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The dynamics associated with certain digital sequences

Peter Hellekalek [1] and Pierre Liardet [2]

Abstract.

We present measure-theoretical dynamical systems behind certain types of digital sequences. This kind of result is a prerequisite if one sets out to study harmonic properties for this type of sequences with the method of skew-product transformations.

§1. Introduction

This paper is on dynamical aspects of certain types of digital sequences which are based on q-additive functions, including the so-called (t, s)-sequences. In particular, we exhibit the dynamical system behind the Sobol'-Faure sequence ([22, 7]) and many other sequences introduced by H. Niederreiter and C.P. Xing [18, 20].

The background of this paper is the following. In the theory of uniform distribution, two types of sequences play a central role. The first type are the "$n\alpha$"-sequences, which are sequences of the form $(\langle n\alpha \rangle)_{n \geq 0}$, where $\alpha = (\alpha_1, \ldots, \alpha_s)$ a fixed vector. The symbol $\langle \cdot \rangle$ denotes the fractional part, and we let $\langle n\alpha \rangle := (\langle n\alpha_1 \rangle, \ldots, \langle n\alpha_s \rangle)$. The second type of sequences are the "digital (t, s)-sequences". The general principle of their construction is due to Niederreiter [17, 18] and incorporates earlier elementary constructions, the so-called van der Corput sequence and the Halton sequences (see L. Kuipers and H.Niederreiter [11] for details and references).

The dynamical system behind any $n\alpha$-sequence is clearly the translation by α on the s-dimensional torus. The cases of van der Corput and

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Halton sequences are also well-understood (see P. Hellekalek [9]). The dynamics involving more advanced types of digital (t, s)-sequences like the Sobol'-Faure sequence are not known. In this work, we will undertake the first steps towards this open problem. In particular, we will exhibit

- the dynamical systems associated with a class of q-additive functions, including the digital (t, s)-sequences in base q constructed from infinite matrices (see Theorems 7 and 10),
- as a special case, the dynamical system behind the Sobol'-Faure sequence in any prime base (see Corollary 2).

This paper is organized into six parts. The next section provides the necessary notation, definitions and background to state and develop our results. The definitions of (t, m, s)-nets and (t, s)-sequences are presented following Niederreiter [17] and classical examples are given. The section ends with the proof that any (t, s)-sequence is well-distributed, a natural result which has already been observed by J. Dick [4] (see Theorem 1.16, p. 31). Our proof is different from Dick's reasoning.

Section 3 shows the natural interplay between digital (t, s)-sequences and q-additive sequences which take their values in specific abelian groups.

Section 4 exposes the construction of a skew product built over a dynamical system with a so-called *cocycle* map and reviews basic tools for proving ergodicity of a given skew product. The aim is to apply the ergodic machinery, previously used by the authors in several places like [9, 13, 14, 15]. Any suitable q-additive sequence leads to a skew-product transformation which is uniformly quasi-continuous (see definition 6) a fact established and exploited in Section 5 (Theorem 7).

Section 6 furnishes a simple criterion to recognize whenever skew products issuing from q-additive sequences are ergodic (Theorem 10) and applications are given. Our criterion is the dynamical version of the algebraic one given by G. Larcher and H. Niederreiter [12] for digital sequences derived from formal Laurent series. Particular examples end this last part.

§2. Definitions, notations and background

The dynamical systems approach to study statistical properties of a sequence $\xi = (x_n)_n$ in a compact metrizable space X proceeds in two steps: in step (i) one has to find the dynamical system behind the given sequence, and in step (ii) one has to employ properties of this system to discover distribution properties of the sequence.

Step (i) usually starts with a probability space $(\Omega, \mathcal{C}, \nu)$, a $\mathcal{C}$-measurable map $S : \Omega \to \Omega$ preserving the probability ν (*i.e.*, $\nu = \nu \circ S^{-1}$) and a measurable *coding map* $c : \Omega \to X$ such that $x_n = c(S^n(\omega_0))$, $n \geq 0$, where ω_0 is an appropriately chosen element of Ω. Additionally, we require the sequence $(S^n(\omega_0))_n$ to be uniformly distributed with respect to the measure ν and, as a counter part, the sequence $(x_n)_n$ should be uniformly distributed with respect to the image probability $\nu \circ c^{-1}$. This will be the case if the coding map c is continuous.

In this paper, we only consider standard measure-theoretical dynamical systems. These are quadruples $\mathcal{T} = (Y, \mathcal{B}(Y), \mu, T)$ where Y is a compact metrizable space, $\mathcal{B}(Y)$ (simply denoted by $\mathcal{B}$ or omitted if the reference to Y is clear) is the Borel σ-algebra of Y, T is a Borel map and μ is a T-invariant Borel probability measure on Y. Recall that a map $f : Y \to Z$ from Y into a topological space Z is said to be μ-continuous if the set of points of discontinuity of f is μ-negligible. The map $\mathbf{1}_E : Y \to \mathbb{R}$ denotes the characteristic function of the subset E of Y, $\mathbf{1}_E(x) = 1$ if $x \in E$ and $\mathbf{1}_E(x) = 0$ otherwise. The subset E of Y is said to be μ-continuous if its characteristic function is μ-continuous. A point y in Y is said to be (T, μ)-generic (or simply $\mathcal{T}$-generic) if for all continuous maps $g : Y \to \mathbb{R}$, one has

$$\lim_N \frac{1}{N} \sum_{0 \leq n < N} g(T^n y) = \int_X g \, d\mu \,. \tag{1}$$

The dynamical system $\mathcal{T}$ is said to be regular if T is μ-continuous.

Step (ii) usually requires $\mathcal{T}$ to be regular, because the restriction to continuous maps is often too restrictive. Moreover, under the assumption of regularity, the limit (1) still holds for any μ-continuous map g if y is $\mathcal{T}$-generic. Recall that a subset E of Y is said to be a *bounded remainder set* (or a subset with *bounded local discrepancy*) for the sequence $\omega = (T^n y)_n$, if there exists $\beta \in [0, 1]$, called admissible frequency, such that the sequence

$$N \mapsto S_N(E, \omega) = \sum_{0 \leq n < N} (\mathbf{1}_E(T^n y) - \beta) \qquad (N \geq 1)$$

is bounded in N. The number $S_N(E, \omega)$ is, by definition, the *local discrepancy* of the first N points of ω relative to the set E. Another reason to work with regular dynamical systems $\mathcal{T} = (Y, \mathcal{B}(Y), \mu, T)$ is that if E ($\subset Y$) is a μ-continuous bounded remainder set for the sequence $\omega = (T^n y)_n$ where y is $\mathcal{T}$-generic, then there exists G in $L^\infty(Y, \mu)$ such that $\mathbf{1}_E - \mu(E) = G \circ T - G$ (see [15], Theorem 2).

Let $\mathbb{T}^s$ denote the s-dimensional torus, $s \geq 1$, and let λ stand for the normalized Haar measure on $\mathbb{T}^s$. We usually identify $\mathbb{T}^s$ with the s-dimensional unit cube $[0,1)^s$. An *interval* in the unit cube will be, by definition, a box of the form $[u,v) := \Pi_{j=1}^s [u_j, v_j)$ with $0 \leq u_j < v_j \leq 1$ for all indices j. A classical, very successful strategy to construct sequences in the unit cube with low discrepancy is first to find subsets Z of N points such that the local discrepancy

$$S(J,\ Z) = \sum_{z \in Z} (\mathbf{1}_J(z) - \lambda(J))$$

is uniformly bounded for a suitable family of intervals J. This leads to the following, now classical definitions of (t,m,s)-sets and (t,s)-sequences in base q for a given integer $t \in \{0, \ldots, m\}$:

Definition 1. *Let t and m be integers such that $0 \leq t \leq m$. A set Z of $N = q^m$ points in $\mathbb{T}^s$ is a (t,m,s)-net in base q if $S(J,\ Z) = 0$ for every interval J of the form*

$$J = \prod_{j=1}^{s} \left[\frac{a_j}{q^{d_j}}, \frac{a_j + 1}{q^{d_j}} \right)$$

with integers d_j, a_j such that $d_j \geq 0$, $0 \leq a_j < q^{d_j}$ $(1 \leq j \leq s)$ and $\sum_{j=1}^s d_j = m - t$ (i.e., $\lambda(J) = q^{t-m}$).

Definition 2. *An infinite sequence $\omega = (\omega_n)_{n \geq 0}$ in $\mathbb{T}^s$ is called a (t,s)-sequence in base q if, for all integers $k \geq 0$ and $m > t$, the sets*

$$Z(k,m) = \{\omega_n\,;\ kq^m \leq n < (k+1)q^m\}$$

are (t,m,s)-nets in base q.

Readily, any interval J that satisfies Definition 1 is a bounded remainder set for any (t,s)-sequence ω. In fact, for any $N \geq 1$, and any m,

$$|S_N(J,\omega)| < q^m\,.$$

For constructions and properties of (t,s)-sequences we refer the reader to the fundamental papers of Niederreiter [17, 18, 19]. In particular, it is known that there are combinatorial obstructions to the existence of (t,m,s)-nets limiting the construction of (t,s)-sequences (for example, $(0,s)$-sequences do not exist if $s > q$, but for $s \leq q$ and q a prime number, such a sequence exists). Also, general lower bounds on t, related to the base q, are given in [20]. For example $t \geq \frac{s}{q} - \log_q \frac{(q-1)s+q+1}{2}$.

