Advanced Studies in Pure Mathematics 49, 2007
Probability and Number Theory — Kanazawa 2005
pp. 133–170

New approach to probabilistic number theory - compactifications and integration

Karl-Heinz Indlekofer

Abstract.

For primes p let

$$A_p := \{n \, : \, n \in \mathbb{N}, \, p|n\}$$

be the set of all natural numbers divisible by p. In his book "*Probabilistic methods in the Theory of Numbers*" (1964) J. Kubilius applies finite probabilistic models to approximate independence of the events A_p. His models are constructed to mimic the behaviour of (truncated) additive functions by suitably defined independent random variables.

Embedding $\mathbb{N}$, endowed with the discrete topology, in the compact space $\beta\mathbb{N}$, the *Stone-Čech compactification* of $\mathbb{N}$, and taking $\overline{A_p} := \mathrm{clos}_{\beta\mathbb{N}} A_p$ leads to *independent events* $\overline{A_p}$. This observation is a motivation for a general integration theory on $\mathbb{N}$ which can be used in various topics of Probabilistic Number Theory. In this paper we present a short compendium of Probabilistic Number Theory concerning the distribution of arithmetical functions. The new model is applied to the result of Erdös and Wintner about the limit distribution of additive functions and to the famous result of Szemeredi in combinatorical number theory. Further applications are given with respect to spaces of limit periodic and almost periodic functions and recent results on q-multiplicative functions.

§1. Introduction

If we say that probabilistic number theory is devoted to solving problems of arithmetic by using (ideas or) the machinery of probability then the subject started *cum grano salis* 1917 [31] with the paper "The normal number of prime factors of a number n" by Hardy and Ramanujan.

Received January 10, 2006.
Revised December 6, 2006.
2000 *Mathematics Subject Classification.* Primary 11K65; Secondary 11N64.

They considered the arithmetical functions ω and Ω, where $\omega(n)$ and $\Omega(n)$ denote the number of different prime divisors and of all prime divisors - i.e. counted with multiplicity - of an integer n, respectively. Introducing the concept "normal order" Hardy and Ramanujan proved that ω and Ω have the normal order "$\log\log n$". Here we say, roughly, that an arithmetical function f has the normal order F, if $f(n)$ is approximately $F(n)$ for almost all values of n.[1] More precisely this means that

$$(1-\varepsilon)F(n) < f(n) < (1+\varepsilon)F(n)$$

for every positive ε and almost all values of n.

The photo has been removed due to copyright issues.

P. Turán

In 1934 Turán [85] gave a new proof of Hardy and Ramanujan's result. It depended on the (readily obtained) estimate

$$\sum_{n\leq x}(\omega(n)-\log\log x)^2 \leq cx\log\log x.$$

[1]A property E is said to hold for almost all n if $\lim\limits_{x\to\infty} x^{-1}\#\{n \leq x :$ E does not hold for $n\} = 0$.

This inequality - reminding us of Tschebycheff's inequality[2] - had a special effect, namely giving Kac the idea of thinking about the role of independence in the application of probability to number theory. Making essential use of the notation of independent random variables, the central limit theorem and sieve methods Kac together with Erdös proved in 1939 [14], 1940 [15]: *For real-valued strongly additive functions f let*

$$A(x) := \sum_{p \leq x} \frac{f(p)}{p} \tag{1}$$

and

$$B(x) := \left(\sum_{p \leq x} \frac{f^2(p)}{p} \right)^{\frac{1}{2}}. \tag{2}$$

Then, if $|f(p)| \leq 1$ and if $B(x) \to \infty$ as $x \to \infty$, the frequencies

$$F_x(z) := \frac{1}{x} \#\{n \leq x : \frac{f(n) - A(x)}{B(x)} \leq z\}$$

converge weakly to the limit law

$$G(z) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{w^2}{2}} dw$$

as $x \to \infty$ (which will be denoted by writing $F_x(z) \Rightarrow G(z)$).

Thus for $f(n) = \omega(n)$ Erdös and Kac obtained a much more general result than Hardy and Ramanujan. For in this case

$$A(x) = \log\log x + O(1)$$

and

$$B(x) = (1 + o(1))(\log\log x)^{\frac{1}{2}}$$

so that

$$x^{-1} \#\{n \leq x : \frac{\omega(n) - \log\log x}{\sqrt{\log\log x}} \leq z\} \Rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{w^2}{2}} dw.$$

[2] At that time Turán knew no probability (see chapter 12 of [10]). The first widely accepted axiom system for the theory of probability, due to Kolmogorov, had only appeared in 1933.

A second effect of the above mentioned paper of Turán was that Erdös, adopting Turán's method of proof showed 1938 [12] that, whenever the three series

$$\sum_{\substack{p \\ |f(p)|>1}} \frac{1}{p}, \quad \sum_{\substack{p \\ |f(p)|\le 1}} \frac{f(p)}{p}, \quad \sum_{\substack{p \\ |f(p)|\le 1}} \frac{f^2(p)}{p}$$

converge then the real-valued strongly additive function f possesses a limiting distribution F, i.e.

$$x^{-1}\#\{n \le x : f(n) \le z\} \Rightarrow F(z)$$

with some suitable distribution function F. It turned out (Erdös and Wintner [16]), that the convergence of these three series was in fact necessary.

All these results can be described as effects of the fusion of (intrinsic) ideas of probability theory and asymptotic estimates. In this context, divisibility by a prime p is an event A_p, and all the $\{A_p\}$ are statistically independent of one another, where the underlying "measure" is given by the *asymptotic density*

$$\delta(A_p) := \lim_{x\to\infty} x^{-1}\#\{n \le x : n \in A_p\}$$

$$= \lim_{x\to\infty} x^{-1} \sum_{\substack{n\le x \\ p|n}} 1 \quad \left(= \frac{1}{p}\right) \tag{3}$$

(If the limit

$$M(f) := \lim_{x\to\infty} x^{-1} \sum_{n\le x} f(n)$$

exists, then we say that the function f possesses an *(arithmetical) mean-value* $M(f)$.)
Then, for strongly additive functions f,

$$f = \sum_p f(p)\varepsilon_p$$

where ε_p denotes the characteristic function of A_p and $M(\varepsilon_p) = \frac{1}{p}$.

The main difficulties concerning the immediate application of probabilistic tools arise from the fact that the arithmetical mean-value (3) defines

The photo has been removed due to copyright issues.

P. Erdős

only a finitely additive measure (or content or pseudo-measure) on the family of subsets of $\mathbb{N}$ having an asymptotic density. To overcome these difficulties one builds a sequence of finite, purely probabilistic models, which approximate the number theoretical phenomena, and then use arithmetical arguments for "taking the limit". This theory, starting with the above mentioned results of Erdös, Kac and Wintner, was developed by Kubilius [62]. He constructed finite probability spaces on which independent random variables could be defined so as to mimic the behaviour of truncated additive functions

$$\sum_{p \leq r} f(p)\varepsilon_p$$

This approach is effective if the ratio $\frac{\log r}{\log x}$ essentially tends to zero as x runs to infinity. For Kubilius was able to give necessary and sufficient conditions in order that the frequencies

$$x^{-1}\#\{n \leq x : f(n) - A(x) \leq zB(x)\}$$

converge weakly as $x \to \infty$, assuming that f belongs to a certain class of additive functions. This result lead to the problem when a given additive function f may be renormalized by functions $\alpha(x)$ and $\beta(x)$, so that as $x \to \infty$ the frequencies

$$x^{-1}\#\{n \leq x : \frac{f(n) - \alpha(x)}{\beta(x)} \leq z\}$$

possess a weak limit (see Kubilius [62], Elliott [10], Levin and Timofeev [65]).

The photos have been removed due to copyright issues.

N. M. Timofeev P. D. T. A. Elliott

All these methods have been developed for and adopted to the investigation of additive functions with its emphasis on sums of independent random variables. The investigation of (real-valued) multiplicative functions goes back to Bakštys [2], Galambos [25], Levin, Timofeev and Tuliaganov [66] and uses Zolotarev's result [89] concerning the characteristic transforms of products of random variables.

We reformulate: A **general problem of probabilistic number theory** is to find appropriate probability spaces where large classes of arithmetic functions can be considered as random variables. In particular, is it possible to write the mean-value $M(f)$ of a function f (if it exits) as an integral $M(f) = \int_X \overline{f}\, d\mu(x)$, when the space X and the integrable function $\overline{f}$ is uniquely determined by $\mathbb{N}$ and f, respectively?

§2. Approximation of independence

In this chapter we have in mind the idea of Kac that, suitably interpreted, divisibility of an integer by differing primes represents independent events. At the beginning we shall consider two examples of algebras of subsets of $\mathbb{N}$. We denote by $\mathcal{A}_1$ the algebra generated by the zero residue classes whereas $\mathcal{A}_2$ is defined as the algebra generated by all residue classes. On both algebras the asymptotic density is finitely but not countably additive. In the case of the algebra $\mathcal{A}_2$ this difficulty will be overcome by the embedding of $\mathbb{N}$ into the polyadic numbers. Concerning the algebra $\mathcal{A}_1$ a solution of the problem will be given by the construction of the model of Kubilius. In chapter 7 we shall formulate a general solution of both of these problems.

For a natural number Q let $E(l, Q)$ denote the set of positive integers n which satisfy the relation $n \equiv l \bmod Q$ where l assumes any value in the range $1 \leq l \leq Q$. Denote by $\mathcal{A}_2$ the algebra generated by all these arithmetic progressions $E(l, Q)$ for $Q = 1, 2, \ldots$ amd $1 \leq l \leq Q$. Observe that each member $A \in \mathcal{A}_2$ possesses an asymptotic density $\delta(A)$ and δ is fully determined by the values

$$\delta(E(l, Q)) = \frac{1}{Q}$$

for each Q and all $1 \leq l \leq Q$. Then δ is finitely additive but not countably additive on the algebra $\mathcal{A}_2$ which will be shown by an example due to Yu. I. Manin (see Postnikov [75]. p. 135).

