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On the speed of convergence to limit distributions for Dedekind zeta-functions of non-Galois number fields

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Abstract.

We evaluate the speed of convergence in the Bohr-Jessen type of limit theorem on the value-distribution of Dedekind zeta-functions of number fields. When K is a Galois number field, the Euler product of the corresponding Dedekind zeta-function $\zeta_K(s)$ is convex, hence the evaluation can be done similarly to the case of the Riemann zeta-function. However, when K is non-Galois, some new ideas (based on the Artin-Chebotarev density theorem etc) are necessary, because the corresponding $\zeta_K(s)$ is not always convex.

§1. Introduction and statement of the result

We begin with recalling the classical result of Bohr and Jessen [1] [2] on the value-distribution of the Riemann zeta-function. Let $s = \sigma + it$ be a complex variable, $\zeta(s)$ the Riemann zeta-function. In the half-plane $\sigma > 1$, there is no difficulty in defining $\log \zeta(s)$. But in the strip $1/2 < \sigma \leq 1$, there is the possibility of the existence of zeros of $\zeta(s)$, because we do not assume the Riemann hypothesis. Therefore, we let

$$\mathcal{G} = \{s = \sigma + it \mid \sigma > 1/2\} - \bigcup_{s_j = \sigma_j + it_j} \{s = \sigma + it_j \mid 1/2 < \sigma \leq \sigma_j\},$$

where the numbers s_j denote the zeros and the pole of $\zeta(s)$ in the region $\sigma > 1/2$. For $s \in \mathcal{G}$ we can define $\log \zeta(s)$ by analytic continuation along the horizontal line segment from $2 + it$.

Let R be any fixed closed rectangle on the complex plane $\mathbb{C}$ with the edges parallel to the axes. Throughout this paper we write $\mu_n(\cdot)$ for

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n-dimensional Lebesgue measure. For any fixed $\sigma > 1/2$, let

$$V(T; R) = \mu_1(\{t \in [1, T] \mid \sigma + it \in \mathcal{G}, \log \zeta(\sigma + it) \in R\}).$$

Bohr and Jessen [1] [2] proved the existence of the limit

$$W(R) = \lim_{T \to \infty} \frac{1}{T} V(T; R), \tag{1.1}$$

which may be regarded as the probability of how many values of $\log \zeta(s)$ on the line $\Re s = \sigma$ belong to the rectangle R.

The speed of convergence on the right-hand side of (1.1) was estimated by the author. In [7] [8] the author proved

$$\begin{aligned} &\frac{1}{T} V(T; R) - W(R) \\ &= O\left(\mu_2(R)(\log \log T)^{-A(\sigma)+\varepsilon} + (\log \log T)^{-B(\sigma)+\varepsilon}\right), \end{aligned} \tag{1.2}$$

where (and throughout this paper) ε denotes an arbitrarily small positive number, not necessarily the same at each occurrence,

$$A(\sigma) = \begin{cases} (\sigma - 1)/7 & (\sigma > 1), \\ (2\sigma - 1)/15 & (1 \geq \sigma > 1/2), \end{cases}$$

and

$$B(\sigma) = \begin{cases} (\sigma - 1)/2 & (\sigma > 1), \\ (2\sigma - 1)/5 & (1 \geq \sigma > 1/2). \end{cases} \tag{1.3}$$

The implied constant on the right-hand side of (1.2) depends only on σ and ε. In [11], the value of $A(\sigma)$ was improved to

$$A(\sigma) = 2\sigma - 1 \tag{1.4}$$

for any $\sigma > 1/2$. Finally, in a joint paper of Harman and the author [3], it has been shown that the $\log \log T$ factor in the error term can be replaced by the $\log T$ factor, that is

$$\frac{1}{T} V(T; R) - W(R) = O\left((\mu_2(R) + 1)(\log T)^{-C(\sigma)+\varepsilon}\right), \tag{1.5}$$

where

$$C(\sigma) = \begin{cases} (\sigma - 1)/(3 + 2\sigma) & (\sigma > 1), \\ 2(2\sigma - 1)/(21 + 8\sigma) & (1 \geq \sigma > 1/2). \end{cases} \tag{1.6}$$

So far this error estimate is the sharpest.

Let p_n be the nth prime number. Then, from the Euler product expression of $\zeta(s)$, we have

$$\log\zeta(\sigma+it) = -\sum_{n=1}^{\infty}\log\left(1-p_n^{-\sigma}e^{-it\log p_n}\right) \tag{1.7}$$

for $\sigma > 1$. In the proof of (1.1) by Bohr and Jessen, it is important to study the behaviour of each term

$$-\log\left(1-p_n^{-\sigma}e^{-it\log p_n}\right).$$

This can be written as $z_n(\{-(t/2\pi)\log p_n\})$, where

$$z_n(\theta_n) = -\log\left(1-p_n^{-\sigma}e^{2\pi i\theta_n}\right) \tag{1.8}$$

and $\{x\} = x - [x]$ is the fractional part of x. When θ_n moves from 0 to 1, $z_n(\theta_n)$ describes a closed convex curve on $\mathbb{C}$, and this fact has been essentially used in the proof of Bohr and Jessen.

However, for more general zeta-functions which have Euler products, the corresponding curve is not always convex. (If the curve is convex, we call the Euler product *convex*; see [10].) Therefore, if one wants to generalize the result of Bohr and Jessen to some wider class of zeta-functions, it is necessary to find a proof which is free from convexity.

The author discovered two such proofs. One of them, based on Prokhorov's theorem, was first published in [9], in which an analogue of (1.1) for certain automorphic L-functions has been proved. Then in [10], the same (actually simplified) method has been applied to a more general class of zeta-functions. Another proof was dicussed in [11] in the case of Dedekind zeta-functions of algebraic number fields, but this method can also be applied to a more general situation, as was pointed out in [12]. Limit theorems in a more probabilistic framework for general zeta-functions introduced in [10] have been studied by Laurinčikas and Kačinskaitė; see, for example, [5] [6]. Some history of this topic and related results are surveyed in [13].

Therefore, now, limit theorems of type (1.1) have been shown for a rather wide class of zeta-functions. Hence it is natural to ask how to generalize quantitative results such as (1.2), (1.5) to the case of such a wide class. However, for those quantitative results, no proof free from convexity has been discovered. Hence the only published result in this direction deals with Dedekind zeta-functions of Galois number fields, because in this case the corresponding curve is convex.

