodies, lattice points, MINKOWSKIs Geometry of Numbers), see, for example, [12], [54].[3]
- Compactification, Topology, topological groups, adéles, idéles.[4]
- Algebra, Algebraic Geometry (Diophantine Analysis).
- Theory of integration, functional analysis ([204], [205]).
- Fourier analysis (see, for example [187]), almost periodic functions, approximation arguments, ergodic theory (see, for example, [64]).

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[3]For a survey of recent results in the theory of lattice points see [113].

[4]See, for example, [70] and [13] for the use of TATEs ideas in algebraic number theory. [13] also contains TATEs Thesis from 1950. – For a more recent example, see [141].

The photos have been removed due to copyright issues.

Figure 1. *A. Ivić, H. Fürstenberg, H. L. Montgomery*

- Probability Theory.

In this article, our interest is mainly in the last three items.

Returning to the 19^{th} and early 20^{th} century, great progress was made possible by using methods from analysis, in particular the theory of complex functions of one variable. RIEMANN defined, in $\Re(s) > 1$, "his" zeta–function ($\zeta(s)$ was already known to L. EULER)

$$\zeta(s) \stackrel{def}{=} \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \text{ where } n^{-s} = e^{-s \cdot \log n}, \tag{1.2}$$

he gave its analytic continuation, proved the functional equation,[5] and made ([202], 1859) several deep conjectures showing an intimate connection of prime number theory and analytic properties of $\zeta(s)$; the famous *Riemann conjecture*, that all non–trivial zeros of $\zeta(s)$ are on the line $\Re(s) = \frac{1}{2}$, is still unsettled.

DIRICHLET ([30], 1837, 1839) showed that there are infinitely many primes in the progression $n \equiv a \bmod q$, if a is coprime with q:

$$\pi(x; q, a) := \#\{p \leq x,\ p \equiv a \bmod q\} \to \infty, \text{ if } \gcd(a, q) = 1.$$

The crucial point was to show that the values of Dirichlet L–functions at $s = 1$ do not vanish,

$$L(1, \chi) \neq 0 \text{ for any character } \chi \neq \chi_0,$$

[5]Concerning the history of the functional equation, see [144].

where χ is a character on the group $(\mathbb{Z}/qZ)^{\times}$, and χ_0 is the character constant equal to 1 („Hauptcharakter"). The Dirichlet L–functions are given as Dirichlet series

$$(1.3) \qquad L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}.$$

J. HADAMARD ([73], 1893–1896) was able to sharpen the WEIERSTRASS factorization theorem considerably, and obtained a product representation of functions connected with the Riemann zeta function,

$$(1.4) \qquad \pi^{-\frac{1}{2}s}(s-1)\cdot\zeta(s)\cdot\Gamma\left(\frac{1}{2}s+1\right) = ae^{bs}\prod_\rho \left(1-\frac{s}{\rho}\right)e^{\frac{s}{\rho}}$$

$\big(\rho$ runs over the non–trivial zeros of $\zeta(s)\big)$, and he was able to show that there are no zeros of the zeta function in some region

$$(1.5) \qquad \left\{s \in \mathbb{C},\ s=\sigma+it,\ \sigma \geq 1 - c_1 \cdot \frac{1}{|t|+2}\right\}.$$

This implied the prime number theorem

$$(1.6) \qquad \pi(x) = \sum_{p\leq x} 1 = \int_2^x \frac{du}{\log u} + \mathcal{O}\left(x\cdot e^{-\gamma\sqrt{\log x}}\right)$$

with a reasonably good remainder term. CH. DE LA VALLÉE–POUSSIN proved the prime number theorem at the same time ([239]).[6]

G. H. HARDY, in collaboration with S. RAMANUJAN, later with J. E. LITTLEWOOD, used the "circle method" in order to obtain deep results on the partition function and the Waring problem. This method is based on the simple idea that the coefficients of a power series $\mathcal{F}(z) = \sum a_n z^n$ are given by a contour integral

$$a_n = \frac{1}{2\pi i}\cdot\oint \frac{\mathcal{F}(\zeta)}{\zeta^{n+1}}\,d\zeta.$$

[6]The [early] development of prime number theory is carefully presented in NARKIEWICZs monograph [189]. The remainder term in (1.6) was improved by J. E. LITTLEWOOD, ..., finally by N. M. KOROBOV and I. M. VINOGRADOV [see [112], p. 347, with a correction by H.–E. RICHERT (see [243], p.226)].

A comparison of the behaviour of the function $\pi(x;q,a)$ in different residue classes is the object of the "Comparative Prime Number Theory", with important contributions of P. TURÁN (see [237]), then also by S. KNAPOWSKI, J. PINTZ and others.

The integral is approximated by highly ingenious ideas; the main terms of the asymptotic formula aimed at come from contributions near the singularities of the function (on $|z| = 1$), and the remaining parts of the integral can be estimated to be small in comparison with the main term.

E. LANDAU ([145], 1911) and G. HARDY and S. RAMANUJAN ([82], 1917; see also [53]) obtained results for the number $\pi_r(x)$ of integers composed of exactly r prime factors.[7] By induction the estimate

$$\pi_r(x) \le c_1 \frac{x}{\log x} \cdot \frac{(\log\log x + c_2)^{k-1}}{(r-1)\,!}$$

was obtained, and it follows that the *normal order* of $\omega(n)$ is $\log\log n$:

If $\psi(n)$ is any real-valued function tending to ∞ as $x \to \infty$, then the inequality

$$(1.7) \qquad |\omega(n) - \log\log n| \le \psi(n)\, \sqrt{\log\log n}$$

is true for "almost all" positive integers n. The same result is true for the function $\Omega(n)$, the total number of prime factors of n.

"Almost all integers n have property P" means that for any $\varepsilon > 0$ there are at most $\varepsilon \cdot x$ integers $n \le x$ for which property P does not hold. Speaking of "almost all" integers is a new idea in number theory, and it is related to similar concepts in the theory of integration or in the theory of probability.[8]

1.2. Beginnings of Probability Theory

Probability theory was not well developped at the time before 1900 or 1910, as may be seen from the Introduction of KRENGELs article [126],

[7]The problems become difficult and interesting, if one asks for results which are uniform with respect to r in some range. See A. HILDEBRANDT [87]. Here it is important to apply analytic methods to the function

$$F(z,s) = \sum_{n=1}^{\infty} \frac{z^{\omega(n)}}{n^s}.$$

See also [206], [218]. — By the way, H.–E. RICHERT ([200] gave asymptotic formulae for the number of integers with exactly r prime factors in residue classes $n \equiv a \bmod q$, with good error terms.

[8]J.–L. MAUCLAIRE [184] mentions that the idea of using Probability Theory in Number Theory shows up already in papers by E. CESÀRO [14] before 1889. — Later, formula (1.7) was greatly improved, see section 3, subsection 3.2.

p. 458. The sixth problem of D. HILBERT (1900) (in English from F. E. BROWDER, [9]) states:

> *Investigations of the foundations of geometry suggest the problem: To treat in the same manner, by means of axioms, those physical sciences in which mathematics plays an important part; first of all, the theory of probability and mechanics.*
>
> *As to the axioms of the theory of probability, it seems to me desirable that their logical investigation should be accompanied by a rigorous and satisfactory development of the method of mean values in mathematical physics* ....

There were some starts to deal with this question by BOHLMANN (1908) and UGO BROGGI (1907), some ideas came from E. BOREL, S. N. BERNSTEIN, LOMNICKI (1923) and STEINHAUS (1923) (see [126], p.459ff, see also [208]). Also, RICHARD VON MISES' paper *Grundlagen der Wahrscheinlichkeitsrechnung*, 1919, Math. Zeitschr., should be mentioned (see [126], p.461ff).

HILBERTs **desideratum** concerning probability theory was finally [satisfactorily] fulfilled by A. N. KOLMOGOROV in 1933 when his famous monograph *"Grundbegriffe der Wahrscheinlichkeitsrechnung"* [124] appeared in print. The concepts of probability, probability space and events were defined rigorously. PAUL TURÁN (see the photo to the left; the author is deeply indepted to P. TURÁN for his helpfulness. A photograph of TURÁNs grave is given on the next page) had not seen KOLMOGOROVs book in 1934, he even did not know Tchebycheff's inequality (see [38] II, p.18). Nevertheless, TURÁN [235] gave a new, important, "probabilistic" proof of the result of HARDY & RAMANUJAN concerning $\omega(n)$. He showed that

$$\sum_{n\leq x}(\omega(n)-\log\log x)^2=\mathcal{O}\left(x\cdot\log\log x\right),$$

and this easily implies the result (1.7) of HARDY & RAMANUJAN.[9] TURÁNs proof uses elementary calculations from number theory; his

[9] The relationship to TCHEBYCHEFFs inequality is obvious: If ξ is a random variable with expectation $E(\xi)=\int_\Omega \xi(w)dP(w)$ and standard deviation $D(\xi)=$

formula is probabilistic in nature, it may be interpreted as an estimate of the variance, the square of $\omega(n)$ minus its expectation value.

§2. The Turán–Kubilius Inequality

2.1. The Results of Turán and Kubilius

TURÁNs method of proof is applicable not only to $\omega(n)$, but also to strongly *additive* functions $w : \mathbb{N} \to \mathbb{C}$ [these satisfy $w(n) = \sum_{p|n} w(p)$], which are uniformly bounded at the primes ([236]). J. KUBILIUS [129] (see the photograph on the left) realized that TURÁNs inequality can be extended to a much larger class of [strongly] additive functions and so he obtained a considerably more general result. *For a given strongly additive function $w : \mathbb{N} \to \mathbb{C}$ there exists a [positive, universal] constant C_1 with the property*

$$\frac{1}{x} \cdot \sum_{n \le x} |w(n) - A(x)|^2 \le C_1 \cdot D^2(x). \tag{2.1}$$

Here, the "expectation" $A(x)$ and the "variance" $D(x)$ are defined as

$$\begin{aligned} A(x) &= \sum_{p \le x} \frac{w(p)}{p}, \\ D^2(x) &= \sum_{p \le x} \frac{|w(p)|^2}{p}. \end{aligned} \tag{2.2}$$

Some work has been done to give an asymptotic evaluation of the constant C_1 in (2.1), uniformly for all additive functions. This work is described in KUBILIUS' paper [138].