Let $Q_i = 3^i$, $i = 1, 2, \ldots$, and put $E_1 = E(0, Q_1)$ and $E_2 = E(1, Q_2)$. For $j \geq 3$ choose l_j to be the smallest positive integer not occuring in $E_1 \cup E_2 \ldots \cup E_{j-1}$. Put $E_j = E(l_j, Q_j)$. It is clear that $\mathbb{N} = \bigcup_{i=1}^{\infty} E_i$. Further $E_i \cap E_j = \emptyset$ if $i \neq j$. For this suppose $j > i$ and $l_j + m_j Q_j = l_i + m_i Q_i$. We see that $l_j = l_i + Q_i(m_i - 3^{j-i} m_j)$ and, since $l_j > l_i$, $l_j \in E_i$ which contradicts the choice of l_j. Since

$$\sum_{i=1}^{\infty} \delta(E_i) = \sum_{i=1}^{\infty} 3^{-i} = \frac{1}{2} < 1 = \delta(\bigcup_{i=1}^{\infty} E_i)$$

the asymptotic density is not a measure on $\mathcal{A}_2$.

Concerning the definition of $\mathcal{A}_1$ we choose, for each prime p, the sets $A_{p^k} = E(0, p^k)$ of natural numbers which are divisible by p^k ($k = 1, 2, \ldots$). Then $\mathcal{A}_1$ will be the smallest algebra containing all the sets A_{p^k}. Obviously, $\mathcal{A}_1$ is a subalgebra of $\mathcal{A}_2$ and the asymptotic density δ is finitely additive. It is not difficult to show by an example that δ is not countably additive on $\mathcal{A}_1$.

In his book [62] Kubilius applies finite probabilistic models to approximate independence of the events A_p for primes p. The study of arithmetic functions within the classical theory of probability, with its emphasis on sums and products of independent random variables, involves a careful balance between the convenience of a measure, with respect to which appropriate events are independent, and the loss of generality for the class of functions which may be considered.

The models of Kubilius are constructed to mimic the behaviour of (truncated) additive functions by suitably defined independent random variables. The construction may run as follows (see [10], p. 119).
Let $2 \le r \le x$, let $S := \{n : n \le x\}$ and put $D = \prod_{p \le r} p$. For each prime p dividing D let $\widetilde{E}(p) := S \cap E(0,p)$ and $\bar{E}(p) = S \backslash E(p)$. If we define, for each positive integer k which divides D, the set

$$E_k = \bigcap_{p|k} E(p) \bigcap_{p|\frac{D}{k}} \widetilde{E}(p)$$

then these sets are disjoint for differing values of k. Further, if $\mathcal{A}$ denotes the σ-algebra which is generated by the $E(p)$, $p \le r$, then each member of $\mathcal{A}$ is a union of finitely many of the E_k. On the algebra $\mathcal{A}$ one defines a measure ν: If

$$A = \bigcup_{j=1}^{m} E_{k_j}$$

then

$$\nu(A) := \sum_{j=1}^{m} [x]^{-1} |E_{k_j}|.$$

Since $\nu(S) = 1$ the triple $(S, \mathcal{A}, \nu)$ forms a finite probability space.
A second measure μ will be defined by

$$\mu(E_k) := \frac{1}{k} \prod_{p|\frac{D}{k}} (1 - \frac{1}{p})$$

where $k|D$. It is clear that $\mu(S) = 1$, and thus the triple $(S, \mathcal{A}, \mu)$ is also a finite probability space. By an application of the Selberg sieve method one can show that

$$\nu(A) = \mu(A) + O(L)$$

holds uniformly for all sets A in the algebra $\mathcal{A}$ with

$$L = \exp\left(-\frac{1}{8} \frac{\log x}{\log r} \log \left(\frac{\log x}{\log r} \right) \right) + x^{-1/15}.$$

The Kubilius model can directly be applied to obtain, in particular, the celebrated theorem of Erdös and Kac. For this we confine our attention in the moment to (real-valued) strongly additive functions f and recall the definitions (1) and (2) of $A(x)$ and $B(x)$. Following Kubilius, we shall say that f belongs to the class H if there exists a function $r = r(x)$ so that as $x \to \infty$,

$$\frac{\log r}{\log x} \to 0, \quad \frac{B(r)}{B(x)} \to 1, \quad B(x) \to \infty.$$

As an archetypal result we mention (see Elliott [10], Theorem 12.1)

Proposition 1. (Kubilius [62]) *Let f be a strongly additive function of class H. Then the frequencies*

$$x^{-1}\#\{n \leq x : f(n) - A(x) \leq zB(x)\} \tag{4}$$

converge to a limit with variance 1 as $x \to \infty$, if and only if there is a nondecreasing function K of unit variation such that at all points at which $K(u)$ is continuous

$$\frac{1}{B^2(x)} \sum_{\substack{p \leq x \\ f(p) \leq uB(x)}} \frac{f^2(p)}{p} \to K(u)$$

as $x \to \infty$. When this condition is satisfied the characteristic function ϕ of the limit law will be given by Kolmogorov's formula

$$\log \phi(t) = \int_{-\infty}^{\infty} (e^{itu} - 1 - itu)u^{-2} dK(u)$$

and the limit law will have mean zero, and variance 1.
Whether the frequencies (4) converge or not,

$$\frac{1}{xB(x)} \sum_{n \leq x} (f(n) - A(x)) \to 0, \tag{5}$$

$$\frac{1}{xB^2(x)} \sum_{n \leq x} (f(n) - A(x))^2 \to 1,$$

holds as $x \to \infty$.

Bearing in mind that in the Kolmogorov representation of the characteristic function of the normal low with variance 1 we have

$$K(u) = \begin{cases} 1 & \text{if } u \geq 0 \\ 0 & \text{if } u < 0 \end{cases}$$

we arrive at (see Elliott) [10], Theorem 12.3)

Proposition 2. (Erdös-Kac [14]) *Let f be a real valued strongly additive function which satisfies $|f(p)| \leq 1$ for every prime p. Let $B(x) \to \infty$ as $x \to \infty$. Then*

$$x^{-1}\#\{n \leq x : f(n) - A(x) \leq zB(x)\} \Longrightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-w^2/2} dw.$$

Remark 1. The value distribution of positive-valued arithmetic function h may be studied in terms of

$$x^{-1}\#\{n \leq x : \log h(n) - \alpha(x) \leq z\beta(x)\}$$

with renormalising constants $\alpha(x)$, $\beta(x) > 0$. For those functions which grow rapidly there is another perspective. We say that the values of positive valued function h are *uniformly distributed in* $(0, \infty)$ if $h(n)$ tends to infinity as $n \to \infty$ and if there exists a positive constant c such that as $y \to \infty$

$$N(h, y) := \sum_{\substack{n \\ h(n) \leq y}} 1 = (c + o(1))y.$$

General results for multiplicative functions h in connection with the existence of the limiting distribution of h/id, where $id(n) = n$ for all $n \in \mathbb{N}$, can be found in Indlekofer [36]. A detailed account concerning multiplicative functions is given by Diamond and Erdös [9].

§3. Uniform Summability

There are three results concerning the asymptotic behaviour of multiplicative functions $g : \mathbb{N} \to \mathbb{C}$ with $|g(n)| \leq 1$ for all $n \in \mathbb{N}$ which have become classical:

1. Delange [7] proved that the mean value $M(g)$ exists and is different from zero if and only if the series

$$\sum_{p} \frac{1 - g(p)}{p} \tag{6}$$

converges, and for some positive r, $g(2^r) \neq -1$.

2. Assuming that g is real-valued and the series (6) diverges, Wirsing [88] proved that g has mean-value $M(g) = 0$. In particular this means that the mean value $M(g)$ always exists for real-valued multiplicative functions of modulus ≤ 1.

3. Halász [30] proved that the divergence of the series

$$\sum_p \frac{1 - Re\ g(p)p^{-it}}{p}$$

for each $t \in \mathbb{R}$ implies that a complex-valued multiplicative g has mean value $M(g) = 0$. Furthermore, he gave a complete description of the means $M(g, x) := x^{-1} \sum_{n \le x} g(n)$ as $x \to \infty$.

Remark 2. If we set $g(n) = \mu(n)$, the Möbius function, then we are precisely concerned with the case where the series $\sum_p p^{-1}(1 - g(p))$ diverges. Moreover, the validity of the assertion $M(\mu) = 0$ was shown by Landau [63] to be equivalent to the prime number theorem. The (first) elementary proof of the prime number theorem by Selberg appeared in 1949. In 1943 Wintner [87], in his book on Erathostenian Averages, asserted that if a multiplicative function g assumes only values ± 1, then the mean value $M(g)$ always existed. But the sketch of his proof could not be substantiated, and the problem remained open as the Erdös-Wintner conjecture. We shall not repeat the story concerning the prize which Erdös offered for a solution of this problem (cf. Elliott [10], p. 254) but in 1967 Wirsing, by his mentioned result, solved this problem. His proof was done by elementary methods (and thus he gave another elementary proof of the prime number theorem) but he could not handle the complex-valued case in its full generality. Only by an analytic method, found by Halász in 1968, and exposed by him in his paper [30], the asymptotic behaviour of $\sum_{n \le x} g(n)$ could be fully determined for all complex-valued multiplicative functions g of modulus smaller or equal to one. As in the case of Wirsing's proof of the Erdös-Wintner conjecture it took again twenty-four years until Daboussi and Indlekofer [6] produced an elementary proof of Halász's theorem. In a subsequent paper Indlekofer [49] following the same lines of the proof gave a more elegant version which served as a model in the book of Schwarz and Spilker [79]. This ends the remark.

The wish to abandon the restriction on the size of g led to the investigation of multiplicative functions which belong to the class $\mathcal{L}^\alpha$, $\alpha > 1$. Here, for $1 \le \alpha \le \infty$,

$$\mathcal{L}^\alpha := \{f : \mathbb{N} \to \mathbb{C},\ ||f||_\alpha < \infty\}$$

denotes the linear space of arithmetic functions with bounded seminorm

$$||f||_\alpha := \Big\{ \limsup x^{-1} \sum_{n\le x} |f(n)|^\alpha \Big\}^{1/\alpha}.$$

Obviously the functions considered by Delange, Wirsing and Halász belong to every class $\mathcal{L}^\alpha$.

A characterisation of multiplicative functions $g \in \mathcal{L}^\alpha$ ($\alpha > 1$) which possess a non-zero mean-value $M(g)$ was independently given by Elliott [11] and using a different method, by Daboussi [5]. These results were the starting point for me to introduce the notation of *uniformly summable functions.*
The underlying motivations for this were the facts

(i) that, if the mean-value $M(f)$ of an arithmetic function f corresponds to an integral over an (finite) integrable function, then it can be approximated by its truncation f_K at height K, i.e.

$$f_K(n) = \begin{cases} f(n) & \text{if } |g(n)| \le K \\ 0 & \text{if } |g(n)| > K, \end{cases}$$

(ii) and that, on the other hand, the partial sums $\Big\{N^{-1} \sum_{n\le N} g(n)\Big\}_{N\in\mathbb{N}}$ converge to $M(g)$.

