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A survey of complex Finsler geometry

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§1. Introduction

Classically geometric problems on complex manifolds are investigated and solved by choosing appropriate Hermitian (or Kähler) metrics. In most cases the results depend, not just on the complex structure but on the choices of the Hermitian/Kähler metrics. On the other hand, naturally arisen *intrinsic* metrics are, almost always, not Hermitian but only Finsler in nature. The term "intrinsic" here refers to objects (or properties) that depend only on the complex structure of the manifold. The most well-known ones are the Kobayashi and the Caratheodory pseudo-metrics ([39], [41]) with the property that Kobayashi is the largest while Caratheodory is the smallest among all intrinsic pseudo-metrics. Essentially all intrinsic (pseudo) metrics arise as solutions of naturally posed extremal problems and, except in very special cases, only the Finsler character is preserved in the minimizing/maximizing process. This renders the deep and beautiful theory of Hermitian/Kähler geometry powerless in dealing with these metrics. Naturally, it is desirable to have a good differential geometric theory for Finsler metrics which, thanks to

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the effort of many people, is now well established in the case of smooth metrics. However, solutions of extremal problems do not usually have very good regularity and the most pressing problem in Finsler geometry is to find ways of dealing with non-smooth metrics. This is a very promising area as, for many of the intrinsic metrics, we do have "convexity" or "subharmonicity, for which we do have some way of extracting geometric information. Moreover we have, as models, the infusion of analysis in Riemannian and Kählerian geometry.

For the category of algebraic varieties we have at our disposal, besides the differential geometric approach, also the algebraic geometric approach. The algebraic geometric concepts and techniques are, in most part, also intrinsic in nature. This suggests that there should be deep and interesting relationship between algebraic geometry and Finsler geometry. First and foremost, at this time, is to find the algebraic geometric counterparts of all the Finsler geometric concepts and, vice versa. With this we can then investigate the same problem from two different angles and, depending on the problem, one of the approach may be more natural and easier than the other. For example, in algebraic geometry, it was discovered by Grothendieck/Serre ([32], [44], [11]) that the most natural way to deal with the Chern classes of vector bundles of higher rank is to work with the tautological line bundle over the projectivization of the vector bundles. From the point of view of Finsler geometry the Chern connection and curvature of a Finsler metric on a vector bundle E naturally reside on the projectivized bundle $\mathbb{P}(E)$ and the Chern classes can be computed from the Chern curvature. Furthermore, the Euler-Poincaré characteristic can be expressed, via the Riemann-Roch Theorem, in terms of the Chern numbers and this extends the Gauss-Bonnet Theorem in Finsler geometry.

The purpose of this article is to briefly present some of the interplay among analysis, algebraic geometry and Finsler geometry from my personal view point. It is my sincere hope that this will attract more people to this most interesting area of mathematics. The article is organized as follows.

In section 2 we introduce the standard facts about complex Finsler geometry in the smooth case. The main theme is the correspondence between smooth strictly pseudo-convex Finsler metrics of a holomorphic vector bundle E with the Hermitian metrics along the Serre line bundle $\mathcal{L}$ over $\mathbb{P}(E)$. Note that for line bundles Hermitian and Finsler metrics are the same thing. The important observation is the fact that a bundle E is *ample* is equivalent to the existence of a Finsler metric on E with *positive bisectional curvature*. On the other hand, the dual bundle $E^\vee$

is ample is equivalent to the existence of a Finsler metric on E with *negative bisectional curvature.* For $E = TM$ where M is compact the famous theorem of Mori [49] asserts that M is isomorphic to $\mathbb{P}^n$ if TM is ample. From the equivalence above we see that this is equivalent to saying that M is isomorphic to $\mathbb{P}^n$ if M admits a smooth Finsler metric with positive bisectional curvature. It should be remarked that Siu and Yau [62] showed that a compact *Kähler* manifold with positive bisectional curvature is isomorphic to $\mathbb{P}^n$.

In section 3 we give a brief explanation and presentation of the relationship between the approach of projectivization with the complex homogeneous Monge-Amperé equation and how this leads to a definition of the Chern curvature *current* for certain class of non-smooth Finsler metrics. Using this we show that the concept of big bundles in algebraic geometry is equivalent to the existence of Finsler metric, not necessarily smooth, such that the bisectional curvature current is strongly positive in the sense of currents (see [11]). There is also the dual formulation that the condition that $E^\vee$ being big is equivalent to the existence of Finsler metric with strongly negative bisectional curvature current. In particular, a compact complex manifold is *Moishezon* if and only if there exists a holomorphic line bundle with strongly positive bisectional curvature current (see [16], [37], [38], [48]).

In section 4 we introduce the readers to one of the most fruitful, beautiful and deepest branch of complex analysis: the Oka Principle [51]. Oka's Principle was modernized and extended by Grauert ([26], [27]) to what is now known as Stein manifolds (we now understand, through the works of Grauert and others, that the original definitions of Oka is one way of characterizing a Stein manifold). The Oka-Grauert Principle was later further extended by Gromov [30], [31] to, what is now known as, the h-Principle or homotopy Principle. The Oka-Grauert Principle was also extended by Cornalba-Griffiths [13] in a different direction: Oka-Grauert Principle with growth condition. Their starting point was the observation that the classical Chow's Theorem for projective varieties does not generalize readily to affine algebraic varieties (these are non-compact but admits compactification by adding ample varieties of codimension one at infinity). Chow's Theorem implies that a holomorphic vector bundle over a projective varieties admits an algebraic structure on the affine part. This can be equivalently expressed in terms of the existence of a Hermitian structure with "algebraic" curvature (see section 4 for details). This is not possible in general over affine varieties and Cornalba-Griffiths conjectured that the next best possibility should be true; namely, every holomorphic vector bundles admits a finite order (of exponential type) structure. They succeeded in establishing this in the case of line bundles by establishing the exitence of a Hermitian metric whose curvature is finite order (of exponential type). For vector bundles of higher rank they developed many techniques but were not quite successful. In his doctoral thesis my student Maican [47] extended partially the theory to the Finsler settings, by going up to the projectivized bundle and transfer the problem to the Serre line bundle. This procedure is somewhat more successful. Maican's approach is refined by the author in [72] and [74].

The results of section 2 and 3 are valid for compact manifolds. We show in section 5 that the theory extends to the situation where the manifold M admits a compactification $\overline{M}$ such that $D = \overline{M} \setminus M$ is of codimension one. Actually, for simplicity, we did this only for the case of surfaces using the recent results in [75]. The theory can be extended to higher dimensions but there is not yet explicit references. Instead of using the bundle T^*M over M we use the bundle $T^*M(\log D)$, the bundle of logarithmic forms. The theory of meromorphic logarithmic forms is natural as it is compatible with residue theory, one of the main technique in complex analysis. The systematic study of logarithmic forms was initiated by Deligne in [15] with deep and important results obtained later by Iitaka [33], [34] and Sakai [56]. The first main observation is that holomorphic forms on M are not necessarily closed if M is non-compact (recall that holomorphic forms are closed on compact Kähler manifolds) but those with logarithmic singularities at infinity, in this case D, are. In this section we also indicate the relevance and importance of Finsler geometry in dealing with problems in complex hyperbolic geometry. Since affine varieties can be compactified hence the techniques of logarithmic forms should be combined with those of section 4 and hopefully can shed more light on the problem of Cornalba-Griffiths.

In recent years more and more differential geometers, algebraic geometers and complex analysts, in order to tackle non-linear problems, examine in details the structure and properties of the jet bundles. There are many types of jet bundles, the one that we are interested in here are what is called the parameterized jet bundles and what the algebraic geometers called the arc spaces (see section 6 for details). In general the bundles J^kX are not vector bundles for $k \geq 2$ ($J^1X = TX$) but they do come with a naturally defined $\mathbb{C}^*$-actions and so the projectivization $\mathbb{P}(J^kX) = J^kX/\mathbb{C}^*$ is defined. For $k \geq 2$ the fibers are *weighted projective spaces* rather than the usual projective spaces. These are well-understood spaces and behave (though more complicated) much the same way as the usual projective spaces. As these spaces do not carry any vector space structure, Hermitian metric no longer make sense but Finsler metric can still be defined via the $\mathbb{C}^*$-action. Thus Finsler geometry is indispensable in the theory of jets. Sections 6 and 7 are designed as a brief introduction of jet bundles and the presentation is based on [28] and [11]. We also introduce briefly, Nevanlinna Theory (see [8], [13], [59], [60], [72], [73]), one of the analytic tool that is extremely useful in Finsler geometry. The logarithmic version of jet bundles ($J^k X(\log D)$ can also be defined in a similar fashion as logarithmic forms discussed in section 5. Due to space limitation we shall present this in more details at another occasion.

Throughout the article we listed a number of open problems. Hopefully these will generate further interests and activities in complex Finsler geometry. We also provide some references to the topics discussed here. Obviously this is a very partial list; many important articles are not mentioned due to the limitation of space.

§2. Connections and Curvatures of Smooth Finsler Metrics

2.1. Basic Concepts of Finsler Geometry

The basic objects in complex Finsler geometry are holomorphic fiber bundles over complex manifolds. We shall first consider the case of vector bundles. The main references for this section are [1], [5], [6], [9], [10] and [11].

Definition 2.1. Let E be a holomorphic vector bundle over a complex manifold M. A non-negative upper semi-continuous function $\phi : E \longrightarrow \mathbb{R}_{\geq 0}$ satisfying the condition $\phi(z; \lambda v) = |\lambda|\phi(z; v)$ for all $\lambda \in \mathbb{C}$ and $v \in E_z$ is said to be a *Finsler pseudo-metric.* It is said to be a *Finsler metric* if, ind addition, $\phi(z; v) > 0$ for all $v \in E_* = E \setminus \{\text{zero} - \text{section}\}$. A Finsler pseudo-metric is said to be *convex* if $\phi(z; v_1 + v_2) \leq \phi(z; v_1) + \phi(z; v_2)$ for all $v_i \in E_z, i = 1, 2$ and for all $z \in M$. A Finsler pseudo-metric is said to be *smooth* if ϕ^2 is of class $\mathcal{C}^\infty$ on E_* and for each z fixed the function $\lambda \mapsto \phi^2(z; \lambda v)$ is of class $\mathcal{C}^\infty$ even at $\lambda = 0$. A smooth Finsler pseudo-metric is said to be *strictly pseudoconvex* if $\phi^2|_{E_z \setminus \{0\}}$ is strictly pseudoconvex, i.e., the Levi-form:

$$H = \left(H_{i\bar{j}} = \frac{\partial^2 \phi^2(z; v)}{\partial v_i \partial \bar{v}_j} \right)_{1 \leq i,j \leq r} \tag{2.1}$$

(where $v_1, ..., v_r$ are the Euclidean coordinates of the fiber E_z) is positive definite for all $z \in M$. □

Remark 2.2. (i) If h is a *Hermitian* metric on E then its norm

$$\phi(z;v) = ||v||_h = \Big(\sum_{i,j} H_{i\bar{j}}(z)v^i\overline{v^j}\Big)^{1/2}$$

is a Finsler metric such that ϕ^2 is of class $\mathcal{C}^\infty$ on E (not merely on E_*) and that

$$\frac{\partial^2\phi^2(z;v)}{\partial v^i \partial \bar{v}^j} = H_{i\bar{j}}(z)$$

is independent of the fiber coordinates.

(ii) Note that in the definition of a smooth Finsler metric we do not require that $\phi^2(z,v)$ be of class $\mathcal{C}^\infty$ along the zero-section. The space of smooth Finsler metrics on E shall be denoted by $\mathcal{F}_{\mathcal{C}^\infty}(E)$.

(iii) The set of strictly pseudoconvex Finsler metrics on E is denoted by $\mathcal{F}_{\text{spsc}}(E)(\subset \mathcal{F}_{\mathcal{C}^\infty}(E))$. Strictly pseudoconvex Finsler metrics that are not Hermitian certainly exist; for example we may take arbitrary strictly pseudoconvex Finsler, but not Hermitian, metrics on local trivialization neighborhoods $E|_U$ then globalize by taking a partition of unity. However, it is known that either of the conditions below is equivalent to the condition that ϕ is the norm of a Hermitian metric:

(a) $\phi^2(z,v)$ is of class $\mathcal{C}^\infty$ along the zero-section;

(b) $H = \big(\partial^2\phi^2(z;v)/\partial v_i \partial \bar{v}_j\big)$ is independent of the fiber coordinates.

□

Pseudoconvexity is a weaker notion than convexity but it is a notion that is invariant by holomorphic change of coordinates while the later is not. If $E = TM$ the indicatrix of a Finsler metric ϕ is defined by

$$I(\phi) = \{(z;v) \in TM \mid \phi(z;v) < 1\}.$$

A Finsler metric F is convex if and only if $I_z(\phi) = \{v \in T_zM \mid \phi(z;v) < 1\}$ is a convex domain in $T_zM \cong \mathbf{C}^n$ for all $z \in M$. It is clear that $I(\phi)$ is circular, i.e., if $v \in I(\phi)$ then $e^{\sqrt{-1}\theta}v \in I(\phi)$. Convexity is not a very stringent assumption as the following result (due to S. Kobayashi, see also [41], [57] and [10]):

Lemma 2.3. *Let ϕ be a Finsler metric on a complex manifold. Then the double dual*

$$\phi^{**}(z;v) = \sup\{|\omega(v)| \mid \omega \in T_z^*M, \phi^*(\omega) < 1\}$$

is a convex Finsler metric. Indeed, at each point $z \in M$, the indicatrix $I_z(\phi^{**})$ is the smallest circular convex set containing $I_z(\phi)$. Moreover, if M is compact then ϕ is positive definite if and only if ϕ^{**} is positive definite.

Denote by $p : E \to M$ the projection and $p_* : TE \to TM$ the differential. The sequence:

$$0 \to \mathcal{V} = \ker p_* \to TE \xrightarrow{p_*} TM \to 0 \tag{2.2}$$

is exact and the kernel $\mathcal{V}$ shall be referred to as the *vertical sub-bundle* of TE. There is a naturally defined global holomorphic section of $\mathcal{V}$, the *position vector field* (also known as the *Liouville vector field*):

$$P = \sum_{i=1}^{r} v^i \frac{\partial}{\partial v^i}. \tag{2.3}$$

For $\phi \in \mathcal{F}_{spsc}(E)$ the Levi form H as defined in (2.1) is identified with a *Hermitian* metric along the fibers of $\mathcal{V}$:

$$< V, W >_H = \sum_{i,j} \frac{\partial^2 \phi^2}{\partial v_i \partial \bar{v}_j} a_i \bar{b_j} \tag{2.4}$$

where $V = \sum a_i \partial/\partial v_i$, $W = \sum b_j \partial/\partial v_j$ are sections of $\mathcal{V}$.