We will employ Niederreiter's building block construction method (see [16]) in a simplified version. To this aim we introduce

(i) $R_q = \{0, 1, \dots, q-1\}$, the set of digits in base q which will be identified with the ring $\mathbb{Z}/q\mathbb{Z}$ of residues mod q.

(ii) $\mathbb{Z}_q$, the totally disconnected compact group of q-adic integers, equipped with its normalized Haar measure μ_q. An element in $\mathbb{Z}_q$ will be represented by an infinite sequence $\alpha = (a_0, a_1, a_2, \dots)$, $a_j \in R_q$ for any $j \geq 0$ (see [5]).

(iii) φ, the radical-inverse map (also called Monna map, see [11]), $\varphi : \mathbb{Z}_q \to [0,1]$,

$$\varphi(a_0, a_1, a_2, \dots) = \sum_{j=0}^{\infty} \frac{a_j}{q^{j+1}}\,.$$

Note that φ is continuous. For any $t \in [0,1)$ there exists a unique element $\hat{t} := (t_0, t_1, t_2, \dots)$ in $\mathbb{Z}_q$ such that $t = \sum_{k=0}^{\infty} t_k q^{-k-1}$ with $t_k \neq q-1$ for all indices k large enough. The map $t \mapsto \hat{t}$ defined on $[0,1)$ is one-to-one, the right inverse of φ and continuous at any point t such that $q^n t \not\equiv 0 \pmod q$ for any integer n.

An integer $n \geq 0$ will be identified with the element $\tilde{n} = (n_0, n_1, \dots)$ of $\mathbb{Z}_q$ via its q-adic representation

$$n \;=\; n_0 + n_1 q + n_2 q^2 + \cdots + n_j q^j + \cdots\,,$$

hence

$$\varphi(\tilde{n}) = \frac{n_0}{q} + \frac{n_1}{q^2} + \frac{n_2}{q^3} + \cdots + \frac{n_j}{q^{j+1}} + \cdots$$

For a given dimension $s \geq 1$, we choose

(iv) bijections $\psi_j^{(i)} : R_q \to R_q$, for $j \geq 0$, $1 \leq i \leq s$ and define the permutations $\psi^{(i)} : \mathbb{Z}_q \to \mathbb{Z}_q$,

$$\psi^{(i)}(a_0, a_1, a_2, \dots) = (\psi_0^{(i)}(a_0), \psi_1^{(i)}(a_1), \psi_2^{(i)}(a_2), \dots)\,. \tag{2}$$

In the sequel we assume that $\psi_j^{(i)}(0) = 0$ for all sufficiently large j and we emphasize this fact by saying that $\psi^{(i)}$ is a *q-digit permutation*. Any q-digit permutation $\psi : \mathbb{Z}_q \to \mathbb{Z}_q$ induces a natural q-digit permutation on $\mathbb{N}$, namely $n \mapsto \sum_{j\geq 0} \psi_j(n_j) q^j$, that we still denote by ψ but we will set $\tilde{\psi}$ to emphasize that the value of ψ is considered in $\mathbb{Z}_q$.

(v) $\mathbb{N} \times \mathbb{N}$-matrices $C^{(1)}, \dots, C^{(s)}$ over R_q.

For $n \in \mathbb{N}$, $\tilde{n} = (n_0, n_1, n_2, \dots)$, we set

$$x_n^{(i)} = \varphi\left(\tilde{\psi}^{(i)}(n) \cdot C^{(i)}\right) \qquad (1 \leq i \leq s)\,. \tag{3}$$

Note that the product, row $\psi^{(i)}(\tilde{n})$ by matrix $C^{(i)}$, is well defined in $\mathbb{Z}_q$. Explicitly, if $C^{(i)} = (c_{k\ell}^{(i)})_{\substack{k\geq 0\\ \ell\geq 0}}$, then

$$\tilde{\psi}^{(i)}(n)\cdot C^{(i)} = \left(\sum_{k=0}^{\infty} \psi_k^{(i)}(n_k)c_{k\ell}^{(i)}\right)_{\ell} .$$

This construction yields a sequence $\omega = (\mathbf{x}_n)_{n\geq 0}$, $\mathbf{x}_n = (x_n^{(1)}, \dots x_n^{(s)})$, in the torus $\mathbb{T}^s$, which is called a digital (t,s)-sequence in base q if ω is a (t,s)-sequence. The integer $t \geq 0$ is the quality parameter of this sequence. The distribution properties of ω depend on the choice of the digit permutations $\psi^{(i)}$ and the choice of the matrices $C^{(i)}$.

Example 1. (Sobol'-Faure sequences) Assume that q is a prime number and let C be equal to the infinite Pascal triangular matrix

$$(4)\quad C=\begin{pmatrix} \binom{0}{0} & 0 & \dots & \dots & \dots & \dots & 0 & \dots \\ \binom{1}{0} & \binom{1}{1} & 0 & \dots & \dots & \dots & 0 & \dots \\ \vdots & \vdots & \ddots & \ddots & & \vdots & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & & & \ddots \quad \ddots & \vdots & \vdots & \vdots \\ \binom{m-2}{0} & \binom{m-2}{1} & \dots & \dots & \binom{m-2}{m-2} & 0 & \vdots & \vdots \\ \binom{m-1}{0} & \binom{m-1}{1} & \dots & \dots & \binom{m-1}{m-2} & \binom{m-1}{m-1} & 0 & \vdots \\ \vdots & \vdots & \dots & \dots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

where each entry is taken modulo q. The sequence

$$(5)\qquad H_q : n \mapsto \left(\varphi(\tilde{n}\cdot C^0), \varphi(\tilde{n}\cdot C^1), \dots, \varphi(\tilde{n}\cdot C^{q-1})\right)$$

will be called the (generalized) Sobol'-Faure sequence in base q on the torus $\mathbb{T}^q$ (see Sobol' [22] for $q = 2$ and Faure [7] for arbitrary prime bases q). Note that $n \mapsto \varphi(\tilde{n})$ is the classical Van der Corput sequence in base q. Actually, H_q is a $(0,q)$-sequence (see [7]).

Discrepancy of (t,s)-sequences ω in base q has been intensively studied by H. Niederreiter in [18]. For example, he gave the following simplified bound for the star discrepancy

$$ND_N^*(\omega) \leq C(s,q)q^t(\log N)^s, \qquad (N \geq 2)$$

where the constant $C(s,q)$ depends only on q and s. We shall need a qualitative result in the following form:

Theorem 1. *Any (t,s)-sequence ω is well-distributed.*

Proof. We show that $\lim_N \left|1/N \sum_{n=k}^{k+N-1} f(\omega_n) - \int_X f\,d\lambda\right| = 0$ for all continuous functions $f : \mathbb{T}^s \to \mathbb{R}$.

Any such f will be uniformly continuous on the compact space $\mathbb{T}^s$. For this reason, for a given $\varepsilon > 0$, there exists a suitable partition of $\mathbb{T}^s$ into q-adic boxes $J_a = \prod_{j=1}^s \left[a_j/q^{d_j}, (a_j+1)/q^{d_j}\right)$ such that, for any index $a = (a_1, \ldots, a_s)$,

$$\forall\, x \in J_a, \quad \left|f(x) - f(v_a)\right| < \varepsilon \tag{6}$$

where $v_a = (a_1/q^{d_1}, \ldots, a_s/q^{d_s})$ denotes the lower left endpoint of the interval J_a. This property can be obtained easily if we choose sufficiently large positive integers d, m, and d_j, $1 \le j \le s$, such that $d_j > d\ \forall j$, $sd > t$, and $m - t = \sum_{j=1}^s d_j$. In other words, we consider sufficiently small boxes J_a to obtain (6).

We observe that

$$\begin{aligned}
&\frac{1}{N}\sum_{n=k}^{k+N-1} f(\omega_n) - \int_{\mathbb{T}^s} f\,d\lambda \\
&= \frac{1}{N}\sum_n\sum_a \Big(f(\omega_n)\mathbf{1}_{J_a}(\omega_n) - f(v_a)\mathbf{1}_{J_a}(\omega_n)\Big) \\
&\quad + \frac{1}{N}\sum_n\sum_a \Big(f(v_a)\mathbf{1}_{J_a}(\omega_n) - f(v_a)\lambda(J_a)\Big) \\
&\quad + \Big(\sum_a f(v_a)\lambda(J_a) - \int_{\mathbb{T}^s} f\,d\lambda\Big),
\end{aligned}$$

where $\sum_n$ stands for $\sum_{n=k}^{k+N-1}$ and $\sum_a$ indicates summation over all $a = (a_1, \ldots, a_s)$, $0 \le a_j < q^{d_j}$, $1 \le j \le s$.