This suggested to involve the concept of *uniform integrability.* In 1980 [33] I introduced the following

Definition 1. A function $f \in \mathcal{L}^1$ is said to be *uniformly summable* if

$$\lim_{K\to\infty} \sup_{N\ge1} N^{-1} \sum_{\substack{n\le N\\ |f(n)|>K}} |f(n)| = 0$$

and the space of all uniformly summable functions is denoted by $\mathcal{L}^\star$.

It is easy to show that, if $\alpha > 1$,

$$\mathcal{L}^\alpha \subsetneqq \mathcal{L}^\star \subsetneqq \mathcal{L}^1.$$

Further, we note that $\mathcal{L}^\star$ is nothing else but the $||\cdot||_1$-closure of l^∞, the space of all bounded functions on $\mathbb{N}$. In the same way we can define the spaces

$$\mathcal{L}^{\star\alpha} := ||\cdot||_\alpha\text{ -closure of } l^\infty.$$

The idea of uniform summability turned out to provide the appropriate tools for describing the mean behaviour of multiplicative functions and gave insight into exactly which additive functions belong to $\mathcal{L}^1$. As typical results we mention generalisations of the results of Delange, Wirsing and Halász.

Proposition 3. (see Indlekofer [33]) (A generalisation of Delange's result) *Let $g : \mathbb{N} \to \mathbb{C}$ be multiplicative and $\alpha \geq 1$. Then the following two assertions hold.*

(i) *If $g \in \mathcal{L}^\star \cap \mathcal{L}^q$ and if the mean-value $M(g) := \lim\limits_{x\to\infty} x^{-1} \sum\limits_{n\leq x} g(n)$ of g exists and is non-zero, then the series*

(7)
$$\sum_p \frac{g(p)-1}{p}, \sum_{\substack{p \\ |g(p)|\leq \frac{3}{2}}} \frac{|g(p)-1|^2}{p}, \sum_{\substack{p \\ |g(p)-1|\geq \frac{1}{2}}} \frac{|g(p)|^\lambda}{p}, \sum_p \sum_{k\geq 2} \frac{|g(p^k)|^\lambda}{p^k}$$

converge for all λ with $1 \leq \lambda \leq \alpha$ and, for each prime p,

$$1 + \sum_{k=1}^{\infty} \frac{g(p^k)}{p^k} \neq 0. \tag{8}$$

(ii) *If the series (7) converge for all λ with $1 \leq \lambda \leq \alpha$ then $g \in \mathcal{L}^\star \cap \mathcal{L}^\alpha$ and the mean-values $M(g)$, $M(|g|^\lambda)$ exist for all λ with $1 \leq \lambda \leq \alpha$. If in addition (8) holds, then $M(g) \neq 0$.*

Note that the membership of $\mathcal{L}^\alpha \cap \mathcal{L}^\star$ and the existence of a non-zero mean value are together equivalent to a set of explicit conditions on the prime powers. Further observe that these conditions imply the existence of the mean values $M(|g|^\lambda)$ for all $1 \leq \lambda \leq \alpha$.

Proposition 4. (see Indlekofer [36]) (A generalisation of Wirsing's result) *Let $g \in \mathcal{L}^\star$ be a real-valued multiplicative function. Then the existence of the mean value $M(|g|)$ implies the existence of $M(g)$.*

Note that Proposition 4 is an appropriate generalisation of Wirsing's result for, if g is multiplicative and $|g| \leq 1$, the mean value of $M(|g|)$ always exists.

A complete characterisation of the means $M(g, x)$ for complex-valued multiplicative functions $g \in \mathcal{L}^\star$ was given in 1980 by Indlekofer (see [38]). As a special result we have

Proposition 5. (A generalisation of Halász's result) *If the complex-valued multiplicative function g belongs to $\mathcal{L}^\star$, and for each $t \in \mathbb{R}$, the* *series*

$$\sum_{\substack{p \\ ||g(p)|-1|\le\frac{1}{2}}} \frac{1 - \mathrm{Re}\, g(p)(|g(p)|p^{it})^{-1}}{p}$$

diverges, then g has mean value zero.

Thus the idea of uniformly summable functions proved to be a successful concept in the investigation of multiplicative functions (and in particular, of additive functions, too). To come back to the methodological aspect and as an *a posteriori* justification of the underlying motivation we turn to the connections between mean values and integrals for multiplicative and additive functions (see Indlekofer [36] and [40]).

Proposition 6. (Indlekofer [36]) *Let the real-valued multiplicative function g be uniformly summable. Then*

(i) *g possesses a limiting distribution G if and only if the mean value $M(|g|)$ exists, and*

(ii) *this limiting distribution is degenerate if and only if $M(|g|)$.*

Moreover, in both cases

$$M(g) = \int_{\mathbb{R}} y\, dG(y)$$

$$M(|g|) = \int_{\mathbb{R}} |y|\, dG(y).$$

Proposition 7. (Indlekofer [40]) *Let $\alpha \ge 1$. For any (real-valued) additive function f the following three propositions are equivalent.*

(i) *The limiting distribution F of f exists and*

$$\int_{\mathbb{R}} |y|^\alpha\, dF(y) < \infty.$$

(ii) *$f \in \mathcal{L}^\alpha$ and the mean value $M(f)$ of f exists.*

(iii) *The series*

$$\sum_p \frac{f(p)}{p}, \qquad \sum_{\substack{p \\ |f(p)|\le 1}} \frac{f^2(p)}{p}, \qquad \sum_{\substack{p \\ |f(p^m)|\ge 1}} \sum_m \frac{|f(p^m)|^\alpha}{p^m}$$

converge.

Moreover, if one of these conditions is satisfied

$$M(f) = \int_{\mathbb{R}} y \, dF(y)$$

$$M(|f|^{\alpha}) = \int_{\mathbb{R}} |y|^{\alpha} \, dF(y).$$

Remark 3. The “reason” for the difference between the additive and multiplicative functions may be found in the fact that there is no additive function in $\mathcal{L}^1 \backslash \mathcal{L}^{\star}$ but there are “many” multiplicative functions in $\mathcal{L}^1$ which are not uniformly summable.

§4. First attempts of a general theory: Polyadic numbers and almost even functions

The ring of polyadic numbers was first introduced by Prüfer [76]. We briefly recall its construction.

Let $\mathbb{Z}$ denote the ring of integers. Then the system $\sum$ consisting of the ideals $(m) := m\mathbb{Z}$ can be taken as a complete system of neighborhoods of zero in the additive group of integers and it generates a topology which we denote by τ. Obviously, the addition is continuous in this topology and the arithmetic progressions $a + (m)$ $(a \in \mathbb{Z})$ build up a complete system of neighborhoods in $\mathbb{Z}$. The multiplication is continuous in the topology, too. For, if $a, b \in \mathbb{Z}$ and if W is any neighborhood of ab, for example $W = ab + (m)$, then one can choose $U = a + (m)$ and $V = b + (m)$ as neighborhoods of a and b, respectively, such that $UV \subset W$. Therefore, $\mathbb{Z}$ endowed with the topology τ forms a topological ring $(\mathbb{Z}, \tau)$. The topological ring $(\mathbb{Z}, \tau)$ is metrizable. It is not difficult to show

Proposition 8. *The function* $\varrho : \mathbb{Z} \times \mathbb{Z} \to [0, 1]$,

$$\varrho(x, y) = \sum_{m=1}^{\infty} \frac{1}{2^m} \left(\frac{x - y}{m} \right),$$

where (t) *denotes the distance from* t *to the nearest integer, defines a metric on* $\mathbb{Z}$ *which metrizes* $(\mathbb{Z}, \tau)$.

Next we give a short review how the the polyadic numbers can be defined. Let S be the set of sequences $\{a_i\}$ of integers such that, given $\varepsilon > 0$, there exists an N such that $\varrho(a_i, a_j) < \varepsilon$ if both $i, j > N$. We call two such Cauchy sequences $\{a_i\}$ and $\{b_i\}$ equivalent if $\varrho(a_i, b_i) \to 0$

as $i \to \infty$. We define the set S of polyadic numbers to be the set of equivalence classes of Cauchy sequences.
One can define the sum (and the product) of two equivalence classes of Cauchy sequences by choosing a Cauchy sequence in each class, defining addition (and multiplication) term-by-term, and showing that the equivalence class of the sum (and the product) only depends on the equivalence class of the two summands (and of the two factors). This enables us to turn the set S of polyadic numbers into a ring. $\mathbb{Z}$ can be identified with a subring of S consisting of equivalence classes containing a constant Cauchy sequence. Finally, it is easy to prove that S is complete with respect to the (unique) metric which extends the metric ϱ on $\mathbb{Z}$. S is a compact space since $\mathbb{Z}$ is totally bounded. Thus on the additive group of the ring S, as a compact group there exists a normalized Haar measure P defined on a σ-algebra $\mathcal{A}$ which contains the Borel sets in S such that $(S, \mathcal{A}, P)$ is a probability space. The measure of an arithmetic progression $\alpha + \beta D$ where $\alpha, \beta \in S$ and D is a natural number, is $1/D$. Therefore, embedding $\mathbb{Z}$ in S eliminates the difficulty associated with the fact that asymptotic density is not countably additive. This enabled Novoselov [70] to develop an "integration theory" for arithmetic functions f which can be approximated by periodic functions with integer period.

Remark 4. The arithmetic in the ring S and certain aspects of polyadic analysis were investigated by Novoselov in a series of papers ([70], ... ,[74]).

An arithmetic function f is called

$$r\text{-periodic, if } f(n+r) = f(n) \text{ for every } n \in \mathbb{N},$$

$$r\text{-even, if } f(n) = f(gcd(n,r)) \text{ for every } n \in \mathbb{N}.$$

It can be shown that the vector space B_r of r-even functions can be generated by the Ramanujan-functions c_d defined by

$$c_d(n) := \sum_{t|\,\gcd(d,n)} t\mu\left(\frac{n}{t}\right)$$

where $d|r$. i.e.

$$B_r = Lin_{\mathbb{C}}[c_d : d|r]$$

whereas each element of the vector space D_r of r-periodoc functions can be written as a linear combination of exponential functions, i.e.

$$D_r := Lin_{\mathbb{C}}[e_{a/r} : a = 1, 2, \dots, r]$$

where $e_{a/r}$ is defined by

$$e_{a/r}(n) = \exp\left(2\pi i \, \frac{a}{r} \, n\right).$$

We put

$$B := \bigcup_{r=1}^{\infty} B_r$$

and

$$D := \bigcup_{r=1}^{\infty} D_r$$

for the vector space of all even and all periodic functions, respectively. Finally, we define the vector space

$$A := Lin_{\mathbb{C}}[e_\beta : \beta \in [0,1)].$$

Obviously

$$B \subset D \subset A.$$

The $||\cdot||_\alpha$-closure of B, D and A leads to

- the space of α-*almost even functions*,
- the space of α-*limit-periodic functions*

and

- the space of α-*almost periodic functions*, respectively.