$\sqrt{E((\xi - E(\xi))^2)}$, then

$$P\Big(|\xi - E(\xi)| > \lambda \cdot D(\xi)\Big) \le \frac{1}{\lambda^2}.$$

Higher analogues of the TURÁN–KUBILIUS inequality are due to ELLIOTT [39]. *Given $\beta \geq 2$, there is a positive constant c_2, so that uniformly for $x \geq 2$ and all additive functions*

(2.3)

$$\frac{1}{x}\sum_{n\leq x}|w(n)-A(x)|^\beta \quad \leq \quad \begin{cases} c_2 D^\beta(x), & \textit{if } 0\leq\beta\leq 2, \\ c_2 D^\beta(x)+c_2\cdot\sum\limits_{p^k\leq x} p^{-k}\left|w(p^k)\right|^\beta, & \textit{if } 2\leq\beta. \end{cases}$$

2.2. Dualization, New Interpretation, Generalizations

A *dual inequality*[10] is: *For a sequence w_n of complex numbers the inequality*

$$\frac{1}{x}\cdot\sum_{p^k\leq x} p^k \left|\sum_{n\leq x, p^k\|n} w_n - p^{-k}\sum_{n\leq x} w_n\right|^2 \leq c_1\cdot\sum_{n\leq x}|w_n|^2 \tag{2.4}$$

is true. In ELLIOTTs monograph [47], p.18ff, a dual of the high–power–analogue of the TURÁN–KUBILIUS inequality (2.3) is given. In his conference report [46], P. D. T. A. ELLIOTT (see the photo to the left) described the position of the TURÁN–KUBILIUS inequality in the framework of Elementary Functional Analysis (see also ELLIOTT's survey article [41], and [47]). ELLIOTTs result (2.3) was generalized, for example, by K.-H. INDLEKOFER ([101]). *If $\phi:\mathbb{R}^+\to\mathbb{R}^+$ tends to ∞ and satisfies*

$$\phi(x+y)\leq\tfrac{1}{2}c\left(\phi(x)+\phi(y)\right),$$

for some $c>0$, and for all x, y, then

$$\frac{1}{x}\sum_{n\leq x}\phi\left(|w(n)-A(x)|\right) \ll \phi(x) + \sum_{p^k\leq x, |w(p^k)|\geq B(x)} \phi\left(\left|w(p^k)\right|\right)\cdot p^{-k}.$$

[10] The method of dualization (from linear algebra) is explained, for example, in ELLIOTT's book [37], pp. 150ff, and the whole monograph [47] is concerned with duality.

A *localized form* of the TURÁN–KUBILIUS inequality is due to P. D. T. A. ELLIOTT [42].

The TURÁN–KUBILIUS inequality may be looked for in *arithmetical semigroups* A, too.[11] It is given in MAUCLAIRE ([182]), for example, and the method of ELLIOTT (see [37]) leads to a proof of an ERDÖS–WINTNER theorem in arithmetical semigroups. J.-L. MAUCLAIRE [179] showed, that a condition $\sum_{a\in A, N(a)\leq x} 1 = L\cdot x + o(x)$ is sufficient for the validity of the TURÁN–KUBILIUS inequality. In contrast, in [182] he stated that *any arithmetical semi-group G is contained in another arithmetical semi-group $\mathcal{G}$, for which the* TURÁN–KUBILIUS *inequality is not valid.*

For more results on the TURÁN–KUBILIUS inequality in semigroups see the dissertation of REIFENRATH ([195], [162], and the Paderborn dissertations [244] and [157] of S. WEHMEIER and of Y.–W. LEE.

§3. The Theorems of Erdös–Wintner and Erdös–Kac

3.1. The Erdös–Wintner Theorem

An important problem, solved more than sixty years ago, is the question of the existence of a limit law for real–valued additive functions $w : \mathbb{N} \to \mathbb{R}$; asymptotically, a limit law describes the distribution of the values of the function w, more exactly, it gives (asymptotically, as $n \to \infty$) the number of integers $n \leq x$ for which $w(n) < z$. Consider, more generally, for subsets $E \subset \mathbb{R}$, the expressions

$$\mu_n(E) = \frac{1}{n}\cdot \#\left\{m \in \mathbb{N} : m \leq n, w(m) \in E\right\},$$

in particular the [finite] "distribution functions"

$$\nu_n(t) = \mu_n(]-\infty, t]) = \frac{1}{n}\cdot \#\Big\{m \in \mathbb{N} : m \leq n, w(m) \leq t\Big\}. \tag{3.1}$$

Then one asks for conditions ensuring the convergence of the sequence of distribution functions $\nu_n(t)$ to some limit distribution $K(t)$, $\nu_n(t) \Longrightarrow$

[11] For arithmetical semigroups see [121] and [123].

The photos have been removed due to copyright issues.

Figure 2. *P. Erdös (with A. Schinzel), P. Erdös*

$K(t)$, as $n \to \infty$ ("$\Longrightarrow$" means convergence at all points of continuity of the limit distribution).

One answer is provided by the famous ERDŐS–WINTNER theorem, modelled in analogy with the KOLMOGOROV three–series theorem of probability theory:[12] *An additive real–valued function w has a limit distribution if and only if the three series*

$$\sum_{p,\,|w(p)|\leq 1} \frac{w(p)}{p}, \qquad \sum_{p,\,|w(p)|>1} \frac{1}{p}, \quad \text{and} \quad \sum_{p,\,|w(p)|\leq 1} \frac{w^2(p)}{p} \tag{3.2}$$

are convergent.

Historically, P. ERDÖS showed in 1938 that the convergence of the three series in (3.2) implies the existence of a limit distribution; a new proof for this result is due to A. RÉNYI [196]. Previously, H. DAVENPORT (1933) and I. J. SCHOENBERG ([209], 1936) proved similar results for the multiplicative functions $n \mapsto \frac{\sigma(n)}{n}$ and $n \mapsto \frac{\varphi(n)}{n}$. The other implication (the existence of a distribution function implies the convergence of the three series) was proved by P. ERDŐS and A. WINTNER [60].

The proof of the ERDÖS–WINTNER theorem can be achieved by an application of the "Continuity Theorem for Characteristic Functions" (see, for example, [163], pp. 47ff): *Let $\{F_n(x)\}$ be a sequence of distribution functions, and denote by $\{f_n(t)\}$ the sequence of the corresponding characteristic functions*

$$f_n(t) = \int_{-\infty}^{\infty} e^{itx}\, dF_n(x). \tag{3.3}$$

[12] See A. RÉNYI [198], p. 420.

Then the sequence $\{F_n(x)\}$ *converges weakly to a distribution function* $F(x)$ *if and only if the sequence* $\{f_n(t)\}$ *converges for every* t *to a function* $f(t)$, *which is continuous at* $t = 0$.

Characteristic functions of arithmetical functions on the range $[1, N]$ are finite sums, and so the problem of convergence of characteristic functions is a question about the existence of mean–values (see later, subsection 4.1) for the multiplicative functions $n \mapsto \exp(2\pi i t w(n))$. DELANGE's theorem, to be treated later (see 4.1), relates the existence of mean–values with the convergence of the series $\sum_p \frac{1}{p} \cdot (1 - e^{2\pi i \alpha w(p)})$, and this helps in proving the convergence of the series (3.2).

The characterization of real–valued additive functions w with limit distributions with finite mean and variance is a result of P. D. T. A. ELLIOTT [33].

Limit distributions of additive functions "modulo 1" were treated by P. D. T. A. ELLIOTT [32]. Denote by $\{\beta\}$ the fractional part $\beta - [\beta]$ of $\beta \in \mathbb{R}$, and $\|\beta\|$ is the distance to the nearest integer. *If* w *is additive, then*

$$\frac{1}{n} \# \{m \le n; \{w(j)\} \le x\} \Longrightarrow F(x)$$

in $0 \le x \le 1$, *as* $n \to \infty$, *if and only if for every positive integer* m *at least one of the following conditions holds:*

(1) $\sum_p \frac{1}{p} \, \|mw(p) - \frac{t}{2\pi}\|^2$ *is divergent.*

(2) $m \cdot w(2^r) \in \frac{1}{2}\mathbb{N}$ *for every integer* $r > 0$

(3) Both series $\sum_p \frac{1}{p} \, \|mw(p)\|^2$, $\sum_p \frac{1}{p} \, \|mw(p)\| \cdot \operatorname{sgn}(\frac{1}{2} - \{mw(p)\})$ *are convergent.*

3.2. Around the Erdös–Kac Theorem

3.2.1. *The Erdös–Kac Theorem.* The Erdős–Kac theorem was proved in 1939 ([55], [56]).[13]

For a real–valued strongly additive function $w : \mathbb{N} \to \mathbb{R}$ define $A(x)$ and $B(x)$ by (2.2). Then P. ERDÖS and M. KAC proved in 1939:

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[13] A fore–runner is ERDŐS' paper [49], where he proved that the number of integers $n \le x$, for which $\omega(n) > \log\log n$, is $\frac{1}{2}\, x + o(x)$, using BRUNs sieve and an asymptotic estimate of the number of integers $n \le x$ for which $\omega(n) = k$ in some [small] range of k.

Let w be a strongly additive function satisfying $|w(p)| \le 1$ for all primes p. Assume that $B(x) \to \infty$ as $x \to \infty$. Then

$$\frac{1}{x}\,\#\Big\{n \le x;\ w(n) - A(x) \le zB(x)\Big\} \Longrightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}u^2}\,du. \tag{3.4}$$

In particular, for $\omega(n) = \sum_{p|n} 1$,

$$\frac{1}{x}\,\#\{n \le x;\ \frac{\omega(n) - \log\log x}{\sqrt{\log\log x}} \le z\} \Longrightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}u^2}\,du.$$

These results also can be used (M. Kac [114]) to obtain value–distribution results for the multiplicative function $\tau(n) = \sum_{d|n} 1$,

$$\frac{1}{x} \cdot \#\left\{n \le x;\ \tau(n) \le 2^{\log\log x + z\sqrt{\log\log x}}\right\} \Longrightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}u^2}\,du.$$

J. Kubilius defined a reasonably large "class H" of additive functions, to which equation (3.4) can be extended. The additive function w is in "class H", *if there exists a function* $r :]0,\infty[\to]0,\infty[$ *such that*

$$\frac{\log r(x)}{\log x} \to 0, \quad \frac{B(r(x))}{B(x)} \to 1, \quad \text{and } B(x) \to \infty, \text{ as } x \to \infty. \tag{3.5}$$

J. Kubilius extended the Erdős–Kac result as follows:

Let $w : \mathbb{N} \to \mathbb{R}$ be a strongly additive function of class H. Then the frequencies

$$\frac{1}{x}\,\#\Big\{n \le x;\ w(n) - A(x) \le zB(x)\Big\} \tag{3.6}$$

converge weakly to a limit distribution as $x \to \infty$ if and only if there is a distribution function $K(u)$, so that almost surely in u

$$\frac{1}{B^2(x)} \sum_{\substack{p \le x \\ w(p) \le uB(x)}} \frac{w^2(p)}{p} \to K(u), \quad \text{as } x \to \infty. \tag{3.7}$$

The characteristic function $\Phi(t)$ of the limit law will be given by

$$\log \Phi(t) = \int_{-\infty}^{\infty} \frac{e^{itu} - 1 - itu}{u^2}\, dK(u), \tag{3.8}$$

and the limit law has mean zero and variance 1.