Note that we have (compare Remark 2.2 (i)):

$$||P(z,v)||_H^2 = \sum_{1 \le i, \le r} H_{ij}(z;v) v^i \bar{v}^j$$

which implies that

$$\sum_{1 \le i, \le r} \frac{\partial^2 ||P(z,v)||_H^2}{\partial v^i \partial \bar{v}^j} = H_{ij} = \frac{\partial^2 \phi^2}{\partial v^i \partial \bar{v}^j}$$

hence

$$||P(z,v)||_H^2 = \sum_{1 \le i, \le r} \frac{\partial^2 ||P(z,v)||_H^2}{\partial v^i \partial \bar{v}^j} v^i \bar{v}^j. \tag{2.5}$$

The Finsler condition implies that $||P(z,v)||_H = \phi$. It is well-known (see for example [48]) that the preceding equation is equivalent to the complex homogeneous Monge-Ampere equation along each fiber (outside of the origin):

$$(d_v d_v^c \log \phi^2)^r = (d_v d_v^c \log ||P(z,v)||_H^2)^r \equiv 0 \tag{2.6}$$

where $d_v = \partial_v + \bar{\partial}_v$ and $d_v^c = \sqrt{-1}(\bar{\partial}_v - \partial_v)$ are the usual differential operators in the fiber direction. This implies that $d_v d_v^c \log \phi^2$ descends to a well-defined form of type $(1,1)$ on the projectivized vector bundle

$\mathbb{P}(E) = E_*/\mathbb{C}^*$, consisting of lines through the origin in each fiber. Let $\mathcal{L}$ be the Serre bundle over $\mathbb{P}(E)$, namely, the line bundle over $\mathbb{P}(E)$ whose restriction to each fiber $\mathbb{P}(E)_z \cong \mathbb{P}^{r-1}$ is the hyperplane bundle $\mathcal{O}_{\mathbb{P}^{r-1}}(1)$ (see [44], [9] and [11]). The next Proposition provides the first indication of a strong relationship between Finsler geometry and algebraic geometry.

Proposition 2.4. (*i*) *The bundle space of $\mathcal{L}^{-1}$, the dual Serre line bundle, is obtained by blowing up the zero section of E. In particular, $\mathcal{L}^{-1}_* = \mathcal{L}^{-1} \setminus \{\text{zero} - \text{section}\}$ is canonically isomorphic to $E_* = E_* = E \setminus \{\text{zero} - \text{section}\}$.*

(*ii*) *The blowing up map $\beta : \mathcal{L}^{-1} \to E$ induces an one to one correspondence $h_\phi \leftrightarrow \phi \circ \beta$ between the family $\mathcal{H}^\infty(\mathcal{L}^{-1})$ of smooth Hermitian fiber metrics on $\mathcal{L}^{-1}$ and the family $\mathcal{F}_{spsc}(E)$ of smooth strictly pseudoconvex Finsler metrics on E.*

We shall refer to H as the Chern metric on the vertical bundle $\mathcal{V}$. The Hermitian connection θ and curvature Θ of the Hermitian bundle $(\mathcal{V}, H)$ are known as the *Chern connection* and *Chern curvature* of the Finsler bundle (E, ϕ). By definition $\theta = (\partial H)H^{-1} = (\theta_i^k)_{1 \le i,k \le r}$ which may be decomposed into the vertical and horizontal parts, more precisely:

$$\theta_i^k = \sum_{j=1}^{r} (\partial H_{i\bar{j}}) H^{\bar{j}k} = \sum_{\alpha=1}^{n} \Gamma_{i\alpha}^k dz^\alpha + \sum_{l=1}^{r} \gamma_{il}^k dv^l \tag{2.7}$$

(where $\Gamma_{i\alpha}^k$) and (γ_{il}^k) are respectively the horizontal and vertical Christoffel symbols:

$$\Gamma_{i\alpha}^k = \sum_{j=1}^{r} \frac{\partial H_{i\bar{j}}}{\partial z^\alpha} H^{\bar{j}k} \text{ and } \gamma_{il}^k = \sum_{j=1}^{r} \frac{\partial H_{i\bar{j}}}{\partial v^l} H^{\bar{j}k}.$$

The Hermitian curvature of a Hermitian connection is by definition $\Theta = \bar{\partial}\theta = (\Theta_i^k)$:

$$\Theta_i^k = d\theta_i^k - \sum_{j=1}^{r} \theta_i^l \wedge \theta_l^k = d\theta_i^k + \sum_{j=1}^{r} \theta_l^k \wedge \theta_i^l = \bar{\partial}\theta_i^k \tag{2.8}$$

which may be decomposed into the horizontal, vertical and mixed components K, κ, μ, ν:

$$\Theta_i^k = K_i^k + \kappa_i^k + \mu_i^k + \nu_i^k \tag{2.9}$$

where the components are given explicitly as follows:

$$\begin{cases} K_i^k = \sum_{\alpha,\beta=1}^n K_{i\alpha\bar{\beta}}^k dz^\alpha \wedge d\bar{z}^\beta, & K_{i\alpha\bar{\beta}}^k = -\partial\Gamma_{i\alpha}^k/\partial\bar{z}^\beta, \\ \kappa_i^k = \sum_{j,l=1}^r \kappa_{ij\bar{l}}^k dv^j \wedge d\bar{v}^l, & \kappa_{ij\bar{l}}^k = -\partial\gamma_{ij}^k/\partial\bar{v}^l, \\ \mu_i^k = \sum_{\alpha=1}^n \sum_{l=1}^r \mu_{i\alpha\bar{l}}^k dz^\alpha \wedge d\bar{v}^l, & \mu_{i\alpha\bar{l}}^k = -\partial\Gamma_{i\alpha}^k/\partial\bar{v}^l, \\ \nu_i^k = \sum_{j=1}^r \sum_{\beta=1}^n \nu_{ij\bar{\beta}}^k dv^j \wedge d\bar{z}^\beta, & \nu_{ij\bar{\beta}}^k = -\partial\gamma_{ij}^k/\partial\bar{z}^\beta. \end{cases}$$

Remark 2.5. (*i*) In general the curvature of a connection is given by

$$\Theta_i^k = d\theta_i^k - \sum_{j=1}^r \theta_i^l \wedge \theta_l^k.$$

For a Hermitian connection, we have relative to holomorphic frames:

$$\Theta_i^k = d\theta_i^k - \sum_{j=1}^r \theta_i^l \wedge \theta_l^k = d\theta_i^k + \sum_{j=1}^r \theta_l^k \wedge \theta_i^l = \bar{\partial}\theta_i^k$$

which is equivalently to the condition that

$$\partial\theta_i^k - \sum_{j=1}^r \theta_i^l \wedge \theta_l^k = \partial\theta_i^k + \sum_{j=1}^r \theta_l^k \wedge \theta_i^l = 0.$$

(*ii*) As observed earlier in Remark 2.2, in the case of a Hermitian metric, the Chern metric H is independent of the fiber metric and coincides with the original Hermitian metric on E. The Chern connection and Chern curvature are just the usual Hermitian connection and curvature of E. □

The Chern connection $\nabla^{\mathcal{V}}$ defines a surjection (see [1] and [9] for details):

$$\gamma : TE \to \mathcal{V}, \ \ \gamma(X) = \nabla_X^{\mathcal{V}} P$$

for any $X \in TE$ and where P is the position vector field. The horizontal sub-bundle of TE is defined to be

$$\mathcal{H} = \ker\gamma. \tag{2.10}$$

The horizontal lifts of the local basis $\{\partial/\partial z^\alpha\}$:

$$\{\partial_\alpha^{\mathcal{H}} = \frac{\partial}{\partial z^\alpha} - \sum_{j,k=1}^r \Gamma_{j\alpha}^k v^j \frac{\partial}{\partial v^k} \,|\, \alpha = 1, ..., n\} \tag{2.11}$$

is a local basis of $\mathcal{H}$ and these together with $\{\partial_i^{\mathcal{V}} = \partial/\partial v^i, i = 1, ..., r\}$ form a local basis for TE. Denote by $K(\cdot,\cdot) : \mathcal{V} \to \mathcal{V}$ the curvature operator of the Chern curvature:

$$K(X,Y)V = \sum_{i,k=1}^{r} \sum_{\alpha,\beta=1}^{n} \Theta_i^k(X,Y)V^i \frac{\partial}{\partial v^k} \tag{2.12}$$

where $X, Y \in TE$ and $V = \sum V^i \partial/\partial v^i$ is a vertical vector field. Note that

$$< K(\cdot,\cdot)V, V >_H = \sum_{i,j=1}^{r} H_{k\bar{j}} v^i \overline{v^j} \Theta_i^k \tag{2.13}$$

is a form of type $(1,1)$ on TE (as the forms Θ_i^k are of type $(1,1)$).

Remark 2.6. If $E = TM$ and if ϕ is the norm of a Hermitian g then (see Remark 2.2) the Chern metric is just the original metric g and the curvature is just the Hermitian curvature of g. The curvature operator can be considered as an operator $K(\cdot,\cdot) : E = TM \to E = TM$. Classically, for non-zero tangent vectors $X, Y \in TM$

$$k(X,Y) = \frac{< K(X,X)Y, Y >_g}{||X||_g^2 ||Y||_g^2}$$

is known as the *holomorphic bisectional curvature* of g. For this reason we shall refer to

$$< K(\cdot,\cdot)P, P >_H$$

as, for lack of a better terminology, the *generalized holomorphic bisectional curvature form* of the Finsler metric ϕ on E. □

2.2. Chern Form, Bisectional Curvature On Ample Bundles

By a direct computation, we have (see [9]):

Lemma 2.7. *With respect to a normal frame for the vertical bundle $\mathcal{V}$ at a point $(z; v) \in E$, we have*

$$dd^c \log \phi^2 = dd^c \log ||P||_H^2 = \pi^* \omega_{FS} - \sqrt{-1} \frac{< K(\cdot,\cdot)P, P >_H}{||P||_H^2}$$

at the point $(z; v)$ where P is the position vector field and ω_{FS} is the Fubini-Study metric on $\mathbb{P}^{r-1}$.

This provides a relationship between the curvature of the Serre line bundle and the bisectional curvature ([9]):

Theorem 2.8. *Let $\phi \in \mathcal{F}_{\text{spsc}}(E)$ and H the Chern metric on the vertical sub-bundle $\mathcal{V}$ of TE. Then the first Chern form $c_1(\mathcal{L}, h_\phi)$ of the Serre line bundle $\mathcal{L}$ over $\mathbb{P}(E) = E_*/\mathbb{C}^*$ is positive definite if and only if the curvature of the Chern curvature $\sqrt{-1} < K(\cdot,\cdot)P, P >_H$ is negative definite.*

The following definition is standard (see [44]):

Definition 2.9. A holomorphic line bundle L over M is said to be *ample* if there exists a hermitian metric h along the fibers such that the first Chern form $c_1(L, h)$ is positive definite. For holomorphic vector bundle E of rank $r \geq 2$ its dual is denoted by $E^\vee$ (the fiber $E_z^\vee$ is the space of hyperplanes through the origin of E_z). The bundle E is said to be ample if the Serre line bundle $\mathcal{L}_{\mathbb{P}(E^\vee)}$ over $\mathbb{P}(E^\vee) = E_*^\vee/\mathbb{C}^*$ is ample. The *dual* bundle $E^\vee$ is said to be ample if the Serre line bundle $\mathcal{L}_{\mathbb{P}(E)}$ over $\mathbb{P}(E) = E/\mathbb{C}^*$ is ample. $\square$

Theorem 2.8 simply means that the ampleness of the dual bundle $E^\vee$ is equivalent to the positivity of the bisectional curvature ([9]):

Corollary 2.10. *Let E be a holomorphic vector bundle over a compact complex manifold M. Then the dual bundle $E^\vee$ is ample if and only if there exists a strictly pseudoconvex Finsler metric on E such that the curvature $\sqrt{-1} < K(\cdot,\cdot)P, P >_H$ is negative definite.*

The preceding has a dual formulation, namely, the ampleness of the bundle E is equivalent to the negativity of the bisectional curvature ([9]):

Corollary 2.11. *Let E be a holomorphic vector bundle over a compact complex manifold M. Then E is ample if and only if there exists a strictly pseudoconvex Finsler metric on E such that the curvature $\sqrt{-1} < K(\cdot,\cdot)P, P >_H$ is positive definite.*

A famous theorem of Mori ([49] valid over any algebraically closed field of characteristic zero) asserts that

Theorem 2.12. *A compact complex manifold with ample tangent bundle is biholomorphic to $\mathbb{P}^n$.*

The differential geometric version is due to Siu and Yau [62]:

Theorem 2.13. *A compact Kähler manifold (M, g) with positive bisectional curvature is biholomorphic to $\mathbb{P}^n$.*

Corollary 2.11 with $E = TM$ implies that:

Theorem 2.14. *A compact complex manifold admitting a strictly pseudoconvex Finsler metric with positive holomorphic bisectional curvature is biholomorphic to $\mathbb{P}^n$.*

The complex projective space is parabolic in the sense that there exists many non-trivial holomorphic curves $f : \mathbb{C} \to \mathbb{P}^n$; on the other extreme we have:

Theorem 2.15. *Let M be a complex manifold admitting a strictly pseudoconvex Finsler metric with strongly negative holomorphic bisectional curvature. Then X is Kobayashi hyperbolic, in particular, every holomorphic map $f : \mathbb{C} \to M$ is constant.*

Strong negativity means that the holomorphic bisectional curvature is bounded above by a strictly negative constant. If M is compact then strong negativity is the same as negativity.

The question (see [29]) whether an ample bundle actually admits a *Hermitian* metric with positive bisectional curvature was investigated in [9] and the answer is provided in the theorem below.