The photos have been removed due to copyright issues.

J. Spilker **W. Schwarz**

We note that Schwarz and Spilker [78] introduced a compactification $\mathbb{N}^\star$ of $\mathbb{N}$ by

$$\mathbb{N}^\star = \prod_{\text{p prime}} \bar{\mathrm{N}}_{\mathrm{p}}$$

where $\bar{N}_p$ denotes the one-point-compactification of the discrete topological spaces $N_p = \{1, p, p^2, \ldots\}$. By this compactification they could describe an "integration theory" of almost even functions.
The above mentioned construction of the polyadic numbers was used for the investigation of limit-periodic functions whereas Mauclaire [69] used the Bohr compactification of $\mathbb{Z}$ for the corresponding investigation of almost periodic functions. In [79] Schwarz and Spilker presented another construction of the compact space $\mathbb{N}^\star$ and the compact ring of polyadic numbers (or Prüfer ring) via Gelfand's theory of commutative Banach algebras.

Some comments are called for in connection with these examples. First of all, the special role played by the asymptotic (or logarithmic) density should be emphasized. Further, it is important to note that despite of the *ad hoc* construction of the compactifications the "size" of these spaces is very restricted; the Möbius μ function, for example, is not an element of any of these spaces.

To abandon all these restrictions we shall make use of the Stone-Čech compactification of $\mathbb{N}$ which enables us to deal with arbitrary algebras of subsets of $\mathbb{N}$ together with arbitrary additive functions on these algebras.

§5. Pseudomeasures on $\mathbb{N}$ and measures on the Stone-Čech compactification

Suppose that $\mathcal{A}$ is an *algebra of subsets of* $\mathbb{N}$, i.e.

(i) $\mathbb{N} \in \mathcal{A}$,
(ii) $A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A}$,
(iii) $A, B \in \mathcal{A} \Rightarrow A \backslash B \in \mathcal{A}$.

Then, if $\mathcal{E}$ denotes the family of simple functions on $\mathbb{N}$, the set

$$\mathcal{E}(\mathcal{A}) := \{s \in \mathcal{E},\ s = \sum_{j=1}^{m} \alpha_j 1_{A_j};\ \alpha_j \in \mathbb{C},\ A_j \in \mathcal{A},\ j = 1, \ldots, m\}$$

of *simple functions on* $\mathcal{A}$ is a vector space. In [48], I investigated the $||\cdot||_q$-closure of $\mathcal{E}(\mathcal{A})$, the space of $\mathcal{L}^{\star q}(\mathcal{A})$-*uniformly summable functions for the algebras* $\mathcal{A}$ whose elements possess an asymptotic density.

These results performed the initial steps towards the idea which can be described as follows: $\mathbb{N}$, endowed with the discrete topology, will be embedded in a compact space $\beta\mathbb{N}$, the Stone-Čech compactification of $\mathbb{N}$, and then any algebra $\mathcal{A}$ in $\mathbb{N}$ with an arbitrary finitely additive set function, a content or pseudomeasure on $\mathbb{N}$, can be extended to an algebra $\bar{\mathcal{A}}$ in $\beta\mathbb{N}$ together with an extension of this pseudomeasure, which turns out to be a premeasure on $\bar{\mathcal{A}}$. The basic necessary concepts are summarized in the following three propositions, where the ring of all real-valued continuous functions defined on a topological space X will be denoted by $C(X)$ and the subring of all bounded members of $C(X)$ will be denoted by $C^b(X)$.

Proposition 9. *There exists a compactification $\beta\mathbb{N}$ of $\mathbb{N}$ with the following equivalent properties.*

(i) *Every mapping f from $\mathbb{N}$ into any compact Hausdorff space Y has a continuous extension $\bar{f}$ from $\beta\mathbb{N}$ into Y.*

(ii) *Every bounded real-valued function on $\mathbb{N}$ has an extension to a function in $C(\beta\mathbb{N})$.*

(iii) *For any two subsets A and B of $\mathbb{N}$,*

$$\overline{A \cap B} = \overline{A} \cap \overline{B},$$

where $\overline{A} = cl_{\beta\mathbb{N}}A$ and $\overline{B} = cl_{\beta\mathbb{N}}B$ are the closures of A and B in $\beta\mathbb{N}$, respectively.

(iv) *Any two disjoint subsets of $\mathbb{N}$ have disjoint closures in $\beta\mathbb{N}$.*

Stone and Čech (see for example Gillman and Jerison [29]) have investigated the compactification βX for completely regular spaces X. The above proposition contains their results for $X = \mathbb{N}$. An immediate consequence of (iii) is

Proposition 10. *The compactification $\beta\mathbb{N}$ of $\mathbb{N}$ has the following property:*

(v) *For any algebra $\mathcal{A}$ in $\mathbb{N}$ the family*

$$\bar{\mathcal{A}} := \{\bar{A} : A \in \mathcal{A}\}$$

is an algebra in $\beta\mathbb{N}$. This property is equivalent to properties (i),...,(iv) of Proposition 9.

It should be observed that $\beta\mathbb{N}$ is unique in the following sense: if a compactification $\overline{\mathbb{N}}$ of $\mathbb{N}$ satisfies any one of the listed conditions, then there exists a homeomorphism of $\beta\mathbb{N}$ onto $\overline{\mathbb{N}}$ that leaves $\mathbb{N}$ pointwise fixed.

As a consequense of property (i) we obtain:

The identity mapping $\imath : \mathbb{N} \to \beta\mathbb{N}$ is a continuous monomorphism, which sends $\mathbb{N}$ onto a dense subset of $\beta\mathbb{N}$, such that the adjoint homomorphism

$$\imath^\star : C(\beta\mathbb{N}) \to C^b(\mathbb{N})$$

$$\imath^{\star}(\overline{f}) = \overline{f} \circ \imath$$

maps $C(\beta\mathbb{N})$ isomorphically and isometrically (relative to the uniform metric) onto $C^b(\mathbb{N})$.

We are now in position to formulate the following fundamental

Proposition 11. *Let $\mathcal{A}$ be an algebra in $\mathbb{N}$ and $\delta : \mathcal{A} \to [0,\infty)$ be a content on $\mathcal{A}$, (i.e. a finitely additive measure). Then the map*

$$\bar{\delta} : \bar{\mathcal{A}} \to [0,\infty)$$

$$\bar{\delta}(\bar{A}) = \delta(A)$$

is σ-additive on $\bar{\mathcal{A}}$ and can uniquely be extended to a measure on the minimal σ-algebra $\sigma(\bar{\mathcal{A}})$ over $\bar{\mathcal{A}}$.

Proof. Obviously $\bar{\delta}$ is a content on $\bar{\mathcal{A}}$. Therefore we only have to show that $\bar{\delta}$ is continuous from above at the empty set $\emptyset$. Suppose $\{\overline{A_n}\}, \overline{A_n} \in \bar{\mathcal{A}}$, is a monotone decreasing sequence converging to $\emptyset$. Then, by the compactness of $\beta\mathbb{N}$, there exists $n_0 \in \mathbb{N}$ such that $\overline{A_n} = \emptyset$ for all $n \geq n_0$ and thus Proposition 11 holds.

The extension of $\bar{\delta}$ is also denoted by $\bar{\delta}$. We remark, as an immediate implication of the above construction

Proposition 12.

(i) *Every finitely additive function on an algebra $\mathcal{A}$ in $\mathbb{N}$ can be extended to a finitely additive function on the algebra of all subsets of $\mathbb{N}$.*

(ii) *Every linear functional on the vector space $\mathcal{E}(\mathcal{A})$ can be extended to a linear functional on l^{∞}.*

In the following we shall concentrate on the following topics

- candidates for measures
- spaces of arithmetic functions
- integration theory for uniformly $\mathcal{A}_\delta$-summable functions
- applications to spaces of almost-even, limit-periodic, almost periodic and almost multiplicative functions, and measure preserving systems.

We should have in mind that these results can be generalized in many directions.

Candidates for measures

Let $\Gamma = (\gamma_{nk})$ be a Toeplitz matrix, i.e. an infinite matrix $\Gamma = (\gamma_{nk})_{n,k\in\mathbb{N}}$ with non-negative real elements γ_{nk} satisfying the following conditions:

$$(i) \qquad \sup_n \sum_{k=1}^{\infty} \gamma_{nk} < \infty,$$

$$(ii) \quad \gamma_{nk} \to 0 \quad (n \to \infty, k \text{ fixed}),$$

$$(iii) \qquad \sum_{k=1}^{\infty} \gamma_{nk} \to 1 \quad (n \to \infty).$$

For a given Toeplitz matrix Γ we define $\delta_\Gamma(A)$ for $A \subset \mathbb{N}$ by

$$\delta(A) := \delta_\Gamma(A) := \lim_{n\to\infty} \sum_{k=1}^{\infty} \gamma_{nk} 1_A(k)$$

if the limit exists. Then, if $\mathcal{A}_\delta$ is an algebra in $\mathbb{N}$ such that $\delta(A)$ exists for all $A \in \mathcal{A}_\delta$ the above construction leads to the probability space $(\beta\mathbb{N}, \sigma(\bar{\mathcal{A}}_\delta), \bar{\delta})$. We observe that

$$||f|| := ||f||_\Gamma := \limsup_{n\to\infty} \sum_{k=1}^{\infty} \gamma_{nk} |f(k)|$$

defines a *seminorm* on the space of functions f for which $||f|| < \infty$.
Remark 5. Toeplitz showed that (i),(ii) and (iii) characterize all those infinite matrices which map the linear space of convergent sequences into itself, leaving the limits of each convergent sequence invariant.

Examples.