If $w \in H$, and

$$\frac{1}{B^2(x)} \sum_{\substack{p \le x \\ |w(p)| > \varepsilon B(x)}} \frac{1}{p} w^2(p) \to 0 \tag{3.9}$$

for every $\varepsilon > 0$, then the frequencies in (3.7) converge to the Gaussian law (as in (3.4)).

3.2.2. *The Elliott–Levin–Timofeev Theorem.* More generally, given two normalizing functions $\alpha(x), \beta(x)$, one can ask if there exists a distribution function $F(z)$ with the property

$$\nu_x\{n \le x, w(n) \le z\beta(x) + \alpha(x)\} \Longrightarrow F(z), \tag{3.10}$$

as $x \to \infty$. An answer is given by the ELLIOTT–LEVIN–TIMOFEEV Theorem (see [38], II, Chapter 16): *Assume that w is a real–valued additive function, and α, β are real–valued [measurable] functions, satisfying*

$$\beta(x) \to \infty \text{ as } x \to \infty, \quad \sup_{1 \le t \le 2} \left| \frac{\beta(x^t)}{\beta(x)} - 1 \right| \to 0 \text{ as } x \to \infty. \tag{3.11}$$

Then (3.10) holds if and only if there exists a constant $A > 0$ such that

$$P\left\{ \sum_{p \le x} X_p \le z\beta(x) + \alpha(x) - \lambda \log x \right\} \Longrightarrow F(z) \tag{3.12}$$

for some distribution function $F(z)$, where X_p are independent random variables defined by $X_p = f(p) - A \log p$ with probability $\frac{1}{p}$, and $= 0$ with probability $1 - \frac{1}{p}$.

3.2.3. *Moments.* In 1955, H. HALBERSTAM [77] calculated *moments* for additive, real–valued functions w elementarily,

$$\lim_{n \to \infty} \sum_{m \le n} \frac{(w(m) - A(n))^k}{nD(n)^{k/2}} = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} x^k e^{-\frac{1}{2}x^2} dx, \tag{3.13}$$

and he deduced the ERDÖS–KAC theorem from equation (3.13). A further extension of this "method of moments" is due to H. DELANGE [25], who also gave a new [analytic] proof for HALBERSTAMs result (3.13).

3.2.4. *Remainder Terms.* Asking for good remainder terms in ((3.4)), A. RÉNYI and P. TURÁN [199] proved, in the special case where $w = \omega$, equation ((3.4)) with a best–possible remainder term $\mathcal{O}\left(\frac{1}{\sqrt{\log \log n}}\right)$, and so a conjecture of W. J. LEVEQUE ([158]) was established for the special additive function $w = \omega$.

Asymptotic expansions for the frequencies $\nu_n\left\{\frac{\omega(m)-\log\log n}{\sqrt{\log\log n}} < x\right\}$ with remainder term were given, for example, by J. KUBILIUS ([131]) and H. DELANGE ([27]):[14]

$$\nu_n\left\{\frac{\omega(m)-\log\log n}{\sqrt{\log\log n}} < x\right\} = G(x)\cdot e^{Q_n(x)}\cdot\left\{1+\mathcal{O}\left(\frac{|x|+1}{\sqrt{\log\log n}}\right)\right\}.$$

More information on the rate of convergence to the Normal Law can be found in ELLIOTTs monograph [38], Chapter 20.

3.2.5. *Composed Functions.* ERDÖS and POMERANCE [57] proved an ERDÖS–KAC theorem for the composed function $n \mapsto \Omega(\varphi(n))$:

$$\lim_{x\to\infty}\frac{1}{x}\#\left\{n\le x; \Omega(\varphi(n))-\frac{1}{2}(\log\log x)^2 \le \frac{1}{\sqrt{3}}\cdot z\,(\log\log x)^{\frac{3}{2}}\right\} = G(z).$$

A similar result ist true for $\omega(\varphi(n))$.

3.2.6. *Brownian Motion.* A connection between additive arithmetic functions and Brownian motion is given, for example, in KUBILIUS' paper [136], and in the survey article [167] of MANSTAVIČIUS.

3.2.7. *Multiplicative Functions.* A result of M. KAC for the multiplicative [divisor–] function τ was mentioned earlier. More general limit laws for multiplicative arithmetical functions were proved by A. BAKŠTYS [4], and by J. KUBILIUS & Z. JUŠKYS [140]. These authors proved for multiplicative real–valued functions g, under suitable assumptions on g (g belongs to some class $M_0(c,\lambda)$, which will not be defined here; $\log_2 n = \log\log n$):

$$\begin{aligned}\tfrac{1}{n}\cdot\#\left\{m\le n, g(m) < |x|^{\lambda\sqrt{\log_2 n}}\cdot\log^{\lambda} n\cdot\operatorname{sgn}(x)\right\}\\ = \phi(x)+\mathcal{O}\left(\tfrac{1}{\sqrt{\log_2 n}}\right),\end{aligned}$$

where $\phi(x)$ is connected with the GAUSS integral.

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[14] $\nu_n\{P_m(x)\}$, for some property $P_m(x)$, is defined as

$$\nu_n\{P_m(x)\} = \frac{1}{n}\cdot\#\Big\{m\le n;\ P_m(x)\Big\}.$$

3.3. Generalizations

The problem of the distribution of the values $w(n)$ of additive functions for $n \le x$ was generalized to [thin] subsequences of $\{1, 2, 3, \dots\}$, for example to the sequence of shifted primes $\{p+1,\ p \text{ prime}\}$ or to the sequence $\{Q(n),\ n = 1, 2, 3, \dots\}$ with a monic polynomial $Q(x) > 0$ with integer coefficients.

3.3.1. *Moments for thin sequences.* H. HALBERSTAM [77] proved: *if w is a strongly additive function, then* (for the definition of A_Q, B_Q see (3.14)):

$$\sum_{n\le x} (w(Q(n)) - A_Q(x))^q = \mu_q \cdot x \cdot B_Q^q(x) + o(x \cdot B_Q^q(x)),$$

if $\max_{p\le x} |w(p)| = o(B_Q(x))$. *When p runs over primes, then*

$$\sum_{p\le x} (w(Q(p)) - A_Q(x))^q = \mu_q \cdot \pi(x) \cdot B_Q^q(x) + o(\pi(x) \cdot B_Q^q(x)),$$

if $|w(p)| \le M$ *and* $\frac{B_Q(x)}{\log\log\log x} \to \infty$.

The definition of $A_Q(x)$ and $B_Q(x)$ is similar as in (2.2), but a factor $\rho(p)$, the number of solutions of the congruence $Q(n) \equiv 0 \bmod p$, has to be inserted. So

$$A_Q(x) = \sum_{p\le x} \frac{w(p)\rho(p)}{p}, \quad B_Q^2(x) = \sum_{p\le x} \rho(p) \cdot \frac{w^2(p)}{p}. \tag{3.14}$$

These results lead to a generalized ERDŐS – KAC–theorem,

$$\frac{1}{x} \# \{n \le x; w(|Q(n)|) - A_Q(x) \le zB_Q(x)\} \Longrightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}w^2}\, dw,$$

as $x \to \infty$, under the assumption $\mu_x = \max_{p\le x} \dfrac{|w(p)|}{B_Q(x)} \to 0$. A corresponding result, where n is restricted to primes, is due to BARBAN (see [5]).

In 1988 H. DELANGE proved the result

$$\lim_{x\to\infty} \frac{1}{\#(S_x)} \cdot \sum_{n\in S_x} \left(\frac{w(n) - A(x)}{B(x)} \right)^q = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u^q \cdot e^{-\frac{1}{2}u^2}\, du,$$

where the sets S_x are a family of finite sequences, satisfying

$$\#(S_x) \to \infty, \ \max_{n\in S_x} n \ll x^{\Delta}, \ \Delta \ge 1,$$

and satisfying some condition guaranteeing the possibility of an application of the sieve method.

K.-H. INDLEKOFER & I. KÁTAI ([107]) calculated,[15] for strongly additive functions, "moments over shifted primes". They showed

$$\lim_{x\to\infty} \frac{\pi(x)}{B^k(x)} \cdot \sum_{p\le x} \Big(f(p+1) - A(x) \Big)^k = \int_{-\infty}^{\infty} z^k \, dF(z)$$

if and only if

$$\limsup_{x\to\infty} \frac{1}{B^k(x)} \cdot \sum_{p\le x, |f(p)|>B(x)} \frac{1}{p} \cdot |f(p)|^k < \infty.$$

3.3.2. *Polynomials.* H. HALBERSTAMs result ([77], see § 3.3.1) was already given. M.B.BARBAN, R. V. UZDAVINIS, P. D. T. A. ELLIOTT and others gave corresponding results on the frequencies (the definitions (3.14) are slightly changed)

$$\frac{1}{x} \# \left\{ p \le x, \; w(|Q(p)|) - A_Q^*(x) \le z \cdot B_Q^*(x) \right\}$$

An ERDÖS–KAC theorem for shifted primes similar to KUBILIUS's result is due to M. B. BARBAN et al. (1965). See ELLIOTT's book (1980), Vol. II, p. 27. E. MANSTAVIČIUS [165] (see the photo on the right), using a result of A. BIKELIS, gave remainder term estimates in the ERDÖS–KAC theorem (improving results of I. N. ORLOV [192] considerably). P. D. T. A. ELLIOTT [43], [45], and K.-H. INDLEKOFER [103] proved ERDÖS–KAC theorems in short intervals: $x - y < n \le x, \; y(x) = x^{1+o(1)}$.

Generalizing a result of A. HILDEBRAND [88], P. D. T. A. ELLIOTT gave an ERDÖS–KAC theorem for pairs of real–valued additive functions. There exists an $\eta(x)$, so that

$$\frac{1}{[x]} \cdot \{ n \le x; f_1(an+b) - f_2(An+B) - \eta(x) \le z \} \to \text{ a distribution function}$$

[15] Earlier results of this kind were given by BARBAN et al. [6], and B. V. LEVIN & A. S. FAĬNLEĬB [159].

The photos have been removed due to copyright issues.

Figure 3. *H. Delange, E. Wirsing*

if and only if there exist real α_j such that the series

$$\sum_{|f_j(p)-\alpha_j \log p|>1} \frac{1}{p}, \quad \sum_{|f_j(p)-\alpha_j \log p|\le 1} \frac{(f_j(p)-\alpha_j \log p)^2}{p}$$

are convergent.