Let K be an algebraic number field, $\ell = [K : \mathbb{Q}]$, $L = \max\{\ell, 2\}$, and $\zeta_K(s)$ the Dedekind zeta-function of K. Define

$$V_K(T; R) = \mu_1(\{t \in [1, T] \mid \sigma + it \in \mathcal{G}_K, \log \zeta_K(\sigma + it) \in R\}),$$

where $\mathcal{G}_K$ is the set defined for $\zeta_K(s)$ analogously to $\mathcal{G}$. The results in [10] [11] imply that, for any number field K, the limit

$$W_K(R) = \lim_{T \to \infty} \frac{1}{T} V_K(T; R) \tag{1.9}$$

exists for $\sigma > 1 - L^{-1}$.

Denote by $N\mathfrak{p}$ the norm of an ideal $\mathfrak{p}$ of K. By $\mathfrak{p}_n^{(1)}, \ldots, \mathfrak{p}_n^{(g(n))}$ we mean the prime divisors of p_n, with norm $N\mathfrak{p}_n^{(j)} = p_n^{f(j,n)}$ $(1 \le j \le g(n))$. Then for $\sigma > 1$ we have

$$\zeta_K(s) = \prod_{\mathfrak{p}} \left(1 - (N\mathfrak{p})^{-s}\right)^{-1} = \prod_{n=1}^{\infty} \prod_{j=1}^{g(n)} \left(1 - p_n^{-f(j,n)s}\right)^{-1}, \tag{1.10}$$

hence

$$\log \zeta_K(\sigma + it) = \sum_{n=1}^{\infty} z_{n,K}\left(\left\{-\frac{t}{2\pi} \log p_n\right\}\right), \tag{1.11}$$

where

$$z_{n,K}(\theta_n) = -\sum_{j=1}^{g(n)} \log\left(1 - p_n^{-f(j,n)\sigma} e^{2\pi i f(j,n)\theta_n}\right). \tag{1.12}$$

If K is a Galois extension of $\mathbb{Q}$, then $f(1, n) = \cdots = f(g(n), n)$ $(= f(n)$, say), hence

$$z_{n,K}(\theta_n) = -g(n) \log\left(1 - p_n^{-f(n)\sigma} e^{2\pi i f(n)\theta_n}\right) \tag{1.13}$$

which is clearly convex as in the case of $z_n(\theta_n)$. In [11], we have used this convexity to obtain

$$\begin{aligned} &\frac{1}{T} V_K(T; R) - W_K(R) \\ &= O\left(\mu_2(R)(\log\log T)^{-A(\sigma)+\varepsilon} + (\log\log T)^{-B(\sigma)+\varepsilon}\right) \end{aligned} \tag{1.14}$$

with the values (1.4) for $A(\sigma)$ and (1.3) for $B(\sigma)$. In [3] it is noted that for $\zeta_K(s)$ of a Galois extension K, an improvement similar to (1.5) (with (1.6)) is possible.

It is the purpose of the present paper to consider the speed of convergence of (1.9) for non-Galois number fields. In this case, the corresponding curve

$$\Gamma_{n,K} = \{z_{n,K}(\theta_n) \mid 0 \leq \theta_n < 1\} \tag{1.15}$$

is not always convex. Nevertheless, we can prove the following result.

Theorem. *Let K be an arbitrary (Galois or non-Galois) algebraic number field. For any $\sigma > 1 - L^{-1}$,*

$$\frac{1}{T}V_K(T;R) - W_K(R) = O\left((\mu_2(R)+1)(\log T)^{-C(\sigma)+\varepsilon}\right) \tag{1.16}$$

with the value (1.6) *for* $C(\sigma)$.

As mentioned above, when K is Galois, this theorem can be shown by a direct generalization of the method in [3]. In the non-Galois case, however, some new ideas are necessary. A key fact for the proof is that, for any fixed K, there are only finitely many patterns of the decomposition of primes into prime ideals in K. This is the reason why we can apply Lévy's inversion formula successfully. Another important tool is the Artin-Chebotarev density theorem, by which we can reduce some part of the proof to the convex case.

§2. The structure of the proof

Let N be a positive integer. It is fundamental in our argument to approximate the Euler product expression (1.10) of $\zeta_K(s)$ by its finite truncation

$$\zeta_{N,K}(s) = \prod_{n=1}^{N}\prod_{j=1}^{g(n)}\left(1 - p_n^{-f(j,n)s}\right)^{-1}. \tag{2.1}$$

Then

$$\log \zeta_{N,K}(\sigma+it) = \sum_{n=1}^{N} z_{n,K}\left(\left\{-\frac{t}{2\pi}\log p_n\right\}\right) \tag{2.2}$$

and, analogously to $V_K(T;R)$, we define

$$V_{N,K}(T;R) = \mu_1\{t \in [1,T] \mid \log\zeta_{N,K}(\sigma+it) \in R\}. \tag{2.3}$$

Let

$$Q_N = \{\boldsymbol{\theta} = (\theta_1,\ldots,\theta_N) \mid 0 \leq \theta_n < 1\}$$

be the N-dimensional unit cube in $\mathbb{R}^N$, and define the mapping $S_{N,K}$ from Q_N to $\mathbb{C}$ by

$$S_{N,K}(\boldsymbol{\theta}) = \sum_{n=1}^{N} z_{n,K}(\theta_n). \tag{2.4}$$

For any subset $A \subset \mathbb{C}$, we put

$$\Omega_{N,K}(A) = \{\boldsymbol{\theta} \in Q_N \mid S_{N,K}(\boldsymbol{\theta}) \in A\},$$

$$\mathbf{x}(t) = \left(\left\{-\frac{t}{2\pi}\log p_1\right\}, \ldots, \left\{-\frac{t}{2\pi}\log p_N\right\}\right).$$

Then $\log \zeta_{N,K}(\sigma + it) = S_{N,K}(\mathbf{x}(t)) \in R$ if and only if $\mathbf{x}(t) \in \Omega_{N,K}(R)$. Noting this fact, and using the Kronecker-Weyl theorem, we can prove (see Section 2 of [11]) that the limit

$$W_{N,K}(R) = \lim_{T\to\infty} \frac{1}{T} V_{N,K}(T; R) \tag{2.5}$$

exists, and is equal to $\mu_N(\Omega_{N,K}(R))$. Hence $W_{N,K}$ is a probability measure on $\mathbb{C}$. Moreover we can show (Sections 3 and 4 of [11]) that the value $W_K(R)$ in (1.9) is given by

$$W_K(R) = \lim_{N\to\infty} W_{N,K}(R). \tag{2.6}$$

Therefore, to prove our theorem, it is necessary to evaluate the speed of convergence of both (2.5) and (2.6).