Theorem 2.16. *Let E be a holomorphic vector bundle over a compact complex manifold M and for any positive integer k let $\mathcal{L}_{\mathbb{P}(\odot^k E)}$ be the Serre line bundle over $\mathbb{P}(\odot^k E) = (\odot^k E \setminus \{\text{zero} - \text{section}\}/\mathbb{C}^*$. Then the following statements are equivalent:*

(1) *$E^\vee$ is ample;*

(2) *$\mathcal{L}_{\mathbb{P}(E)}$ is ample;*

(3) *$\odot^k E^*$ is ample for some positive integer k;*

(4) *$\mathcal{L}_{\mathbb{P}(\odot^k E)}$ is ample for some positive integer k;*

(5) *$\odot^k E^*$ is ample for all positive integer k;*

(6) *$\mathcal{L}_{\mathbb{P}(\odot^k E)}$ is ample for all positive integer k;*

(7) *there exists a strictly pseudoconvex Finsler metric along the fibers of E with negative generalized holomorphic bisectional curvature;*

(8) *there exists a positive integer k and a strictly pseudoconvex Finsler metric along the fibers of $\odot^k E$, with negative generalized holomorphic bisectional curvature;*

(9) *for any positive integer k there exists a strictly pseudoconvex Finsler metric along the fibers of $\odot^k E$ with negative generalized holomorphic bisectional curvature;*

(10) *there exists a positive integer m and a Hermitian metric along the fibers of $\odot^m E$ with negative holomorphic bisectional curvature.*

We have also the dual formulation ([9]):

Theorem 2.17. *Let E be a holomorphic vector bundle over a compact complex manifold M and for any positive integer k let $\mathcal{L}_{\mathbb{P}(\odot^k E^\vee)}$ be the Serre line bundle over $\mathbb{P}(\odot^k E^\vee)$. Then the following statements are equivalent:*

(1) *E is ample;*

(2) $\mathcal{L}_{\mathbb{P}(E^\vee)}$ *is ample;*

(3) $\odot^k E$ *is ample for some positive integer* k*;*

(4) $\mathcal{L}_{\mathbb{P}(\odot^k E^\vee)}$ *is ample for some positive integer* k*;*

(5) $\odot^k E$ *is ample for all positive integer* k*;*

(6) $\mathcal{L}_{\mathbb{P}(\odot^k E^\vee)}$ *is ample for all positive integer* k*;*

(7) *there exists a Finsler metric along the fibers of* E *with positive generalized holomorphic bisectional curvature;*

(8) *for some positive integer* k *there exists a Finsler metric along the fibers of* $\odot^k E$ *with positive generalized holomorphic bisectional curvature;*

(9) *for all positive integer* k *there exists a Finsler metric along the fibers of* $\odot^k E$ *with positive generalized holomorphic bisectional curvature;*

(10) *there exists a positive integer* m *and a Hermitian metric along the fibers of* $\odot^m E$ *with positive generalized holomorphic bisectional curvature.*

One of the beautiful and deep result in Hermitian geometry is the relationship between Hermitian Einstein vector bundles and stable bundles in algebraic geometry (see [52], [40] for further references in this direction). Due to page limitation we shall not discuss this in any detail except to pose the following general problem:

Open Problem 1. *Extend the theory of Hermitian Einstein bundles to Finsler Einstein bundles.*

The concept of Kähler Finsler metric was first introduced by Royden [55]. The concept was later refined by Abate and Patrizio [1].

Open Problem 2. *Extend the topological theorems of classical Kähler manifolds to Kähler Finsler manifolds.*

Open Problem 3. *Extend the pinching theorem for bisectional curvature of Kähler manifolds to Kähler Finsler manifolds.*

See Kobayashi-Nomizu [35] for the precise statement of the pinching theorem. See also Rademacher [53] in the case of Finsler curvature over the real numbers.

§3. Curvature Currents of Non-Smooth Finsler Metrics

3.1. Finsler Metrics and The Monge-Amperé Equation

Before we deal with the case of non-smooth Finsler metrics we reexamine the smooth case from the point of view of the Monge-Amperé equation. This would provide us with the idea of defining the curvature of non-smooth Finsler metrics. The main references for this section are [1], [9] and [68]. We begin from the simplest case of $\mathbb{C}^r, r \geq 2$, which is the fiber of a vector bundle of rank r.

Denote by $< u, v >= \sum u_i \bar{v}_i$ the Euclidean inner product on $\mathbb{C}^r$ with norm $||v||$. The bundle space of the tautological line bundle $\mathcal{O}_{\mathbb{P}^{r-1}}(-1)$ is the blow up of $\mathbb{C}^r$ at the origin, hence $\mathcal{O}_{\mathbb{P}^{r-1}}(-1) \setminus \{\text{zero} - \text{section}\}$ may be identified with $\mathbb{C}^r_* = \mathbb{C}^r \setminus \{0\}$. The Euclidean norm $||v||$ is identified as a Hermitian metric h_0 along the fibers of the tautological line bundle $\mathcal{O}_{\mathbb{P}^{r-1}}(-1)$. The $(1,1)$-form $-dd^c \log ||v||^2$ is invariant by the $\mathbb{C}^*$-action, i.e., $-dd^c \log ||\lambda v||^2 = -dd^c \log ||v||^2, \lambda \in \mathbb{C}^*$, hence descends to a well-defined form $c_1(\mathcal{O}_{\mathbb{P}^{r-1}}(-1), h_0)$ (the first Chern form of the tautological line bundle $\mathcal{O}_{\mathbb{P}^{r-1}(-1)}$) on $\mathbb{P}^{r-1}$. If we equip the dual bundle $\mathcal{O}_{\mathbb{P}^{r-1}}(1)$ with the dual metric h_0^* then $dd^c \log ||v||^2$ descends to $c_1(\mathcal{O}_{\mathbb{P}^{r-1}}(1), h_0^*) = -c_1(\mathcal{O}_{\mathbb{P}^{r-1}}(-1), h_0)$. It is well known that $c_1(\mathcal{O}_{\mathbb{P}^{r-1}}(1), h_0^*)$ is the Fubini-Study form on $\mathbb{P}^{r-1}$. The fact that $dd^c \log ||v||^2$ descends to $\mathbb{P}^{r-1}$ implies that (as $\dim \mathbb{P}^{r-1} = r - 1$)

$$(dd^c \log ||v||^2)^r \equiv 0$$

on $\mathbb{C}^r \setminus \{0\}$. This is known as the complex homogeneous equation.

The next theorem is due to Stoll (see [68] and the references there):

Theorem 3.1. *Let M be a complex manifold M of complex dimension r. and $\tau : M \to \mathbb{R}_{\geq 0} = [0, \infty)$ be a strictly pseudoconvex exhaustion of class $\mathcal{C}^k, k \geq 5$. Assume that* $\log \tau$ *satisfies the complex homogeneous Monge-Ampère equation*

$$(dd^c \log \tau)^r \equiv 0$$

on $M_* = M \setminus \{\tau = 0\}$. Then $\{\tau = 0\}$ consists of a single point $\{0\}$ and the exponential map (of the Kähler metric $dd^c\tau$)

$$\exp_0 : \mathbb{C}^r = T_0 M \to M$$

is a biholomorphic map with the property that $\exp_0^* \tau = ||v||^2$. *In fact*

$$\exp_0 : (\mathbb{C}^r, dd^c ||v||^2) \to (M, dd^c \tau)$$

is an isometry.

Remark 3.2. If we take an arbitrary Hermitian metric H on $\mathbb{C}^r$ with norm $h = ||v||_H$ then h may be identified with a Hermitian metric on $\mathcal{O}_{\mathbb{P}^{r-1}}(1)$. Just as in the case of the Euclidean norm, the $(1,1)$-form

$dd^c \log h^2$ on $\mathbb{C}^r \setminus \{0\}$ descends to the Chern form $c_1(\mathcal{O}_{\mathbb{P}^{r-1}}(1), h^*)$. This again implies that

$$(dd^c \log h^2)^r \equiv 0$$

on $\mathbb{C}^r \setminus \{0\}$. A Finsler metric ϕ on $\mathbb{C}^r$ is of the form $\phi(v) = \rho([v])||v||$ where ρ is a well-defined function on $\mathbb{P}^{r-1}$. Thus $\log \phi$ satisfies also the complex homogeneous Monge-Amperé equation (as both $\rho([v])$ and $||v||$ satisfy the equation)

$$(dd^c \log \phi)^r \equiv 0$$

on $\mathbb{C}^r \setminus \{0\}$. However it is known that Theorem 3.1 is not valid for a general Finsler metric as such a metric do not satisfy the regularity assumption of the theorem at the distinguished point $\{0\}$. In fact, it is well-known that a Finsler metric ϕ such that ϕ^2 is smooth at the origin is actually the norm of a Hermitian metric. Thus Theorem 3.1 is applicable only to the norm of a Hermitian metrics. □

The position (or radial) vector field is by definition:

$$P = \sum_{i=1}^{r} v^i \frac{\partial}{\partial v^i}$$

and a vector field X is said to be transversal to the radial direction if $< P, X >= 0$ where $< \cdot, \cdot >$ is the Euclidean inner product. The radial direction can also be described as the kernel of the map $\pi_* : T\mathbb{C}^r_* \to T\mathbb{P}^{r-1}$.

Proposition 3.3. *A Finsler metric ϕ on $\mathbb{C}^r$ is necessary of the form $\phi(v) = \rho([v])||v|| = (\rho \circ \pi)(v)||v||$ where ρ is a function defined on $\mathbb{P}^{r-1}$. Consequently, the following conditions are equivalent:*

(i) ϕ is strictly pseudoconvex.

(ii) $dd^c \log ||v||^2 > dd^c \log(\rho \circ \pi)$ in the transversal direction on $\mathbb{C}^r \setminus \{0\}$.

(iii) $dd^c \log \phi^2$ descends to a positive definite $(1,1)$-form on $\mathbb{P}^{r-1}$.

Proof. Since $\rho \circ \pi$ is a function on $\mathbb{P}^{r-1}$, it is clear ($\rho \circ \pi$ is constant along any line through the origin) that the radial derivative vanishes, i.e.,

$$P(\rho \circ \pi) = \sum_{i=1}^{r} v^i \frac{\partial(\rho^2 \circ \pi)}{\partial v^i} = 0.$$

This implies that $d \log \phi^2 = d \log \rho^2 + d \log ||v||^2 = d \log ||v||^2$ in the radial direction, that is:

$$d \log \phi^2(P) = P \log \phi^2(P) = P \log ||v||^2 = d \log ||v||^2(P).$$

Analogously $d^c \log \phi^2 = d^c \log ||v||^2$ in the radial direction as well. The Monge-Amperé equation implies that $dd^c \log \phi^2 = dd^c \log \rho^2 = dd^c \log ||v||^2 = 0$ in the radial direction.

Since $dd^c \log \phi^2 = dd^c \log \rho^2 + dd^c \log ||v||^2$ we conclude that $dd^c \log \phi^2$ is positive in the directions transversal to the radial direction if and only if $dd^c \log \rho^2 + dd^c \log ||v||^2$ is positive in these directions. A direct computation shows that

$$dd^c \phi^2 = \phi^2 dd^c \log \phi^2 + d \log \phi^2 \wedge d^c \log \phi^2.$$

We see from the preceding discussion that

$$dd^c \phi^2 = d \log ||v||^2 \wedge d^c \log ||v||^2 = \frac{d||v||^2 \wedge d^c ||v||^2}{||v||^4}$$

is automatically positive in the radial direction (and off the origin). On the other hand, in the transversal directions

$$dd^c \phi^2 = \phi^2 dd^c \log \phi^2 = \phi^2 (dd^c \log \rho^2 + dd^c \log ||v||^2)$$

hence it is positive if and only if the right hand side above is positive in the transversal directions. □

Corollary 3.4. *Let $\phi = (\rho \circ \pi)||v||$ be a Finsler metric of class $C^k, k \geq 2$ on $\mathbb{C}^r$. Then there exists a constant $c > 0$ such that $(\rho^{2(c-1)} \circ \pi)\phi^2$ and $||v||^{(2-2c)/c}\phi^2$ are strictly pseudoconvex.*

Proof. The forms $dd^c \log ||v||^2, dd^c \log(\rho^2 \circ \pi)$ descend to forms on $\mathbb{P}^{r-1}$. By compactness there exist a constant $c > 0$ such that

$$dd^c \log ||v||^2 > c dd^c \log(\rho \circ \pi).$$

This last condition is equivalent to the condition that the Finsler metric

$$(\rho^{2(c-1)} \circ \pi)\phi^2 = (\rho^{2c} \circ \pi)||v||^2$$

is strictly pseudoconvex. Alternatively the condition is also equivalent to the condition that the Finsler metric

$$||v||^{(2-2c)/c}\phi^2 = (\rho^{2c} \circ \pi)||v||^{2/c}$$

is strictly pseudoconvex. □

Suppose that $\phi = \gamma ||v||^2$ is a strictly pseudoconvex Finsler metric. where ρ is constant along each of the lines through the origin. We shall denote by $H_0 = (\delta_i^j)_{1 \leq i,j \leq r}$ the Euclidean inner product and

$$H = (\phi^2_{i\bar{j}})_{1 \leq i,j \leq r}$$

(where partial derivatives are denoted by subscripts) the Hermitian inner product defined by ϕ. Let $g = ||v||^2$ and P the radial vector field then we have

$$P(\gamma^2 g) = (P\gamma^2)g + \gamma^2 Pg = \gamma^2 Pg$$

as the radial directive of γ vanishes and, for the same reason

$$P\overline{P}\phi^2 = \gamma^2 P\overline{P}g.$$

It is easily verified that

$$||P||^2_{H_0} = \sum_{i,j} v^i \bar{v}^j \delta^i_j = \sum_i |v^i|^2 = ||v||^2 = g$$

which implies that

$$||P||^2_H = \sum_{i,j} v^i \bar{v}^j \phi^2_{i\bar{j}} = P\overline{P}(\phi^2) = \gamma^2 P\overline{P}g = \gamma^2 ||P||^2_{H_0} = \gamma^2 ||v||^2 = \phi^2.$$

Thus we have

$$dd^c \log ||P||^2_H = dd^c \log \phi^2 = dd^c \log \gamma^2 + dd^c \log ||v||^2.$$

Identifying $||v||$ and ϕ with Hermitian metrics h_0 and h along the fibers of $\mathcal{O}_{\mathbb{P}^{r-1}}(-1)$, we see before that $dd^c \log ||P||^2_H$ descends to $c_1(\mathcal{O}_{\mathbb{P}^{r-1}}(1), h^*)$ and $dd^c \log ||v||^2$ descends to $c_1(\mathcal{O}_{\mathbb{P}^{r-1}}(1), h_0^*) = \omega_{\mathrm{FS}}$. The function $\gamma = \rho \circ \pi$ so $dd^c \log \gamma^2$ descends to the form $dd^c \log \rho^2$ on $\mathbb{P}^{r-1}$. Thus we have:

$$c_1(\mathcal{O}_{\mathbb{P}^{r-1}}(1), h^*) = c_1(\mathcal{O}_{\mathbb{P}^{r-1}}(1), h_0^*) + dd^c \log \rho^2 = \omega_{\mathrm{FS}} + dd^c \log \rho^2.$$

The preceding discussion, with minor modification, carries over to any smooth Finsler metric on a holomorphic vector bundle over a complex manifold. More precisely, we have:

Proposition 3.5. *A Finsler metric ϕ on E is necessary of the form $\phi(z; v) = \rho(a; [v])||v||_{H_0}$ where ρ is a function defined on $\mathbb{P}(E)$ and H_0 is a fixed Hermitian metric on E. Consequently, ϕ is strictly pseudoconvex if and only if $d_v d_v^c \log ||v||^2_H > d_v d_v^c \log(\rho \circ \pi)$ in the transversal direction on each fiber $E_z \setminus \{0\}$ if and only if $d_v d_v^c \log \phi^2$ descends to a positive definite $(1,1)$-form on each fiber $\mathbb{P}(E)_z$.*

For a Hermitian metric $H_0 = \left(h^0_{ij}\right)_{1 \le i,j \le r}$ and a strictly pseudoconvex Finsler metric ϕ on E, let h_0 and h be the respective Hermitian metric on the tautological $\mathcal{L} = \mathcal{O}_{\mathbb{P}(E)}(-1)$ over $\mathbb{P}(E)$ defined by H_0 and ϕ. We also let

$$(3.1) \qquad H = \big(H_{ij} = \frac{\partial \phi^2}{\partial v_i \partial \bar{v}_j}\big)_{1 \le i,j \le r}$$

be the Hermitian metric along the fibers of TE defined by ϕ. The Hermitian metric on H_0 defines in an obvious (tautological) way a Hermitian metric on TE which we continue to denote by H_0. Then we have:

$$||P||_H^2 = \sum_{1 \le i,j \le r} H_{ij} v^i \bar{v}^j = \phi^2(z;v) = \rho^2(z;[v])||v||_{H_0}^2 = \rho^2(z;[v])||P||_{H_0}^2$$

where $P = \sum v^i \partial/\partial v^i$ is the position vector field. *What this says is that, even though H and H_0 may not be conformal ($H \neq \rho^2(z;[v])H_0$ in general) they are conformal in the radial direction.* Denote by $\gamma(z;v) = \rho(z;v)$ then

Proposition 3.6. *With the notations and assumptions above, we have*

$$dd^c \log ||P||_H^2 = dd^c \log \phi^2 = dd^c \log \gamma^2 + dd^c \log ||P||_{H_0}^2$$

where P is the position vector field. Moreover, the identity descends to the following identity

$$c_1(\mathcal{O}_{\mathbb{P}(E)}(1), h^*) = c_1(\mathcal{O}_{\mathbb{P}(E)}(1), h_0^*) + dd^c \log \rho^2$$

on $\mathbb{P}(E)$.