§4. Arithmetical Functions

Important for the deduction of the results in § 3 is information on the existence of mean–values $M(f) = \lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} f(n)$ of arithmetical functions $f : \mathbb{N} \to \mathbb{C}$.

4.1. Mean–Value Theorems for Multiplicative Functions

In number theory, many results on the existence of mean–values and on asymptotic formulae for special arithmetical functions ($\varphi(n)$, $\tau(n)$, $\dots$) were proved, often with emphasis on good or best–possible remainder terms; but there were also rather early general results on mean–values for certain classes of arithmetical functions (for example Axer [1] and Wintner [247]). The condition $\sum_p \frac{|p-1|}{p} < \infty$, much stronger than Delanges condition (4.2), is crucial in Wintners theorem.

By skilfull methods from number theory H. Delange (1961, [26]) was able to prove an elegant result on multiplicative arithmetical functions. This theorem – as well as E. Wirsings theorems on multiplicative functions [[248], [249]) — expressed the heuristic idea that knowledge on the values of multiplicative functions at primes has consequences on the behaviour of multiplicative functions in general, as may be guessed from the Euler product of the generating Dirichlet series,

$$\sum_{1}^{\infty} \frac{f(n)}{n^s} = \prod_p \left(1 + \frac{f(p)}{p^s} + \frac{f(p^2)}{p^{2s}} + \dots\right). \tag{4.1}$$

If $f : \mathbb{N} \to \mathbb{C}$ is multiplicative, $|f| \le 1$, then there is a non–zero mean–value $M(f) = \lim\limits_{x\to\infty} \frac{1}{x} \cdot \sum_{n\le x} f(n)$ if and only if the series

$$\sum_p \frac{1 - f(p)}{p} \text{ is [conditionally] convergent,} \tag{4.2}$$

and if, for alle primes p,

$$\sum_{k=0}^{\infty} \frac{f(p^k)}{p^k} \ne 0. \tag{4.3}$$

Condition 4.3 is equivalent with $f(2^k) \ne -1$ for some $k \ge 1$. A Delange theorem with remainder terms is due to Postnikov [193], in improved form to Elistratov [31].

General asymptotic formulae for a large class of non–negative multiplicative functions are due to Eduard Wirsing.

Theorem of E. Wirsing [248]. *If $f \ge 0$ is multiplicative, $\tau > 0$,*

$$\sum_{p\le x} f(p)\log p = \big(1 + o(1)\big)\tau \cdot x, \quad f\left(p^k\right) \le \gamma_1 \cdot \gamma_2^k$$

for any $k \ge 2$, where $0 < \gamma_2 < 2$, $\Longrightarrow$

$$\sum_{n\le x} f(n) = \big(1 + o(1)\big) \cdot \frac{x}{\log x} \cdot \prod_{p\le x} \left(1 + \frac{f(p)}{p} + \frac{f(p^2)}{p^2} + \dots\right). \tag{4.4}$$

In 1967, Wirsing [249] gave other theorems, weakening the hypothesis on $\sum_p f(p)$ considerably, and allowing for complex–valued functions. In particular, this result contains the prime number theorem as a special case. The deepest theorem in this connection was obtained by an ingenious variation of classical analytic methods; it is the

Theorem of G. Halász, [74]. *If $f : \mathbb{N} \to \mathbb{C}$ is multiplicative, and if $|f| \le 1$, then:*

If f is real-valued, then $\exists$ $M(f)$, and for complex-valued f,

$$\frac{1}{x} \cdot \sum_{n \leq x} f(n) = (c + o(1)) \cdot x^{ia_0} \cdot L(\log x), \tag{4.5}$$

where L is a slowly oscillating function, $|L| = 1$, and a_0 is a real constant, which can be given explicitly.

Elementary proofs of the HALÁSZ theorem are due to DABOUSSI & INDLEKOFER [23].

4.1.1. *The Elliott–Daboussi–Theorem.* The condition $|f| \leq 1$ from DELANGEs theorem was removed by ELLIOTT [35], and the result was extended later by ELLIOTT [40], and by H. DABOUSSI [16], [17]. The condition $|f| \leq 1$ is replaced by a condition on the size of the values $f(p^k)$ in the mean (see (iii) below.

Elliott–Daboussi's Theorem. *If $f : \mathbb{N} \to \mathbb{C}$ is multiplicative, if $q > 1$, and if (see (4.9)) $\|f\|_q < \infty$, then a non-zero mean-value $M(f) \neq 0$ exists if and only if*

(i) Delange's series $\displaystyle\sum_p \frac{f(p) - 1}{p}$ *is convergent,*

(ii) $\displaystyle\sum_{\substack{p \\ |f(p)-1| < \frac{5}{4}}} \frac{|f(p) - 1|^2}{p} < \infty, \qquad \sum_{\substack{p \\ |f(p)-1| > \frac{5}{4}}} \frac{|f(p)|^q}{p} < \infty,$

(iii) $\displaystyle\sum_p \sum_{k \geq 2} \frac{|f(p^k)|^q}{p^k} < \infty$, *and*

(iv) $\displaystyle\sum_{k \geq 1} \frac{f(p^k)}{p^k} \neq 0$ *for any prime p.*

4.1.2. *Mean–value theorems in multiplicative arithmetical semigroups.* The mean–value theorems mentioned (due to DELANGE, WIRSING, HALÁSZ) were generalized to multiplicative arithmetical semigroups, starting with the work of J. KNOPFMACHER [121]. Some results are surveyed in the paper [162] by L. LUCHT & K. REIFENRATH. More details may be found in REIFENRATHs dissertation [195]. There are many results, concerning mean–values of additive and of multiplicative functions in semigroups, in the Paderborn dissertation [244] of S. WEHMEIER.

4.1.3. *Mean–value theorems in additive arithmetical semigroups.* The mean–value theorems mentioned, in particular HALÁSZ' theorem, were generalized to additive arithmetical semigroups; the deepest results are due to W. B. ZHANG. We cannot give his results here; the interested reader is referred to ZHANGs papers in Math. Z. **229** (1998), 195–233, Illinois J. **42** (1988), 189–229, Math. Z. **235** (2000), 747–816, and to [250], [252] and [253]. In [252] there is also a generalization of the Elliott–Daboussi theorem (see later, p.386) to additive arithmetical semigroups. Mean–value theorems for q–additive and q–multiplicative functions are given in YI–WEI LEE–STEINKÄMPERs dissertation [157].

4.2. Using the Turán–Kubilius Inequality

In 1965, A. RÉNYI gave a simple proof for the existence of $M(f)$, if the Delange series (4.2) is convergent. His idea of proof is to use an approximation of $\log f$ by truncated additive functions, and the TURÁN–KUBILIUS Inequality allows a sufficiently good estimate of the difference.

4.2.1. *The Relationship Theorem.* A useful tool for reducing the proof to the simplest cases is the "relationship theorem" (E. HEPPNER & W. SCHWARZ (1978) [84]; weaker theorems of this kind were given previously by H. DELANGE and L. LUCHT).

Assume that the multiplicative functions f and g are "related", i.e.

$$\sum_p \frac{|f(p)-g(p)|}{p} < \infty,$$

and that f, $g \in \mathcal{G}$, where

$$\mathcal{G} \stackrel{def}{=} \left\{ F \text{ multiplicative}, \quad \sum_p \frac{|F(p)|^2}{p^2} < \infty, \sum_p \sum_{k\ge 2} \frac{|F(p^k)|}{p^k} < \infty \right\},$$

and that all the factors of the generating Dirichlet series (4.1)

$$\sum_1^\infty \frac{f(n)}{n^s} = \prod_p \varphi_f(p,s), \qquad \varphi_f(p,s) = 1 + \frac{f(p)}{p^s} + \frac{f(p^2)}{p^{2s}} + \dots,$$

do not vanish in $\Re(s) \ge 1$. Then there is a [small] multiplicative function h, satisfying

$$g = f * h, \quad \textit{and} \quad \sum_1^\infty \frac{1}{n}\,|h(n)| < \infty.$$

Corollary. *If $M(f)$ exists, then $M(g)$ exists, too.*

The proof uses a Wiener type lemma for Dirichlet series, which was proved by E. HEWITT & R. WILLIAMSON in 1957.[16]

The [absolutely convergent] Dirichlet series $\sum_1^\infty \frac{a_n}{n^s}$, *where* $\sum_1^\infty |a_n| < \infty$, *has an [absolutely convergent] inverse*

$$\sum_1^\infty \frac{b_n}{n^s} = \left(\sum_1^\infty \frac{a_n}{n^s}\right)^{-1}, \quad \textit{satisfying} \ \sum_1^\infty |b_n| < \infty,$$

if and only if there is some lower bound $\delta > 0$ *for* $\left|\sum_1^\infty \frac{a_n}{n^s}\right|$ *in the half-plane* $\Re(s) \geq 0$.

4.2.2. *Sketch of Rényi's Proof.* By relationship arguments one may assume that the values $f(p)$ have real part $\geq \frac{3}{4}$ and that f is strongly multiplicative. Then, approximate the multiplicative function f by a "truncated" strongly multiplicative function f_K^*, $f_K^*(n) = \prod_{\substack{p|n \\ p\leq K}} f(p)$. The mean-value $M(f_K^*)$ is easily calculated. And it can be expected that $M(f_K^*)$ is near $M(f)$, if K is large. This is made precise by the estimate

$$\Delta_N = \frac{1}{N}\sum_{n\leq N} |f(n) - f_K^*(n)| \leq \frac{1}{N}\sum_{n\leq N} |f_K^*(n)| \cdot \left|\prod_{p|n,\ p>K} f(p) - 1\right|.$$

Using the TURÁN–KUBILIUS inequality for the strongly additive function $w(n) = \sum_{p|n,\ p>K} \log f(p)$, one obtains

$$\left|\prod_{p|n,\ p>K} f(p) - 1\right| = \left|e^{w(n)} - 1\right| \leq |w(n)| \cdot \left(1 + |e^{w(n)}|\right).$$

By CAUCHYs inequality and the convergence of the Delange series (4.2), the estimate $\Delta_N \to 0$ (as $N \to \infty$) is obtained.

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[16]For an elementary proof see [213]. For a relationship theorem for functions of several variables see E. HEPPNER [83]. For important generalizations see [161].

4.2.3. *Spaces of Arithmetical Functions.* Using RÉNYIs method and the relationship theorem, DELANGEs theorem can be extended to larger classes of multiplicative (and additive) functions (see [216], [104]).