Concerning (2.5), let

$$E_{N,K}(T; R) = W_{N,K}(R) - \frac{1}{T} V_{N,K}(T; R).$$

Then our result is

Proposition 1. *Let N be sufficiently large, and let m and r be large positive integers with $2rN \le m$. Then we have*

$$E_{N,K}(T; R) \ll \frac{N^{1/2}}{r} + \frac{Nr}{m} + \frac{1}{T}(6r \log m)^N \exp(mN \log N). \tag{2.7}$$

This is a generalization of Proposition 2 in [3]. On the other hand, as for (2.6), we have

Proposition 2. *For any sufficiently large N, we have*

$$|W_K(R) - W_{N,K}(R)| \ll \mu_2(R) N^{1-2\sigma} (\log N)^{-2\sigma}. \tag{2.8}$$

When K is Galois, this is (6.4) in [11]. The basic structure of the proof of Proposition 2 is the same as in [11], but some additional difficulty arises because the pattern of the decomposition of primes is not so simple as in the non-Galois case.

In Section 3 we will prove Proposition 2. In the course of the proof we will state and use a lemma on the evaluation of certain integrals, which will be proved in Section 4. Section 5 will be devoted to the proof of Proposition 1, and finally in the last section we will combine these two propositions to complete the proof of the theorem.

At the end of this section we show a preparatory lemma, which will be used in Section 5. Let $\Theta_n = \partial S_{N,K}(\boldsymbol{\theta})/\partial \theta_n$. Then we have

Lemma 1. *There exists a positive constant $C = C(\sigma, \ell)$ for which the inequality*

$$\left(\sum_{n=1}^{N} (\Re \Theta_n)^2 \right)^{1/2} \leq C \tag{2.9}$$

holds for any N.

In fact, by straightforward calculations we obtain

$$\Re \Theta_n = -2\pi \sum_{j=1}^{g(n)} \frac{p_n^{-f(j,n)\sigma} f(j,n) \sin(2\pi f(j,n)\theta_n)}{1 - 2p_n^{-f(j,n)\sigma} \cos(2\pi f(j,n)\theta_n) + p_n^{-2f(j,n)\sigma}}. \tag{2.10}$$

Since

$$1 - 2p_n^{-f(j,n)\sigma} \cos(2\pi f(j,n)\theta_n) + p_n^{-2f(j,n)\sigma} \geq (1 - p_n^{-f(j,n)\sigma})^2,$$

we have

$$\begin{aligned} |\Re \Theta_n| &\leq 2\pi \sum_{j=1}^{g(n)} \frac{p_n^{-f(j,n)\sigma} f(j,n)}{(1 - p_n^{-f(j,n)\sigma})^2} \\ &\leq 2\pi \frac{p_n^{-\sigma}}{(1 - p_n^{-\sigma})^2} \sum_{j=1}^{g(n)} f(j,n) \leq 2\pi \ell \frac{p_n^{-\sigma}}{(1 - p_n^{-\sigma})^2}. \end{aligned}$$

Hence

$$\sum_{n=1}^{N}(\Re\Theta_n)^2 \leq (2\pi\ell)^2\sum_{n=1}^{N}\frac{p_n^{-2\sigma}}{(1-p_n^{-\sigma})^4} \leq (2\pi\ell)^2\sum_{n=1}^{\infty}\frac{p_n^{-2\sigma}}{(1-p_n^{-\sigma})^4},$$

from which the lemma immediately follows.

In the case of the Riemann zeta-function, (2.9) has been given in Section 4 of a joint paper of Miyazaki and the author [14]. The above proof is a direct generalization of the argument given there.

§3. Proof of Proposition 2

The fundamental tool for the proof of Proposition 2 is, similarly to the proof of (6.4) in [11], Lévy's inversion formula. Therefore it is necessary to consider the Fourier transform

$$\Lambda_{N,K}(w) = \int_{\mathbb{C}} e^{i<z,w>} dW_{N,K}(z), \tag{3.1}$$

where $< z, w >= \Re(z)\Re(w) + \Im(z)\Im(w)$. Then

$$\Lambda_{N,K}(w) = \int_{Q_N} \exp(i < S_{N,K}(\boldsymbol{\theta}), w >) d\mu_N(\boldsymbol{\theta}) = \prod_{n=1}^{N} K_{n,K}(w), \tag{3.2}$$

where

$$K_{n,K}(w) = \int_0^1 \exp(i < z_{n,K}(\theta_n), w >) d\theta_n. \tag{3.3}$$

Substituting definition (1.12) into the above, we have

$$K_{n,K}(w) = \int_0^1 \exp\left(i < F_n(p_n^{-\sigma}e^{2\pi i\theta_n}), w >\right) d\theta_n, \tag{3.4}$$

where

$$F_n(z) = -\sum_{j=1}^{g(n)} \log\left(1 - z^{f(j,n)}\right). \tag{3.5}$$

Since

$$\sum_{j=1}^{g(n)} e(j,n)f(j,n) = \ell, \tag{3.6}$$

where $e(j,n)$ is the ramification index of $\mathfrak{p}_n^{(j)}$ over p_n, we see that $1 \leq g(n) \leq \ell$. For any integer g satisfying $1 \leq g \leq \ell$, let $\mathcal{F}_g$ be the set of all integer vectors $\mathbf{f} = (f(1), \ldots, f(g))$ for which there exists an n such that $g = g(n)$ and $f(j) = f(j,n)$ $(1 \leq j \leq g)$. Then

$$\mathcal{F} = \bigcup_{1 \leq g \leq \ell} \mathcal{F}_g \tag{3.7}$$

is a finite set because of (3.6). For each $\mathbf{f} \in \mathcal{F}$, define

$$F_{\mathbf{f}}(z) = -\sum_{j=1}^{g} \log\left(1 - z^{f(j)}\right). \tag{3.8}$$

Let $\mathbb{N}$ be the set of positive integers. For any $n \in \mathbb{N}$, there exists a unique $\mathbf{f} \in \mathcal{F}$ for which $F_n = F_{\mathbf{f}}$ holds. Hence $\mathbb{N}$ can be decomposed into

$$\mathbb{N} = \bigcup_{\mathbf{f} \in \mathcal{F}} \mathcal{N}(\mathbf{f}), \tag{3.9}$$

where

$$\mathcal{N}(\mathbf{f}) = \{n \in \mathbb{N} \mid F_n = F_{\mathbf{f}}\}. \tag{3.10}$$