Proposition 3.6 together with the result in section 2 (see Theorem 2.8) and the first part of this section imply that

$$(3.2) \qquad \frac{< K(\cdot,\cdot)P, P >_H}{||P||_H^2} = \frac{< K_0(\cdot,\cdot)P, P >_{H_0}}{||P||_{H_0}^2} + dd^c \log \gamma^2 - d_v d_v^c \log \gamma^2$$

where d_v, d_v^c are operators in the fiber directions of E. This is the identity that we shall extend to the case of non-smooth Finsler metrics.

3.2. Singular Finsler Metrics and Big Bundles

The main references of this section are [16], [10], [37], and [38].

Definition 3.7. A Finsler metric ϕ on a holomorphic vector bundle E over a complex manifold is said to be *weakly regular* if $dd^c[\log \phi]$ exist as currents. The space of weakly regular Finsler metrics shall be denoted by $\mathcal{F}_{\mathrm{wr}}(E)$. We use the notation $dd^c[\,]$ to indicate that the differentiation is taken in the sense of distribution. □

We have, as in Proposition 2.4, the following correspondence:

Proposition 3.8. *The isomorphism between E_* and $\mathcal{L}_*^{-1}$ induces a one to one correspondence between the family of weakly regular Finsler metrics on E and the family of weakly regular Hermitian metrics on the Serre line bundle $\mathcal{L}$ over $\mathbb{P}(E) = E_*/\mathbb{C}^*$.*

Now, for $\phi \in \mathcal{F}_{\mathrm{wr}}(E)$ we still have the identity:

$$\phi(z; v) = \rho(z; [v])||v||_{H_0} = \gamma \circ \pi(z; v)||v||_{H_0}.$$

where H_0 is a smooth Hermitian metric and $\pi(z; v) = (z; [v])$ is the projection map. The assumption that $dd^c[\log \phi^2]$ exists as a current implies that the currents $dd^c[\log \rho^2]$ and $dd^c[\log \gamma^2]$ exist. Moreover, since the bisectional curvature of H_0 exists for a Hermitian metric, we conclude that the right hand side of (3.2) exists as a current. We summarize these in the corollary below:

Corollary 3.9. *An arbitrary Finsler metric on a holomorphic vector bundle is of the form $\phi(z; v) = \rho(z; [v])||v||_{H_0} = \gamma \circ \pi(z; v)||v||_{H_0}$ where H_0 is a smooth Hermitian metric on E. If $\phi \in \mathcal{F}_{\mathrm{wr}}$ then*

$$dd^c[\log ||P||_H^2] = dd^c[\log \phi^2] = dd^c[\log \gamma^2] + dd^c \log ||P||_{H_0}^2$$

as currents. The identity descends to the following identity

$$c_1(\mathcal{O}_{\mathbb{P}(E)}(1), h^*) = c_1(\mathcal{O}_{\mathbb{P}(E)}(1), h_0^*) + dd^c[\log \rho^2]$$

on $\mathbb{P}(E)$ with $c_1(\mathcal{O}_{\mathbb{P}(E)}(1), h^*)$ the Chern current of the weakly singular metric h corresponding to the Finsler metric ϕ.

The bisectional curvature current of ϕ is by definition the current

$$\frac{< K_0(\cdot, \cdot)P, P >_{H_0}}{||P||_{H_0}^2} + dd^c[\log \gamma^2] - d_v d_v^c[\log \gamma^2] \tag{3.3}$$

and shall be denoted symbolically denoted by

$$\frac{< K(\cdot, \cdot)P, P >_\phi}{\phi^2}.$$

Definition 3.10. A positive (p, p)-current T on a complex manifold X is said to be *strictly positive* if there exists a smooth (p, p)-form on X such that $T - \gamma > 0$. A (p, p)-current T is *strictly negative* if $-T$ is strictly positive. □

The concept of big bundles is standard in algebraic geometry:

Definition 3.11. A holomorphic line bundle $\mathcal{L}$ over a complex variety of complex dimension n is said to be *big* if $\dim H^0(M, \mathcal{L}^m) = O(m^n)$. A holomorphic vector bundle E is said to be *big* if the Serre line bundle $\mathcal{L}$ over $\mathbb{P}(E^\vee)$ is big. □

Remark 3.12. By a theorem of Grothendieck/Serre (see [32] and [11]):

$$H^0(M, sym^m E) \cong H^0(\mathbb{P}(E^\vee), \mathcal{L}^m)$$

where $\mathcal{L}$ is the Serre line bundle over $\mathbb{P}(E^\vee)$. Since $\dim E^\vee = n + r - 1$ where $r = \text{rank} E$ and $n = \dim M$ we see that that E is big if and only if

$$h^0(M, sym^m E) = \dim H^0(M, sym^m E) = O(m^{n+r-1}).$$

□

Definition 3.13. A compact complex manifold M is said to be *Moishezon* if the transcendence degree of its meromorphic function field $\mathcal{M}(M) = \dim_{\mathbb{C}} M$. □

By Chow's theorem, for a projective variety, the rational function field is the same as the meromorphic function field and that the transcendence degree of the rational function field is equal to its dimension. Thus Moishezon manifolds are generalization of projective manifolds. A famous theorem of Moishezon ([48]) asserts that a Moishezon manifold M is projective if and only if it is Kähler.

If $\mathcal{L}$ is a big line bundle then $h^0(M, \mathcal{L}^m) = \dim H^0(M, \mathcal{L}^m) = O(m^n)$. Let $\Phi_m = [\sigma_0, ..., \sigma_N] : M \to \mathbb{P}(N)$ be the map defined by a basis $\{\sigma_0, ..., \sigma_N\}$ of $H^0(M, \mathcal{L}^m)$. For a big bundle the map Φ_m is bimeromorphic onto its image for $m >> 0$. The closure $\overline{\Phi_m(M)}$, being a projective subvariety, is Moishezon. Since the meromorphic function field of M is isomorphic to that of $\overline{\Phi_m(M)}$ we see that M is Moishezon. The converse is also valid. Indeed we have the following result (see [16], [37] and [38]):

Theorem 3.14. *Let M be a compact complex manifold. Then the following statements are equivalent:*

(1) *M is Moishezon.*

(2) *There is a closed strictly positive Hodge* $(1,1)$*-current on M.*

(3) *There is a holomorphic line bundle $\mathcal{L}$ over M with a singular metric $h \in \mathcal{H}_{\text{wr}}(\mathcal{L})$ such that the Chern current $c_1(\mathcal{L}, h)$ is strictly positive* (*Definition 3.10*).

(4) *There is a holomorphic line bundle $\mathcal{L}$ over M with a singular metric $h \in \mathcal{H}_{\rm wr}(\mathcal{L})$ which is smooth outside a subvariety of lower dimension such that the Chern current $c_1(\mathcal{L}, h)$ is strictly positive.*

(5) *There is a big holomorphic line bundle over M.*

(6) *M is bimeromorphic to a projective manifold.*

Recall that a closed positive $(1,1)$-current ω is said to be a *Hodge-current* if $[\omega]$ is integral, i.e., $[\omega] \in H^2(M, \mathbb{Z})$. The preceding Theorem is the analogue of Kodaira's characterization of projective varieties via the existence of an ample line bundle. Using this we have the following characterization of big vector bundles in terms of the strict positivity (see Corollary 3.9) of the bisectional curvature current ((3.3)) of a Finsler metric (see [10]).

Theorem 3.15. *Let M be a compact complex manifold and let E be a holomorphic vector bundle of rank ≥ 2 over M. Then the following statements are equivalent:*

(1) *The vector bundle E is big.*

(2) *The line bundle $\mathcal{L}$ over $\mathbf{P}(E^\vee)$ is big.*

(3) *There exists $h \in \mathcal{H}_{\rm wr}(\mathcal{L})$ on the line bundle $\mathcal{L}$ over $\mathbb{P}(E^\vee)$ such that $c_1(\mathcal{L}, h)$ is a strictly positive current.*

(4) *There is a singular Finsler metric $F \in \mathcal{F}_{\rm wr}(E)$ such that the curvature current is strictly positive.*

We have also the dual formulation ([10]):

Theorem 3.16. *Let M be a compact complex manifold and let E be a holomorphic vector bundle of rank ≥ 2 over M. Then the following statements are equivalent:*

(1) *The vector bundle $E^\vee$ is big.*

(2) *The line bundle $\mathcal{L}$ over $\mathbb{P}(E)$ is big.*

(3) *There exists $h \in \mathcal{H}^{\rm wr}(\mathcal{L})$ on the line bundle $\mathcal{L}$ over $\mathbb{P}(E)$ such that $c_1(\mathcal{L}, h)$ is a strictly positive current.*

(4) *There is a singular Finsler metric $F \in \mathcal{F}^{\rm wr}(E)$ such that the bisectional curvature current is strictly negative relative to horizontal forms.*

Corollary 3.17. *Under any of the condition of the preceding Theorem, $M, \mathbb{P}(E)$ and $\mathbb{P}(E^\vee)$ are Moishezon.*

§4. Oka's Principle

4.1. The Oka-Grauert-Gromov Principle

The main references of this sections are [51], [25], [26], [30], [31], [23] and [73].

Let X and Y be complex manifolds. The pair (X, Y) is said to be an *Oka pair* if the Oka Principle holds, namely,

Oka Principle: *Every continuous mapping $f : X \to Y$ is homotopic to a holomorphic mapping $g : X \to Y$.*

It was first discovered by Oka that Stein manifolds provide the most natural settings for the resolution of Oka's principle. Extrinsically, Stein manifolds are simply closed submanifolds of $\mathbb{C}^N$. The following theorem is a partial list of various different characterization of Stein manifolds:

Theorem 4.1. *Let X be a complex manifold. Then the following conditions are equivalent.*

(1) *X is Stein.*

(2) *Global holomorphic functions separate points and X is holomorphically convex.*

(3) *For any discrete sequence, finite or infinite, $\{x_i \mid i \in I\}$ and any sequence of complex numbers $\{c_i \mid i \in I\}$ there exists a holomorphic function on X such that $f(x_i) = c_i$ for all $i \in I$.*

(4) *For any point $x \in X$ there exists a sequence of holomorphic functions $\{f_i\}$ on X such that x is an isolated point of $A = \cap_i\{z \in X \mid f_i(z) = 0\}$ and X is holomorphically convex.*

(5) *X contains no compact complex subvariety of strictly positive dimension and X is holomorphically convex.*

(6) *The sheaf cohomology groups $H^i(X, \mathcal{S}) = 0$ for all $i \geq 1$ and for all coherent sheaf $\mathcal{S}$ on X.*

(7) *There exists a proper real valued function $\tau : X \to \mathbb{R}$ such that the Levi form $dd^c\tau = \sqrt{-1}\partial\bar{\partial}\tau$ is positive definite.*

(8) *There is a holomorphic embedding of X as a closed complex submanifold of $\mathbb{C}^N$ for some N.*

Remark 4.2. A function τ satisfying the conditions in (7) above is said to be a strictly plurisubharmonic (or pseudoconvex) exhaustion. On $\mathbb{C}^n$ there is a distinguished exhaustion $\tau = ||z||^2 = |z_1|^2 + \cdots + |z_n|^2$ such that $dd^c\tau$ is the Euclidean metric. Moreover (see Remark 3.2), on $\mathbb{C}^n \setminus \{0\}$ the function $\phi = \log \tau$ satisfies the complex homogeneous Monge-Ampère equation. For any (closed) complex submanifold X of $\mathbb{C}^n$ the function $\tau = ||z||^2|_X$ is a strictly plurisubharmonic exhaustion of X and $dd^c\tau$ is the metric induced by the Euclidean metric. If X is defined by polynomials then it is said to be affine algebraic and the volume (in the metric $dd^c\tau$) of X grows polynomially:

$$\int_{X \cap B^n(r)} (dd^c\tau)^m = O(r^m)$$

where $B^n(r)$ is the ball of radius r, centered at the origin, in $\mathbb{C}^n$ and $m = \dim_{\mathbb{C}} X$.

The theory of Oka's principle originated from the most insightful work of Oka ([50]) in 1939 that, on a domain of holomorphy (= Stein domain) in $\mathbb{C}^n$, the Second Cousin Problem is holomorphically solvable if and only if it is continuously solvable:

Theorem 4.3. (Oka 1939) *Let X be a complex space satisfying the condition $H^1(X, \mathcal{O}_X) = 0$ (this is the case if X is Stein) and $\{U_i\}$ be an open cover of X by Stein open subsets. Let c_i be continuous non-vanishing functions such that $c_i/c_j = f_{ij}$ is non-vanishing and holomorphic on $U_i \cap U_j$. Then there exist non-vanishing holomorphic functions f_i on U_i such that $f_i/f_j = c_i/c_j$ on $U_i \cap U_j$ for all i, j.*

The result of Oka was extended by Grauert in the famous articles in 1957 [22] and 1958 [23]. The main results are:

Theorem 4.4. *Let X be a Stein manifold and Y a complex homogeneous manifold then (X, Y) is an Oka pair.*

Theorem 4.5. *Let X be a Stein space and G a complex Lie group. Then*

(a) Every continuous principal G-bundle admits a structure of a holomorphic principal G-bundle.

(b) Two holomorphic principal G-bundles are holomorphically isomorphic if and only if they are continuously isomorphic. In fact every continuously isomorphism can be homotopically deformed to a holomorphic isomorphism.