Denote the set of linear combinations of (the "even", and so periodic) RAMANUJAN sums

$$c_r(n) = \sum_{d|gcd(r,n)} d\mu\left(\frac{r}{d}\right) = \sum_{1\le a\le r,\ (a,r)=1} \exp(2\pi i \cdot \frac{a}{r} \cdot n) \tag{4.6}$$

resp. exponential functions $n \mapsto \exp(2\pi i\alpha n), \alpha$ rational, resp. α irrational, by $\mathcal{B}$, resp. $\mathcal{D}$, resp. $\mathcal{A}$. The closures of these $\mathbb{C}$ –vector–spaces with respect to the ("uniform" or supremum) norm

$$\|f\|_u = \sup_{n\in\mathbb{N}} |f(n)| \tag{4.7}$$

are the spaces

$$\mathcal{B}^u, \text{ resp. } \mathcal{D}^u, \text{ resp. } \mathcal{A}^u \tag{4.8}$$

of *uniformly–even, uniformly–limit–periodic,* and *uniformly–almost periodic* functions. These vector–spaces are in fact Banach algebras.

The closures of $\mathcal{B}$, $\mathcal{D}$, $\mathcal{A}$, with respect to the semi–norm

$$\|f\|_q = \left(\limsup_{x\to\infty} \frac{1}{x} \cdot \sum_{n\le x} |f(n)|^q\right)^{\frac{1}{q}}, \quad q \ge 1, \tag{4.9}$$

are denoted by

$$\mathcal{B}^q, \text{ resp. } \mathcal{D}^q, \text{ resp. } \mathcal{A}^q, \tag{4.10}$$

the spaces of *q–almost–even*, *q–limit–periodic*, and *q–almost–periodic* functions.

4.2.4. *Properties of these Spaces.* These spaces have convenient properties useful for approximation arguments.

(1) $\mathcal{B} \subset \mathcal{B}^u \subset \mathcal{B}^q \subset \mathcal{D}^q \subset \mathcal{A}^q \subset \mathcal{A}^1$, $q \ge 1$.
(2) Functions in $\mathcal{A}^1$ do have a *mean–value*, *Fourier coefficients* $\hat{f}(\alpha)$ and *Ramanujan coefficients* $a_r(f)$,

$$\hat{f}(\alpha) \stackrel{\text{def}}{=} M(f \cdot \mathbf{e}_{-\alpha}), \qquad a_r(f) \stackrel{\text{def}}{=} \frac{1}{\varphi(r)} \cdot M(f \cdot c_r).$$

$$\begin{tikzcd}[column sep=small, row sep=small]
 & & \mathcal{A}^1 \arrow[dl, dash] \arrow[dd, dash] & \\
 & \mathcal{D}^1 \arrow[dl, dash] \arrow[dd, dash] & & \\
\mathcal{B}^1 \arrow[dd, dash] & & \mathcal{A}^q,\ (q \geq 1) \arrow[dl, dash] \arrow[dd, dash] & \\
 & \mathcal{D}^q \arrow[dl, dash] \arrow[dd, dash] & & \\
\mathcal{B}^q \arrow[dd, dash] & & \mathcal{A}^u \arrow[dl, dash] \arrow[dr, dash] & \\
 & \mathcal{D}^u \arrow[dl, dash] \arrow[dr, dash] & & \mathcal{A} \arrow[dl, dash] \\
\mathcal{B}^u \arrow[dr, dash] & & \mathcal{D} \arrow[dl, dash] & \\
 & \mathcal{B} & &
\end{tikzcd}$$

$$\|f\|_u = \sup_{n=1,2,\ldots} |f(n)|,$$

$\mathcal{B}^u = \|.\|_u$ –closure of $\mathcal{B}$,

$\mathcal{D}^u$, $\mathcal{A}^u$ analogously.

$$\|f\|_q = \limsup_{x\to\infty} \frac{1}{x} \cdot \left(\sum_{n\leq x} |f(n)|^q \right)^{\frac{1}{q}}, \quad q \geq 1.$$

Spaces of Arithmetical Functions

(3) $\mathcal{B}^u \cdot \mathcal{B}^q \subset \mathcal{B}^q, \qquad \mathcal{B}^q \cdot \mathcal{B}^{q'} \subset \mathcal{B}^1$, if $\dfrac{1}{q} + \dfrac{1}{q'} = 1$.
(4) $f \in \mathcal{B}^q \Longrightarrow \Re(f),\ \Im(f),\ |f| \in \mathcal{B}^q$.
(5) $f, g \in \mathcal{B}^q$ real–valued $\Longrightarrow \max(f,g) \in \mathcal{B}^q$, $\min(f,g) \in \mathcal{B}^q$.
(6) $f \in \mathcal{B}^1$, $\|f\|_q < \infty \Longrightarrow f \in \mathcal{B}^r$, if $1 \leq r < q$.
(7) $f \geq 0, \alpha, \beta \geq 1 \Longrightarrow \left\{ f^\alpha \in \mathcal{A}^\beta \iff f \in \mathcal{A}^{\alpha\cdot\beta} \right\}$. (H. Daboussi)
(8) Additive resp. multiplicative shifts map $\mathcal{A}^q$ into itself.

4.2.5. *Indlekofer's Spaces.* K.–H. INDLEKOFER defined spaces (see [104], [105], [106])

$$\begin{aligned} \mathcal{L}^q &= \left\{ f : \mathbb{N} \to \mathbb{C},\ \|f\|_q < \infty \right\}, \quad 1 \leq q < \infty, \\ \mathcal{L}^* &= \left\{ f : \mathbb{N} \to \mathbb{C},\ f \text{ uniformly summable} \right\}. \end{aligned} \tag{4.11}$$

Here, f is called "uniformly summable", if large values of $|f|$ are rare, more precisely

(4.12) *f is uniformly summable, if* $\displaystyle \lim_{K\to\infty} \sup_{N\geq 1} \frac{1}{N} \cdot \sum_{\substack{n\leq N \\ |f(n)|>K}} |f(n)| = 0.$

$\mathcal{L}^*$ is the $\|.\|_1$–closure of ℓ^∞, the space of bounded arithmetical functions, and for $q > 1$ the inclusions

$$\mathcal{L}^q \subset \mathcal{L}^* \subset \mathcal{L}^1$$

hold. Then INDLEKOFER [94] generalized the DELANGE–ELLIOTT–DABOUSSI result for *multiplicative* functions to

Theorem. *Let $q \geq 1$, and $f : N \to \mathbb{C}$ is multiplicative. Then:*

(1) *If $f \in \mathcal{L}^* \cap \mathcal{L}^q$, and if $M(f)$ exists and is $\neq 0$, then the series*

$$(4.13) \quad \sum_p \frac{f(p)-1}{p}, \qquad \sum_{p,\,|f(p)|\leq\frac{3}{2}} \frac{|f(p)-1|^2}{p},$$

$$\sum_{p,\,|f(p)-1|\geq\frac{1}{2}} \frac{|f(p)|^\lambda}{p}, \quad \sum_p \sum_{k\geq 2} \frac{1}{p^k}\cdot|f(p^k)|^\lambda$$

do converge for all λ, $1 \leq \lambda \leq q$, and

$$(4.14) \qquad \sum_{k=0}^{\infty} \frac{f(p^k)}{p^k} \neq 0 \text{ for every prime } p.$$

(2) *If the four series (4.13) converge, then $f \in \mathcal{L}^* \cap \mathcal{L}^q$, and the mean–values $M(f^\lambda)$ exist for any λ, $1 \leq \lambda \leq q$. If (4.14) holds in addition, then $M(f) \neq 0$.*

INDLEKOFER also extended the result of G. HALÁSZ. *If $f : \mathbb{N} \to \mathbb{C}$ belongs to $\mathcal{L}^*$, and if the series*

$$\sum_{p,\,|\,|f(p)|-1|\leq\frac{1}{2}} \frac{1}{p}\cdot\left(1 - \Re f(p)(|f(p)|p^{it})^{-1}\right) \text{ diverges for any real } t,$$

then f possesses a mean–value $M(f) = 0$.

For real–valued additive functions w there is a limit distribution F (and $\int_{-\infty}^{\infty} |u|^q dF(y) < \infty$) if and only if $w \in \mathcal{L}^q$ and the mean–value exists ([98]).

4.2.6. *Characterization of some classes of arithmetical functions in $\mathcal{B}^q$.*

1) Multiplicative functions in $\mathcal{B}^q$ with mean–value $M(f) \neq 0$ are characterized exactly by the conditions of the ELLIOTT – DABOUSSI theorem (H. DABOUSSI, W. SCHWARZ & J. SPILKER, K.–H. INDLEKOFER).

1a) A characterization of multiplicative funktions in $\mathcal{D}^q$ having at least one non–vanishing Fourier coefficient is possible by similar conditions. (DABOUSSI, SCHWARZ & SPILKER). Under suitable conditions, multiplicative functions f don't have Fourier–coefficients $\hat{f}(\alpha) \neq 0$ for *irrational* α., according to a result of H. DABOUSSI; this was generalized by INDLEKOFER & KÁTAI [108], and further in [111] to

If f is a uniformly summable function with a void Fourier–Bohr spectrum (so $\limsup_{x\to\infty} \frac{1}{x}\left|\sum_{n\le x} f(n)e(-n\alpha)\right| = 0$ for $\alpha \in \mathbb{R}$), and if g is a q–multiplicative functions satisfying $|g(n)| = 1$, then

$$\frac{1}{x}\sum_{n\le x} f(n)g(n) \to 0, \;\; \text{as } x \to \infty.$$

2) Additive funktions in $\mathcal{B}^q$ are characterized by similar conditions on the convergence of certain infinite series over primes (A. HILDEBRANDT & J. SPILKER [90], K.–H. INDLEKOFER). We quote the theorem of A. HILDEBRAND and J. SPILKER (1980), which was proved independently by P. D. T. A. ELLIOTT too, and which was improved by K.-H. INDLEKOFER.

The photos have been removed due to copyright issues.

Figure 4. *A. Hildebrandt, K.-H. Indlekofer, G. Tenenbaum*

Assume that $f : \mathbb{N} \to \mathbb{C}$ is additive, and $q \ge 1$. Then the following conditions are equivalent:

(1) $f \in \mathcal{B}^q$.
(2) $M(f)$ *exists,* $\|f\|_q < \infty$.