The first and third sums are, in absolute value, both smaller than ε. This is due to (6), and holds independently of k. The absolute value of the second sum is bounded from above by $(2/N)\,||f||_\infty\, q^{2m-t}$. This estimate is also independent from k and follows from

$$\left|\sum_{n=k}^{k+N-1} \left(\mathbf{1}_{J_a}(\omega_n) - \lambda(J_a)\right)\right| < 2q^m$$

for any index a. We obtain the bound

$$\left|\frac{1}{N}\sum_{n=k}^{k+N-1} f(\omega_n) - \int_{\mathbb{T}^s} f\,d\lambda\right| \le 2\varepsilon + \frac{2}{N}\,||f||_\infty\, q^{2m-t},$$

which yields the theorem. □

§3. q-additive sequences and related (t,s)-sequences

We begin by introducing the familiar notion of q-additive functions (or sequences). Let A be an arbitrary but fixed additive abelian group with neutral element 0_A. For any integer n and any element $\alpha \in A$ we set $n{\cdot}\alpha := \sum_{j=1}^{n} \alpha$ and $(-n){\cdot}\alpha = n{\cdot}(-\alpha)$, as usual.

Definition 3. *A sequence $f : \mathbb{N} \to A$ is said to be a q-additive function if*

$$f(0) = 0_A \qquad \textit{and} \qquad f(n) = \sum_{j\geq 0} f(n_j q^j),$$

where $n = n_0 + n_1 q + \cdots + n_j q^j + \ldots$ is the q-adic expansion of the nonnegative integer n. In addition, f is said to be strongly q-additive if $f(aq^j) = a{\cdot}f(q^j)$ for any $j \geq 0$ and $a = 0, 1, \ldots, q-1$.

In case A is a subgroup of $\mathbb{U}$, the group of complex numbers of modulus 1, a q-additive function will be called q-multiplicative, in accordance with common practice. Note that if ψ is a digital permutation of $\mathbb{N}$ and $f : \mathbb{N} \to A$ is a q-additive function, then there is a sequence $g : \mathbb{N} \to A$ periodic of period a power of q, such that the sequence $n \mapsto f(\psi(n)) - g(n)$ is q-additive.

Example 2. *(Sum-of-digits function)* The classical example of a q-additive function is the sum-of-digits function $s_q : \mathbb{N} \to \mathbb{Z}/q\mathbb{Z}$,

$$s_q(n) = \sum_{j\geq 0} n_j \pmod q.$$

Note that s_q is strongly q-additive and, for $q = 2$, is called the Thue-Morse sequence in the literature.

Example 3. *(General 2-additive functions)* Any A-valued 2-additive function f is strongly 2-additive and is determined by the sequence $(f(2^k))_{k\geq 0}$. Readily, any sequence $(\varepsilon_k)_{k\geq 0}$ in $A^{\mathbb{N}}$ determines a unique 2-additive function $f_\varepsilon : \mathbb{N} \to A$ by setting $f(2^k) = \varepsilon_k$. These sequences, also called generalized binary Morse sequences, were introduced by M. Keane [10] from combinatorial and dynamical points of view.

Let $\mathbb{A}_q$ be the compact product group $\mathbb{A}_q = (\mathbb{Z}/q\mathbb{Z})^{\mathbb{N}}$, equipped with its normalized Haar measure $\lambda_{\mathbb{A}_q}$, which is also equal to the Haar measure μ_q on $\mathbb{Z}_q$. From a purely topological point of view, $\mathbb{A}_q$ and $\mathbb{Z}_q$ are the same space and so, using the same formula, the Monna map φ is also defined on $\mathbb{A}_q$. We use the symbol Π_j both for the j-th projection $(a_0, a_1, a_2, \dots) \mapsto a_j$ from $\mathbb{Z}_q$ to $\mathbb{Z}/q\mathbb{Z}$ and the similar j-th projection from $\mathbb{A}_q$ to $\mathbb{Z}/q\mathbb{Z}$.

Example 4. *(Digital sequences and q-additive functions)* Assume that $f : \mathbb{N} \to \mathbb{A}_q$ is a q-additive function and set $f_j = \Pi_j \circ f$. The maps $f_j : \mathbb{N} \to \mathbb{Z}/q\mathbb{Z}$ are q-additive and by definition

$$f(n) = (f_0(n), f_1(n), f_2(n), \dots). \tag{7}$$

Let $\psi^{(i)}$, $i = 1, 2$ denote arbitrary q-digit permutations. Then we associate with f and $\psi = (\psi^{(1)}, \psi^{(2)})$ the sequence $[\psi, f]$ in $\mathbb{T}^2$ given by

$$[\psi, f](n) := \Big(\varphi(\psi^{(1)}(\tilde{n})), \varphi(f(\psi^{(2)}(\tilde{n})))\Big). \tag{8}$$

Going back to the general construction (3), let $\mathcal{F} = \{f_k : \mathbb{N} \to \mathbb{Z}/q\mathbb{Z}\,;\ k \geq 0\}$ be a family of $\mathbb{Z}/q\mathbb{Z}$-valued strongly q-additive functions and let $f : \mathbb{N} \to \mathbb{A}_q$ be the strongly q-additive function (7). The sequence (3) is obtained from (8) by taking $C^{(1)}$ equal to the identity matrix and $C^{(2)}$ equal to the following matrix $C(\mathcal{F})$:

$$C(\mathcal{F}) = \begin{pmatrix} f_0(1) & f_1(1) & \dots & f_k(1) & \dots \\ f_0(q) & f_1(q) & \dots & f_k(q) & \dots \\ \dots & \dots & \dots & \dots & \dots \\ f_0(q^k) & f_1(q^k) & \dots & f_k(q^k) & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

§4. Skew product associated to a q-additive function

4.1. Generalities on skew products

Let $\mathcal{T} = (Y, \mathcal{B}(Y), \mu, T)$ be a standard dynamical system that we assume to be invertible, let A be a compact metrizable abelian group equipped with its Haar measure λ_A and let $\gamma : Y \to A$ be a Borel function. The transformation $T_\gamma : Y \times A \to Y \times A$ is given by

$$T_\gamma(y, a) = (Ty, a + \gamma(y)). \tag{9}$$

Readily, T_γ is invertible and the product measure $\mu \otimes \lambda_A$ is T_γ-invariant. In fact, this constitutes a measure-theoretical dynamical system $\mathcal{T}_\gamma = (Y \times A, \mathcal{B}(Y \times A), \mu \otimes \lambda_A, T_\gamma)$, called skew product over $\mathcal{T}$ with γ. Iterating T_γ introduces the cocycle $\Gamma : (n, y) \mapsto \gamma_n(y)$ $((n, y) \in \mathbb{Z} \times Y)$ for T with

$$\gamma_n(y) := \begin{cases} \gamma(y) + \dots + \gamma(T^n y) & \text{if } n > 0, \\ 0_A & \text{if } n = 0, \\ \gamma(T^n y) + \gamma(T^{n+1} y) + \dots + \gamma(T^{-1} y) & \text{if } n < 0, \end{cases} \tag{10}$$

so that $T_\gamma(y,a) = (T^n y, a+\gamma_n(y))$. The cocycle Γ is said to be a coboundary for T, if there exists a Borel map $c : Y \to A$ such that for all rational integers n,

$$\Gamma(n,y) = c(T^n y) - c(y) \quad \mu\text{-a.e.}$$

There are at least three general basic tools to prove the ergodicity of the skew product $\mathcal{T}_\gamma$, whenever T is already ergodic. The first tool involves functional equations associated with all nontrivial characters of A, and has emerged previously in the literature in ([1, 3, 23, 24]). The general case where A is not commutative, can be found in [14]. Actually, the criterion can be stated as follows:

Theorem 2. *Assume that $\mathcal{T}$ is ergodic. The skew product T_γ is ergodic if and only if for any nontrivial character χ of A the functional equation*

$$V(y) = \chi(\gamma(y))V(T(y)) \quad \mu\text{-a.e.} \tag{11}$$

has no Borel map solution $V : Y \to \mathbb{U}$.

The second tool is related to the notion of essential values introduced by K. Schmidt [21]:

Definition 4. *An element a in A is called an essential value of the cocycle Γ for T associated with γ if for any Borel set B of Y such that $\mu(B) > 0$ and for any neighborhood V of a, one has*

$$\mu\Big(\bigcup_{n\in\mathbb{Z}} (B \cap T^{-n}B \cap \{y \in Y\,;\ \gamma_n(y) \in V\})\Big) > 0\,. \tag{12}$$

To simplify, we will speak of essential values of γ for T. Note that condition (12) can be replaced by the following one which might appear to be stronger but is in fact equivalent:

$$\bigcup_{n\in\mathbb{Z}} (B \cap T^{-n}B \cap \{y \in Y\,;\ \gamma_n(y) \in V\} \neq \emptyset\,. \tag{13}$$

The set of essential values forms a closed subgroup $E(\gamma)$ of A and, as a corollary of a general result of K. Schmidt [21] we have:

Theorem 3. *$\mathcal{T}_\gamma$ is ergodic if and only if $E(\gamma) = A$.*

We need also the following result ([21], Theorem 3.9):

Theorem 4. *With the above notations, Γ is a coboundary for T if and only if $E(\gamma) = \{0_A\}$.*

The third tool is related to the notion of generic points for standard measure-theoretic dynamical systems.