Let $\mathcal{F}_1$ be the set of all $\mathbf{f} \in \mathcal{F}$ for which $\mathcal{N}(\mathbf{f})$ has infinitely many elements, and $\mathcal{F}_2 = \mathcal{F} \setminus \mathcal{F}_1$. Then for any $\mathbf{f} \in \mathcal{F}_2$, there exists the largest positive integer belonging to $\mathcal{N}(\mathbf{f})$, which we denote by $n_2(\mathbf{f})$. In order to study $\mathcal{F}_1$, we use the following lemma.

Lemma 2. *Let $\rho > 0$, and assume that the series*

$$F(z) = \sum_{n=h}^{\infty} a_n z^n \qquad (a_n \in \mathbb{C},\ a_h \neq 0) \tag{3.11}$$

is convergent absolutely in $|z| < \rho$. Let Γ be the closed curve on the complex plane defined by

$$\Gamma = \Gamma(r) = \{F(re^{2\pi i\theta}) \mid 0 \leq \theta < 1\} \qquad (0 < r < \rho).$$

Then we have

(i) *There exists a $\rho_0 = \rho_0(h, \rho, F)$ with $0 < \rho_0 < \rho$ such that $\Gamma(r)$ is a closed convex curve for any r satisfying $0 < r \leq \rho_0$.*

(ii) *There exists a* $\rho_1 = \rho_1(h, \rho, F)$ *with* $0 < \rho_1 < \rho$ *for which*

$$\int_0^1 \exp\left(i < F(re^{2\pi i\theta}), w >\right) d\theta = O(r^{-h/2}|w|^{-1/2}) \tag{3.12}$$

holds for any $w \in \mathbb{C}$ *and for any* r *satisfying* $0 < r \leq \rho_1$, *where the implied constant depends on* h, ρ *and* F.

The proof of this lemma will be given in the next section. Here we apply the assertion (ii) of this lemma to $F = F_{\mathbf{f}}$. Then $\rho = 1$, and we obtain

$$\int_0^1 \exp\left(i < F_{\mathbf{f}}(re^{2\pi i\theta}), w >\right) d\theta = O(r^{-h(\mathbf{f})/2}|w|^{-1/2}) \tag{3.13}$$

for any $w \in \mathbb{C}$ and any positive $r \leq \rho_1 = \rho_1(\mathbf{f}) < 1$, where $h(\mathbf{f}) = \min\{f(1), \ldots, f(g)\}$.

When $\mathbf{f} \in \mathcal{F}_1$, there are infinitely many elements in $\mathcal{N}(\mathbf{f})$, hence we can find sufficiently large $n_1(\mathbf{f}) \in \mathcal{N}(\mathbf{f})$ such that $p_n^{-\sigma} \leq \rho_1(\mathbf{f})$ for any $n > n_1(\mathbf{f})$. For those n, (3.13) is valid with $r = p_n^{-\sigma}$. Hence, combining with (3.4), we find that

$$|K_{n,K}(w)| \leq \beta(\mathbf{f}) p_n^{\sigma h(\mathbf{f})/2}|w|^{-1/2} \qquad (n \in \mathcal{N}(\mathbf{f}),\ n > n_1(\mathbf{f})) \tag{3.14}$$

where $\beta(\mathbf{f})$ is a constant depending on $\mathbf{f}$.

Now define

$$n_0 = \max\left\{\max_{\mathbf{f}\in\mathcal{F}_1} n_1(\mathbf{f}),\ \max_{\mathbf{f}\in\mathcal{F}_2} n_2(\mathbf{f})\right\}. \tag{3.15}$$

Then any $n > n_0$ is an element of some $\mathcal{N}(\mathbf{f})$, $\mathbf{f} \in \mathcal{F}_1$, hence inequality (3.14) is valid for those n. Therefore

$$|K_{n,K}(w)| \leq \beta p_n^{\sigma\ell/2}|w|^{-1/2} \tag{3.16}$$

for any $n > n_0$, where $\beta = \max\{\beta(\mathbf{f}) \mid \mathbf{f} \in \mathcal{F}_1\}$. This bound (3.16) is a generalization of the inequality stated in line 4, p.206 of [11]. The argument how to deduce the assertion of Proposition 2 from (3.16) is the same as on p.204 and p.206 of [11], so we omit it.

Remark 1. It is to be noticed that n_0 can be determined because $\mathcal{F}$ is a finite set, that is, there are only finitely many patterns of the decomposition of primes into prime ideals for any fixed field K. This fact is essential in our proof.

Remark 2. The first assertion of Lemma 2 is actually not necessary for the purpose of the present paper. But it (with its proof) clarifies the geometric meaning of the lemma, especially it implies that the second assertion is essentially a property of convex curves.

§4. Proof of Lemma 2

The purpose of this section is to prove Lemma 2, stated in the preceding section. The case $h = 1$ of this lemma is due to Jessen and Wintner [4] (see Theorems 12 and 13 of their paper). The following proof is a (simplified) generalization of their argument.