(i) Choosing

$$\gamma_{nk} = \begin{cases} \frac{1}{n} & \text{if } k \le n, \\ 0 & \text{if } k > n \end{cases}$$

defines Cesaro's summability method and leads to *asymptotic density* and to the seminorm

$$||f|| := \limsup_{n\to\infty} n^{-1} \sum_{k\le n} |f(k)|.$$

(ii) If we put

$$\gamma_{nk} := \begin{cases} \frac{1}{\log n} \cdot \frac{1}{k} & \text{if } k \le n, \\ 0 & \text{if } k > n, \end{cases}$$

we obtain *logarithmic density* with the seminorm

$$||f|| = \limsup_{n\to\infty} \frac{1}{\log n} \sum_{k\le n} \frac{|f(k)|}{k}.$$

(iii) Let $\{I_n\}$ be a sequence of non-empty intervals in $\mathbb{N}$, $I_n = [a_n, b_n]$ such that $b_n - a_n \to \infty$ if $n \to \infty$. We define

$$\gamma_{nk} = \begin{cases} \frac{1}{b_n - a_n} & \text{if } k \in I_n, \\ 0 & \text{otherwise.} \end{cases}$$

If $A \subset \mathbb{N}$ is given and, for some sequences $\{I_n\}$ of such intervals, the limit

$$\delta(A) := \lim_{n\to\infty} \frac{|(A \cap I_n)|}{b_n - a_n}$$

exists, we say that A possesses a *Banach-density*.

(iv) Let $g: \mathbb{N} \to \mathbb{R}^+$ be a nonnegative function with $g(1) > 0$. We put

$$\gamma_{nk} = \begin{cases} \left(\sum\limits_{m\le n} g(m)\right)^{-1} \cdot g(k) & \text{if } k \le n, \\ 0 & \text{if } k > n \end{cases}$$

and assume that $\gamma_{nk} \to 0$ as $n \to \infty$ (k fixed). If the limit

$$M_g(f) := \lim_{n\to\infty} \left(\sum_{m\le n} g(m)\right)^{-1} \sum_{k\le n} f(k)g(k)$$

exists we say that f *possesses a mean-value with weight* g and denote this mean by $M_g(f)$.

§6. Spaces of arithmetic functions and integration theory

Let δ be a set function defined by some Toeplitz matrix Γ and let $\mathcal{A} = \mathcal{A}_\delta$ be an algebra in $\mathbb{N}$ such that $\delta(A)$ is defined for all $A \in \mathcal{A}$, i.e. if $\Gamma = (\gamma_{nk})$,

$$\delta(A) := \lim_{n\to\infty} \sum_{k=1}^{\infty} \gamma_{nk} 1_A(k)$$

exists for every $A \in \mathcal{A}$. Further, let $||\cdot|| = ||\cdot||_\Gamma$ be the corresponding seminorm. Then we introduce the following spaces.

Definition 2. Denote by $\mathcal{L}^{\star 1}(\mathcal{A})$ the $||\cdot||$-closure of $\mathcal{E}(\mathcal{A})$. A function $f \in \mathcal{L}^{\star 1}(\mathcal{A})$ is called uniformly $(\mathcal{A})$-summable. By $L^{\star 1}(\mathcal{A})$ we denote the quotient space $\mathcal{L}^{\star 1}(\mathcal{A})$ modulo null-functions (i.e. functions f with $||f|| = 0$).

Definition 3.

(i) A nonnegative arithmetic function f is called $\mathcal{A}$-*measurable* in case each truncation $f_K = \min(K, f)$ lies in $\mathcal{L}^{\star 1}(\mathcal{A})$ and f is tight, i.e. for every $\varepsilon > 0$ the estimate

$$\limsup_{n\to\infty} \sum_{\substack{k=1 \\ |f(k)|>K}}^{\infty} \gamma_{nk} < \varepsilon$$

holds for some K.

(ii) A real-valued arithmetic function is called $\mathcal{A}$-*measurable* in case its positive and negative parts f^+ and f^- are $\mathcal{A}$-measurable.

(iii) A complex-valued arithmetic function f is called $\mathcal{A}$-*measurable* in case Ref, Imf are $\mathcal{A}$-measurable. The space of all $\mathcal{A}$-measurable functions is denoted by $\mathcal{L}^\star(\mathcal{A})$. Further we define $L^\star(\mathcal{A})$ as $\mathcal{L}^\star(\mathcal{A})$ modulo null-functions, i.e. functions f for which $\delta(\{m : f(m) \neq 0\}) = 0$.

A first consequence of Proposition 11 is that, for all $s \in \mathcal{E}(\mathcal{A})$,

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} \gamma_{nk} s(k) = \int_{\beta\mathbb{N}} \bar{s}\, d\bar{\delta},$$

where $\bar{s} : \beta\mathbb{N} \to \mathbb{C}$ denotes the extension of s.
Starting from this we consider measurable and integrable functions on the probability space $(\beta\mathbb{N}, \sigma(\bar{\mathcal{A}}), \bar{\delta})$ and relate these to the functions from $\mathcal{L}^\star(\mathcal{A})$.

The probability space $(\beta\mathbb{N}, \sigma(\bar{\mathcal{A}}), \bar{\delta})$ leads to the wellknown space

$$L(\bar{\delta}) := L(\beta\mathbb{N}, \sigma(\bar{\mathcal{A}}), \bar{\delta})$$

$$= \{\overline{f} : \beta\mathbb{N} \to \mathbb{C}, \sigma(\bar{\mathcal{A}}) - \text{measurable}\} \text{ modulo null-functions}$$

and

$$L^1(\bar{\delta}) := L^1(\beta\mathbb{N}, \sigma(\bar{\mathcal{A}}), \bar{\delta})$$

$$= \{\overline{f} : \beta\mathbb{N} \to \mathbb{C}, ||\overline{f}|| < \infty\} \text{ modulo null-functions}$$

with norm

$$||\overline{f}|| := \int_{\beta\mathbb{N}} |\overline{f}| \, d\bar{\delta}.$$

A connection between the spaces $\mathcal{L}^\star(\mathcal{A})$ and $\mathcal{L}^{\star 1}(\mathcal{A})$ and the spaces L and L^1, respectively, is given by

Proposition 13.

(i) *There exists a vector-space isomorphism*

$$^-\colon L^\star(\mathcal{A}) \to L(\bar{\delta})$$

such that

$$\bar{s} = \imath^{\star -1}(s) \quad \textit{for every} \;\; s \in \mathcal{E}(\mathcal{A}).$$

(ii) *There exists a norm-preserving vector-space isomorphism*

$$^-\colon L^{\star 1}(\mathcal{A}) \to L^1(\bar{\delta})$$

such that

$$\bar{s} = \imath^{\star -1}(s) \quad \textit{for every} \;\; s \in \mathcal{E}(\mathcal{A}).$$

Proof.(i) By Definition 3 we may restrict to nonnegative functions. Assume that $f \in \mathcal{L}^\star(\mathcal{A})$ is nonnegative, and let $\{s_n\}$ be a sequence of nonnegative simple functions from $\mathcal{E}(\mathcal{A})$ which define f (see Definition 3). Then $\bar{s}_n$ converge on $\beta\mathbb{N}$ to a $\bar{\delta}$-measurable function $\overline{f}$, which is finite $\bar{\delta}$-almost everywhere.
Therefore, by reducing modulo null-functions one obtains a well-defined $1-1$ linear map $^-\colon L^\star(\mathcal{A}) \to L(\bar{\delta})$ whose restriction to $\mathcal{E}(\mathcal{A})$ is given by $\imath^{\star -1}$. The map $^-$ preserves the distribution function, which means that the (limit) distribution of $f \in L^\star(\mathcal{A})$ coincides with the distribution of $\overline{f} \in L(\bar{\delta})$. Lastly, in order to show that $^-$ is onto, we choose for a given nonnegative $\overline{f} \in L(\bar{\delta})$ a sequence $\{\bar{s}_n\}$ of simple functions from $\mathcal{E}(\bar{\mathcal{A}})$ such that $\bar{s}_n$ converges to $\overline{f}$ $\bar{\delta}$-everywhere. (This choice is possible because $\sigma(\bar{\mathcal{A}})$ is generated by $\bar{\mathcal{A}}$). The restrictions s_n to $\mathbb{N}$ converge to some $f \in \mathcal{L}^\star(\mathcal{A})$ and (i) is proved for nonnegative functions. The general case follows then immediately. The proof of (ii) runs on the same lines as above. The map $^-$ is constructed in the following way: Given $f \in L^\star(\mathcal{A})$, choose a sequence $\{s_n\}$ of simple functions from $\mathcal{E}(\mathcal{A})$ such that $||f - s_n|| \to 0$ as $n \to \infty$. Then the functions $\bar{s}_n = L^{\star -1}(s_n)$ form a Cauchy sequence in L^1 and the limit $\overline{f}$ is the desired image of f in L^1. These remarks end the proof of Proposition 13.

§7. Applications

Almost even functions

The algebra $\mathcal{A}_1$ introduced in §2 can be defined as the algebra in $\mathbb{N}$ which is generated by the sets

$$A_{p^k} := \{n : p^k || n\}$$

(p prime, $k = 0, 1, \ldots$). The construction of §6 together with the asymptotic density δ leads to the space

$$L(\beta\mathbb{N}, \sigma(\overline{\mathcal{A}}_1), \overline{\delta})$$

which corresponds to the space $L^*(\mathcal{A}_1)$ of *almost-even* functions.

Distribution of additive functions

If a real-valued additive function f is given, we can put

$$f = \sum_p f_p$$

where f_p is defined by

$$f_p(n) = \begin{cases} f(p^k) & \text{if } p^k || n, \\ 0 & \text{otherwise.} \end{cases}$$

Obviously, every f_p is uniformly $\mathcal{A}_1$-summable, and we denote by $\overline{f}_p$ its unique extension to an integrable function on $\beta\mathbb{N}$. Then $\{\overline{f}_p\}_{p \text{ prime}}$ is a set of independent random variables and $\sum_p \overline{f}_p$ converges a.s. if and only if f possesses a limit distribution. This result can be seen as another *a posteriori* justification of the mentioned idea of Kac connected with the role of independence in probabilistic number theory.

Concerning the *renormalization of additive functions* (see Proposition 1) we consider the increasing sequence $\sigma(\bar{\mathcal{A}}_1^{(n)})$ of σ-algebras where $\bar{\mathcal{A}}_1^{(n)}$ is generated by

$$\{\bar{A}_{p^k} : p \leq n,\ k \in \mathbb{N}\}.$$

Obviously

$$\bigcup_{n \in \mathbb{N}} \sigma(\bar{\mathcal{A}}_1^{(n)}) = \sigma(\bar{\mathcal{A}}_1).$$

Centering the independent random variables $\{\overline{f}_p\}$ at expectations leads to the martingale $\{\bar{S}_n\}_{n=1,2,\ldots}$, where

$$\bar{S}_n = \sum_{i=1}^{n} (\overline{f}_{p_i} - \mathbb{E}(\overline{f}_{p_i})).$$

Using the Lindeberg-Levy theorem for martingales (see [23]) one can prove Proposition 1.