(3) *The following series are convergent:*

$$\sum_{p,|f(p)|\le 1} \frac{f(p)}{p}, \quad \sum_{p,|f(p)|\le 1} \frac{|f(p)|^2}{p}, \quad \sum_{p,\ k\ge 1,|f(p^k)|>1} \frac{|f(p^k)|^q}{p^k}.$$

If one of these conditions is satisfied, then the RAMANUJAN *expansion* $F = \sum_r a_r c_r$, $a_r = \frac{1}{\varphi(r)} \cdot M(f \cdot c_r)$ *of* f *is pointwise convergent. This expansion is absolutely convergent, if* $\sum_{p,|f(p)|\le 1} \frac{f(p)}{p}$ *is absolutely convergent.*

3) Another class of functions, investigated for example by J. COQUET, H. DELANGE, M. PETER, J. SPILKER and others, is the class of q–additive or q–multiplicative functions, and there are similar results.

The most complete results on this topic are due to YI–WEI LEE–STEINKÄMPER [157] in her dissertation (Paderborn 2005), supervised by K.–H. INDLEKOFER.

For example, for a q–multiplicative function f *the following assertions are equivalent:*

(i) f *is uniformly summable and* $\|f\|_1 > 0$.

(ii) *For any* $\alpha > 0$ $f \in \mathcal{L}^\alpha$ *and* $\|f\|_\alpha > 0$.

(iii) *For any* $\alpha > 0$ *the series* $\sum_{r\ge 0} \frac{1}{q} \sum_{a=1}^{q-1} \Big(|f(aq^r)|^\alpha - 1\Big)^2$ *is convergent, and there are real constants* $c_j(\alpha)$ *and a sequence* $\{R_i\} \ne \infty$, *so that*

$$\sum_{r<R} \frac{1}{q} \sum_{a=1}^{q-1} \Big(|f(aq^r)|^\alpha - 1\Big)^2 \le c_1(\alpha), \quad \sum_{r<R_i} \frac{1}{q} \sum_{a=1}^{q-1} \Big(|f(aq^r)|^\alpha - 1\Big)^2 \ge c_2(\alpha).$$

In LEEs dissertation there are also results on q–additive functions, on the Turán–Kubilius inequality for these, and a result on the limit–distribution of such functions.

4.3. Gelfand's Theory and Almost–Even Functions with Prescribed Values

4.3.1. *Interpolation Problem, Gelfand's Theory.* The spaces $\mathcal{B}^u$ and $\mathcal{D}^u$ are small. Nevertheless, the next result, which is due to J.–CHR. SCHLAGE–PUCHTA, J. SPILKER, & W. SCHWARZ (see [207], extending

[186]), shows, that there are "many" functions in $\mathcal{B}^u$. The interpolation problem is, to give conditions such that for given integers $0 < a_1 < a_2 < \ldots$ and given bounded complex numbers $b_1, b_2, \ldots$, there exists a function $f \in \mathcal{B}^u$ satisfying $f(a_n) = b_n$, for all $n \in \mathbb{N}$.

For the sake of completeness we state elementary facts from GELFANDs Theory (see [204], p. 268ff). For a commutative Banach–algebra $\mathcal{X}$ (with unit element e and norm $\|\cdot\|$) denote by

$$\Delta_{\mathcal{X}} = \{h : \mathcal{X} \to \mathbb{C},\ h \text{ is an algebra–homomorphism }\}$$

the set of algebra–homomorphisms on $\mathcal{X}$. Any $h \in \Delta_{\mathcal{X}}$ is continuous, and any maximal ideal in $\Delta_{\mathcal{X}}$ is the kernel of some $h \in \Delta_{\mathcal{X}}$. The Gelfand–transform $\hat{x}$ of $x \in \mathcal{X}$ is

$$\hat{x} : \Delta_{\mathcal{X}} \to \mathbb{C},\ \hat{x}(h) \stackrel{def}{=} h(x),$$

and so $\hat{}$ is a map $\hat{} : \mathcal{X} \to \hat{\mathcal{X}} = \{\hat{x} : \Delta_{\mathcal{X}} \to \mathbb{C},\ x \in \mathcal{X}\}$. Under the weakest topology, which makes every $\hat{h}$ continuous, $\Delta_{\mathcal{X}}$ becomes a compact topological Hausdorff space. If $\mathcal{X}$ is a semi–simple[17] B^*–algebra,[18] then the Gelfand–transform $\hat{}$ is an isometric isomorphism of $\mathcal{X}$ onto $\mathcal{C}(\Delta_{\mathcal{X}})$, the algebra of complex–valued continuous functions on $\Delta_{\mathcal{X}}$ with the sup–norm.

4.3.2. *The Maximal Ideal Space of $\mathcal{B}^u$.* All the homomorphisms h from the "maximal ideal space" $\Delta_{\mathcal{B}}$ of $\mathcal{B}^u$ are given as follows ([127], [186]): For any vector $\mathcal{K} = \left(e_p\right)_{p\in\mathbb{P}}$, where e_p is an integer from $[0,\infty[$ or equal to ∞, and any function $f \in \mathcal{B}^u$, define a "function value"

$$f(\mathcal{K}) = \lim_{r\to\infty} f\left(\prod_{p\le r} p^{\min\{r,e_p\}}\right).$$

For $f \in \mathcal{B}^u$, this limit does exist.[19] Define

$$h_{\mathcal{K}} : \Delta_{\mathcal{B}} \to \mathbb{C} \text{ by } h_{\mathcal{K}}(f) = f(\mathcal{K}).$$

[17] The radical of $\mathcal{X}$ (the intersection of all maximal ideals) equals (0).

[18] there is an involution $^* : \mathcal{X} \to \mathcal{X}$ satisfying $\|x \cdot x^*\| = \|x\|^2$.

[19] If $\mathcal{K}$ has only finitely many entries $e_p \neq 0$, and if none of these is equal to ∞, then $f(\mathcal{K}) = f\left(\prod_p p^{e_p}\right)$.

Then the maximal ideal space $\mathcal{B}^u$ of $\mathcal{B}$ is[20] the set of all $h_{\mathcal{K}}$, where $\mathcal{K} = \left(e_p\right)_{p\in\mathbb{P}}$. If $n = \prod_p p^{\nu_p(n)}$ is an integer, then the evaluation–homomorphismus $h_n : f \mapsto f(n)$ equals $h_{\mathcal{K}_n}$, where $\mathcal{K}_n = \{\nu_p(n),\ p \in \mathbb{P}\}$. A subbasis of the topology on $\Delta_{\mathcal{B}}$ is given by the vectors

$(*, \dots, *, e_p, *, *, \dots)$, where e_p is fixed and finite, or $e_p \geq$ some constant, and $*$ are arbitrary integers from $[0, \infty]$.

The solution of the Interpolation Problem is given by the

Theorem. *Let a strictly increasing sequence $\{a_n\}_{n\in\mathbb{N}}$ of positive integers and a bounded sequence $\{b_n\}_{n\in\mathbb{N}}$ of complex numbers be given with the following property:*

> *If $\{n_k\}_{k\in\mathbb{N}}$ is any strictly increasing sequence of positive integers such that for any $r \in \mathbb{N}$ the sequence $\{\gcd(a_{n_k}, r\,!)\}_{k\in\mathbb{N}}$ is eventually constant, then $\lim\limits_{k\to\infty} b_{n_k}$ exists, and, in the case that, with some integer m [not depending on r], $\lim\limits_{k\to\infty} \gcd(a_{n_k}, r\,!) = \gcd(a_m, r\,!)$ for every r, its value is b_m.*

Then there is a function $f \in \mathcal{B}^u$ with values $f(a_n) = b_n$ for all $n \in \mathbb{N}$.

4.3.3. *Sketch of the Proof.* Define $\mathcal{E} \subset \Delta_{\mathcal{B}}$ as the [discrete] set of evaluation homomorphisms $\mathcal{E} = \{h_{a_n},\ n = 1, 2, \dots\}$; denote its set of accumulation points by $\mathcal{H}$. The union $K = \mathcal{E} \cup \mathcal{H} \ \subset \Delta_{\mathcal{B}}$ is closed, therefore compact. Define $F : K \to \mathbb{C}$, for points $h_{a_n} \in \mathcal{E}$ by $F\left(h_{a_n}\right) = b_n$, and for points $\eta = h_{\mathcal{K}} \in \mathcal{H}$ as follows: choose a sequence $\left\{h_{a_{n_k}}\right\}_k$ converging to η, and define $F\left(h_{\mathcal{K}}\right) = \lim_{k\to\infty} b_{n_k}$. This limit exists, F is well–defined and continuous on K. Therefore, by the TIETZE extension theorem there is a continuous function $F^* : \Delta_{\mathcal{B}} \to \mathbb{C}$, extending F. By GELFANDs theory, F^* is the image of some function $f \in \mathcal{B}^u$, $F^* = \hat{f}$, and due to

$$f\left(a_n\right) = h_{a_n}(f) = \hat{f}\left(h_{a_n}\right) = F^*\left(h_{a_n}\right) = F\left(h_{a_n}\right) = b_n$$

the function f solves the interpolation problem $f(a_n) = b_n$.

A similar result (with a similar proof) is true for the space $\mathcal{D}^u$.

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[20] $\Delta_{\mathcal{B}}$ may be described as the topological product $\prod\limits_p \{1, p^1, p^2, \dots, p^\infty\}$, where $\{1, p^1, p^2, \dots, p^\infty\}$ is the one–point–compactification of the discrete space $\{1, p^1, p^2, \dots\}$.

§5. Kubilius Model

For the application of methods of probability theory, for example the Berry–Esseen Theorem,

Let $X_1, \dots, X_n$ be independent random variables (with distribution functions F_ν) with mean zero, variance D_ν and third moment

$$L_\nu = \int_{-\infty}^{\infty} z^3 dF_\nu(z),\ \nu = 1, 2, \dots, n;$$

then, uniformly for real z,

$$P\left(\frac{1}{\sigma}\sum_{nu=1}^{n} X_\nu \le z\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{1}{2}w^2} dw + \mathcal{O}\left(\frac{1}{\sigma^3}\sum_{\nu=1}^{n} L_\nu\right),$$

with an absolute $\mathcal{O}$–constant, and where $\sigma^2 = D_1 + \cdots + D_n$,

to strongly additive functions f, $f(n) = \sum_{p|n} f(p)$, one might try to use random variables X_p on some suitable space $(\Omega, \mathcal{A}, P)$, where

$$\begin{aligned} X_p &= f(p) \text{ with probability } \frac{1}{p}, \\ X_p &= 0 \text{ with probability } 1 - \frac{1}{p}. \end{aligned}$$

Unfortunately, this approach does not work, because the "events" one naturally would like to choose for dealing with additive functions, the zero–residue–classes [the set $E(p^k)$ of integers $n \le x$ divisible by a prime power p^k], are only "nearly" independent. KUBILIUS (see [132]) constructed finite probabilistic models to mimic the behaviour of truncated additive functions by appropriately defined independent random variables. A possible construction is described in ELLIOTTs monograph [37], Chapt. 3.