Corollary 4.6. *Let X be a contractible Stein manifold. Then every holomorphic vector bundle E is holomorphically trivial.*

Grauert's Theorem that the homotopy principle holds for mappings from a Stein manifold into a homogeneous manifold was extended by Gromov [28, 29] to the case where the target space contains sufficiently many holomorphic copies of the complex line $\mathbb{C}$. For instance to the cases where the target space is (i) a *spray* or, is (ii) subelliptic (we refer the readers to Gromov [28], [29] and the more recent works of Forstnerič [21] for details, a brief survey of results can be found in [73]).

4.2. Oka Principle with Growth Condition

There is another extension of the Oka/Grauert principle due to Cornalba and Griffiths [13] which may be thought of as Oka's Principle with growth conditions (see also [47], [72] and [74]). Their theory is based on the fact that a holomorphic vector bundle over a projective algebraic variety is necessarily algebraic (Chow's Theorem) yet the analogue is false over *affine* algebraic varieties (i.e., subvarieties of $\mathbb{C}^N$ defined by *polynomials* and, a priori, Stein); namely, there exist holomorphic vector bundles such that the transition functions cannot be chosen as polynomials (which means that the bundle does not admit an algebraic structure). Cornalba and Griffiths looked for the next best possibility: whether the transition functions can be chosen as *holomorphic functions of exponential type of finite order* (we shall simply say functions of finite order. All classical functions, e.g., theta functions zeta functions, Weierstrauss $\wp$-functions, Bessel functions etc., belong to this category). They succeeded in establishing this in the case of line bundles (see [13]):

Theorem 4.7. *Every holomorphic line bundle $\mathcal{L}$ over an affine algebraic manifold X admits a finite order structure. More precisely, there exits an affine open cover $\{U_i\}$ of X such that $\mathcal{L}|_{U_i}$ is trivial and on any of the intersection $U_i \cap U_j$ the transition function $g_{ij} = \exp P_{ij}$ where P_{ij} is a polynomial on the affine open set $U_i \cap U_j$.*

The group of equivalence classes of holomorphic line bundles is known as the Picard group, denoted $\mathrm{Pic}(X)$. Those that admit a finite order structure shall be denoted by $\mathrm{Pic}_{\mathrm{f.o.}}(X)$. It is well-known that, on a Stein manifold, the Chern map:

$$c_1 : \mathrm{Pic}(X) \to H^2(X, \mathbb{Z})$$

(sending a line bundle to its first Chern class) is an isomorphism. The theorem of Cornalba-Griffiths asserts that, for an affine manifold:

$$c_1 : \mathrm{Pic}_{\mathrm{f.o.}}(X) \to H^2(X, \mathbb{Z})$$

is an isomorphism. We remark in passing that, for a Stein manifold, $H^{2,0}(X, \mathbb{Z}) = H^{0,2}(X, \mathbb{Z}) = 0$ hence $H^2(X, \mathbb{Z}) = H^{1,1}(X, \mathbb{Z})$.

The method of Cornalba-Griffiths does not extend to the case of vector bundles of higher rank. The proof in the case of line bundle is based on equivalent reformulations of the finite order condition (see [13], [47] and [72]):

Theorem 4.8. *Let $\mathcal{L}$ be a line bundle over an affine algebraic manifold. Then the following conditions are equivalent.*

(1) *$\mathcal{L}$ admits a finite order structure, namely, there exists an open affine local trivializing cover $\{U_i\}$ such that the transition g_{ij} is holomorphic and of finite order on the affine open set $U_i \cap U_j$.*

(2) *There exists a hermitian metric h on $\mathcal{L}$ such that the first Chern form of the metric satisfies the condition*

$$|c_1(\mathcal{L}, h)| \leq C\phi^\lambda dd^c\tau$$

for positive constants C and λ, where the function $\tau = ||z||^2|_X$ is obtained via an embedding of X in $\mathbb{C}^N$ as an affine algebraic subvarieties and $||z||^2 = |z_1|^1 + \cdots + |z_N|^2$(see Remark 4.2).

(3) *There exists a finite order holomorphic map $f : X \to \mathbb{P}^N$ such that $f^*\mathcal{O}_{\mathbb{P}^N}(1) = \mathcal{L}$ where $\mathcal{O}_{\mathbb{P}^N}(1)$ is the hyperplane line bundle.*

(4) *There exists a global holomorphic section σ of $\mathcal{L}$ such that the zero divisor $Y = [\sigma = 0]$ is of finite order.*

The concept of a holomorphic map and subvarieties of finite order are defined, in a natural way, as in Nevanlinna Theory (see [13], [59] and [72] for the precise definitions). Indeed the proof of the preceding theorem uses analytic techniques in Nevanlinna Theory as well as techniques in Hörmander's L^2-theory for the $\bar{\partial}$ problem.

The analogue of the notions of finite order can also be formally extended to vector bundles of higher rank:

(I) *A holomorphic vector bundle E over an affine algebraic manifold X is said to be of finite order if there exists an open affine local trivializing cover $\{U_i\}$ such that the transition g_{ij} is of finite order on the affine open set $U_i \cap U_j$ for all i and j.*

(IIa) *A holomorphic vector bundle over an affine algebraic manifold X is said to be of finite order if there exists a Hermitian metric h on E such that the mixed bisectional curvature Θ satisfies the condition that*

$$|\Theta| \leq C\tau^{\lambda} dd^c \tau$$

for some positive constants C and λ.

(III) *A holomorphic vector bundle E over X is said to be of finite order if there exists a holomorphic map of finite order $f : X \to \mathrm{Gr}(r, N)$ for some m such that $E = f^*\mathcal{U}$ where $\mathcal{U}$ is the dual universal bundle over $\mathrm{Gr}(r, N)$.*

(VI) *A holomorphic vector bundle E of rank r on X is of finite order if there exists global holomorphic sections $\sigma_1, ..., \sigma_r$ of E such that $Y_q = [\sigma_1 \wedge \cdots \wedge \sigma_{r-q+1} = 0]$ is a subvariety of codimension $\min\{q, n\}$ and of finite order for $q = 1, ..., r$.*

For E of rank ≥ 2 Cornalba-Griffiths established only that (II) implies (III). There is a well-known counter-example, due to Cornalba and Shiffman, to the naive form of the analytic version of the classical algebraic Bezout Theorem. They constructed a holomorphic map of order *zero* for which the the growth of the number of common zeros is of *infinite order.* This means that definition (VI) requires modification.

In the thesis of my student Maican [47], the Hermitian metric in definition (IIa) was replaced by a Finsler metric:

(IIb) *A holomorphic vector bundle over an affine algebraic manifold X is said to be of finite order if there exists a Finsler metric h on E such that the bisectional curvature Θ satisfies the condition that*

$$|\Theta| \leq C\tau^{\lambda} dd^c \tau$$

for some positive constants C and λ.

The idea is to work on the projectivization of the bundle, transferring the problem to that about the Serre line bundle $\mathcal{L}$. This is the same circle of ideas used in the previous sections. The advantage being that we only have to deal with line bundles. The disadvantage is that we have to work on the bundle space rather than the original affine algebraic manifold X. The first technical problem is that $\mathbb{P}(E^{\vee})$ is neither affine algebraic manifold nor projective algebraic. The classical theory (for instance, vanishing theorems and embedding theorems are established for projective varieties and for Stein varieties; this is also the case for Nevanlinna theory) is no longer applicable and has to be extended. Fortunately, the classical $\bar{\partial}$-theory works well for Kähler manifolds (in fact Finsler-Kähler is enough) and in our case, it turns out that $\mathbb{P}(E^{\vee})$ is Kähler. An affine manifold X is Kähler with the Kähler metric ($dd^c\tau$ where $\tau = ||z||^2|_X$) induced by the Euclidean metric via an embedding of X in $\mathbb{C}^N$ as an algebraic submanifold. By a result in [10], we know that, for any hermitian metric h_0 along the fibers of the Serre line bundle $\mathcal{L}$ there is a convex increasing function χ such that $\chi(\rho)$ is a strictly plurisubharmonic exhaustion with the property that

$$\omega = c_1(\mathcal{L}, h_0) + p^* dd^c \chi(\tau) = c_1(\mathcal{L}, p^*(\chi(\tau))h_0) > 0$$

is a Kähler metric on $\mathbb{P}(E^{\vee})$ where $p : \mathbb{P}(E^{\vee}) \to X$ is the projection map. In fact this is true whenever the base manifold is Stein and for an algebraic submanifold we have the added benefit that χ can be chosen to be a polynomial.

To deal with the problem we first transfer the finite order condition (IIb) to the Serre line bundle $\mathcal{L}$ by using the correspondence between the Finsler metrics on E and the Hermitian metrics along $\mathcal{L}$ (Proposition 2.4) as well as the relationship between the the first Chern form $c_1(\mathcal{L}, h)$ and the bisection curvature Θ_h of the corresponding Finsler metric:

(IIc) *A holomorphic vector bundle E of rank $r \geq 2$ over an affine algebraic manifold X is said to be of finite order if there exists a Hermitian* *metric h along the Serre line bundle $\mathcal{L}$ on $\mathbb{P}(E^\vee)$ such that*

$$c_1(\mathcal{L}, h) \leq dd^c \tau^\lambda \tag{4.1a}$$

for some $\lambda > 0$, the Ricci curvature satisfies the condition:

$$c_1(\mathcal{L}, h) - \operatorname{Ric} c_1(\mathcal{L}, h) \geq \epsilon c_1(\mathcal{L}, h) > 0 \tag{4.1b}$$

for some positive constant ϵ and moreover,

$$\int_0^r \frac{dt}{t} \int_{\{\tilde{\tau} \leq r\}} c_1(\mathcal{L}, h) \wedge dd^c (dd^c \log \tilde{\tau})^{n-1} \wedge \omega^r_{\mathbb{P}(E)} = O(r^\lambda). \tag{4.1c}$$

The last condition is formulated so that Nevanlinna Theory can be conveniently applied. The formulation above emphasis that conditions on the bisectional curvature of a Finsler metric can always be equivalently formulated in terms of conditions on the first Chern form of the Serre line bundle.

Define for $p, q \geq 0$:

$$A^{p,q}_{\text{f.o.}}(\mathbb{P}(E^\vee), \mathcal{L}) = \{\eta \mid \eta \text{ finite order } \mathcal{L}\text{–value form of type } (p,q)\}$$

where finite order means

$$\int_{\mathbb{P}(E^\vee)} |\eta|^2_{hg} e^{-C\tilde{\tau}^\lambda} \omega^{n+r-1}_{\mathbb{P}(E^\vee)} < \infty$$

and

$$\int_{\mathbb{P}(E^\vee)} |\bar{\partial}\eta|^2_{hg} e^{-C\tilde{\tau}^\lambda} \omega^{n+r-1}_{\mathbb{P}(E^\vee)} < \infty$$

for some positive constants C, λ; g is the Kähler metric on $\mathbb{P}(E^\vee)$, h is the finite order metric on $\mathcal{L}$ (see (IIc)) and $\tilde{\tau} = p^*\tau$. For $p \geq 0, q \geq 1$ let

$$\begin{cases} Z^{p,q}_{\text{f.o.}}(\mathbb{P}(E^\vee), \mathcal{L}) = \{\eta \in A^{p,q}_{\text{f.o.}}(\mathbb{P}(E^\vee), \mathcal{L}) \mid \bar{\partial}\eta = 0\}, \\ B^{p,q}_{\text{f.o.}}(\mathbb{P}(E^\vee), \mathcal{L}) = \bar{\partial} A^{p,q-1}_{\text{f.o.}}(\mathbb{P}(E^\vee), \mathcal{L}) \end{cases}$$

and, for $q \geq 0$, define the finite order cohomology group:

$$H^q_{\text{f.o.}}(\mathbb{P}(E^\vee), \mathcal{L}) = Z^{0,q}_{\text{f.o.}}(\mathbb{P}(E^\vee), \mathcal{L}) / B^{0,q}_{\text{f.o.}}(\mathbb{P}(E^\vee), \mathcal{L}).$$

It can be shown via standard L^2-theory that

$$H^0_{\text{f.o.}}(\mathbb{P}(E^\vee), \mathcal{L}) = \{\sigma \mid ||\sigma||^2_{h,\lambda} < \infty \text{ for some } \lambda > 0\} \tag{4.2}$$

is the space of L^2*-finite order holomorphic sections of* $\mathcal{L}$ with integral norm

$$||\sigma||^2_{h,\lambda} = \int_{\mathbb{P}(E^\vee)} |\sigma|^2_h e^{-C\tilde{\tau}^\lambda} \omega^{n+r-1}_{\mathbb{P}(E^\vee)} < \infty.$$

We have a vanishing theorem with growth condition (see [47], [72]):

Theorem 4.9. *Let X be an affine manifold and E a vector bundle of rank $r \geq 2$. Assume that E is of finite order in the sense of Definition V. Then*

$$H^q_{\text{f.o.}}(\mathbb{P}(E^\vee), \mathcal{L}) = 0$$

for all $q \geq 1$ where $\mathcal{L}$ is the Serre line bundle on $\mathbb{P}(E^\vee)$.

Proof. (Sketch) By the Weitzenbock formula we have the estimate:

$$||\bar{\partial}\eta||^2_{hg,\lambda} + ||\bar{\partial}^*\eta||^2_{h,\lambda} \geq \left((c_1(\mathcal{L}, h) - \text{Ric}\, \omega_{\mathbb{P}(E^\vee)} \wedge \eta, \eta\right)_{hg,\lambda}$$

where $\text{Ric}\, \omega_{\mathbb{P}(E^\vee)} = dd^c \log \det g$ and $(\ ,\ )_{hg,\lambda}$ is the integral inner product defined by hg with weight $e^{-C\tilde{\rho}^\lambda}$. By (4.1a) and (4.1b) we arrive at the a priori estimate

$$||\bar{\partial}\eta||^2_{hg,\lambda} + ||\bar{\partial}^*\eta||^2_{hg,\lambda} \geq \epsilon ||\eta||^2_{hg,\lambda}$$

and the standard harmonic theory implies the vanishing of the cohomology groups for $q \geq 1$. □

We also modified condition (IV) by going up to the projectivized bundle:

(IVc) A holomorphic vector bundle E, of rank $r \geq 2$, over a special affine algebraic manifold X is said to be of finite order if there exists an injective holomorphic immersion $F : \mathbb{P}(E^\vee) \to \mathbb{P}^N$ such that $F^*(\mathcal{O}_{\mathbb{P}^N}(1)) = \mathcal{L}_{\mathbb{P}(E^\vee)}$ and satisfying the following estimate

$$\int_0^R \frac{dt}{t} \int_{\tilde{\rho} \leq t} (dd^c \log \tilde{\rho})^{n-1} \wedge F^*\omega^r_{FS} = O(R^\lambda).$$

The vanishing theorem above implies, as in the classical situation of Stein Embedding Theorem and Kodaira Embedding Theorem, the following Embedding Theorem (see [72]):

Theorem 4.10. *There exists a finite dimensional linear subspace W of the space of finite order sections $H^0_{\text{f.o.}}(\mathbb{P}(E^\vee), \mathcal{L})$ as defined in* (4.2) *such that the map*

$$F = [\sigma_0, ..., \sigma_N] : \mathbb{P}(E^\vee) \to \mathbb{P}(W)$$

defined by a basis $\sigma_0, ..., \sigma_N$ of W is an injective holomorphic immersion and that

$$F^*(\mathcal{O}_{\mathbb{P}(W)}(1)) = \mathcal{L}$$

where $\mathcal{O}_{\mathbb{P}(W)}(1)$ is the hyperplane bundle over $\mathbb{P}(W) \cong \mathbb{P}^N$.