Remark 6. In the case of multiplicative functions we proceed in a similar manner. If a real-valued multiplicative function g is given we put

$$g = \prod_p g_p$$

where

$$g_p(n) = \begin{cases} g(p^k) & \text{if } p^k \| n, \\ 1 & \text{otherwise.} \end{cases}$$

The unique extension $\bar{g}_p$ of g_p build a set $\{\bar{g}_p\}$ of independent random variables, and an application of Zolotarev's result [89] concerning the characteristic function of products of random variables gives necessary and sufficient conditions for the convergence of the product $\prod\limits_p \bar{g}_p$ which turns out to be equivalent to the existence of the limit distribution of g.

Erdös-Wintner Theorem

Let $\mathcal{A}_1'$ be the algebra generated by the sets

$$A_p' := \{n : p|n\}$$

and let $\varepsilon_p, \overline{\varepsilon}_p$ denote the characteristic function of A_p' and $\overline{A_p'}$, respectively. Then the real-valued function

$$f = \sum_p f(p)\varepsilon_p$$

satiesfies, since the random variables $\overline{\varepsilon}_p$ are independent (with respect to $\overline{\delta}$) the following assertions

(i) $f = \sum\limits_p f(p)\varepsilon_p$ *possesses a limit distribution*

$\Longleftrightarrow$

(ii) $\overline{f} = \sum\limits_p f(p)\overline{\varepsilon}_p$ *converges $\overline{\delta}$-almost everywhere*

$\Longleftrightarrow$

(iii)

$$\sum_{\substack{p \\ |f(p)|>1}} \overline{\delta}(\overline{\varepsilon}_p), \quad \sum_{\substack{p \\ |f(p)|\leq 1}} \mathbb{E}[f(p)]\overline{\varepsilon}_p, \quad \sum_{\substack{p \\ |f(p)|\leq 1}} Var[f(p)\overline{\varepsilon}_p]$$

converge (Three series theorem)

$\Longleftrightarrow$

(iv) $$\sum_{\substack{p \\ |f(p)|>1}} \frac{1}{p}, \quad \sum_{\substack{p \\ |f(p)|\leq 1}} \frac{f(p)}{p}, \quad \sum_{\substack{p \\ |f(p)|\leq 1}} \frac{f^2(p)}{p} \textit{ converge.}$$

For the implication (i) $\Rightarrow$(ii) we apply Delange's theorem and the equivalence theorem for the sum of independent random variables and the product of the corresponding characteristic functions (c.f. [67], p.263). The equivalence of (ii), (iii) and (iv), respectively, is the Three Series Theorem, whereas the relation (iv) $\Rightarrow$ (i) follows again from Delange's theorem.

Euclid's Theorem

Euclid proved in 300 B.C. "*There are infinitely many primes*". Ribenboim's *The New Book of Prime Number Records* contains eleven proofs of the Euclidean result. His "favorite" is a *topological proof* which is due to Fürstenberg (1955). Here we add a *probabilistic* proof.

For this let $\mathcal{A}'_1$ be again the algebra generated by (A'_p) and let δ be the asymptotic density on $\mathcal{A}'_1$.
We have $\mathbb{N} = \cup A'_{p_i} \cup \{1\}$, where p_i runs through the set of primes and, for every finite set $\mathfrak{I} \subset \{1, 2, \ldots\}$,

$$\beta\mathbb{N} \setminus \bigcup_{i\in\mathfrak{I}} \overline{A'_{p_i}} = \bigcap_{i\in\mathfrak{I}} \left(\beta\mathbb{N} \setminus \overline{A'_{p_i}}\right).$$

Since the $\overline{A'_{p_i}}$ are independent events,

$$\overline{\delta}\left(\bigcap_{i\in\mathfrak{I}} \left(\beta\mathbb{N} \setminus \overline{A'_{p_i}}\right)\right) = \prod_{i\in\mathfrak{I}} \left(1 - \frac{1}{p_i}\right) > 0.$$

The assumption that there are only a finite number of prime numbers leads to a contradiction.

Limit periodic functions

The algebra $\mathcal{A}_2$ already introduced in §2, is generated by the sets

$$A_{l,p^k} := \{l + n : n \in A_{p^k}\}$$

(p prime,k=1,2,...) with $l = 1, 2, ..., p^k$. We choose again the asymptotic density δ as a suitable pseudomeasure and arrive, by the construction of §6, at the space

$$L(\beta\mathbb{N}, \sigma(\overline{\mathcal{A}}_2), \overline{\delta})$$

which is isomorphic to the space $L^*(\mathcal{A}_2)$ of *limit periodic* functions.

Remark

Let $\mathcal{D}$ be the set of the step functions on $\mathcal{A}_2$. Denote by $\mathcal{D}_u$ the completion of $\mathcal{D}$ concerning the sup-norm. Let further S be the set of the multiplicative linear not identically vanishing functions on $\mathcal{D}_u$. The system of the sets $\overline{A}^S, A \in \mathcal{A}_2$ is an algebra. In S we take the weak topology $\mathcal{T}(\mathcal{D}_u)$ with $U_{\in,f_1,\ldots,f_r}(\varphi_0) = \{\varphi \in S : \max_{1\le j\le r} |f_j(\varphi) - f_j(\varphi_0)| < \varepsilon\}$ as neighborhood basis. With this topology is S compact. Let $\overline{\delta}_S(\overline{A}^S) = \delta(A), \delta$ the asymptotic density. $\overline{\delta}_S$ is on $\overline{\mathcal{A}}_2^S = \{\overline{A}_2^S : A \in \mathcal{A}_2\}$ is a premeasure (compactness argument). $\overline{\delta}_S$ can be uniquely extended (to a measure) on $\sigma(\overline{\mathcal{A}}_2^S)$.

Since $\mathcal{A}_2$ is countable we have $\sigma(\overline{\mathcal{A}}_2^S) = \mathcal{B}(S)$, where $\mathcal{B}(S)$ is the σ-algebra of the Borel-sets.

Let $\tau_{\mathcal{A}_2}$ be the canonical mapping from $\beta\mathbb{N}$ onto S (remark $\tau_{\mathcal{A}_2}|_{\mathbb{N}} = id_{\mathbb{N}}$) with $\tau_{\mathcal{A}_2}(\varphi) = \varphi|_{T(\mathcal{A}_2)}$ whereby φ is a multiplicative linear function on $\ell^\infty(\mathbb{N})$ and $T(\mathcal{A}_2)$ denotes the set of the step functions concerning $\mathcal{A}_2$ and S the space of the multiplicative functions on $T(\mathcal{A}_2)_u$. Then $\overline{\delta}_S$ is the image measure of $\overline{\delta}$ constructed above (See [3]).

Almost multiplicative functions

Let f be a multiplicative function which assumes only the values $-1, -0$ and 1, and define the sets

$$A_f^+ := \{n : f(n) = 1\}$$

$$A_f^0 := \{n : f(n) = 0\}$$

and

$$A_f^- := \{n : f(n) = -1\}$$

with characterisic functions f^+, f^0 and f^-, respectively. Obviously

$$\begin{aligned} f^{+} &= \tfrac{1}{2}(|f|+f) \\ f^{0} &= 1 - f^{+} - f^{-} \\ f^{-} &= \tfrac{1}{2}(|f|-f). \end{aligned}$$

We define the algebra $\mathcal{A}_3$ to be the algebra generated by the sets A_f^+, A_f^0, A_f^- for all multiplicative f with $f(\mathbb{N}) \subset \{-1, 0, 1\}$. An arbitary element A of $\mathcal{A}_3$ has a characteristic function which is a linear combination of such multiplicative functions, and thus the asymptotic density $\delta(A)$ exists. Then §6 yields the space

$$L(\beta\,\mathbb{N}, \sigma(\overline{\mathcal{A}}_3), \bar{\delta})$$

which is isomorphic to the space $L^*(\mathcal{A}_3)$ of *almost multiplicative functions.*

Measure preserving system

Let

$$S : \mathbb{N} \to \mathbb{N}$$

$$S(n) = n + 1$$

be the *shift operator* on $\mathbb{N}$, and let $\bar{S}$ be its unique extension to $\beta\mathbb{N}$. Let B be a subset of $\mathbb{N}$ with positive *upper Banach density*, i.e.

$$\limsup_{|I| \to \infty} \frac{|B \cap I|}{|I|} > 0$$

where I ranges over intervals of $\mathbb{N}$. Let the algebra $\mathcal{A}_4$ be generated by the translates

$$\{S^n B : n = 0, 1, 2, \ldots\}.$$

The algebra $\mathcal{A}_4$ is countable, and thus there exists a sequence of intervals $\{I_n\}$, $I_n = [a_n, b_n]$, $b_n - a_n \to \infty$ such that

$$\delta(A) := \lim_{n \to \infty} \frac{|A \cap I_n|}{b_n - a_n}$$

exists for all $A \in \mathcal{A}_4$. Then the extension according to Proposition 13 leads to the measure preserving system

$$(\beta\mathbb{N}, \sigma(\bar{\mathcal{A}}_4), \bar{\delta}, \bar{S}). \tag{9}$$

For this we obtain by a result of Fürstenberg.

Proposition 14. (Fürstenberg [21]) *Let $\bar{\delta}(\bar{B}) > 0$. Then, for any $k > 1$, there exists $n \neq 0$ with*

$$\bar{\delta}(\bar{B} \cap \bar{S}^n \bar{B} \cap \ldots \cap \bar{S}^{(k-1)n} \bar{B}) > 0.$$

This implies the result of Szemeredi [82]

Let $B \subset \mathbb{N}$ possess a positive Banach density. Then B contains arbitrarily long arithmetic progressions.