Assume that $2 \le r \le x$. Define, for any prime p dividing $\prod_{p \le r} p$, the residue class

$$E(p) = \{n \le x, n \equiv 0 \bmod p\},\ \text{and } \overline{E}(p) = S \setminus E(p),$$

where $S = \{n \in \mathbb{N},\ n \le x\}$, and, for $k \,\Big|\, \prod_{p \le r} p$, write

$$E_k = \bigcap_{p|k} E(p) \qquad \bigcap_{p \,\left|\, \left((\prod_{p \le r} p)/k\right)\right.} \overline{E}(p).$$

Let $\mathcal{F}$ be the least σ–algebra containing all the $E(p)$, and define a [finitely additive] measure ν on $\mathcal{F}$:

$$\nu(A) = \sum_{1}^{m} \frac{1}{[x]} \cdot \#\left(E_{k_j}\right), \text{ for } A = \bigcup_{1}^{m} E_{k_j},$$

to obtain a finite probability space $(S, \mathcal{F}, \nu)$.

Now some ideas from number theory are involved. SELBERGs sieve–method gives

$$\#\left(E_k\right) = \left(1 + \mathcal{O}\left(L\right)\right) \cdot \frac{x}{k} \cdot \prod_{p \,\left|\, \left(\left(\prod_{p \le r} p\right) / k\right)\right.} \left(1 - \frac{1}{p}\right),$$

as long as $k \le x^{\frac{1}{2}}$, where

$$L = \exp\left(-\frac{1}{8} \frac{\log x}{\log r} \log\left(\frac{\log x}{\log r}\right)\right) + x^{-\frac{1}{15}}.$$

Define a second measure μ on $\mathcal{F}$ by

$$\mu(E_k) = \frac{1}{k} \prod_{p \,\left|\, \left(\left(\prod_{p \le r} p\right) / k\right)\right.} \left(1 - \frac{1}{p}\right).$$

Then μ and ν are "close",

$$\nu E_k = \left(1 + \mathcal{O}\left(L\right)\right) \mu E_k, \qquad \nu A = \mu A + \mathcal{O}\left(L\right), \text{ uniformly in } \mathcal{F}.$$

For the "truncated" additive function $g(n) = \sum_{p|n,\ p \le r} f(p)$ we obtain

$$\frac{1}{x} \#\{n \le x;\ g(n) \le u\} = P\left(\sum_{p \le r} X_p \le u\right) + \mathcal{O}\left(L\right).$$

To deduce a result for the original function f, it is necessary to give a good estimate for the frequencies

$$\frac{1}{x}\ \#\{n \le x;\ |f(n) - A(x) - (f_r(n) - A(r))| > \varepsilon\, B(x)\},$$

which is done by the TURÁN–KUBILIUS inequality and the fact that $f \in H$ (the class H was defined via formula (3.5)).

§6. Integration

This section deals very sketchily with the problem of "integrating" arithmetical functions[21] and to use these theories in order to obtain results on arithmetical functions.

Since "naturally" defined subsets of $\mathbb{N}$ (like arithmetic progressions) do not form a σ–algebra (in the sense of measure theory), one has to proceed in another way; the general idea for these investigations is to associate to an arithmetical function f some other function f^* defined on some suitably chosen compact topological space (or semi–group).

The first effective theory of integration for arithmetical functions is due to E. V. NOVOSELOV 1962–1964, see [191]. A good description of this method can be found in MAUCLAIREs paper [180]. The techniques of E. V. NOVOSELOV are strong enough to give a proof of DELANGE's result (see §4.1).

6.1. J. Knopfmacher, W. Schwarz, J. Spilker

A rather simple theory of integration for arithmetical functions was developed in papers of SCHWARZ & SPILKER, in 1971 and 1976 ([211], [212], [216]). Unfortunately this theory is definitely weaker than NOVOSELOVs theory. Define countable sets $\{1, p, p^2, \dots\}$ with discrete topology, and form the ALEXANDROFF–one–point–compactification $\mathbb{N}_p$ by adding one point p^∞. Define a measure μ_p, $\mu_p(p^k) = p^{-k} \cdot \left(1 - \frac{1}{p}\right)$, $\mu_p(p^\infty) = 0$, on $\mathbb{N}_p$. Then the product measure $\mu = \prod_p \mu_p$ on the compact space $\mathbb{N}^* = \prod_p \mathbb{N}_p$ is the same as the measure coming from the mean–value–functional $f \mapsto M(f)$ [for $f \in \mathcal{B}^u$] via the F. RIESZ representation theorem (see, e.g. [204]), and

$$\mathcal{B}^u \simeq \mathcal{C}(\mathbb{N}^*),$$

the algebra of continuous functions on $\mathbb{N}^*$. Thus, mean–values may be represented as integrals,

$$M(f) = \int_{\mathbb{N}^*} f d\mu.$$

In 1976, J. KNOPFMACHER ([122]) showed, that the quotient space $\mathcal{B}^q$/nullspace is $\simeq L^q(\mathbb{N}^*, \mu)$. And, he showed that the whole theory can be extended to arithmetical semigroups.

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[21]There is an interesting survey paper of J.-L. MAUCLAIRE, *Integration and Number Theory* [180], concerning the subject of the first two subsections.

A simplification of the approach described above is sketched in MAUCLAIREs paper [180].

6.2. J.–L. Mauclaire

It is difficult to sketch the contents of J.-L. MAUCLAIRE's highly interesting *"Intégration et Théorie des Nombres"* in short. MAUCLAIRE uses the BOHR-compactification $\hat{\bar{\mathbb{Z}}}$ of the character group $\hat{\mathbb{Z}}$ of the additive group $\mathbb{Z}$ of integers, and so good knowledge from analysis is necessary to read this book. In this monograph, the DABOUSSI–ELLIOTT theorem is proved (Chapt. III), and the ERDŐS–WINTNER theorem, too.

In MAUCLAIREs survey paper [180] the main ideas of his approach are well readably described. See also [183].

6.3. K.–H. Indlekofer's Integration Theory

INDLEKOFERs theory of integration of arithmetical functions is given in [105] and [106]. We follow this presentation.

Let $\mathcal{A}$ be an algebra[22] of subsets of $\mathbb{N}$ with a finitely additive set function $\delta : \mathcal{A} \to [0, \infty[$ defined for all $A \in \mathcal{A}$.[23]

For example, one can use

$$\delta(\mathcal{A}) = \lim_{n\to\infty} \sum_{k=1}^{\infty} \gamma_{nk} 1_A(k),$$

where $\Gamma = (\gamma_{nk})_{n,k}$ is a TOEPLITZ matrix:

$$\text{(i)} \quad \sup_n \sum_{k=1}^{\infty} \gamma_{nk} < \infty,$$

$$\text{(ii)} \quad \gamma_{nk} \to 0, \text{ if } n \to \infty, \ k \text{ fixed},$$

[22]$\mathbb{N} \in \mathcal{A}$, $A \cup B$ and $B \setminus A$ are in $\mathcal{A}$, if $A, B \in \mathcal{A}$.

[23]A big advantage of INDLEKOFERs approach is that [deep] results obtained by other methods can be built into the construction.

$$\text{(iii)} \qquad \sum_{k=1}^{\infty} \gamma_{nk} \to 1, \text{ as } n \to \infty.$$ [24]

Then, for simple functions

$$s \in \mathcal{E}(\mathcal{A}) = \left\{ s;\ s = \sum_{1}^{m} \alpha_j \, 1_{A_j},\ \alpha_j \in \mathbb{C},\ A_j \in \mathcal{A} \right\},$$

the definition

$$\int_{\beta\mathbb{N}} \overline{s}\, d\overline{\delta} = \lim_{n\to\infty} \sum_{1}^{\infty} \gamma_{nk} s(k)$$

leads to the Lebesgue space

$$L^1(\beta\mathbb{N}, \sigma(\overline{\mathcal{A}}), \overline{\delta}) = \{\overline{f} : \beta\mathbb{N} \to \mathbb{C}, \|\overline{f}\| < \infty\},$$

with the [semi]–norm

$$\|\overline{f}\| = \int_{\beta\mathbb{N}} |\overline{f}|\, d\overline{\delta}.$$

The photo has been removed due to copyright issues.

Figure 5. *L. Murata, K.-H. Indlekofer*

There is a norm–preserving vector space isomorphism

$$\mathcal{L}^{*1}(\mathcal{A}) \text{ (mod null–functions) } \to L^1(\overline{\delta}) \text{ (mod null–functions)},$$

where

$$\mathcal{L}^{*1} = \|.\|_1\text{–closure of } \mathcal{E}(A),$$

and, as in (4.9), $\|f\|_q = \left(\limsup_{x\to\infty} \frac{1}{x} \sum_{n\le x} |f(n)|^q\right)^{\frac{1}{q}}$.

[24]Examples of TOEPLITZ–matrices are provided by $\gamma_{nk} = \frac{1}{n}$, if $k \le n$, otherwise $\gamma_{nk} = 0$ — this leads to asymptotic density, or by $\gamma_{nk} = \frac{1}{k} \cdot \frac{1}{\log n}$, if $k \le n$, otherwise $\gamma_{nk} = 0$ — this leads to logarithmic density.

The photos have been removed due to copyright issues.

Figure 6. *H. Fürstenberg; E. Manstavičius, A. Laurinčikas*

Examples. Starting with the algebra $\mathcal{A}_2$ generated by the zero residue classes $\{n \in \mathbb{N}, n \equiv 0 \bmod q\}$, for $q \in \mathbb{N}$, with asymptotic density, one obtains the theory of KNOPFMACHER, SCHWARZ & SPILKER.

Starting with the algebra $\mathcal{A}_1$ generated by all residue classes $\{n \in \mathbb{N},\ n \equiv a \bmod q\}$, for $a, q \in \mathbb{N}$, with asymptotic density, one obtains the integration theory of E. V. NOVOSELOV.

Using a deep ergodic result of FÜRSTENBERG[25] on the shift operator $S(n) = n+1$ (and with asymptotic density δ on $\mathbb{N}$): *If* $\overline{\delta}(\overline{B}) > 0$, *then for any* $k > 1$ *there exists an integer* $n \neq 0$ *so that*

$$\overline{\delta}\left(\overline{B} \cap \overline{S}^{n}\overline{B} \cap \cdots \cap \overline{S}^{(k-1)n}\overline{B}\right) > 0,$$

then, using the algebra $\mathcal{A}$ generated by the translations $\{S^n B,\ n = 0, 1, 2, \dots\}$, INDLEKOFERs theory gives:

If $B \subset \mathbb{N}$ *has asymptotic density* $\delta(B) > 0$, *then* B *contains arbitrarily long arithmetic progressions* (VAN DER WAERDEN, K. F. ROTH [203], SZEMERÉDI [227]).