Theorem 5. *Assume that $\mathcal{T}$ is regular, ergodic, and $\gamma : Y \to A$ is μ-continuous. Then $\mathcal{T}_\gamma$ is ergodic if and only if for any $\mathcal{T}$-generic point $y \in Y$ and any point a in A, the pair (y, a) is $\mathcal{T}_\gamma$-generic or, equivalently, the sequence*

$$n \mapsto (T^n y, a + \gamma_n(y))$$

is uniformly distributed with respect to the product measure $\mu \otimes \lambda_A$.

Proof. See [13], Theorem (2.1).

4.2. Dynamics from q-additive functions

In order to construct the dynamical system associated with a q-additive function, we will introduce successively the q-adic odometer τ_q, the discrete derivative of a sequence, the cocycle associated with a given q-additive sequence and finally the skew product above τ_q using such a cocycle.

Recall that the q-adic odometer $\tau_q = (\tau, \mathbb{Z}_q, \mu_q)$ is defined by

$$\tau(x) = x + 1 \qquad (x \in \mathbb{Z}_q)\,.$$

The map τ is continuous and it is well-known that τ_q is uniquely ergodic and that μ_q is the unique invariant measure.

A k-tuple $w = (w_0, \cdots, w_{k-1})$ in $\{0, 1, \ldots, q-1\}^k$ will also be denoted as a word $w_0 \cdots w_{k-1}$ of length $|w| := k$. The *cylinder set* $[w]$ defined by w in $\mathbb{Z}_q$ is given by

$$[w] = \{x \in \mathbb{Z}_q\,;\ x_m = w_j,\ 0 \le j < k\},$$

its length being the length of w. In analogy to this definition, we introduce the cylinder set defined by w in $\mathbb{N}$:

$$[w; \mathbb{N}] = \{n \in \mathbb{N}\,;\ n = n_0 + n_1 q + n_2 q^2 + \ldots,\ n_j = w_j,\ 0 \le j < k\}\,.$$

If $w = \wedge$, the empty word, one puts $[\wedge] = \mathbb{Z}_q$ and $[\wedge; \mathbb{N}] = \mathbb{N}$.

Definition 5. *(Discrete derivative) Let A be an abelian group and let $f : \mathbb{N} \to A$ be any sequence. We define the* discrete derivative Δf *of f as the sequence*

$$n \ \mapsto\ \Delta f(n) = f(n+1) - f(n)\,.$$

Let $f : \mathbb{N} \to A$ be any q-additive function. It is easy to see that the discrete derivative Δf is constant on the cylinder set $[(q-1)^k 0; \mathbb{N}] = [(q-1, q-1, \ldots, q-1, i); \mathbb{N}]$ of length $k+1$, $0 \le i < q-1$ and $k \ge 0$, and takes the value

$$c_{k,i} := f((i+1)q^k) - f(q-1) - \cdots - f((q-1)q^{k-1} - f(iq^k) .$$

The set $\mathbb{Z}_q$ can be partitioned in the form

$$\mathbb{Z}_q = \{-1\} \cup \bigcup_{\substack{k \ge 0 \\ 0 \le i < q-1}} [(q-1)^k i],$$

where $[(q-1)^0 0]$ denotes the cylinder set $[0]$ and -1 stands for the element $(q-1, q-1, q-1, \ldots)$. As a consequence, the sequence Δf can be extended to all of $\mathbb{Z}_q$. We use the same notation for this extension:

$$\Delta f(x) = \begin{cases} c_{k,i} & \text{if } x \in [(q-1)^k i],\ 0 \le i < q-1, \\ 0_A & \text{if } x = -1 . \end{cases}$$

If A is a topological group, $\Delta f : \mathbb{Z}_q \to A$ is continuous in every point of $\mathbb{Z}_q \setminus \{-1\}$. The value 0_A of Δf at -1 will be replaced eventually by $\lim_{x \to -1, x \neq -1} \Delta f(x)$, if this limit exists. We associate with Δf the cocycle $(n, x) \mapsto \Delta_n f(x)$ for τ defined by

$$\Delta_n f(x) = \begin{cases} \Delta f(x) + \cdots + \Delta f(x+n-1) & \text{if } n > 0 , \\ 0_A & \text{if } n = 0 , \\ \Delta f(x+n) + \Delta f(x+n+1) + \cdots + \Delta f(x-1) & \text{if } n < 0 , \end{cases}$$

and the skew product

$$\tau_f : \mathbb{Z}_q \times G \to \mathbb{Z}_q \times A,$$
$$\tau_f(x, a) = (x+1,\ a + \Delta f(x)) .$$

In particular, $\tau_f^n(x, a) = (x+n, a + \Delta_n f(x))$ for any $n \in \mathbb{Z}$, and

$$\tau_f^n(0, 0_A) = (n, f(n)),$$

if $n \ge 0$. Note that $f(n)$ is not defined for negative integers n.

From now on, we assume that A is a compact metrizable abelian group. The above skew product $\tau_f := (\mathbb{Z}_q \times A, \mathcal{B}(\mathbb{Z}_q \times A), \mu_q \otimes \lambda_A, \tau_f)$ is regular. In fact, the points of discontinuity of τ_f belong to $\{-1\} \times A$. Due to that fact, Theorem 5 can be rephrased as follows:

Corollary 1. *τ_f is ergodic if and only if any point (x, a) of $\mathbb{Z}_q \times A$ is $(\tau_f, \mu_q \otimes \lambda_A)$-generic. In particular, if τ_f is ergodic, then τ_f is uniquely ergodic.*

Proof. If τ_f is ergodic, the individual ergodic theorem says that $\mu_q \otimes \lambda_A$-almost every couple (x, a) is $(\tau_f, \mu_q \otimes \lambda_A)$-generic. We want a stronger result. Theorem 5 says that for any τ-generic point x and any $a \in A$, the sequence (x, a) is τ_f-generic, but all points in $\mathbb{Z}_q$ are τ-generic. Reciprocally, assume that all points (x, a) are $(\tau_f, \mu_q \otimes \lambda_A)$-generic and let ν be a Borel probability measure, τ_f-invariant and ergodic. Since $\mathbb{Z}_q \times A$ is compact and metrizable, by a classical result ν-almost all points are (τ_f, ν)-generic. Therefore $\int_X f\,d\nu = \int_X f\,d(\mu_q \otimes \lambda_A)$ for all continuous maps $f : X \to \mathbb{R}$, so that $\nu = \mu_q \otimes \lambda_A$, proving that τ_f is uniquely ergodic. $\square$

Remark 1. As long as we are concerned with general q-additive sequences $f : \mathbb{N} \to A$, the introduction of a q-digit permutation ψ is irrelevant because the sequence $f \circ \psi$ is also q-additive up to a periodic sequence of period a power of q. More precisely, let r be an integer such that $\psi_j(0) = 0$ if $j \geq r$ and define $g : \mathbb{N} \to A$ by $g(n) = \sum_{j=0}^{r-1} f(\psi_j(n_j)q^j)$. The sequence g is periodic of period q^r in the sense that $g(n + q^r) = g(n)$ for any integer n. Then, $F = f \circ \psi - g$ is q-additive. The map g extends naturally to a unique continuous map G on $\mathbb{Z}_q$. In fact $G(x) = g\big(\sum_{j=0}^{r} x_j q^j\big)$. Consequently, the discrete derivative of $f \circ \psi$ extends to a continuous map on $\mathbb{Z}_q \setminus \{-1\}$ such that $\Delta(f \circ \psi) = \Delta F + G(x+1) - G(x)$. The skew product $\tau_{f\circ\psi}$ related to $\Delta(f \circ \psi)$ is conjugate to the skew product τ_F by the map $(x, a) \mapsto (x, a - G(x))$. Hence, from a dynamical point of view, there is no restriction to assume that $\psi_j(0) = 0_A$ for all j.

The following result says that a q-additive sequence is well-distributed as soon as the sequence is uniformly distributed. The proof can be found in the literature for particular situations only but was finally proved for general scales of numeration systems by G. Barat in his Doctoral Thesis [2]. We shall give a simple proof for constant base q, inspired by [15].

Theorem 6. *If a q-additive sequence $f : \mathbb{N} \to A$ is uniformly distributed (for the Haar measure of A), then the sequence is well-distributed.*

Proof. By assumption, for any nontrivial character χ of A one has $\lim_N \frac{1}{N}\sum_{n=0}^{N-1} \chi(f(n)) = 0$ and we have to show that

$$\limsup_{N} \sup_{k\geq 0} \left| \frac{1}{N} \sum_{n=k}^{k+N-1} \chi(f(n)) \right| = 0. \tag{14}$$

For any ε, $0 < \varepsilon < 1$, choose $M \geq 1$ such that $|\sum_{n=0}^{q^M-1} \chi(f(n))| \leq q^M\varepsilon/4$. For $N \geq 4q^M/\varepsilon$ and any $k \geq 0$ let a and b the integers defined by the inequalities $aq^M \leq k < (a+1)q^M \leq bq^M \leq N+k < (b+1)q^M$. Put $S(u,v) := \sum_{n=u}^{v-1} \chi(f(n))$ for short and note that $S(rq^M, (r+1)q^M) = \chi(f(rq^M))S(0,q^M)$. From the equality

$$S(k, N+k) = -S(aq^M, k) + S(aq^M, bq^M) + S(bq^M, N+k)$$

and the inequality $(b-a)q^M \leq N + q^M$, we obtain

$$\begin{aligned} |S(k,k+N)| &\leq 2q^M + (b-a)|S(0,q^M)| \\ &\leq 2q^M + (b-a)q^M\varepsilon/4, \\ &< \varepsilon N(1/2+1/4+1/4) = \varepsilon N. \end{aligned}$$

Assertion (14) follows. From now on, it is a standard routine to conclude that $n \mapsto f(n)$ is well-distributed. □

§5. Applications to (t,s)-sequences

In this section we prove the following general result.