Let $0 < |z| = r \leq \rho/2$. Then series (3.11) is convergent uniformly in z. Put $z(\theta) = F(re^{2\pi i\theta})$, $\xi(\theta) = \Re z(\theta)$, $\eta(\theta) = \Im z(\theta)$. Writing $a_n = |a_n|e^{2\pi i\omega_n}$ we have

$$\xi(\theta) = \sum_{n=h}^{\infty} |a_n| r^n \cos(2\pi(\omega_n + n\theta)), \tag{4.1}$$

$$\eta(\theta) = \sum_{n=h}^{\infty} |a_n| r^n \sin(2\pi(\omega_n + n\theta)). \tag{4.2}$$

The gradient of the tangential line for Γ at $z(\theta)$ is $\eta'(\theta)/\xi'(\theta)$ (where the prime denotes the differentiation with respect to θ). Using (4.1) and (4.2) we have

$$\left(\frac{\eta'(\theta)}{\xi'(\theta)}\right)' = \frac{|a_h|^2 r^{2h}(2\pi h)^3 + O(r^{2h+1})}{|a_h|^2 r^{2h}(2\pi h)^2 \sin^2(2\pi(\omega_h + h\theta)) + O(r^{2h+1})} \tag{4.3}$$

where the implied constants depend on h, ρ and F. Let

$$I = \{\theta \in [0,1) \mid |\sin(2\pi(\omega_h + h\theta))| \geq 1/\sqrt{2}\}.$$

Then, for any $\theta \in I$, we have

$$\left(\frac{\eta'(\theta)}{\xi'(\theta)}\right)' = \frac{2\pi h}{\sin^2(2\pi(\omega_h + h\theta))} + O(r), \tag{4.4}$$

which is positive when r is sufficiently small. Hence the gradient constantly increases when $\theta \in I$.

When $\theta \notin I$, we have $|\cos(2\pi(\omega_h + h\theta))| \geq 1/\sqrt{2}$. In this case we change the role of the real axis and the imaginary axis. Then the gradient we should consider is $-\xi'(\theta)/\eta'(\theta)$. we see that

$$(4.5) \qquad \left(-\frac{\xi'(\theta)}{\eta'(\theta)}\right)' = \frac{2\pi h}{\cos^2(2\pi(\omega_h + h\theta))} + O(r),$$

which is again positive for any sufficiently small r. Therefore the first assertion of Lemma 2 follows.

We proceed to the proof of the second assertion. Let $\tau = \arg w$, so $w = |w|e^{i\tau}$. Then

$$(4.6) \qquad \int_0^1 \exp\left(i < F(re^{2\pi i\theta}), w >\right) d\theta = \int_0^1 \exp\left(ig_\tau(\theta)|w|\right) d\theta,$$

where $g_\tau(\theta) = \xi(\theta)\cos\tau + \eta(\theta)\sin\tau$. Using (4.1) and (4.2) we have

$$(4.7) \qquad g_\tau(\theta) = \sum_{n=h}^{\infty} |a_n| r^n \cos(2\pi(\omega_h + h\theta) - \tau),$$

and

$$(4.8) \qquad g'_\tau(\theta) = -r^h \left\{|a_h| 2\pi h \sin(2\pi(\omega_h + h\theta) - \tau) + O(r)\right\},$$

$$(4.9) \qquad g''_\tau(\theta) = -r^h \left\{|a_h|(2\pi h)^2 \cos(2\pi(\omega_h + h\theta) - \tau) + O(r)\right\},$$

$$(4.10) \qquad g'''_\tau(\theta) = r^h \left\{|a_h|(2\pi h)^3 \sin(2\pi(\omega_h + h\theta) - \tau) + O(r)\right\},$$

with the implied constants depending on h, ρ and F. Let

$$I_\tau = \{\theta \in [0,1) \mid |\sin(2\pi(\omega_h + h\theta) - \tau)| \geq 1/\sqrt{2}\}.$$

The set I_τ consists of $2h$ disjoint intervals of length $1/4h$. (The interval including 0 and the interval including the neighbourhood of 1 are to be combined.) We denote each of those intervals by $I_\tau(k)$ $(1 \leq k \leq 2h)$. If r is sufficiently small, then from (4.8) we have

$$(4.11) \qquad |g'_\tau(\theta)| \gg r^h \qquad (\theta \in I_\tau(k),\ 1 \leq k \leq 2h)$$

with the implied constant depending on h, ρ, F. Moreover from (4.10) we see that, for each k, the sign of $g'''_\tau(\theta)$ does not change when θ moves in the interval $I_\tau(k)$. Hence $g''_\tau(\theta)$ is monotonic in $I_\tau(k)$, so there is at most one point $\theta = \theta_0(k) \in I_\tau(k)$ at which $g''_\tau(\theta) = 0$. This $\theta_0(k)$, if exists, divides $I_\tau(k)$ into two subintervals on which $g'_\tau(\theta)$ is monotonic. If $\theta_0(k)$ does not exist, $g'_\tau(\theta)$ is monotonic on $I_\tau(k)$.

On the other hand, the set $[0,1) \setminus I_\tau$ also consists of $2h$ disjoint intervals, which we denote by $J_\tau(k)$ $(1 \le k \le 2h)$. If $\theta \in J_\tau(k)$ then $|\cos(2\pi(\omega_h + h\theta) - \tau)| \ge 1/\sqrt{2}$. Hence, if r is sufficiently small, then from (4.9) we obtain

$$|g_\tau''(\theta)| \gg r^h \qquad (\theta \in J_\tau(k),\ 1 \le k \le 2h) \tag{4.12}$$

with the implied constant depending on h, ρ, F.

Now we divide the right-hand side of (4.6) as

$$\int_0^1 \exp\left(ig_\tau(\theta)|w|\right) d\theta = \sum_{k=1}^{2h} \int_{I_\tau(k)} + \sum_{k=1}^{2h} \int_{J_\tau(k)} . \tag{4.13}$$

Because of (4.11) and the monotonicity mentioned above, we can apply Lemma 4.2 of Titchmarsh [16] to the integrals on $I_\tau(k)$. The integrals on $J_\tau(k)$ are estimated by (4.12) and Lemma 4.4 of [16]. The result is that

$$\int_0^1 \exp\left(ig_\tau(\theta)|w|\right) d\theta \ll \frac{1}{r^h|w|} + \frac{1}{(r^h|w|)^{1/2}}. \tag{4.14}$$

When $r^h|w| \ge 1$, then the right-hand side of (4.14) is $\ll (r^h|w|)^{-1/2}$, which implies (3.12). When $r^h|w| < 1$, inequality (3.12) holds trivially because the left-hand side is ≤ 1. The proof of Lemma 2 is now complete.