It was shown in [70] that the definitions (IIb), (IIc) and (IVc) for finite order vector bundles are equivalent. For this we need some analytic machinery from Nevanlinna Theory which we now briefly describe.

Definition 4.11. A Kähler manifold (X, ω) of dimension n is said to be a *generalized parabolic manifold* if there is a plurisubharmonic exhaustion ψ such that

(i) $\{\psi = -\infty\}$ is a closed subset of strictly lower dimension,

(ii) ψ is smooth outside $\{\psi = -\infty\}$ and satisfies the *generalized homogeneous Monge-Ampere equation*

$$(dd^c\psi)^{k-1} \wedge \omega^{n-k} \not\equiv 0, \quad (dd^c\psi)^k \wedge \omega^{n-k} \equiv 0 \tag{4.8}$$

on $X \setminus \{\psi = -\infty\}$ for some integer $1 \leq k \leq n$. Such an exhaustion is said to be a *generalized parabolic exhaustion.* If $k = n$ the exhaustion function τ is said to be *parabolic* and the manifold is said to be a *parabolic manifold.* □

Nevanlinna theory on parabolic manifolds is well-known (see for example [59], [60] and [69]). The Euclidean space $\mathbb{C}^n$ is parabolic with parabolic exhaustion $\psi = \log ||z||^2$ (see Remark 4.2). More generally, an affine algebraic variety is also parabolic with a parabolic exhaustion function obtained as follows. An affine algebraic variety of complex dimension n may be exhibited as a finite branched cover $\pi : X \to \mathbb{C}^n$. Then $\psi = \pi^* \log ||z||^2$ is a parabolic exhaustion. We are interested in vector bundles and projectivized vector bundles over affine algebraic manifolds. These are not parabolic manifolds in the classical sense, however, they are generalized parabolic manifolds ([72] and [74]):

Theorem 4.12. *Let E be a holomorphic vector bundle over an affine algebraic manifold. Then $E, E^\vee, \mathbb{P}(E), \mathbb{P}(E^\vee)$ are generalized parabolic manifolds.*

Theorem 4.13. *All standard results in Nevanlinna theory that are valid on parabolic manifolds are also valid on generalized parabolic manifolds.*

It would take much space to present Nevanlinna theory in a proper way so we simply refer the readers to the references cited above for the details. Theorem 4.13 for parabolic manifolds was used by Cornalba-Griffiths in the proof of Theorem 4.8 for line bundles over affine manifolds. Using Theorem 4.13 to the Serre line bundle over the generalized parabolic manifold $\mathbb{P}(E^\vee)$ we get analogously:

Theorem 4.14. *Let E be holomorphic vector bundle over an affine algebraic manifold then the three definitions, $(IIb), (IIc)$ and (IVc), of finite order structure on E are equivalent.*

Remark 4.15. We remark that, in his thesis, M. Maican ([47]) established the implications

$$(IIc) \implies (IVc) \implies (IIb).$$

The proof in the preceding can be made more natural and less technical if one can solve a certain Monge-Ampére Type Equation on $\mathbb{P}(E^\vee)$. We assume that $\mathbb{P}(E^\vee)$ is of finite order in the sense of definition (IIb). We have a Kähler metric on $\mathbb{P}(E^\vee)$ given by $\omega = F^*\omega_{FS} = c_1(\mathcal{L}, F^*h_{FS})$. We do not have a good estimate on $dd^c \log\det(g_{i\bar{j}})$, the Ricci of ω^{n+r-1} where $(g_{i\bar{j}})$ are the components of ω. On the other hand, we have:

$$K^{-1}_{\mathbb{P}(E^\vee)} = \mathcal{L}^r \otimes p^* \det E \otimes K_X^{-1}.$$

Definition (IVc) implies that there is a finite order metric on $\mathcal{L}^r$ and $\det E$ and K_X^{-1}, being line bundles on X, also admit finite order metric. Thus the anti-canonical bundle $K^{-1} = K^{-1}_{\mathbb{P}(E^\vee)}$ admits a metric H such that $-c_1(K, H) = c_1(K^{-1}, H) > 0$ and is of finite order. Let dV be a volume form on $\mathbb{P}(E^\vee)$ so

$$dV = \sqrt{-1}^{n+r-1}\Gamma dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n \wedge dv_1 \wedge d\bar{v}_1 \cdots \wedge dv_{r-1} \wedge d\bar{v}_{r-1}$$

in terms of coordinates. The following problem is open.

Open Problem 4. *Is there a Kähler metric ω such that*

$$\operatorname{Ric} \omega^{n+r-1} = dd^c \log\det(g_{i\bar{j}}) = dd^c \log G = \operatorname{Ric} dV$$

where $g_{i\bar{j}}$ are the components of ω?

We may fix a Kähler metric ω_0 on $\mathbb{P}(E^\vee)$. Then

$$dV = e^{f_0}\omega_0^{n+r-1}$$

for some real-valued function f_0. We seek a Kähler metric

$$\omega = \omega_0 + dd^c\gamma$$

where γ is a real-valued function such that

$$(\omega_0 + dd^c\gamma)^{n+r-1} = e^{f_0}\omega_0^{n+r-1}.$$

□

§5. Finsler Geometry and Hyperbolic Geometry

5.1. Holomorphic Sectional Curvature of Finsler Metrics

The main references of this section are [1], [55], [65] and [10]. Classically it is well-known that $\mathbb{P}^1 \setminus \{3 \text{ points}\}$ is covered by the unit disc and so it admits a Hermitian metric of constant negative curvature. Very little is known in higher dimension. In the special case of complements of *hyperplanes* in $\mathbb{P}^n$, it is known that $\mathbb{P}^n \setminus \{2^n+1$ hyperplanes in general position$\}$ admits a *continuous Finsler* metric with *hsc* (= holomorphic sectional curvature) $\leq -c^2 < 0$ where c is a constant. We shall simply refer to this by saying that the *hsc is strongly negative.* On a compact manifold strongly negativity is the same as negativity.

The concept of holomorphic sectional curvature of a Finsler metric is defined as follows (see [11]). Let h be a *smooth* Finsler metric on a complex manifold M and X be a tangent vector at a point $x \in M$. Then

$$\mathrm{hsc}(X) = \sup_C \{G_{h|_C}(X)\}$$

where the supremum is taken over all local smooth complex curve (Riemann surface) C through the point x and $G_{h|_C}(X)$ is the Gaussian curvature of the metric on C induced by h (for a Riemann surface, Finsler = Hermitian = Kähler). The definition is equivalent to the usual concept as defined in [34] if h is Hermitian (via the Hermitian curvature) and can be expressed via the bisectional curvature (*abbrev. hbsc from now on*)

$$k(X,Y) = \frac{< K(X,X)Y, Y >}{||X||^2||Y||^2}$$

by simply allowing the two sections X and Y be equal. Since the bisectional curvature is defined, as a current, for Finsler metric that are only weakly regular (see section 3 above) the hsc are also defined by taking $X = Y$ (see also section 4 in [10]). As we have seen earlier the bisectional curvature can be completely understood from the point of view of algebraic geometry. Namely, on a compact manifold, the condition that hbsc > 0 (resp. < 0) corresponds to the concept of "ampleness" of the tangent (resp. cotangent) bundle. By Theorem 2.15 the existence of a pseudoconvex Finsler metric with strongly negative bhsc implies that the manifold is hyperbolic. In fact the same is true under the weaker condition (goes back to Ahlfors):

Theorem 5.1. *A complex manifold is hyperbolic if there exists a continuous Finsler metric with strongly negative hsc.*

From the point of view of complex differential geometry the holomorphic sectional curvature is the most natural concept of curvature and it would be very important to find the algebraic analogue:

Open Problem 5. *Is there an algebraic geometric notion corresponding to the notion of the holomorphic sectional curvature?*

5.2. Logarithmic Vector Bundles

The main references for this section are [15], [18], [19], [33], [34], [56] and [75].

It is well-known that the space $\mathbb{P}^n \setminus \{2n+1$ hyperplanes in general position$\}$ is hyperbolic. The question is whether it admits strongly negatively curved continuous Finsler metric. We consider the case $n = 2$ searching for curves C, not just hyperplanes in $\mathbb{P}^2$ such that $M = \mathbb{P}^2 \setminus C$ admit Finsler metrics with strongly negative holomorphic *bisectional* curvature. *In view of the results in section 2 this is equivalent to the condition that the logarithmic tangent bundle* $E = T^*\mathbb{P}^2(\log C)$ *is ample.* Global sections of E are simply meromorphic 1-forms on $\mathbb{P}^2$ which are holomorphic on $\mathbb{P}^2 \setminus C$ and the singularities along the curve C is of logarithmic type, i.e., of the form df/f locally where $C = [f = 0]$. The reason for using such forms is based on the fact that $\mathbb{P}^2$ has no global holomorphic forms and, on the other hand, meromorphic forms with complicated singularities are not closed forms. It is a result of Deligne [13] that logarithmic forms preserve a very important property of global holomorphic forms on compact varieties: they are closed forms (∂-closed and $\bar{\partial}$-closed). The singularities along C also implies that the Finsler metric with strongly negative bisectional curvature is a *complete* metric on $\mathbb{P}^2 \setminus C$. We present below a few basic facts about logarithmic forms and refer the readers to the deep and important works [15], [33], [34] and [56] for the general theory.

Let M be a compact complex manifold of dimension n and $C \subset M$ be a divisor. We assume that $C = \sum_{i=1}^{q} C_i$ where each C_i is irreducible. Given a local meromorphic function f on an open set $U \subset M$ with zeros and poles contained in $U \cap C$ then $d \log f$ is a meromorphic 1-form on U with poles in $U \cap C$. Denote by $\mathcal{M}_C(M)^*$ the sheaf of germs of non-zero meromorphic functions on M with zeros and poles contained in D. Then

$d\log(\mathcal{M}_C(M)^*)$ is a subsheaf of the sheaf of germs of meromorphic 1-forms, $\Omega^1_M(C)$, with poles along C. It is clear that the following sequence is exact

$$0 \to \mathbb{C}^* \to \mathcal{M}_C(M)^* \xrightarrow{d\log} d\log(\mathcal{M}_C(M)^*) \to 0.$$

We have an induced long exact sequence

$$0 \to H^0(M, \mathbb{C}^*) \to H^0(M, \mathcal{M}_C(M)^*) \to H^0(M, d\log \mathcal{M}_C(M)^*) \to$$
$$\xrightarrow{\delta} H^1(M, \mathbb{C}^*) \to \cdots$$

with connecting homomorphism δ. Note that we have

$$H^1(M, \mathbb{C}^*) = \mathrm{Hom}(\pi_1(M), \mathbb{C}^*) = \mathrm{Hom}(H_1(M, \mathbb{Z}), \mathbb{C}^*).$$

At a *regular* point $x \in C_i$ there exists local coordinate $(U, z_1, ..., z_n)$ such that $C \cap U = \{z_1 = 0\}$ and any $\omega \in H^0(M, d\log \mathcal{M}(M)^*)$ is of the form

$$\omega|_U = a_i \frac{dz_1}{z_1} + \eta_U$$

where a_i is an integer (independent of U) and $\eta_U = \sum_{i=2}^n a_i(z) dz_i$ is a regular 1-form on U. The integer a_i is the residue of ω along C_i and we define a divisor

$$\mathrm{div}\,(\omega) = \sum_{i=1}^q \mathrm{Res}_\omega(C_i) C_i.$$

The divisor $\mathrm{div}\,(\omega)$ naturally decomposed as

$$\mathrm{div}\,(\omega) = D_0(\omega) + D_\infty(\omega)$$

where D_0 and D_∞ are respectively, the zero and pole divisors of ω:

$$D_0(\omega) = \sum_{\mathrm{Res}_\omega(D_i) > 0} \mathrm{Res}_\omega(C_i) C_i, \;\; D_\infty(\omega) = \sum_{\mathrm{Res}_\omega(D_i) < 0} \mathrm{Res}_\omega(C_i) C_i.$$

Let $\pi : \tilde{M} \to M$ be the universal cover of M. Fix a point $\tilde{x}_0 \in \tilde{M} \setminus \pi^{-1}(C)$ and define

$$\theta(\tilde{x}) = \exp(\int_{\tilde{x}_0}^{\tilde{x}} \pi^* \omega).$$

This function is related to the connecting homomorphism:

$$H^0(M, d\log \mathcal{M}_C(M)^*) \to\xrightarrow{\delta} H^1(M, \mathbb{C}^*)$$

by the formula:

$$\delta(\omega)(\gamma) = \frac{\theta(\gamma\tilde{x})}{\theta(\tilde{x})} = \exp(\int_\gamma \omega),\ \gamma \in \pi_1(M).$$

Definition 5.2. *The sheaf of germs of logarithmic* 1*-forms with poles along a divisor* C*, denoted* $\Omega^1_M(\log C)$*, is the sheaf generated by* $d\log(\mathcal{M}_C(M)^*)$ *over* $\mathcal{O}_M$*, i.e., it is the sheaf of germs of rational* 1*-forms of the type*

$$\sum f_i\omega_i$$

where $f_i \in (\mathcal{O}_M)_x, \omega_i \in (d\log\mathcal{M}_C(M)^*)_x$.

The next result is of fundamental importance:

Theorem 5.3. *Every global logarithmic* 1*-form is* ∂*-closed and* $\bar{\partial}$*-closed.*

Proposition 5.4. *Let* $\beta : (\tilde{M}, \tilde{C}) \to (M, C)$ *be a desingularization of* C *via a succession of monodial transformations with the properties that*

(*i*) *the restriction* $\beta : \tilde{M} \setminus \tilde{C} \to M \setminus C$ *is biholomorphic;*
(*ii*) $\tilde{C}$ *is a divisor of simple normal crossings.*

Then the induced maps

$$\beta^* : H^0(M, d\log\mathcal{M}_D(M)^*)) \to H^0(\hat{M}, d\log\mathcal{M}_{\tilde{C}}(\tilde{M})^*))$$

and

$$\beta^* : H^0(M, \Omega^1_M(\log C)) \to H^0(\hat{M}, \Omega^1_{\tilde{M}}(\log\tilde{C}))$$

are isomorphisms.