Theorem 7. *For each $i = 1, \ldots, s-1$ let $f^{(i)} : \mathbb{N} \to \mathbb{A}_q$ be a q-additive function and let $\psi^{(i)}$ be a q-digit permutation. Set $f := (f^{(1)}, \ldots, f^{(s-1)})$, $\psi = (\psi^{(1)}, \ldots \psi^{(s-1)})$ and*

$$f \bullet \psi : n \mapsto (f^{(1)} \circ \psi^{(1)}(n), \ldots, f^{(s-1)} \circ \psi^{(s-1)}(n)).$$

Assume there exists an integer $t \geq 0$ such that

$$n \mapsto (\varphi(\tilde{n}), \varphi(f^{(1)} \circ \psi^{(1)}(n)), \ldots, \varphi(f^{(s-1)} \circ \psi^{(s-1)}(n)))$$

is a (t,s)-sequence. Set $A = (\mathbb{A}_q)^{s-1}$. Then the skew product

$$\tau_{f\bullet\psi}(x,a) = (\tau x, a + \Delta(f \bullet \psi)(x)) \quad ((x,a) \in \mathbb{Z}_q \times A)$$

is uniquely ergodic.

Before beginning the proof we introduce the notion of uniform quasi-continuous transformations

Definition 6. *Let X be a compact metrizable space. A Borel transformation $T : X \to X$ is said to be uniformly quasi-continuous if the set D of its discontinuity points verifies the following property: for all*

$\varepsilon > 0$, there exists a continuous map $g : X \to [0,1]$ and an integer $N(\varepsilon)$ such that $\mathbf{1}_D \leq g$ and for any integer $N \geq N(\varepsilon)$, the inequality

$$\sup_{x \in X} \frac{1}{N} \sum_{0 \leq n < N} g \circ T^n(x) \leq \varepsilon \tag{15}$$

holds.

If T is uniformly quasi-continuous, then T is μ-continuous for any T-invariant measure μ. The next result explains the reason why we have introduced this definition.

Lemma 1. *Let X and $T : X \to X$ be as in the definition above. Let $\xi \in X$ be such that the sequence $(T^n\xi)_{n \geq 0}$ is dense in X and assume that for any continuous map $f : X \to \mathbb{R}$, the sequence of means*

$$M_k(f) : N \mapsto \frac{1}{N} \sum_{0 \leq n < N} f \circ T^{k+n}(\xi)$$

converges uniformly in $k \in \mathbb{N}$. Then, T is uniquely ergodic.

Proof. The lemma is classical if T is continuous. To prove the general case, we introduce the space $\mathcal{C}_{\mathbb{R}}(X)$ of real continuous maps on X, equipped with the supremum norm $||\cdot||_\infty$, and we consider the linear map $L : \mathcal{C}_{\mathbb{R}}(X) \to \mathbb{R}$, $L(f) := \lim_N \frac{1}{N} \sum_{0 \leq n < N} f \circ T^n(\xi)$. Clearly, L is positive, continuous and $L(\mathbf{1}_X) = 1$. By the Riesz representation theorem, there exists a unique positive Borel probability measure ν which represents L. Let us prove that ν is T-invariant. To this aim, it is enough to show that for any non negative continuous map $f : X \to \mathbb{R}$, the equality $\int_X f \, d\nu = \int_X f \circ T \, d\nu$ holds. Let D be the set of discontinuity points of T; the set of discontinuity points of $f \circ T$ is contained in D. Choose an arbitrary $\varepsilon > 0$ and let $g : X \to [0,1]$ be a continuous map such that (15) holds as soon as N is large enough, say $N \geq N(\varepsilon)$. The function

$$f_1 = \max\{f \circ T, 2||f||_\infty g\}$$

takes the value $2||f||_\infty g$ on the open set $U := g^{-1}(]\frac{1}{2}, \infty))$ while the closed set $g^{-1}(1)$ is included in U and contains D. Therefore, f_1 is continuous at any point x of X. By construction, $0 \leq f \circ T \leq f_1 \leq f \circ T + 2||f||_\infty g$ and, after integration,

$$0 \leq \int_X f_1 \, d\nu - \int_X f \circ T \, d\nu \leq 2||f||_\infty \varepsilon.$$

But we also have

$$\begin{aligned} 0 &\le \frac{1}{N}\sum_{0\le n<N} f_1\circ T^n(\xi) - \frac{1}{N}\sum_{0\le n<N} f_1\circ T^{n+1}(\xi) \\ &\le 2||f||_\infty \frac{1}{N}\sum_{0\le n<N} g\circ T^n(\xi). \end{aligned}$$

Passing to the limit, one gets $0 \le \int_X f_1\,d\nu - \int_X f\,d\nu \le 2||f||_\infty\varepsilon$. These inequalities lead to the estimate

$$\Big|\int_X f\,d\nu - \int_X f\circ T\,d\nu\Big| \le 2||f||_\infty\varepsilon\,.$$

Consequently, $\int_X f\,d\nu = \int_X f\circ T\,d\nu$.

Next we show that for any $y\in X$,

$$\lim_N \frac{1}{N}\sum_{0\le n<N} f\circ T^n(y) = \int_X f\,d\nu\,,$$

the convergence being uniform in y. To this goal, let us introduce the set $D_N(y) = \{n\in\mathbb{N}\,;\ 0\le n<N \text{ and } T^n y\in D\}$ and its complementary set $C_N(y) = \{0,1,2,\ldots,N-1\}\setminus D_N(y)$. The choice of g implies that

$$\operatorname{card}(D_N(y)) \le N\varepsilon \tag{16}$$

for any $N\ge N(\varepsilon)$ (and any $y\in X$). Take $\eta>0$; independently of ε and y, there exists $N_1(\eta)$ such that, from the assumption of uniform convergence of the family of sequences $M_k(f)$, the inequality

$$\sup_{k\in\mathbb{N}}\Big|\frac{1}{N}\sum_{0\le n<N} f\circ T^{n+k}(\xi) - \int_X f\,d\nu\Big| \le \eta/4$$

holds if $N\ge N_1(\eta)$. Without lost of generality, we may replace f by $f-\int_X f\,d\nu$. This choice simplifies the proof by making use of the property $\int_X f\,d\nu = 0$. For any integer $M\ge N_1(\eta)$, set

$$D_{M,\varepsilon}(y) = \bigcup_{n\in D_M(y)} (n-N_1(\eta),n]\cap\mathbb{N}\,.$$

The complementary set of $D_{M,\varepsilon}(y)$ in $\{0,1,2,\ldots,M-1\}$ is the union of disjoint intervals, namely $I^\ell_{M,\varepsilon}(y)$, $\ell=1,\ldots,L=L(M,\varepsilon,y)$, with

$$\sum_{\ell=1}^{L}\operatorname{card}\big(I^\ell_{M,\varepsilon}(y)\big) \ge (1-\varepsilon N_1(\eta))M \tag{17}$$

in case $M \geq N(\varepsilon)$. We may force $\varepsilon \leq \frac{1}{2N_1(\eta)}$ in order to have non empty intervals $I^{\ell}_{M,\varepsilon}(y)$ and set (omitting the reference to M, ε and y to simplify the notation)

$$J^{\ell} := I^{\ell}_{M,\varepsilon}(y) \cup (\max I^{\ell}_{M,\varepsilon}(y), \max I^{\ell}_{M,\varepsilon}(y) + N_1(\eta)]\,.$$

Write $J^{\ell} = [a_{\ell}, b_{\ell})$ (interval in $\mathbb{N}$) for short. By construction, $b_{\ell} - a_{\ell} \geq N_1(\eta)$ and $T^{a_{\ell}}(y)$ is a continuity point of the maps T^m for integers $m \geq 0$ such that $a_{\ell} + m < b_{\ell}$. Hence, there exists a neighborhood W_{ℓ} of $T^{a_{\ell}}(y)$ such that for any z in W_{ℓ} and any integer m, $0 \leq m < b_{\ell} - a_{\ell}$, one has

$$\left| f \circ T^{a_{\ell}+m}(y) - f(T^m z) \right| \leq \frac{\eta}{4}.$$

For any integer $\ell = 1, 2, 3, \ldots, L$, select an integer k_{ℓ} satisfying $T^{k_{\ell}}(\xi) \in W_{\ell}$. This choice is allowed due to the fact that $\{T^n \xi\,;\; n \in \mathbb{N}\}$ is dense in X. Summing the above inequalities leads to

$$\Big| \sum_{n \in J^{\ell}} f \circ T^n(y) - \sum_{m=0}^{b_{\ell}-a_{\ell}-1} f(T^{m+k_{\ell}}(\xi)) \Big| \leq (b_{\ell} - a_{\ell}) \frac{\eta}{4}$$

while

$$\Big| \sum_{m=0}^{b_{\ell}-a_{\ell}-1} f \circ T^{m+k_{\ell}}(\xi) \Big| \leq (b_{\ell} - a_{\ell}) \frac{\eta}{4}\,.$$