§5. Proof of Proposition 1

In this section we describe how to prove Proposition 1. The basic structure of the proof is similar to that developed in [3], hence we omit the details except for some key points of the proof.

For a point $\mathbf{n} \in \mathbb{Z}^N$ we put

$$Q_N(\mathbf{n}) = \{\boldsymbol{\theta} \in Q_N \mid r\boldsymbol{\theta} \in \mathbf{n} + Q_N\},$$

where r is a large positive integer, and define

$$D_1 = \bigcup_{Q_N(\mathbf{n}) \cap \Omega_N(R) \ne \emptyset} Q_N(\mathbf{n}), \qquad D_2 = Q_N \setminus \bigcup_{Q_N(\mathbf{n}) \subset \Omega_N(R)} Q_N(\mathbf{n}).$$

First, similarly to the inequalities given in p.22 of [3], we can show

$$E_{N,K}(T;R) \le |\mu_N(D_1) - W_{N,K}(R)| + B_1 + O\left(\frac{Nr}{m}\right) \tag{5.1}$$

and

(5.2)

$$E_{N,K}(T;R) \geq -|\mu_N(D_2) - 1 + W_{N,K}(R)| - B_2 + O\left(\frac{Nr}{m}\right) + O\left(\frac{1}{T}\right),$$

where m is a large positive integer satisfying $2rN \leq m$, and B_j $(j = 1, 2)$ is the same as in [3] and satisfies the estimate

$$B_j \ll \frac{1}{T}(6r \log m)^N \exp(mN \log N). \tag{5.3}$$

The proof of these results is exactly the same as the argument in Section 2 of [3], which is based on the ideas in [7] and a lemma of Vinogradov.

Proposition 1 will clearly follow from (5.1), (5.2) and (5.3), if we can show the following lemma.

Lemma 3. *For any sufficiently large N, we have*

$$|\mu_N(D_1) - W_{N,K}(R)| \ll N^{1/2} r^{-1}, \tag{5.4}$$

$$|\mu_N(D_2) - 1 + W_{N,K}(R)| \ll N^{1/2} r^{-1}. \tag{5.5}$$

This lemma is a generalization of Lemma 2 of [3]. In [3], we studied the case of the Riemann zeta-function, hence the associated curves are convex. In order to use the convexity in the present general situation, we rearrange the summation with respect to n as follows. Applying the Artin-Chebotarev density theorem (see, e.g., Proposition 7.15 of Narkiewicz [15]) we see that there exist infinitely many primes p_n for which $g(n) = \ell$ and $f(j,n) = 1$ $(1 \leq j \leq \ell)$ hold. Denote the first three of such primes by $p_{n(1)}$, $p_{n(2)}$, and $p_{n(3)}$. Define p_n^* by $p_1^* = p_{n(1)}$, $p_2^* = p_{n(2)}$, $p_3^* = p_{n(3)}$, and

$$p_n^* = \begin{cases} p_{n-3} & (4 \leq n \leq n(1) + 2), \\ p_{n-2} & (n(1) + 3 \leq n \leq n(2) + 1), \\ p_{n-1} & (n(2) + 2 \leq n \leq n(3)), \\ p_n & (n(3) + 1 \leq n). \end{cases}$$

Similarly we define θ_n^*, $z_{n,K}^*$, $\Gamma_{n,K}^*$, and put $\boldsymbol{\theta}^* = (\theta_1^*, \ldots, \theta_N^*)$,

$$\Omega_{N,K}^*(A) = \{\boldsymbol{\theta}^* \in Q_N \mid S_{N,K}(\boldsymbol{\theta}^*) \in A\},$$
$$W_{N,K}^*(A) = \mu_N(\Omega_{N,K}^*(A)).$$

Then $W^*_{N,K}(A) = W_{N,K}(A)$ for all $N \geq n(3)$, so we may consider $W^*_{N,K}(A)$ instead of $W_{N,K}(A)$.

The merit of this rearrangement is that the first three curves $\Gamma^*_{\nu,K}$ ($\nu = 1, 2, 3$) are convex. In fact, we have

$$z^*_{\nu,K}(\theta^*_\nu) = z_{n(\nu),K}(\theta_{n(\nu)}) = -\ell \cdot \log(1 - p_{n(\nu)}^{-\sigma} e^{2\pi i \theta_{n(\nu)}}) \tag{5.6}$$

($\nu = 1, 2, 3$), which is just a constant multiple of $z_{n(\nu)}(\theta_{n(\nu)})$ (defined by (1.8)). Therefore the analogue of Lemma 3 of [3] is valid for these $z^*_{\nu,K}(\theta^*_\nu)$ ($\nu = 1, 2, 3$), and hence the analogue of Lemma 4 of [3] holds for $W^*_{3,K}$.

We use the notation $d(\mathbf{x}, B)$ for the distance from a point $\mathbf{x} \in Q_N$ to a subset $B \subset Q_N$, and ∂B for the boundary of B. Our next aim is to show

$$\mu_N(\{\boldsymbol{\theta} \in Q_N \mid d(\boldsymbol{\theta}, \partial\Omega^*_{N,K}(R)) \leq \delta\}) \ll \delta \tag{5.7}$$

for any $\delta > 0$. In the case of the Riemann zeta-function, this inequality has been proved as (21) of [3], by using Lemmas 3 and 4 of [3] and formula (4.1) of [14]. We have already noted that the analogues of Lemmas 3 and 4 of [3] are valid in our present situation. Therefore, in order to generalize the argument in [3] to obtain a proof of (5.7), the remaining task is to establish the following analogue of (4.1) of [14]: For any $k \in \mathbb{R}$ and any small $\varepsilon > 0$, there exists a positive constant $C = C(\sigma, \ell)$ for which

$$\{\boldsymbol{\theta} \in Q_N \mid d(\boldsymbol{\theta}, \partial\Omega^*_{N,K}(k)) \leq \varepsilon\} \subset \bigcup_{k - C\varepsilon \leq t \leq k + C\varepsilon} \Omega^*_{N,K}(t) \tag{5.8}$$

holds, where

$$\Omega^*_{N,K}(t) = \Omega^*_{N,K}(\{z \mid \Re z = t\}).$$

To prove (5.8) by the method explained in Section 4 of [14], it is enough to show that

$$\left(\sum_{n=1}^{N} (\Re\Theta^*_n)^2 \right)^{1/2} \leq C \tag{5.9}$$

for any N, where $\Theta^*_n = \partial S_{N,K}(\boldsymbol{\theta}^*)/\partial\theta^*_n$ and C is the same as in Lemma 1. We have already proved this inequality in Lemma 1. Lemma 1 is stated for Θ_n, but the argument for Θ^*_n is the same. Therefore we obtain (5.8), hence (5.7).