§8. Further applications: q-multiplicative functions

The starting point of the definition of (classical) multiplicative functions is the unique representation of the natural numbers $n = \prod_{p \in \mathbb{P}} p^{\alpha_p(n)}$, $\alpha_p(n) = \max\{\alpha : p^\alpha | n\}$ as a product of prime numbers. Then $f : \mathbb{N} \to \mathbb{C}$ is called *multiplicative* in case

$$f(n) = \prod_{p \in \mathbb{P}} f(p^{\alpha_p(n)}).$$

Now, let $q \geq 2$ be an integer and $\mathbb{A} = \{0, 1, \ldots, q-1\}$. The *$q$-ary expansion* of some $n \in \mathbb{N}_0$ is defined as the unique sequence $\varepsilon_0(n), \varepsilon_1(n), \ldots$ for which

$$n = \sum_{j=0}^{\infty} \varepsilon_j(n) q^j\,, \quad \varepsilon_j(n) \in \mathbb{A} \tag{10}$$

holds. $\varepsilon_0(n), \varepsilon_1(n), \ldots$ are called the *digits* in the q-ary expansion of n. In fact, $\varepsilon_r(n) = 0$ if $r > \log n / \log q$. A function $f : \mathbb{N}_0 \to \mathbb{C}$ is called *q-multiplicative* if $f(0) = 1$, and for every $n \in \mathbb{N}_0$,

$$f(n) = \prod_{j=0}^{\infty} f(\varepsilon_j(n) q^j). \tag{11}$$

In a recent paper Indlekofer, Lee and Wagner [54] could give a complete characterisation of q-multiplicative uniformly summable functions. They proved

Proposition 15. (see [54]), Theorem 1) *Let f be a q-multiplicative function. Then the following assertions are equivalent.*

(i) *$f \in \mathcal{L}^*$ and $||f||_1 > 0$.*
(ii) *Let $\alpha > 0$. The series*

$$\sum_{r=0}^{\infty} \frac{1}{q} \sum_{a=0}^{q-1} (|f(aq^r)|^\alpha - 1)^2 \tag{12}$$

is convergent, and for some constants $c_1(\alpha), c_2(\alpha) \in \mathbb{R}$, for all R and for some sequence $\{R_i\}, R_i \to \infty$, the inequalities

$$\sum_{r<R} \frac{1}{q} \sum_{a=0}^{q-1} (|f(aq^r)|^\alpha - 1) \leq c_1(\alpha) < \infty \tag{13}$$

and

$$\sum_{r<R_i} \frac{1}{q} \sum_{a=0}^{q-1} (|f(aq^r)|^\alpha - 1) \geq c_2(\alpha) > -\infty \tag{14}$$

hold.
(iii) *$f \in \mathcal{L}^\alpha$ and $||f||_\alpha > 0$ for all $\alpha > 0$.*

The mean behaviour of such functions is given by

Proposition 16. (see [54], Theorem 2) *Let $f \in \mathcal{L}^*$ be a q-multiplicative function and $||f||_1 > 0$. Further, let $q^{R-1} \leq N < q^R, R \in \mathbb{N}$. Then, as $N \to \infty$,*

$$\frac{1}{N} \sum_{n<N} f(n) = \prod_{r<R} \left(1 + \frac{1}{q} \sum_{a=1}^{q-1} (f(aq^r) - 1)\right) + o(1)$$

and, for every $\alpha > 0$,

$$\frac{1}{N} \sum_{n<N} |f(n)|^\alpha = \prod_{r<R} \left(1 + \frac{1}{q} \sum_{a=1}^{q-1} (|f(aq^r)|^\alpha - 1)\right) + o(1).$$

Remark 7. The case $|f| \leq 1$ has been treated by Delange [8].

An immediate consequence of Proposition 16 is the following

Proposition 17. (see [54], Corollary 1) *Let f be q-multiplicative. Then the following assertions hold.*

(i) *Let $f \in \mathcal{L}^*$. If the mean-value $M(f)$ of f exists and is different from zero then the series*

$$\sum_{r=0}^{\infty}\sum_{a=0}^{q-1}(f(aq^r)-1) \tag{15}$$

and

$$\sum_{r=0}^{\infty}\sum_{a=0}^{q-1}|f(aq^r)-1|^2 \tag{16}$$

converge and

$$\sum_{a=0}^{q-1} f(aq^r) \neq 0 \ \text{ for each } r \in \mathbb{N}_0.$$

(ii) *If the series (15) and (16) converge then $f \in \mathcal{L}^*$, the mean-value $M(f)$ of f exists,*

$$M(f) = \prod_{r=0}^{\infty}\left(\frac{1}{q}\sum_{a=0}^{q-1} f(aq^r)\right)$$

and $||f - f_R||_1 \to 0$ as $R \to \infty$, where

$$f_R(n) = \prod_{r \le R} f(\varepsilon_r(n)q^r).$$

(iii) *Let $f \in \mathcal{L}^*$. If the mean-value $M(f)$ of f exists and is different from zero then the mean-value $M(|f|^\alpha)$ of $|f|^\alpha$ exists for each $\alpha > 0$ (and is different from zero).*

The case of mean-value zero is contained in

Proposition 18. (see [54], Corollary 2) *Let $f \in \mathcal{L}^*$ be q-multiplicative. Then the mean-value $M(f)$ of f is zero if and only if $\Pi_R = o(1)$ as $R \to \infty$.*

For $\beta \in \mathbb{R}$ the function $e_\beta : \mathbb{N}_0 \to \mathbb{C}$ defined by $e_\beta(n) := \exp(2\pi i \beta n)$ $(n \in \mathbb{N}_0)$ is q-multiplicative, and the mean value $M(e_\beta)$ of e_β exists.

q-ary almost even functions

Let the algebra $\mathcal{A}_5$ be generated by the sets

$$A_{j,a} := \{n : \varepsilon_j(n) = a\}$$

($j = 0, 1, 2, \ldots ; a = 0, \ldots, q-1$). Every $A \in \mathcal{A}_5$ possesses an asymptotic density $\delta(A)$, and following the assertions of §6 gives the space

$$\mathcal{L}\left(\beta\mathbb{N}, \sigma\left(\overline{\mathcal{A}_5}\right), \overline{\delta}\right)$$

and thus the space $\mathcal{L}^*(\mathcal{A}_5)$ of *q-ary almost even* functions.

Almost periodic functions

We recall that the functions $e_\beta (\beta \in \mathbb{R})$ are q-multiplicative and possess a mean-value.

Now let $\mathcal{C}$ be the family of all half-open intervals in $\mathbb{R}/\mathbb{Z}$ and denote with $\mathcal{A}_6$ the algebra generated by the sets

$$A(\beta, E) := \{n : e_\beta(n) \in E\}$$

$(\beta \in \mathbb{R}/\mathbb{Z}, E \in \mathcal{C})$.

Every $A \in \mathcal{A}_6$ has an asymptotic density and thus the space

$$\mathcal{L}\left(\beta\mathbb{N}, \sigma\left(\overline{\mathcal{A}_6}\right), \overline{\delta}\right)$$

corresponds to the space $\mathcal{L}^*(\mathcal{A}_6)$ of *almost periodic* functions.

Remark 8. In his paper [35] Indlekofer could give a complete characterisation of almost periodic multiplicative functions. Lee [64] used Proposition 17 and 18 to obtain similar results for q-multiplicative functions generalizing results by Spilker [80]. Essentially she could determine the possible values of the Fourier-Bohr spectrum of uniformly summable q-multiplicative functions.

References

[1] G. J. Babu, Probabilistic methods in the theory of arithmetic functions, Lectures in Math., **2**, Macmillan of India, Delhi, 1978.

[2] A. Bakštys, On limit distribution laws for multiplicative arithmetical functions. I (Russian), Litovsk. Mat. Sb., **8** (1968), 5–20.

[3] A. Barat, K.-H. Indlekofer and R. Wagner, On some compactifications of $\mathbb{N}$, preprint.

[4] H. Daboussi, Caractérisation des fonctions multiplicatives p. p. B^λ à spectre non vide, Ann. Inst. Fourier (Grenoble), **30** (1980), 141–166.

[5] ———, Sur les fonctions multiplicatives ayant une valeur moyenne non nulle, Bull. Soc. Math. France, **109** (1981), 183–205.

[6] H. Daboussi and K.-H. Indlekofer, Two elementary proofs of Halász's theorem, Math. Z., **209** (1990), 43–52.

[7] H. Delange, Sur les fonctions arithmétiques multiplicatives, Ann. Sci. École Norm. Sup. (3), **78** (1961), 273–304.

[8] ———, Sur les fonctions q-additives on q-multiplicatives, Acta Arith., **21** (1972), 285–298.

[9] H. Diamond and P. Erdös, Multiplicative functions whose values are uniformly distributed in $(0, \infty)$, In: Proc. Queen's Number Theory, 1979, (ed. P. Ribenboim), Queen's Papers in Pure and Appl. Math., Queen's Univ., Kingston, Ont., 1980, pp. 329–378.

[10] P. D. T. A. Elliott, Probabilistic number theory. I: Mean value theorems. II: Central limit theorems, Springer-Verlag, 1979, 1980.

[11] ———, Mean value theorems for multiplicative functions bounded in mean α-power, $\alpha > 1$, J. Australian Math. Soc. (Ser. A), **29** (1980), 177–205.

[12] P. Erdös, On the density of some sequences of numbers. III, J. London Math. Soc., **13** (1938), 119–127.

[13] P. Erdös and K.-H. Indlekofer, Multiplikative Funktionen auf kurzen Intervallen, J. Reine Angew. Math., **381** (1987), 148–160.

[14] P. Erdös and M. Kac, On the Gaussian law of errors in the theory of additive functions, Proc. Nat. Acad. Sci. U. S. A., **25** (1939), 206–207.

[15] ———, The Gaussian law of errors in the theory of additive number-theoretic functions, Amer. J. Math., **62** (1940), 738–742.

[16] P. Erdös and A. Wintner, Additive arithmetical functions and statistical independence, Amer. J. Math., **61** (1939), 713–721.

[17] J. Fehér, Über die Eindeutigkeits, Zulässigkeits- und (mod 1) Eindeutigkeitsmengen $\mathbb{R}$-additiver Funktionen, Ann. Univ. Sci. Budapest. Sect. Comput., **14** (1994), 33–37.

[18] ———, On integer valued multiplicative and additive arithmetical functions, Ann. Univ. Sci. Budapest. Sect. Comput., **14** (1994), 39–45.

[19] J. Fehér, K.-H. Indlekofer and L. L. Stacho, On sets of uniqueness for completely additive arithmetic functions, Analysis, **16** (1996), 405–415.

[20] H. Fürstenberg, Ergodic behaviour of diagonal measures and a theorem of Szemeredi on arithmetic progressions, J. d'Analyse Math., **31** (1977), 204–256.

[21] ———, Recurrence in Ergodic Theory and Combinational Number Theory, Princton Univ. Press, 1981.