Taking into account (17), we cut the sum $S := \sum_{n=0}^{M-1} f \circ T^n(y)$ into three parts to obtain

$$\begin{aligned} |S| \;&\leq\; \sum_{\ell=1}^{L} \Big| \sum_{n \in J^{\ell}} f \circ T^n(y) - \sum_{m=0}^{b_{\ell}-a_{\ell}-1} f(T^{m+k_{\ell}}(\xi)) \Big| \\ &\qquad + \sum_{\ell=1}^{L} \Big| \sum_{m=0}^{b_{\ell}-a_{\ell}-1} f \circ T^{m+k_{\ell}}(\xi) \Big| + \varepsilon N_1(\eta) M ||f||_{\infty} \\ &\leq\; \sum_{\ell=1}^{L} (b_{\ell} - a_{\ell}) \frac{\eta}{2} + \varepsilon N_1(\eta) M ||f||_{\infty}. \end{aligned}$$

If we put $\varepsilon = \frac{1}{2(||f||_{\infty}+1)N_1(\eta)}$, then for any $M \geq N(\varepsilon)$ (note that $N(\varepsilon) > N_1(\eta)$ by (17)) we derive

$$\sup_{y \in X} \Big| \frac{1}{M} \sum_{n=0}^{M-1} f \circ T^n(y) \Big| \leq \eta\,.$$

Going back to the general case, we have proved that for any continuous map $f : X \to \mathbb{R}$ the sequence of means $\frac{1}{N}\sum_{n=0}^{N-1} f \circ T^n(y)$ converge uniformly in y to $\int_X f\, d\nu$. It follows that T is uniquely ergodic with the unique invariant measure ν. □

Lemma 2. *Let $A = A_1 \times \cdots \times A_{s-1}$ be a direct product of compact metrizable abelian groups. For any q-additive function $f^{(i)} : \mathbb{N} \to A_i$ and any q-digital permutation $\psi^{(i)} : \mathbb{N} \to \mathbb{N}$ $(0 \leq i \leq s-1)$, the skew product $\tau_{f\bullet\psi}$, where $f \bullet \psi = (f^{(1)} \circ \psi^{(1)}, \ldots, f^{(s-1)} \circ \psi^{(s-1)})$, is uniformly quasi-continuous.*

Proof. The set D of discontinuous points of $\tau_{f\bullet\psi}$ is contained in $\{-1\} \times A$. For any integer $k \geq 1$, the function $g = \mathbf{1}_{[1^k]\times A}$ is continuous and since τ realizes a circular permutation of the q^k cylinder sets in $\mathbb{Z}_q$ of length k, one gets

$$\sup_{x\in\mathbb{Z}_q} \frac{1}{N} \sum_{0\leq n<N} g \circ \tau^n_{f\bullet\psi}(x) \leq \frac{1}{q^k} + \frac{1}{N},$$

proving that $\tau_{f\bullet\psi}$ is uniformly quasi-continuous. □

Proof of Theorem 7

Let $L : [0,1)^s \to \mathbb{Z}_q \times (\mathbb{A}_q)^{s-1}$ be the one-to-one map defined by $L(t_0, t_1, \ldots, t_{s-1}) = (\widehat{t_0}, \widehat{t_1}, \ldots, \widehat{t_{s-1}})$. By construction, the sequence $n \to L(\varphi(\tilde{n}), \varphi(f^{(1)} \circ \psi^{(1)}(n)), \ldots, \varphi(f^{(s-1)} \circ \psi^{(s-1)}(n)))$ is just

$$n \mapsto (\tilde{n}, f \bullet \psi(n)) \tag{18}$$

and satisfies the following property: for any integer $m \geq t$, any s-uple of words $w = (w_0, \ldots w_{s-1})$ on the alphabet $\{0, 1, \ldots, q-1\}$ with $|w_0| + \cdots + |w_{s-1}| = m - t$, and any integer $k \geq 0$, one has

$$\begin{aligned} q^t &= \sum_{kq^m \leq n < (k+1)q^m} \mathbf{1}_{[w_0]\times[w_1]\times\cdots\times[w_{s-1}]}(\tilde{n}, f \bullet \psi(n)) \\ &= q^m \mu_q([w_0]) \cdots \mu_q([w_{s-1}]) . \end{aligned}$$

The linear combinations of characteristic functions $\mathbf{1}_{[w_0]\times[w_1]\times\cdots\times[w_{s-1}]}$ with $\sum_{j=0}^{s-1} |w_j| > t$ being dense in the space of all continuous maps $f : \mathbb{Z}_q \times (\mathbb{A}_q)^{s-1} \to \mathbb{R}$ equipped with the uniform convergence topology, we may easily adapt the proof of Theorem 1 to show that the sequence (18) is well-distributed in the compact group $\mathbb{Z}_q \times (\mathbb{A}_q)^{s-1}$. But the same result holds for the sequence $\tau^n_{f\bullet\psi}(\tilde{0}, 0_{(\mathbb{A}_q)^{s-1}}) = (\tilde{n}, f \bullet \psi(n) - f \bullet \psi(0))$. Then, due to Lemma 2 we may apply Lemma 1 to conclude the unique ergodicity of $\tau_{f\bullet\psi}$. □

Corollary 2. *Let q be a prime number and let H_q denote the digital $(0,q)$-sequence* (5) *of H. Faure. Then the skew product τ_{H_q} defined on $\mathbb{Z}_q \times (\mathbb{A}_q)^{s-1}$ is uniquely ergodic.*

§6. Applications to q-additive functions

6.1. Discrete valued q-additive functions

We begin with a result that describes the structure of a general q-additive function with discrete values and continuous derivative.

Theorem 8. *Let A be a discrete abelian group and let $f : \mathbb{N} \to A$ be a q-additive function. The discrete derivative Δf can be extended continuously to all of $\mathbb{Z}_q$ if and only if there exist an integer s, a q-additive periodic sequence g with period q^s, and an element α in A such that*

$$f(n) = g(n) + \left\lfloor \frac{n}{q^s} \right\rfloor \cdot \alpha \tag{19}$$

for all $n \in \mathbb{N}$.

Proof. It is clear that Formula (19) defines a q-additive function whose discrete derivative can be extended continuously to all of $\mathbb{Z}_q$. Reciprocally, assume that Δf has a continuous extension on $\mathbb{Z}_q$. The group A being a discrete space, Δf must be constant on some cylinder set containing $-1 = (q-1, q-1, q-1, \dots)$. Therefore, there exists an integer $s \geq 0$, such that Δf is equal to a constant c on $[(q-1)^s; \mathbb{N}]$. In particular, for any digit $e \in \{0, 1, \dots, q-2\}$, one has $f((e+1)q^s) - b_s - f(eq^s) = c$ with $b_s = \sum_{0 \leq j < s} f((q-1)q^j)$. Hence the following equalities

$$\begin{aligned} f(q^s) - f(0) &= f(2q^s) - f(q^s) \\ &= f(3q^s) - f(2q^s) \\ &\;\;\vdots \\ &= f((q-1)q^s) - f((q-2)q^s) \end{aligned}$$

hold, ensuring $f(eq^s) = e \cdot f(q^s)$ for $e = 0, \dots, q-1$. In addition, computing Δf on the cylinder set $[(q-1)^{s+1}0]$, we get $f(q^{s+1}) - b_s - f((q-1)q^s) = c$, but we also have $f(q^s) - b_s = c$ hence, with the previous relations, we obtain $f(q^{s+1}) = q \cdot f(q^s)$. The same relations hold if we replace s by $s+k$, $k \geq 0$. Therefore $f(eq^{s+k}) = eq^k \cdot f(q^s)$. To obtain the decomposition (19), we introduce the periodic sequence g of period q^s,

$$g(n) = \sum_{0 \leq j < s} f(n_j q^j),$$

where $n = \sum_{j\geq 0} n_j q^j$ is the standard expansion of n in base q. Now, $(f-g)(n) = \sum_{j\geq s} n_j q^{j-s} \cdot f(q^s)$ but this sum is precisely $\left\lfloor \frac{n}{q^s} \right\rfloor \cdot \alpha$ with $\alpha = f(q^s)$. □

We consider Theorem 8 for the case where A is a finite q-torsion group:

Theorem 9. *Assume that A is a finite group such that every element of A is of order a power of q and let $f : \mathbb{N} \to A$ be a q-additive function. Then the following assumptions are equivalent:*

(i) *Δf can be extended continuously to all of $\mathbb{Z}_q$;*
(ii) *f is periodic and the period is a power of q (i.e., there exists an integer $K \geq 0$ such that $f(n+q^K) = f(n)$ for all $n \in \mathbb{N}$);*
(iii) *Δf, defined on $\mathbb{Z}_q$, is a coboundary for τ.*

Proof. *(i)* $\Rightarrow$ *(ii)*: Due to Theorem 8, we only have to show that $n \mapsto \lfloor n/q^s \rfloor \cdot \alpha$ is periodic with a period equal to a power of q. By assumption there is an integer r such that $q^r \cdot \alpha = 0_A$, hence $\lfloor (n + q^{s+r})/q^s \rfloor \cdot \alpha = \lfloor n/q^s \rfloor \cdot \alpha$ as expected.