A divisor C in a complex manifold is said to be of simple normal crossings if every singular point x of C admits a local open coordinate $(U, z_1, ..., z_n)$ in M such that $U \cap U$ consists of coordinate axes, i.e.,

$$U \cap C = \{z \in U \mid z_1 \cdots z_k = 0\}. \tag{5.1}$$

The integer k may depend on the singular point $x \in C$.

Example 5.5. If the divisor C is of simple normal crossing then a point x with coordinate neighborhood as in (5.1) then a logarithmic 1-form is of the form

$$\sum_{i=1}^{k} a_i(z)\frac{dz_i}{z_i} + \eta$$

where a_i is a holomorphic function on U and η is a holomorphic 1-form on U. Note that it is important that in the expression above the singularity is of the form $dz_i/z_i = d\log z_i$. An expression dz_i/z_j is meromorphic but not of logarithmic type if $i \neq j$. □

For general divisor C we have:

Proposition 5.6. *The vector space $H^0(M, \Omega^1_M(\log C))$ over $\mathbb{C}$ admits a basis consisting of elements of $H^0(M, d\log \mathcal{M}_C(M)^*)$.*

We shall find conditions so that the bundle $E = T^*\mathbb{P}^2(\log C)$ is ample. The next result (see [45], [75]) reduces the problem to a problem on Chern numbers. This reduction is important because computing Chern numbers is much easier than computing curvature (which is quite impossible except for very special cases).

Theorem 5.7. *Let E be a rank 2 vector bundle over a non-singular compact complex surface M. Assume that E is spanned, $c_1^2(E) > c_2(E) > 0$ and $\det E$ is ample. Then E is ample.*

Note that since $\det E$ is a line bundle the ampleness is equivalent to the condition that $c_1(E) > 0$. Theorem 5.6 is obtained via the Riemann-Roch Theorem, the Bogomolov Theorem and a Lemma of Gieseker. In our case these Theorems take the following forms:

Theorem 5.8. (Riemann-Roch) *Let E be a holomorphic vector bundle of rank $r = 2$ over a compact surface X with $c_1^2(E) > c_2(E)$. Then $\chi(\mathrm{sym}^m E) = O(m^3)$.*

Theorem 5.9. (Bogomolov) *Let E be a holomorphic vector bundle of rank $r = 2$ over a compact surface X satisfying the conditions (i) $c_1^2(E) > c_2(E)$ and (ii) there exists a positive integer m_0 such that $K_X^{-1} \otimes (\det E)^{m_0}$ is effective. Then E is big, namely,*

$$h^0(X, \mathrm{sym}^m E) = \dim H^0(X, \mathrm{sym}^m E) = O(m^3).$$

Lemma 5.10. (Gieseker's Lemma) *If E is non-ample then there exists an effective irreducible curve C in X such that $E|_C$ admits a trivial quotient.*

Riemann-Roch and Bogomolov's Theorems imply (under the assumption of Theorem 5.7) that the bundle E is spanned (i.e., global sections span every fiber of E). Spannedness is not enough to guarantee ampleness as the Chern numbers of a trivial bundle are zero but is clearly spanned by global sections. In any case the condition that $\det E$ is ample (i.e., $c_1(E) > 0$ and Giesker's Lemma imply that E is ample.

Theorem 5.7 is then applied to the case $X = \mathbb{P}^2$ and $E = T^*X(\log C)$ where C is a curve in $\mathbb{P}^2$ with simple normal crossings. It remains to compute the Chern numbers. This is accomplished via the work of [56] (see also [19]). The outcome is the following theorem in [75]:

Theorem 5.11. *Let $C = C_1 + \cdots + C_q$ be a curve, of simple normal crossings, in $\mathbb{P}^2$ with smooth irreducible components C_i of degree d_i for $1 \leq i \leq q$. Assume that $K_{\mathbb{P}^2} + C$ is ample. Then $c_1^2(T^*\mathbb{P}^2(\log C)) - c_2(T^*\mathbb{P}^2(\log C)) > 0$ if and only if one of the following cases holds:*

$$\begin{cases} q \geq 5: & 1 \leq d_1 \leq d_2 \leq ... \leq d_q, \\ q = 4: & (i)\ d_1 = d_2 = 1, 2 \leq d_3 \leq d_4, \quad (ii)\ d_1 = 1, 2 \leq d_2 \leq d_3 \leq d_4, \\ & (iii)\ 2 \leq d_1 \leq d_2 \leq d_3 \leq d_4, \\ q = 3: & (i)\ d_1 = 1, d_2 = 3, 4 \leq d_3, \quad (ii)\ d_1 = 1, 4 \leq d_2 \leq d_3, \\ & (iii)\ d_1 = d_2 = 2, 3 \leq d_3, \quad (iv)\ d_1 = 2, 3 \leq d_2 \leq d_3, \\ & (v)\ 3 \leq d_1 \leq d_2 \leq d_3, \\ q = 2: & (i)\ d_1 = 4, 7 \leq d_2, \quad (ii)\ 5 \leq d_1 \leq d_2. \end{cases}$$

In order to get to ampleness we need the condition that $T^*\mathbb{P}(\log C)$ be spanned. This more difficult to verify. The following result can be found in [75]:

Corollary 5.12. *Let $C = C_1 + ... + C_q$ be any of the curves in the list below:*

$$\begin{cases} q \geq 5: & 1 \leq d_1 \leq d_2 \leq ... \leq d_q, \\ q = 4: & (i)\ d_1 = d_2 = 1, 2 \leq d_3 \leq d_4, \\ & (ii)\ d_1 = 1, 2 \leq d_2 \leq d_3 \leq d_4, \\ & (iii)\ 2 \leq d_1 \leq d_2 \leq d_3 \leq d_4. \end{cases}$$

Assume that C is of simple normal crossings and that $\cap[JF_I = 0] = \emptyset$ where I ranges over all subsets of $\{1, 2, ..., q\}$ consisting of 3 distinct elements. Then $T^*\mathbb{P}^2(\log C)$ is ample, equivalently, there exists a complete continuous Finsler metric on $\mathbb{P}^2 \setminus C$ with holomorphic bisectional curvature $\leq -c^2$ where c is a constant.

In the preceding Corollary JF_I is the Jacobian determinant of the map F_I where

$$F_I = (P_{i_0}, P_{i_1}, ..., P_{i_n})$$

with $C_i = [P_i = 0]$, and P_i is a homogeneous polynomial with $\deg P_i = d_i = \deg C_i$ (see section 3 for more details). The point is that a general configuration of $C = C_1 + ... + C_q$ in the list satisfies the condition $\cap[JF_I = 0] = \emptyset$. In other words, those configurations that do not satisfy this condition are Zariski closed and of strictly lower dimension.

We end this section by posing three open problems:

Open Problem 6. *Extend the results of this section to* $\mathbb{P}^n$ *for all* n.

The result presented above deal with strongly negative holomorphic bisectional curvature which implies strongly negative holomorphic sectional curvature. The later condition allows for more examples but we do not know, exactly, how big this class is:

Open Problem 7. *Find curves* C *in* $\mathbb{P}^2$ *such the* $\mathbb{P}^2 \setminus C$ *admits a Finsler metric with strongly negative holomorphic sectional curvature.*

Open Problem 8. *Affine varieties admit compactification by adding divisors at infinity. Can the problem of Cornalba-Griffiths, in section 4, be resolved via logarithmic objects?*

It will be of great significance, and the approach more elegant, if this can be achieved.

§6. Finsler Geometry and Jet Bundles

6.1. Definitions and Examples

In the literature there are various different notions of jet bundles. The readers are referred to the series of papers by Ehresmann [21], [22] and the monograph by Gromov [30] for details. In complex hyperbolic geometry as well as in Nevanlinna Theory, it is most natural to work with the *parameterized jet bundles* $J^k X$. Green and Griffiths ([28]) are the first to provide a systematic study of these bundles. We also refer the readers to [11], [72] and [73] for further development in this direction. In the last few years these bundles also attracted the attention of many algebraic geometers. The algebraic geometers refer to these as arc spaces. In a very short period of time there is an explosion of results concerning these spaces and many applications. The readers are referred to [14] and [64] for these development. In many way there is no surprise of the significance of the jet bundles. For many years we rely mostly on the tangent and cotangent bundles but, as we move in to the territory of non-linear problems, the jet bundles become more and more indispensable.

We begin by recalling the definition of jet bundles. Let $\mathcal{H}_x, x \in X$, be the sheaf of germs of holomorphic curves:

$$\{f : \Delta_r \to X \text{ is holomorphic for some } r > 0 \text{ and } f(0) = x\}$$

where Δ_r is the disc of radius r in $\mathbb{C}$. Define, for $k \in \mathbb{N}$, an equivalence relation by designating two elements $f, g \in \mathcal{H}_x$ as k-equivalent (written $f \sim_k g$) if

$$f_j^{(p)}(0) = g_j^{(p)}(0)$$

for all $1 \leq p \leq k$, where $f_j = z_j \circ f, z_1, ..., z_n$ are local holomorphic coordinates near x and $f_j^{(p)} = \partial^p f_j / \partial\zeta$ is the p-th order derivative relative to the variable $\zeta \in \Delta_r$. The sheaf of *parameterized k-jets* is defined by:

$$J^k X = \cup_{x \in X} \mathcal{H}_x / \sim_k . \tag{6.1}$$

Elements of $J^k X$ will be denoted by $j^k f(0) = (f(0), f'(0), ..., f^{(k)}(0))$.

We set $J^0 X = \mathcal{O}_X$ and it is clear that $J^1 X = TX$. In general, $J^k X$ is not locally free (i.e., not a vector bundle) for $k \geq 2$. There is, however, a natural $\mathbb{C}^*$-action on $J^k X$ defined via parametrization as follows. Define, for $\lambda \in \mathbf{C}^*$ and $f \in \mathcal{H}_x$, a map $f_\lambda \in \mathcal{H}_x$ by $f_\lambda(t) = f(\lambda t)$. Then

$$j^k f_\lambda(0) = (f_\lambda(0), f_\lambda'(0), ..., f_\lambda^{(k)}(0)) = (f(0), \lambda f'(0), ..., \lambda^k f^{(k)}(0))$$

and the $\mathbb{C}^*$-action is given by

$$\lambda \cdot j^k f(0) = (f(0), \lambda f'(0), ..., \lambda^k f^{(k)}(0)). \tag{6.2}$$

For the tangent bundle TX we have the dual $T^*X = \Omega_X^1$ which is the sheaf associated to the presheaf

$$\Omega_U^1 = \{\omega : TX|_U \to \mathbb{C} \text{ holomorphic } \mid \omega(\lambda \cdot j^1 f) = \lambda \omega(j^1 f), \lambda \in \mathbb{C}\}.$$

The homogenity condition means that ω is a homogeneous polynomial of degree one along each of the fibers $T_x X$. In many applications (such as in the Riemann-Roch Theorem) we need also the symmetric product $\mathrm{sym}^m T^*X$. This is the sheaf associated to the presheaf

$$\mathrm{sym}^m \Omega_U^1 = \{\omega : (TX|_U)^n \to \mathbb{C} \mid \omega \text{ is a homogeneous polynomial of degree } m\}.$$

An section of $\mathrm{sym}^m \Omega_U^1 = \{\omega : (TX|_U)^n$ is also called an 1-form of weight m. Analogously, we define for positive integers m, k, *the sheaf of germs of k-jet differentials of weight m*, denoted $\mathcal{J}_k^m X$, to be the sheaf associated to the presheaf

$$\mathcal{J}_k^m U = \{\omega : J^k X|_U \to \mathbb{C} \text{ holomorphic } \mid \omega(\lambda j^k f) = \lambda^m \omega(j^k f), \lambda \in \mathbb{C}\}. \tag{6.3}$$

Note that $\mathcal{J}_1^1 X = T^*X = \Omega_X^1$. We also set $\mathcal{J}_0^m X = \mathcal{O}_X$ for all m.

The dimension of the global sections $H^0(X, T^*X)$, known as the *irregularity* of the manifold M (*genus* if X is a Riemann surface) is obviously a very important invariant of the manifold and so it is not surprising that the non-linear invariants $H^0(X, \mathcal{J}_k^m X)$ carry important information about the manifold. Some basic properties of $J^k X$ and $\mathcal{J}_k^m X$ are listed below (see [28], [11]).

Proposition 6.1. *Let X and Y be complex manifolds and let $F : X \to Y$ be a holomorphic map.*

(*a*) *For any $\ell \leq k$ the map $p_{k\ell} : J^k X \to J^\ell X$ defined by*

$$p_{k\ell}(j^k f(0)) \stackrel{\text{def}}{=} j^\ell f(0)$$

is a well-defined $\mathbb{C}^*$-bundle map (the forgetting map).

(*b*) *The k-th order induced map $J^k F : J^k X \to J^k Y$, defined by*

$$J^k F(j^k f(0)) \stackrel{\text{def}}{=} j^k (F \circ f)(0)$$

is a well-defined $\mathbb{C}^*$-bundle map and $J^1 F : TX \to TY$ is the usual differential of the map F.

(*c*) *Given any holomorphic map $f : \Delta_r \to X$ $(0 < r \leq \infty)$, the map (the k-th order lifting) $j^k f : \Delta_{r/2} \to J^k X$ defined by*

$$j^k f(\zeta) \stackrel{\text{def}}{=} j^k g(0), \;\; \zeta \in \Delta_{r/2}$$

where $g(\xi) = f(\zeta + \xi)$ is holomorphic for $\xi \in \Delta_{r/2}$ and commutes with the projection $p_k : J^k X \to X$, i.e., $p_k \circ j^k f = f$.

(*d*) *The map $\delta : \mathcal{J}_k^m X \to \mathcal{J}_{k+1}^{m+1} X$ defined by $\delta f = df$ if $k = 0$ and for $k \geq 1$*

$$\delta\omega(j^{k+1} f) \stackrel{\text{def}}{=} (\omega(j^k f))', \;\; \omega \in \mathcal{J}_k^m X$$

is a $\mathbb{C}^*$-bundle map (derivation).

(*e*) *For $\ell \leq k$ the natural projection $p_{kl} : J^k X \to J^l X$ induces an injection (the dual forgetting map) $p_{kl}^* : \mathcal{J}_l^m X \to \mathcal{J}_k^m X$ defined by "forgetting" those derivatives higher than l:*

$$p_{kl}^* \omega(j^k f) \stackrel{\text{def}}{=} \omega(p_{kl}(j^k f)) = \omega(j^l f).$$

(We shall simply write $\omega(j^k f) = \omega(j^l f)$ if no confusion arises).

(*f*) *the symmetric product of a k-jet differential of weight m_1 and a ℓ-jet differential of weight m_2 is a $\max\{k, \ell\}$-jet differential of weight $m_1 + m_2$.*

The derivativation δ (defined in (d) above) is not to be confused with exterior differentiation. For exterior differentiation we have $d^2 = d \circ d = 0$ while $\delta^2 = \delta \circ \delta \neq 0$.