§7. Functional Limit Theorems, Universality

7.1. Functional Limit Theorems

There is a far–reaching generalization of the ideas leading to the Erdős–Kac and Erdős–Wintner theorem. Important contributions to this topic are due to E. MANSTAVIČIUS. We refer to the survey paper

[25] [63]; the result is closely connected with SZEMERÉDIs famous result on arithmetical progressions. For a well readable presentation see [64].

[167].[26] This paper starts *from the invariance principle established by P. Erdős and M. Kac in the fourties and from more general functional limit theorems for partial sum processes for independent random variables.* Furthermore, *the development of a parallel theory dealing with those dependent random variables which appear in probabilistic number theory* is described.[27] This survey paper, dedicated to the memory of PAUL ERDŐS (with an extensive bibliography of 89 items) deals with

— partial sum processes for independent random variables,

— additive functions and functionals on them,

— additive functions and Brownian motion (see also [136]),

— models of other processes with independent increments,

— additive functions on sparse sequences,

— multiplicative functions,

— divisors and stochastic processes.

As one example we give one [technical] result due to MANSTAVIČIUS.

Let $h : \mathbb{N} \to \mathbb{R}$ be additive, $\beta(n) \to \infty$, and let X be a stable process with an explicitly given characteristic function (containing the parameters a_1, a_2, α). In order that $G_n \Longrightarrow X$ it is necessary and sufficient that for any $u > 0$

$$\sum_{\substack{p\le n\\ h(p)<-u\beta(n)}} \frac{1}{p} \to a_1 \cdot u^{-\alpha}, \qquad \sum_{\substack{p\le n\\ h(p)>u\beta(n)}} \frac{1}{p} \to a_2 \cdot u^{-\alpha},$$

and that

$$\lim_{\varepsilon\to 0} \limsup_{n\to\infty} \sum_{\substack{p\le n\\ |h(p)|<\varepsilon\beta(n)}} \frac{1}{p}\, h(p) = 0.$$

Here

$$G_n = \frac{1}{\beta(n)} \sum_{p|m,\ p\le z(t)} h(p) - \alpha(n, z(t)),$$

and (in $t \in [0,1]$)

$$z(t) = \max\{u;\ B^2(n,u) \le tB^2(n,n)\},$$

[26]From the review by FILIP SAIDAK in Math. Reviews we quote: *"This excellent, long overdue survey paper, concerning the theory of general functional limit theorems for partial sum processes, fills the gap left by all the existing textbooks and expository papers on the subject".*

[27]From the abstract of [167].

$$B^2(n,u)=\sum_{p\le u}\left(\frac{h(p)}{\beta(n)}\right)^{*2}\frac{1}{p},\quad \alpha(n,u)=\sum_{p\le u}\left(\frac{h(p)}{\beta(n)}\right)^{*}\frac{1}{p}.$$

The star * ist defined by $u^*=u$, if $|u|<1$, and $u^*=sgn(u)[=\pm 1]$ otherwise.

7.2. Number Theory in the Symmetric Group

Starting point of this topic is a paper of E. LANDAU ([143]), reproduced in the „Handbuch von der Lehre der Verteilung der Primzahlen" (1909) on the maximal order $f(n)$ of elements of the symmetric group $\mathcal{S}_n$ with $n!$ elements, so

$$f(n)=\max_{\sigma\in\mathcal{S}_n}\operatorname{ord}(\sigma)=\max_{\substack{r,a_1,\dots,a_r\in\mathbb{N}\\ a_1+\dots+a_r=n}}\operatorname{lcm}[a_1,\dots,a_r]=\max_{\sum p^\beta\le n}\left(\prod_p p^\beta\right).$$

E. LANDAU showed

$$\log f(n)\sim\sqrt{n\cdot\log n}\,,\ \text{ as } n\to\infty.$$

The function $f(n)$ was carefully studied in papers by J. L. NICOLAS (Bull. Soc. Math. France **97** (1969), 129–191; Acta Arithm. **14** (1968) 315–332); see also [173] and [174]); for example,

$$\log f(n)=\sqrt{\operatorname{li}^{-1}(n)}+\mathcal{O}\left(ne^{-\gamma\sqrt{\log n}}\right).$$

The first limit theorem seems to be due to V. L. GONČAROV [71]. *Denote by $g(\sigma)$ the number of cycle–lengths in the canonical decomposition of $\sigma\in\mathcal{S}_n$, then*

$$\lim_{n\to\infty}\frac{1}{n!}\cdot\#\left\{\sigma\in\mathcal{S}_n;\ g(\sigma)\le\log n+t\sqrt{\log n}\right\}$$
$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t}\exp\left(-\frac{1}{2}u^2\right)\,du.$$

The subject was studied by ERDŐS & TURÁN (see the series of papers [59]); for example,

$$\lim_{n\to\infty}\frac{1}{n!}\cdot\#\left\{\sigma\in\mathcal{S}_n\ \log\operatorname{ord}(\sigma)\le\frac{1}{2}\log^2 n+t\log^{\frac{3}{2}}n\right\}$$

$$=\sqrt{\frac{3}{2\pi}}\int_{-\infty}^{t}\exp\left(-\frac{3}{2}u^2\right)\,du.$$

In more recent time, E. Manstavičius started the study of properties of the symmetric group again.

We cannot explain Manstavičius' papers here in short, we refer to [169], [170], and in particular to his paper [171] in these proceedings.

7.3. Universality

This section deals with the value–distribution of zeta–functions, and ideas from measure theory and probability theory are important for investigations on "universality". The first results are due to H. Bohr (see, for example, [8]). Prototype of the results aimed for is S. M. Voronins result [241] (see also [117]):

Let $0 < r < \frac{1}{2}$, and let $s \mapsto g(s)$ be a non–vanishing, in $|z| \le r$ continuous, in $|z| < r$ holomorphic function. For any $\varepsilon > 0$ there are real values τ such that $\sup_{|s|\le r} \left| \zeta\left(s + \frac{3}{4} + i\tau\right) - g(s) \right| < \varepsilon.$

After Voronins paper there were several authors dealing with "universality", for example B. Bagchi, R. Garunkštis, A. Good, R. Kačinskaite, A. Laurinčikas, K. Matsumoto, A. Reich, R. Šleževičienė, J. Steuding.

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Figure 7. *R. Garunkštis, J. Steuding, R. Slečeviciene; K. Matsumoto*

Probability comes into the topic through a method of Bagchi, considerably extended by Laurinčikas [152], see also [148].[28] From Steudings habilitation thesis we give an example of a limit theorem for a

[28]Certainly, the revived interest in "universality" owes much to A. Laurinčikas, who inspired several young mathematicians to work on this subject.

subclass $\tilde{\mathcal{S}}$ of the SELBERG class $\mathcal{S}$; this class $\mathcal{S}$ consists of Dirichlet–series $\sum_1^\infty a_n \cdot n^{-s}$, having an Eulerproduct $\prod_p(\dots)$ and a functional equation of the kind of the functional equation of $\zeta(s)$ (with Gamma–factors), and satisfying $a_n \ll n^\varepsilon$. The subclass $\tilde{\mathcal{S}} \subset \mathcal{S}$ is restricted by the demand for the existence of $\lim_{x\to\infty} \frac{1}{\pi(x)} \sum_{p\le x} |a(p)|^2$ and by some restriction on the shape of the factors $(\dots)$ in the Eulerproduct.

To any Dirichlet series $\mathcal{L} \in \tilde{\mathcal{S}}$ attach a probability measure P_T by

$$P_T(A) = \frac{1}{T} \cdot \textit{Lebesgue–measure of } \{\tau \in [0,T],\ \mathcal{L}(\sigma + i\tau) \in A\},$$

for Borel–sets A in the space $\mathcal{H}(\mathcal{D})$ of functions holomorphic in the strip

$$\mathcal{D} = \left\{ s \in \mathbb{C};\ \max\left(\frac{1}{2},\ 1 - \frac{1}{d_{\mathcal{L}}}\right) < \sigma < 1 \right\}. \tag{7.1}$$

[The "degree" $d_{\mathcal{L}}$ of $\mathcal{L}$ is defined by data from the functional equation of $\mathcal{L}$.] Then ([225], Chapt. 6) *the probability measure P_T converges weakly to some probability measure P, as $T \to \infty$* , and the measure P is explicitly given.

This limit theorem permits the proof of a universality result for Dirichlet–series in the restricted Selberg class $\tilde{\mathcal{S}}$.[29]

Let $\mathcal{K}$ be a compact subset of the strip $\mathcal{D}$ with connected complement, and let $g(s)$ be a non–vanishing function continuous on $\mathcal{D}$, and holomorphic in the interior of $\mathcal{K}$. If $\mathcal{L} \in \tilde{\mathcal{S}}$, then, for any $\varepsilon > 0$

$$\liminf_{T\to\infty} \frac{1}{T} \cdot \textit{L–measure of } \left\{ \tau \in [0,T];\ \max_{s\in\mathcal{K}} |\mathcal{L}(s+i\tau) - g(s)| < \varepsilon \right\} > 0.$$

[29]In the literature there are many universality results, for example for L–functions, for the Lerch zeta–function, the Matsumoto zeta–function, for zeta–functions attached to cusp forms, for Hecke L–functions, ... See, for example, [153], [152], [175], [176], [194], [222], [221], [223], [225], and many others. V. GARBALIAUSKIENĖ, in her Vilnius dissertation [66] gives universality results for L–functions attached to elliptic curves. In [69] there are such results for the Estermann zeta–function.

There are also "joint universality results" — that means that tuples of certain zeta–function can simultaneously approximate given holomorphic function (of course, under suitable assumptions). A first prototype of this phenomenon for Dirichlet L–functions with non–equivalent charactes is also due to VORONIN [242]. See also [117], more recently [68] or [67].

So, VORONINs result on the zeta–function is extended to a much larger class of zeta–functions, and the assertion *"There is some τ"* is made quantitive — these τ's with the universality property do have a positive lower density.

There are survey papers on universality, for example [72], [149], [177], and [148].

§8. Conclusion

In this survey article only some parts of Probabilistic Number Theory could be sketched. The author hopes, that it became clear that Probabilistic Number Theory is an active field of mathematical research, where methods from number theory, analysis and probability theory work together in order to obtain interesting arithmetical results.

The author enjoyed the conference in Kanazawa very much, it was — thanks to the organizers Profs. SUGITA, MATSUMOTO, and MURATA — a pleasant stay. He gratefully acknowledges financial support from the organization committee.

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