(ii) $\Rightarrow$ *(i)* and *(ii)* $\Rightarrow$ *(iii)*: These implications are obvious. In fact, if f is periodic with period a power of q, say q^K, one can extend f to the continuous map G on $\mathbb{Z}_q$ by the formula $G(x) = f(\sum_{j=0}^{K-1} x_j q^j)$.

(iii) $\Rightarrow$ *(i)*: Assume that Δf is a coboundary and cannot be extended by continuity to all of $\mathbb{Z}_q$. In other words, the sequence $n \mapsto c_{n,0}$ (with $c_{n,0} = f(q^n) - f(q-1) - f((q-1)q) - \cdots - f((q-1)q^{n-1}))$ does not converge. Since A is finite, for any integer $k \geq 0$, there exists $n_k > k$ such that $f(q^{n_k}) - f((q-1)q^k) - f((q-1)q^{k+1}) - \cdots - f((q-1)q^{n_k-1}) \neq 0_A$ and we can construct two increasing sequences of integers $(m_k)_k$ and $(n_k)_k$ such that $m_k < n_k$ and $f(q^{n_k}) - f((q-1)q^{m_k}) - f((q-1)q^{m_k+1}) - \cdots - f((q-1)q^{n_k-1}) = c \neq 0_A$. We claim that c is an essential value of the cocycle Δf. Take any Borel set B in $\mathbb{Z}_q$ with $h_q(B) > 0$. There exists $M \geq 0$ such that $h_q(B \cap \tau^{-q^m}(B)) > 0$ for all $m \geq M$. Now observe that the set of $x \in \mathbb{Z}_q$ such that $(x_{m_k}, \ldots, x_{n_k-1}, x_{n_k}) = (q-1, \ldots, q-1, 0)$ for infinitely many integers k has Haar measure 1. Hence, there exists k such that $m_k \geq M$ and $h_q\big(B \cap \tau^{-q^{m_k}}(B) \cap \{x \in \mathbb{Z}_q\,;\ \Delta_{q^{m_k}} f(x) = c\}\big) > 0$. This gives a contradiction with the fact that the set of essential values of a coboundary is reduced to the trivial subgroup $\{0_A\}$ (see Theorem 4). □

This result will provide a simple criterion to recognize whenever τ_f is ergodic.

6.2. Criterion for ergodicity

Let Φ be the $\mathbb{Z}/q\mathbb{Z}$-module of all $\mathbb{Z}/q\mathbb{Z}$-valued q-additive functions and set

$$\Phi_0 = \{g \in \Phi : \exists \ell \in \mathbb{N}, \forall n \in \mathbb{N}\ g(n + q^\ell) = g(n)\}.$$

Definition 7. *A family $\mathcal{F} = \{f_k : \mathbb{N} \to \mathbb{Z}/q\mathbb{Z},\ k \geq 0\}$ of q-additive functions will be called independent modulo Φ_0 if for all $m > 0$ and $(\epsilon_0, \dots, \epsilon_{m-1}) \in (\mathbb{Z}/q\mathbb{Z})^m$ the following implication holds:*

$$\epsilon_0 \cdot f_0 + \cdots + \epsilon_{m-1} \cdot f_{m-1} \in \Phi_0 \ \Rightarrow\ \epsilon_0 = \epsilon_1 = \ldots = \epsilon_{m-1} = 0.$$

Theorem 10. *Let $\mathcal{F}^{(r)} = \{f_k^{(r)} : \mathbb{N} \to \mathbb{Z}/q\mathbb{Z},\ k \geq 0\}$, $r = 1, \dots, s-1$ be $s-1$ families of q-additive functions, let $f^{(r)} : \mathbb{N} \to \mathbb{A}_q$ denote the q-additive function*

$$f^{(r)}(n) = (f_0^{(r)}(n), f_1^{(r)}(n), f_2^{(r)}(n), \dots),$$

and define the q-additive function $F : n \mapsto \mathbb{A}_q^{s-1}$ by

$$F(n) = (f^{(1)}(n), \dots, f^{(s-1)}(n)).$$

The skew product $(\tau_F, \mathbb{Z}_q \times \mathbb{A}_q^{s-1}, \mu_q \otimes \lambda_{\mathbb{A}_q^{s-1}})$ is ergodic (hence uniquely ergodic) if and only if the hypothesis

(H) *the family $\mathcal{F} = \{f_k^{(r)};\ k \geq 0, 1 \leq r < s\}$ is independent modulo Φ_0*

holds.

Proof. We use Theorem 2. Assume that the skew product is not ergodic. There exists a nontrivial character χ of $\mathbb{A}_q^{s-1}$ and a Borel map $V : \mathbb{Z}_q \to \mathbb{U}$ such that $\chi \circ \Delta F = V \circ \tau / V$. Note that $\chi \circ \Delta F = \Delta(\chi \circ F)$ (with a multiplicative law) and by Theorem 4, the group of essential values of $\Delta(\chi \circ F)$ for τ is reduced to $\{1\}$. Moreover, the corresponding cocycle takes its values in $\mathbb{U}_q$, the group of q-th roots of unity in $\mathbb{U}$. Consequently, we may also assume that V takes its values in $\mathbb{U}_q$ (see Theorem 4). Theorem 9 implies that $\chi \circ F$ is periodic with period a power of q. The character χ has the form

$$\chi(\alpha) = \exp\Big(2i\pi \frac{1}{q} \sum_{r=1}^{s-1} \sum_{j=0}^{m_r} \chi_j^{(r)} \cdot \alpha_j^{(r)}\Big) \tag{20}$$

with $\alpha = (\alpha^{(1)}, \dots, \alpha^{(s-1)}) \in \mathbb{A}_q^{s-1}$, the $\chi_j^{(r)}$ being integers in $\{0, 1, \dots, q-1\}$ that correspond to characters $\chi^{(r)}$ of $\mathbb{A}_q$, namely

$$\chi^{(r)}(u) = \exp\big(2i\pi \frac{1}{q} \sum_{j=0}^{m_r} \chi_j^{(r)} \cdot u_j\big) \quad (u = (u_0, u_1, u_2, \dots)).$$

Let g be the $\mathbb{Z}/q\mathbb{Z}$-valued q-additive function given by the nontrivial linear combination

$$g = \sum_{r=1}^{s-1} \sum_{j=0}^{m_r} \chi_j^{(r)} \cdot f_j^{(r)}. \tag{21}$$

By construction $\exp(\frac{1}{q} g) = \chi \circ F$, so that g belongs to Φ_0. This proves that the hypothesis (H) is not verified. Reciprocally, assume that there is a nontrivial finite linear combination (with coefficients in $\mathbb{Z}/q\mathbb{Z}$) between elements of $\mathcal{F}$ which belongs to Φ_0. In other words, one has a nontrivial relation (21) with g in Φ_0. Extend g to a continuous map $G : \mathbb{Z}_q \to \mathbb{U}_q$ and let χ be the character of $\mathbb{A}_q^{s-1}$ defined by (20). By construction, χ is nontrivial and the continuous map $V = \exp(\frac{2i\pi}{q} G)$ verifies the functional equation $V(x) = \chi(\Delta F(x)) V(\tau x)$. Hence, by Theorem 2, the skew product in not ergodic. □

Remark 2. (i) The hypothesis (H) implies that all maps $f_k^{(r)}$ are distinct.

(ii) Up to a coboundary for τ, Theorem 10 contains the case where digit permutations are involved (see Remark 1). We let the interested reader rephrase the theorem for digit permutations.

Corollary 3. *Under the assumption of Theorem 10, the sequence* $n \mapsto (\varphi(n), \varphi(f^{(1)}(n)), \dots, \varphi(f^{(s-1)}(n)))$ *is well-distributed in* $\mathbb{T}^s$.

The proof is clear. □

6.3. Applications

Theorem 10 is a basic and efficient tool to prove ergodicity. Let us illustrate this fact through a few interesting applications.