Lastly, since the length of the longest diagonal of $Q_N(\mathbf{n})$ is $N^{1/2}r^{-1}$, we choose $\delta = N^{1/2}r^{-1}$ in (5.7) to obtain Lemma 3. This completes the proof of Proposition 1.

§6. Completion of the proof of the theorem

Now we are going to complete the proof of our theorem. First we show one more lemma. Let us write the rectangle R as

$$R = \{z \mid \alpha_1 \leq \Re z \leq \alpha_2, \beta_1 \leq \Im z \leq \beta_2\},$$

and define

$$R_i = R_i(\delta) = \{z \mid \alpha_1 + \delta \leq \Re z \leq \alpha_2 - \delta, \beta_1 + \delta \leq \Im z \leq \beta_2 - \delta\},$$
$$R_y = R_y(\delta) = \{z \mid \alpha_1 - \delta \leq \Re z \leq \alpha_2 + \delta, \beta_1 - \delta \leq \Im z \leq \beta_2 + \delta\},$$

where δ is a small positive number. Then we have

Lemma 4. *For any large N we have*

$$|W_{N,K}(R) - W_{N,K}(R_i)| \ll \delta^{1/2}, \quad |W_{N,K}(R) - W_{N,K}(R_y)| \ll \delta^{1/2}.$$

When K is Galois, this is Lemma 7 of [11], which has been proved by using properties of convex curves. To prove the lemma in the non-Galois case, we notice that it is enough to show this lemma for $W^*_{N,K}$. Then the first three curves are convex of the form (5.6), hence we can apply the argument of proving Lemma 7 of [11] to the present case, the details being omitted.

Let $\sigma > 1$. Formula (4.1) of [11] implies

$$\text{(6.1)} \qquad |\log\zeta_K(\sigma+it) - \log\zeta_{N,K}(\sigma+it)| \ll \sum_{n=N+1}^{\infty} p_n^{-\sigma} \ll N^{1-\sigma}(\log N)^{-\sigma},$$

where the implied constants depend only on σ and ℓ. Hence

$$\text{(6.2)} \qquad V_{N,K}(T; R_i(\delta)) \leq V_K(T;R) \leq V_{N,K}(T; R_y(\delta))$$

for $\delta = C_1 N^{1-\sigma}(\log N)^{-\sigma}$ with a positive constant $C_1 = C_1(\sigma, \ell)$. Hence

(6.3)
$$\left|W_K(R) - \frac{1}{T}V_K(T;R)\right| \leq \max\left\{\left|W_K(R) - \frac{1}{T}V_{N,K}(T;R_i)\right|, \left|W_K(R) - \frac{1}{T}V_{N,K}(T;R_y)\right|\right\}.$$

On the other hand, since

$$\left|W_K(R) - \frac{1}{T}V_{N,K}(T;R_i)\right| \le |W_K(R) - W_{N,K}(R)|$$

$$+ |W_{N,K}(R) - W_{N,K}(R_i)| + \left|W_{N,K}(R_i) - \frac{1}{T}V_{N,K}(T;R_i)\right|,$$

by using Proposition 1 (applied to R_i), Proposition 2 and Lemma 4, we obtain

$$\left|W_K(R) - \frac{1}{T}V_{N,K}(T;R_i)\right| \ll \mu_2(R)N^{1-2\sigma}(\log N)^{-2\sigma}$$
$$+ \delta^{1/2} + \frac{N^{1/2}}{r} + \frac{Nr}{m} + \frac{1}{T}(6r\log m)^N \exp(mN\log N) \tag{6.4}$$

with the above choice of δ. We can estimate $|W_K(R) - T^{-1}V_{N,K}(T;R_y)|$ similarly. Substituting these estimates into the right-hand side of (6.3), we obtain

(6.5)

$$\left|W_K(R) - \frac{1}{T}V_K(T;R)\right| \ll \mu_2(R)N^{1-2\sigma}(\log N)^{-2\sigma}$$
$$+ N^{-(\sigma-1)/2}(\log N)^{-\sigma/2} + \frac{N^{1/2}}{r} + \frac{Nr}{m} + \frac{1}{T}(6r\log m)^N \exp(mN\log N).$$

Put $N = (\log T)^\alpha$, $m = (\log T)^\beta$ and $r = (\log T)^\gamma$. How to find the optimal choice of parameters is discussed in the first section of [3]. That is, first assume $\alpha + \beta = 1 - \varepsilon$ to show that the last term on the right-hand side is small. Then require

$$\frac{1}{2}\alpha - \gamma = -\frac{1}{2}\alpha(\sigma - 1) + \varepsilon, \quad \alpha + \gamma - \beta = -\frac{1}{2}\alpha(\sigma - 1) + \varepsilon$$

to obtain $\alpha = 2/(3 + 2\sigma) + \varepsilon$ and

$$\left|W_K(R) - \frac{1}{T}V_K(T;R)\right|$$
$$\ll \mu_2(R)(\log T)^{-\alpha(2\sigma-1)+\varepsilon} + (\log T)^{-\alpha(\sigma-1)/2+\varepsilon}, \tag{6.6}$$

which gives the assertion of the theorem for $\sigma > 1$.