[22] H. Fürstenberg and Y. Katznelson, An ergodic Szemeredi theorem for commuting transformations, J. d'Analyse Math., **34** (1978), 275–291.

[23] P. Gänssler and W. Stute, Wahrscheinlichkeitstheorie, Springer-Verlag, 1977.

[24] J. Galambos, Limit distribution of sums of (dependent) random variables with applications to arithmetical functions, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete, **18** (1971), 261–270.

[25] ———, On the distribution of strongly multiplicative functions, Bull. London Math. Soc., **3** (1971), 307–312.

[26] ———, Distribution of additive and multiplicative functions, In: The theory of Arithmetic Functions, Proceedings of conf., Western Michigan

Univ., 1971, Lecture Notes in Math., **251**, Springer-Verlag, 1972, pp. 127–139.

[27] J. Galambos and K.-H. Indlekofer, Additive functions vanishing outside a given set of primes. Part I: Limiting distribution on short intervals, Publ. Math. Debrecen, **40** (1992), 211–217.

[28] J. Galambos, K.-H. Indlekofer and I. Kátai, A reneval theorem for random walks in multidimensional time, Trans. Mer. Math. Soc., **300** (1987), 759–769.

[29] L. Gillman and M. Jerison, Rings of continuous functions, Springer-Verlag, Heidelberg-New York, 1976.

[30] G. Halász, Über die Mittelwerte multiplikativer zahlentheoretischer Funktionen, Acta. Math. Acad. Sci. Hung., **19** (1968), 365–403.

[31] G. H. Hardy and S. Ramanujan, The normal number of prime factors of a number n, Quart. J. Math., **40** (1917), 76–92.

[32] G. H. Hardy and J. E. Littlewood, Some problems on 'Partitio Numerorum' III: On the expression of a number as a sum of primes, Acta Math., **44** (1923), 1–70.

[33] K.-H. Indlekofer, A mean-value theorem of multiplicative arithmetical function, Math. Z., **172** (1980), 255–271.

[34] ———, Remark on a theorem of G. Halász, Arch. Math., **36** (1980), 145–151.

[35] ———, Some remarks on almost-even and almost-periodic functions, Arch. Math., **37** (1981), 353–358.

[36] ———, Limiting distributions and mean-values of multiplicative arithmetical functions, J. Reine Angew. Math., **328** (1981), 116–127.

[37] ———, On multiplicative arithmetical functions, Colloq. Math. Soc. Janos Bolyai, **34** (1981), 731–748.

[38] ———, Properties of uniformly summable multiplicative functions, Period. Math. Hungar., **17** (1986), 143–161.

[39] ———, Limiting distributions and mean-values of complex-valued multiplicative functions, Prob. Theory and Math. Stat., **1** (1986), 547–552.

[40] ———, Cesáro means of additive functions, Analysis, **6** (1986), 1–24.

[41] ———, Über Grenzverteilungen multiplikativer Funktionen mit logarithmischen Momenten, Lit. Math. Sbornik, **26** (1986), 435–446.

[42] ———, Gleichgradige Summierbarkeit bei verallgemeinerten Momenten additiver Funktionen, Arch. Math., **49** (1987), 508–512.

[43] ———, Über verallgemeinerte Momente additiver Funktionen, Monatshaft Math., **103** (1987), 121–132.

[44] ———, Über Verallgemeinerungen der Turán-Kubilius Ungleichung, Acta Arith., **52** (1989), 67–73.

[45] ———, Limit laws and moments of additive functions in short intervals, Number theory, **1**, Budapest, 1987, 193–220; Colloq. Math. Soc. János Bolyai, **51**, North-Holland, Amsterdam, 1990.

[46] ———, Limiting distributions of additive functions in short intervals, Acta Math. Acad. Sci. Hungar., **56** (1990), 11–22.

[47] ———, On some spaces of arithmetical functions. I, Analysis Math., **18** (1992), 203–221.

[48] ———, A new method in probabilistic number theory, In: Probability Theory and Applications, Math. Appl., **80**, 1992, pp. 299–308.

[49] ———, Remarks on an elementary proof of Hálasz's theorem, Liet. Mat. Rink., **33** (1993), 417–423.

[50] K.-H. Indlekofer and I. Kátai, Momente additiver Funktionen auf der Folge $\{p+1\}$, Liet. Mat. Rink., **28** (1988), 669–679.

[51] ———, Generalized moments of additive functions, J. Number Theory, **32** (1989), 281–288.

[52] ———, Multiplicative functions with small increments. I, II, III, Acta Math. Hungar., **55** (1990), 97–101; Acta Math. Hungar., **56** (1990), 159–164; Acta Math. Hungar., **58** (1991), 121–132.

[53] ———, On the distribution of translates of additive functions, Acta. Math. Hungar., **61** (1993), 343–356.

[54] K.-H. Indlekofer, Y.-W. Lee and R. Wagner, Mean behaviour of uniformly summable q-multiplicative functions, Ann. Univ. Sci. Budapest. Sect. Comput., **25** (2005), 171–194.

[55] K.-H. Indlekofer and N. M. Timofeev, A mean value theorem for multiplicative functions on the set of shifted primes, Ramanujan J., **2** (1998), 153–165.

[56] ———, Estimates for multiplicative functions on the set of shifted primes, Acta Math. Hungar., **81** (1998), 195–221.

[57] ———, On some subsets of the set of shifted primes, Publ. Math. Debrecen, **52** (1998), 429–468.

[58] I. Kátai, On the distribution of arithmetical functions on the set or prime plus one, Compositio Math., **19** (1968), 278, 289.

[59] ———, On the distribution of arithmetical functions, Acta Math. Acad. Sci. Hungar., **20** (1969), 69–87.

[60] ———, Distribution of digits of primes in q-ary canonical form, Acta Math. Hungar., **47** (1986), 341–359.

[61] ———, Distribution of q-additive functions, In: Probability Theory and Apllications, Math. Appl., **80**, 1992, pp. 309–318.

[62] J. Kubilius, Probabilistic methods in the theory of numbers, Transl. Math. Monographs, **11**, Amer. Math. Soc., Providence, R.I., 1964.

[63] E. Landau, Über die Äquivalenz zweier Hauptsätze der analytischen Zahlentheorie, Sitzungsbericht der Kaiserlichen Akad. der Wiss. in Wien, Math. Maturw., 1202^a (1911), 1–16.

[64] Y.- L. Lee, Characterization of almost-periodic q-multiplicative functions, Ann. Univ. Sci. Budapest. Sect. Comput., **22** (2003), 403–411.

[65] B. V. Levin and N. M. Timofeev, On the distribution of values of additive functions, Acta Arith., **26** (1974/75), 333–364.

[66] B. V. Levin, N. M. Timofeev and S. T. Tuliaganov, The distribution of the values of multiplicative functions, Litovsk. Mat. Sb., **13** (1973), 87–100; English transl. in Math. Trans. Acad. Sci. Lithuanian SSR, **13** (1973).

[67] M. Loève, Probability Theory I, Springer-Verlag, 1978.

[68] J. Maddox, Elements of functional analysis, Cambridge Univ. Press, 1970.

[69] J.-L. Mauclaire, Intégration et théorie des nombres, Hermann, Paris, 1986.

[70] E. V. Novoselov, Topological theory of divisibility of integers. (Russian), Elabuzh. Gos. Ped. Inst. Uchen. Zap., **8** (1960), 3–23.

[71] ______, Integration on a bicompact ring and its applications to number theory. (Russian), Izv. Vysš. Učebn. Zaved. Matematika, **1961** (1961), 66–79.

[72] ______, Some formulas connected with a reduced residue system, Dokl. Akad. Nauk SSSR, **143** (1962), 1274–1277; English transl. in Soviet Math. Dokl., **3** (1962).

[73] ______, Foundations of classical analysis and of the theory of analytic functions in a polyadic domain. (Russian), Izv. Vysš. Učebn. Zaved. Matematika, **1963** (1963), 71–88.

[74] ______, A new method in probabilistic number theory, Izv. Akad. Nauk SSSR Ser. Mat. Ser. Mat., **28** (1964), 307–364; English transl. in Amer. Math. Soc. Transl. (2), **52** (1966).

[75] A. G. Postnikov, Introduction to analytic number theory, Transl. Math. Monogr., **68** (1988).

[76] H. Prüfer, Neue Begründung der algebraischen Zahlentheorie, Math. Ann., **94** (1925), 198–243.

[77] P. Ribenboim, The New Book of Prime Number Records, Springer-Verlag, 1996.

[78] W. Schwarz and J. Spilker, Eine Anwendung des Approximationssatzes von Weierstrass-Stone auf Ramanujan-Summen, Nieuw Arch. Wisk. (3), **19** (1971), 198–209.

[79] ______, Arithmetical functions, London Math. Soc. Lecture Note Ser., **184**, Cambridge Univ. Press, 1994.

[80] J. Spilker, Almost-periodicity of g-additive and g-multiplicative functions, Anal. Probab. Methods Number Theory, 2002, pp. 256–264.

[81] E. Szemeredi, On sets of integers containing no four terms in arithmetic progression, Acta Math. Acad. Sci. Hungar., **20** (1969), 89–104.

[82] ______, On sets of integers containing no k elements in arithmetical progressions, Acta. Arith., **27** (1975), 199–245.

[83] N. M. Timoveev, Stable law for additive arithmetic functions, Matem. Zametki, **37** (1985), 465–472.

[84] ______, The Turán-Erdös hypothesis on the distribution of the values of additive functions on the sequences of shifted primes, Acta Arithm., **58** (1991), 113–131.

[85] P. Turán, On the theorem of Hardy and Ramanujan, J. London Math. Soc., **9** (1934), 274–276.

[86] ______, Über einige Verallgemeinerungen eines Satzes von Hardy und Ramanujan, J. London Math. Soc., **11** (1936), 125–133.

[87] A. Wintner, Eratosthenian Averages, Waverly Press, Baltimore, 1943.

[88] E. Wirsing, Das asymtotische Verhalten von Summen über multiplikative Funktionen, II., Acta. Math. Acad. Sci. Hungar., **18** (1967), 411–467.

[89] V. M. Zolotarev, On a general theory of multiplication of independent random variables, Izv. Akad. Nauk, **142** (1962), 788–791; Soviet. Math. (Amer. Math. Soc. Transl.), **3** (1962), 166–170.

Faculty of Computer Science
Electrical Engineering and Mathematics
University of Paderborn
D-33095 Paderborn
Germany