6.3.1. First we pay attention to the criterion given in [12] to ensure that a digital sequence based on formal Laurent series is uniformly distributed. Let $\mathbb{Z}/q\mathbb{Z}[[z^{-1}]]$ be the ring of formal Laurent series with coefficients in $\mathbb{Z}/q\mathbb{Z}$. Actually, the base q is not assumed to be a prime number. We begin by associating to any strongly q-multiplicative function $u : \mathbb{N} \to \mathbb{Z}/q\mathbb{Z}$ the formal Laurent series $[u] = \sum_{j \geq 0} u(q^j) z^{-j}$. If u belongs to Φ_0 with minimal period q^d, then $[u]$ is a polynomial in

z^{-1} of degree d. Choose formal Laurent series $L_r = \sum_{k\geq 0} u_k^{(r)} z^{-k}$ in $\mathbb{Z}/q\mathbb{Z}[[z^{-1}]]$ $(1 \leq r \leq s-1)$ and define the matrices

$$C^{(r)} = \begin{pmatrix} u_0^{(r)} & u_1^{(r)} & \dots & u_k^{(r)} & \dots \\ u_1^{(r)} & u_2^{(r)} & \dots & u_{k+1}^{(r)} & \dots \\ \dots & \dots & \dots & \dots & \dots \\ u_k^{(r)} & u_{k+1}^{(r)} & \dots & u_{2k-1}^{(r)} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

Consider the digital sequence

$$(22) \qquad U : n \mapsto (\tilde{n}C^{(1)}, \dots, \tilde{n}C^{(s-1)})$$

which is also the $\mathbb{A}_q^{s-1}$-valued strongly q-additive function

$$U = (U^{(1)}, \dots, U^{(s-1)})$$

with $\mathbb{A}_q$-valued strongly q-additive functions $U^{(r)} : \mathbb{N} \to \mathbb{A}_q$ given by $U^{(r)} = (U_0^{(r)}, U_1^{(r)}, U_2^{(r)}, \dots)$ where the strongly q-additive functions $U_k^{(r)} : \mathbb{N} \to \mathbb{Z}/q\mathbb{Z}$ are defined by

$$(23) \qquad U_j^{(r)}(n) = \sum_{i\geq 0} n_i \cdot u_{i+j}^{(r)}.$$

In [12] the authors proved (with q prime) that the sequence (22) is uniformly distributed if and only if

(HS) $1, L_1, \dots, L_{s-1}$ *are linearly independent over the ring* $\mathbb{Z}/q\mathbb{Z}[z]$.

To see the equivalence between (HS) and (H) in this particular case, we first observe that $[U_j^{(r)}] = z^j L_r - \sum_{i=0}^{j-1} u_i^{(r)} z^{j-i}$ (and $[U_0^{(r)}] = L_r$). Hence, a nontrivial linear combination of 1 and the L_r over $\mathbb{Z}/q\mathbb{Z}[z]$ gives rise to a nontrivial relation of the form $P_0(z) + \sum_{r=1}^{s-1} \sum_{j=0}^{m_r} \chi_j^{(r)} [U_j^{(r)}]$ with $P_0 \in \mathbb{Z}/q\mathbb{Z}[z]$. In fact P_0 must be constant and the nontrivial linear combination

$$\sum_{r=1}^{s-1} \Big(\sum_{j=0}^{m_r} \chi_j^{(r)} U_j^{(r)} \Big) = c_0$$

over $\mathbb{Z}/q\mathbb{Z}$ shows that the hypothesis (H) is false. Reciprocally, assume that a relation (21) holds between the family of q-multiplicative functions $f_j^{(r)} = U_j^{(r)}$. In other words

$$[g] = \sum_{r=1}^{s-1} \Big(\sum_{j=0}^{m_r} \chi_j^{(r)} z^j \Big) L_r + P(z)$$

for a suitable polynomial $P(z)$ in the variable z. Hence, property (HS) is false. Finally, we have proved

Proposition 1. *Let* $U : n \mapsto (\tilde{n}C^{(1)}, \dots, \tilde{n}C^{(s-1)})$ *be the digital sequence* (22) *constructed from formal Laurent series* $L_1, \dots, L_{s-1}$ *as above. Then the skew product* τ_U *is ergodic (hence uniquely ergodic) if and only if the hypothesis* (HS) *holds.* □

6.3.2. Consider the family of 2-additive functions given by

$$f_k(n) = \sum_{j \geq 0} \binom{j}{k} n_j,$$

with $\binom{j}{k} = 0$ if $j < k$. The sequence F defined as in Theorem 10 with $s = 2$ is nothing else than the Sobol'-Faure sequence H_2 in (5) and we already know (Corollary 2) that the corresponding skew product τ_{H_2} is uniquely ergodic. Let us derive this result from Theorem 10 without using the fact that H_2 is a $(0, 2)$-sequence. To this aim, we have to prove that the given family $\{f_k ;\ k \geq 0\}$ is independent modulo Φ_0. This is due to the easy observation that the matrix (4) contains block of consecutive rows of the form

$$\begin{pmatrix}
1 & 0 & 0 & \dots & \dots & \dots & 0 & \dots \\
* & 1 & 0 & \dots & \dots & \dots & 0 & \dots \\
* & * & 1 & 0 & \dots & \dots & 0 & \dots \\
\vdots & \vdots & \ddots & \ddots & \ddots & & \vdots & \vdots \\
\vdots & \vdots & & \ddots & \ddots & \ddots & \vdots & \vdots \\
* & * & \dots & \dots & * & 1 & 0 & \dots
\end{pmatrix}.$$

A similar proof holds for any prime number q but involves all matrices C^r $(1 \leq r \leq q - 1)$ and requires the nice relation (see [7]):

$$(C^r)_{jk} = r^{j-k} \binom{j}{k} \pmod{q}.$$

6.3.3. From any ergodic skew product τ_f with a q-additive function $f : \mathbb{N} \to \mathbb{A}_q$ we can produce many others. In fact, let $f = (f_0, f_1, f_2, \dots)$ be any $\mathbb{A}_q$-valued q-additive function and set $\mathcal{F} = \{f_0, f_1, f_2, \dots\}$. For each i, $1 \leq i \leq s - 1$, let $\mathcal{F}^{(i)} = \{f_0^{(i)}, f_1^{(i)}, f_2^{(i)}, \dots\}$ be a subfamily of $\mathcal{F}$ such that $\{\mathcal{F}^{(1)}, \dots, \mathcal{F}^{(s-1)}\}$ form a partition of $\mathcal{F}$. Finally, define the $\mathbb{A}_q$-valued q-additive functions $f^{(i)} = (f_0^{(i)}, f_1^{(i)}, f_2^{(i)}, \dots)$ and the $(\mathbb{A}_q)^{s-1}$-valued q-additive function $F = (f^{(1)}, \dots, f^{(s-1)})$. The following proposition and corollary are a straightforward consequence of Theorem 10:

Proposition 2. *With the above notation, the skew product* τ_f : $\mathbb{Z}_q \times \mathbb{A}_q \to \mathbb{Z}_q \times \mathbb{A}_q$ *is ergodic if and only if the skew product*

$$\tau_F : \mathbb{Z}_q \times (\mathbb{A}_q)^{s-1} \to \mathbb{Z}_q \times (\mathbb{A}_q)^{s-1}$$

is ergodic.

Corollary 4. *With the above notation, assume that the sequence* $n \mapsto (\varphi(\tilde{n}), \varphi \circ f(n))$ *is uniformly distributed in* $[0,1)^2$. *Then the sequence* $n \mapsto (\varphi(\tilde{n}), \varphi \circ F(n))$ *is uniformly distributed in* $[0,1)^s$.

Remark 3. When the base q is greater than 2, we can easily produce $\mathbb{Z}/q\mathbb{Z}$-valued q-additive functions g which are not of the form $f \circ \psi$, where f is a strongly q-additive sequence and ψ a q-digit permutation. This is the case, for example, of the q-additive sequence g given by $g(aq^j) = c_j$ for all $a = 1, 2, \ldots, q-1$ and $j \geq 0$, where $(c_j)_j$ is any sequence in $\mathbb{Z}/q\mathbb{Z}$ which is not the null sequence. To continue this illustration, let g_k, $k = 0, 1, 2, \ldots$ be a family of such $\mathbb{Z}/q\mathbb{Z}$-valued q-additive sequences with sequences $(c_{jk})_{j \geq 0}$. Choose each c_{jk} in $\{0,1\}$ satisfying $c_{jk} \equiv \binom{j}{k} \pmod 2$. Then, the structure of the Pascal triangular matrix (4) mod 2, exhibited above, can be used to prove that the family of sequences g_k verifies the hypothesis (H). Hence τ_g with $g = (g_0, g_1, g_2, \ldots)$ is ergodic. In particular, the sequence $n \mapsto (\varphi(\tilde{n}, \varphi(g(n)))$ is well distributed in $[0,1)^2$ but is not a digital $(0,2)$-sequence, due to the fact that this map is not one-to-one.

In a forthcoming paper we will continue this work by considering several coprime basis $q_1, \ldots, q_r$. The main question is to characterize q_i-additive sequences $F_i : \mathbb{N} \to \mathbb{A}_{q_i}$ such that the direct product $T = \tau_{F_1} \times \cdots \times \tau_{F_r}$ is ergodic. Note that T is also a skew product above the ergodic translation $(x_1, \ldots, x_s) \mapsto (x_1 + 1, \ldots, x_r + 1)$ on $\mathbb{Z}_{q_1} \times \cdots \times \mathbb{Z}_{q_r}$ with the cocycle $F : n \mapsto (F_1(n), \ldots, F_r(n))$. Such functions which are also digital (t,s)-sequences are of main interest.

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Peter Hellekalek
Fachbereich Mathematik
Universität Salzburg
Hellbrunnerstr. 34
A-5020 Salzburg
Austria
E-mail address: `peter.hellekalek@sbg.ac.at`

Pierre Liardet
Université de Provence
39, rue F. Joliot-Curie
F-13453 Marseille cedex 13
France
E-mail address: `liardet@cmi.univ-mrs.fr`