Finally we consider the case $1 - L^{-1} < \sigma \le 1$. Let $\delta > 0$, and by $k^\delta_{N,K}(T)$ we mean the measure of the set

$$\{t \in [1,T] \mid \sigma + it \in \mathcal{G},\ |\log\zeta_K(\sigma + it) - \log\zeta_{N,K}(\sigma + it)| \ge \delta\}.$$

Then

(6.7)

$$V_{N,K}(T;R_i(\delta)) - k_{N,K}^{\delta}(T) \le V_K(T;R) \le V_{N,K}(T;R_y(\delta)) + k_{N,K}^{\delta}(T)$$

(this is (4.3) of [11]). Using these inequalities instead of (6.2), this time we have

$$(6.8)\quad \left|W_K(R) - \frac{1}{T}V_K(T;R)\right| \ll \mu_2(R)N^{1-2\sigma}(\log N)^{-2\sigma} + \delta^{1/2} + \frac{N^{1/2}}{r} + \frac{Nr}{m} + \frac{1}{T}(6r\log m)^N \exp(mN\log N) + \frac{1}{T}k_{N,K}^{\delta}(T).$$

In Section 7 of [11] we have shown

$$(6.9)\quad \frac{1}{T}k_{N,K}^{\delta}(T) \ll \delta^{-2}\log(\delta^{-1})\left(N^{-3+\varepsilon} + T^{-1}N^{-2+\varepsilon}\right) + \delta^{-2}\left\{N^{1-2\sigma+\varepsilon} + T^{-1+L(1-\sigma)+\varepsilon}\exp(C\ell N^{1/L})\right\} + \frac{1}{T}\,.$$

This estimate has been deduced from Lemma 5 of [11]. Lemma 5 of [11] has been proved under the assumption that K is Galois, but actually this assumption is not used in the proof. Hence (6.9) holds for any number field K.

We again put $N = (\log T)^{\alpha}$, $m = (\log T)^{\beta}$, $r = (\log T)^{\gamma}$ and assume $\alpha < 1$, $\alpha + \beta = 1 - \varepsilon$. Then, since $-1 + L(1-\sigma) < 0$, the factor

$$T^{-1+L(1-\sigma)+\varepsilon}\exp(C\ell N^{1/L})$$

is small. Hence, substituting (6.9) into the right-hand side of (6.8), we have

$$(6.10)\quad \left|W_K(R) - \frac{1}{T}V_K(T;R)\right| \ll \mu_2(R)N^{1-2\sigma}(\log N)^{-2\sigma} + \delta^{1/2} + \frac{N^{1/2}}{r} + \frac{Nr}{m} + \delta^{-2}\log(\delta^{-1})N^{-3+\varepsilon} + \delta^{-2}N^{1-2\sigma+\varepsilon}.$$

Choose the value of δ by

$$\delta^{1/2} = \delta^{-2}N^{1-2\sigma} = \delta^{-2}(\log T)^{\alpha(1-2\sigma)},$$

so $\delta = (\log T)^{-2\alpha(2\sigma-1)/5}$. Then we require

$$\frac{1}{2}\alpha - \gamma = -\frac{1}{5}\alpha(2\sigma-1) + \varepsilon, \quad \alpha + \gamma - \beta = -\frac{1}{5}\alpha(2\sigma-1) + \varepsilon$$

to obtain $\alpha = 10/(21+8\sigma)+\varepsilon$ and

$$(6.11)\qquad \left|W_K(R) - \frac{1}{T}V_K(T;R)\right| \ll \mu_2(R)(\log T)^{-\alpha(2\sigma-1)+\varepsilon} + (\log T)^{-\alpha(2\sigma-1)/5+\varepsilon}.$$

This implies the theorem for $1 - L^{-1} < \sigma \le 1$.

Remark. We have actually proved (6.6) and (6.11), which are slightly sharper than the statement of the theorem.

Note Added in Proof

A generalization of the result in the present paper to the case of Hecke L-functions associated with ideal class characters has already been published in Analysis **26** (2006), 313–321.

References

[1] H. Bohr and B. Jessen, Über die Wertverteilung der Riemannschen Zetafunktion, Erste Mitteilung, Acta Math., **54** (1930), 1–35.

[2] H. Bohr and B. Jessen, Über die Wertverteilung der Riemannschen Zetafunktion, Zweite Mitteilung, Acta Math., **58** (1932), 1–55.

[3] G. Harman and K. Matsumoto, Discrepancy estimates for the value-distribution of the Riemann zeta-function IV, J. London Math. Soc. (2), **50** (1994), 17–24.

[4] B. Jessen and A. Wintner, Distribution functions and the Riemann zeta function, Trans. Amer. Math. Soc., **38** (1935), 48–88.

[5] R. Kačinskaitė, A discrete limit theorem for the Matsumoto zeta-function on the complex plane, Liet. Mat. Rink., **40** (2000), 475–492 (in Russian); English transl: Lithuanian Math. J., **40** (2000), 364–378.

[6] A. Laurinčikas, Limit theorems for the Matsumoto zeta-function, J. Théor. Nombr. Bordeaux, **8** (1996), 143–158.

[7] K. Matsumoto, Discrepancy estimates for the value-distribution of the Riemann zeta-function I, Acta Arith., **48** (1987), 167–190.

[8] K. Matsumoto, Discrepancy estimates for the value-distribution of the Riemann zeta-function III, Acta Arith., **50** (1988), 315–337.

[9] K. Matsumoto, A probabilistic study on the value-distribution of Dirichlet series attached to certain cusp forms, Nagoya Math. J., **116** (1989), 123–138.

[10] K. Matsumoto, Value-distribution of zeta-functions, In: Analytic Number Theory, (eds. K. Nagasaka and E. Fouvry), Lecture Notes in Math., **1434**, Springer-Verlag, 1990, pp. 178–187.

[11] K. Matsumoto, Asymptotic probability measures of zeta-functions of algebraic number fields, J. Number Theory, **40** (1992), 187–210.

[12] K. Matsumoto, Asymptotic probability measures of Euler products, In: Proceedings of the Amalfi Conference on Analytic Number Theory, (eds. E. Bombieri et al.), Univ. di Salerno, 1992, pp. 295–313.

[13] K. Matsumoto, Probabilistic value-distribution theory of zeta-functions, Sūgaku, **53** (2001), 279–296 (in Japanese); English transl: Sugaku Expositions, **17** (2004), 51–71.

[14] K. Matsumoto and T. Miyazaki, On some hypersurfaces of high-dimensional tori related with the Riemann zeta-function, Tokyo J. Math., **10** (1987), 271–279.

[15] W. Narkiewicz, Elementary and Analytic Theory of Algebraic Numbers, 2nd ed., Springer-Verlag, 1990.

[16] E. C. Titchmarsh, The Theory of the Riemann Zeta-function, Oxford, 1951.

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