Example 6.3. An 1-jet differential is a differential 1-form $\omega = \sum_{i=1}^n a_i(z)dz_i$. Let $f = (f_1, ..., f_n) : \Delta_r \to X$ be a holomorphic map. Then

$$\omega(j^1 f) = \sum_{i=1}^n a_i(f) dz_i(f') = \sum_{i=1}^n a_i(f) f_i'$$

and $\delta\omega$ is a 2-jet differential of weight 2, given by

$$\delta\omega(j^2 f) = (\omega(j^1 f))' = \Big(\sum_{i=1}^n a_i(f) f_i'\Big)' = \sum_{i,j=1}^n \frac{\partial a_i}{\partial z_j}(f) f_i' f_j' + \sum_{i=1}^n a_i(f) f_i''.$$

In particular, if we take $\omega = dz_i$ then

$$\delta dz_i(j^2 f) = f_i''.$$

It is convenient to write $\delta^2 z_i$ instead of δdz_i. This is consistent with (d) in Proposition 6.1. Iterating we get, $\delta^2\omega$ which is a 3-jet differential of weight 3:

$$\delta^2\omega(j^3 f) = \sum_{i,j=1}^n \frac{\partial^2 a_i}{\partial z_j \partial z_k}(f) f_i' f_j' f_k' + 3\sum_{ij=1}^n \frac{\partial a_i}{\partial z_j}(f) f_i'' f_j' + \sum_{i=1}^n a_i(f) f_i'''.$$

For $\omega = dz_i$, we have, $\delta^3 z_i(j^3 f) = \delta^2 dz_i(j^3 f) = f_i'''$. Analogously, we have:

$$\delta^k z_i(j^k f) = f_i^{(k)}$$

for any positive integer k. Using the jet differentials $\{\delta^j z_i\}$ any k-jet differentials may be expressed as follows. For multi-indices $I_j = (i_{1,j}, ..., i_{n,j}), j = 1, ..., k$, set

$$(\delta^j z)^{I_j} = (\delta^j z_1)^{i_{1,j}} \cdots (\delta^j z_n)^{i_{n,j}}$$

where the multiplications are symmetric products. Then a k-jet differential of weight m is of the form:

$$\omega = \sum_{|I_1|+2|I_2|+...+k|I_k|=m} a_{I_1,...,I_k}(z)(\delta^1 z)^{I_1} \cdots (\delta^k z)^{I_k}$$

where $a_{I_1,...I_k}$ are holomorphic functions which is symmetric with respect to the indices in each I_j and $|I_j| = i_{1,j} + \cdots + i_{n,j}$. In other words,

$$\omega(j^k f) = \sum_{|I_1|+2|I_2|+...+k|I_k|=m} a_{I_1,...,I_k}(f')^{I_1}...(f^{(k)})^{I_k}.$$

□

For s Riemann surface the dimension of the space of global k-jet differentials m can be computed via the classical Riemann-Roch formula for curves and a basis can be explicitly constructed as the following example shows:

Example 6.4. (See [11] for details.) Let

$$X = \{[z_0, z_1, z_2] \in \mathbb{P}^2 \mid P(z_0, z_1, z_2) = 0\}$$

be a non-singular curve of degree $d = 4$. The genus of X is 3. Riemann-Roch Theorem implies that

$$h^0(\mathcal{J}_2^2 X) = h^0(\mathcal{K}_X^{\otimes 2}) + h^0(\mathcal{K}_X) = 2g + g = 9$$

(where $h^0(\mathcal{S}) = \dim_{\mathbb{C}} H^0(X, \mathcal{S}), \mathcal{K}_X$ is the canonical line bundle and g is the genus). Since the genus is 3, there are 3 linearly independent 1-forms $\omega_1, \omega_2, \omega_3$ which may be expressed explicitly as (see [2, 3, 4] or [11]):

$$\omega_1 = \frac{z_0(z_0dz_1 - z_1dz_0)}{\partial P/\partial z_2}, \omega_2 = \frac{z_1(z_0dz_1 - z_1dz_0)}{\partial P/\partial z_2}, \omega_3 = \frac{z_2(z_0dz_1 - z_1dz_0)}{\partial P/\partial z_3}.$$

A basis for $H^0(\mathcal{J}_2^2 X)$ is given by

$$\omega_1^{\otimes 2}, \omega_2^{\otimes 2}, \omega_3^{\otimes 2}, \omega_1 \otimes \omega_2, \omega_1 \otimes \omega_3, \omega_2 \otimes \omega_3, \delta\omega_1, \delta\omega_2, \delta\omega_3$$

where δ is the derivation defined in Proposition 6.1. Note that tensor product and symmetric product are the same for line bundles. The first six of these provide a basis of $H^0(\mathcal{K}_X^{\otimes 2})$ and the last three may be identified with a basis of $H^0(\mathcal{K}_X)$. □

6.2. Finsler Jet Metrics

The parameterized jet bundles are, in general, only $\mathbb{C}^*$-bundles but not vector bundles hence (Hermitian metrics does not make sense) can only be equipped with *Finsler* metrics.

Definition 6.5. A *Finsler pseudo-metric* (or a *k-jet pseudo-metric*) on $J^k X$ is a function

$$\rho = \rho_k : J^k X \to \mathbb{R}_{\geq 0}$$

satisfying the condition

$$\rho(\lambda \cdot \mathbf{j}_k) = |\lambda| \rho(\mathbf{j}_k)$$

for all $\lambda \in \mathbb{C}$ and $\mathbf{j}_k \in J^k X$. A k-jet pseudo-metric is said to be a *k-jet metric* if ρ is positive outside the zero-section. □

A $(k-1)$-jet pseudo-metric $(k \geq 2)$ ρ_{k-1} can be considered as a k-jet pseudo-metric via the *forgetting map* (see Proposition 6.1):

$$\rho_{k-1}(\mathbf{j}_k) := \rho_{k-1}(\mathbf{j}_{k-1}). \tag{6.4}$$

where $\mathbf{j}_k = j^k f(0)$ and $\mathbf{j}_{k-1} = j^{k-1} f(0)$.

Example 6.6. Define, for $\mathbf{j}_k \in J^k M, k \geq 1$,

$$\kappa_k(\mathbf{j}_k) = \inf\{1/r\} \geq 0$$

where the infimum is taken over all r such that there exists $f : \Delta_r \to X$ holomorphic with $j^k f(0) = \mathbf{j}_k$. It is clear that it is a Finsler pseudo-metric. For $k = 1$ this is the Kobayashi-Royden pseudo-metric (in general we only know that it is non-negative) on $J^1 X = TX$. Fot this reason we shall refer to κ_k as the *k-th order infinitesimal Kobayashi-Royden pseudo-metric.* We shall say that X is *k-jet hyperbolic* if κ_k is indeed a Finsler metric (positive, not merely non-negative); i.e. $\kappa_k(\mathbf{j}_k) > 0$ for each non-zero k-jet $\mathbf{j}_k$. Thus 1-jet hyperbolic is the same as the well-known concept of Kobayashi hyperbolicity. It is easily seen that k-jet hyperbolic implies $(k+1)$-jet hyperbolic (see [10] and [11] for further information). The notion introduced here is not to be confused with the k-dimensional $(1 \leq k \leq n = \dim X)$ Kobayashi pseudo-metric in the literature if $k = 1$ (see for example Lang [42]); n-dimensional Kobayashi hyperbolicity is more commonly known as measure hyperbolicity. The k-dimensional Kobayashi pseudo-metric is defined only for $k \leq n$ but the k-jet metric introduced here can be defined for any positive integer k. □

Example 6.7. Denote by $\Delta_r = \{z \in \mathbb{C} \mid |z| < r\}$ the disc of radius r in the complex plane. If $r = 1$ we simple write Δ instead of Δ_1. A k-jet $\mathbf{j}_k$ in the fiber $(J^k\Delta)_z$ may be represented by a holomorphic map $f : \Delta_\epsilon \to \Delta$ such that $f(0) = z \in \Delta$, i.e., $\mathbf{j}_k = f^{(k)}(0)$. Define a k-jet metric by

$$\rho_k(\mathbf{j_k}) = |f^{'}(0)|^k + |f^{''}(0)|^{k/2} + \cdots + |f^{(k-1)}(0)|^{k/(k-1)} + |f^{(k)}(0)|$$

where $\mathbf{j_k} = j^k f(0)$. Recall that the classical Poincaré metric is given by

$$ds = \frac{|dz|}{1 - |z|^2}$$

and so

$$ds(\mathbf{j}_1) = ds(j^1 f(0)) = |f'(0)| = \rho_1(\mathbf{j}_1).$$

The *k-th order Caratheodory-Reiffen pseudo-metric* on a complex space X is defined by

$$\chi_k(\mathbf{j_k}) = \sup \rho_k(F_*(\mathbf{j_k}))$$

where ρ_k is as above and the supremum is taken over all holomorphic map $F : X \to \Delta$. For $k = 1$ this is the *classical Caratheodory-Reiffen pseudo-metric* on the tangent bundle. We say that X is k-Caratheodory hyperbolic if χ_k is positive definite. For $k = 1$ this is the usual concept of Caratheodory hyperbolicity. It is clear from the definition that $(k+1)$-Caratheodory hyperbolic implies k-Caratheodory hyperbolic. □

The classical Royden-Kobayashi jet metric belongs to the class of "intrinsic" metrics. Roughly speaking intrinsic means that it depends only on the complex structure; the readers are referred to [41] and [10] for a precise definition and for further references. It is a classical result that, among the intrinsic 1-jet metrics, the classical Kobayashi metric is the largest and classical Cartheodory is the smallest. It is an open problem if this generalizes to the case of the k-the order Kobayashi metric and the k-the order Cartheodory metric. Hence we pose the following problem:

Open Problem 9. *Extend the classical theory on 1-jet intrinsic metrics to k-jet intrinsic metrics for all k.*

There are other interesting intrinsic metrics other than the Kobayashi and the Caratheodory metrics; for example the intrinsic pseudo-distance introduced by Azukawa-Klimek-Sibony (see [11] and the references cited there) is also quite interesting but not that much is known (as compare to the Kobayashi and the Caratheodory metrics) about it.

There is also a very interesting intrinsic "metric" in the literature which is defined on the space of k-cycles which we describe below:

Example 6.8. (Chern-Levine-Nirenberg [12]) There is a very interesting intrinsic Finsler pseudo-metric introduced by Chern, Levine and Nirenberg where the metric is defined not on M but on the space of cycles (for cycle spaces see Barlet and Koziarz [7]). Denote by $\mathcal{P}$ the family of plurisubharmonic functions, of class $\mathcal{C}^2$, on X. Let Γ be a homology class on X and define:

$$\rho(\Gamma) = \sup_{u\in\mathcal{P}} \inf_{\gamma\in\Gamma} \left| \int_\gamma d^c u \wedge (dd^c u)^{2m-1} \right|$$

if $\dim \Gamma = 2m-1$ and

$$\rho(\Gamma) = \sup_{u\in\mathcal{P}} \inf_{\gamma\in\Gamma} \left| \int_\gamma du \wedge d^c u \wedge (dd^c u)^{2m} \right|$$

if $\dim \Gamma = 2m$. $\square$

The integral in the definition above arises naturally and is related to the complex homogeneous equations (see also sections 3 and 4). More precisely, the equation $(dd^c u)^n = 0, n = \dim X$, is the Euler equation of the functional

$$I(u) = \int_M du \wedge d^c u \wedge (dd^c u)^{n-1}.$$

It is known that the regularity assumption on u can be weaken considerably. We refer the readers to the original papers for further refinements. Other than these bit that much about the Chern-Levin-Nirenberg metric is known so we pose the following problem:

Open problem 10. *Investigate the geometry of the Chern-Levin-Nirenberg metric on the space of cycles.*

6.3. Algebraic Version of Jet Spaces

In recent years algebraic geometers are also interested in what is known as the space of arcs (see [14] and [64]). Over the category of complex varieties this is precisely the jet bundles introduced above. We recall here the algebraic definition of arc spaces.

Let X be a non-singular algebraic variety X defined over a field $\mathbb{F}$ of characteristic zero. A *k-jet* at $x \in X$ is a morphism:

$$\gamma : \text{Spec } \mathbb{F}[t]/(t^{k+1}) \to X$$

where $\mathbb{F}[t]$ (resp. $\mathbb{F}[[t]]$) is the polynomial ring (resp. the ring of formal power series) in one variable defined over $\mathbb{F}$ such that $\gamma(\text{Spec } \mathbb{F}) = x$. The space of k-jets at x is isomorphic to the space of polynomials of degree at most k, vanishing at the origin. The k-jet bundle $J^k X$ is the $\mathbb{F}^*$-bundle whose fiber over x is the space of k-jets at x. A *polynomial arc* (resp. *formal arc*) at $x \in X$ is a morphism:

$$\gamma : \text{Spec } \mathbb{F}[t] \to X \quad (\text{respectively } \gamma : \text{Spec } \mathbb{F}[[t]] \to X).$$

The space of algebraic (resp. formal) arcs at x is isomorphic to the space of polynomials (resp. formal power series) vanishing at the origin. The bundle of algebraic (resp. formal) arcs $J^\infty[X]$ (resp. $J^\infty[[X]]$) is the $\mathbb{F}^*$-bundle whose fiber over x is the space of algebraic (resp. formal) arcs at x.

If the field $\mathbb{F}$ is a valuated field, i.e., equipped with an absolute value (Archimed-ean or otherwise) then the ring of *convergent* power series $\mathbb{F}\{t\}$ is defined. Note that an absolute is just a Finsler metric.

The complex number field $\mathbb{C}$ with the usual (Archimedean) absolute and the p-adic number field $\mathbb{C}_p$ with the (non-Archimedean) p-adic absolute valuse are examples of valuated fields. In this case the bundle of arcs, denoted $J^\infty(X)$, are defined with fibers over $x \in X$ being the space of holomorphic arcs, i.e., morphisms:

$$\gamma : \text{Spec } \mathbb{F}\{t\} \to X$$

such that $\gamma(\text{Spec } \mathbb{F}) = x$. The inclusion of rings (for $\ell \leq k$):

$$\mathbb{F}[t]/(t^\ell) \subset \mathbb{F}[t]/(t^k) \subset \mathbb{F}[t] \subset \mathbb{F}\{t\} \subset \mathbb{F}[[t]]$$

induces projections:

$$J^\infty[[X]] \to J^\infty X \to J^\infty[X] \to J^k X \to J^l X.$$

Example 6.9. If $X = \mathbb{F}^n$ is the affine n-space, where $\mathbb{F}$ is a valuated field of characteristic zero, then

$$J^k X = \{(\sum_{i=0}^k a_{i,1}t^i, ..., \sum_{i=0}^k a_{i,n}t^i) \mid a_{i,j} \in \mathbb{F}\},$$

$$J^\infty[X] = \cup_{k=0}^\infty \{(\sum_{i=0}^k a_{i,1}t^i, ..., \sum_{i=0}^k a_{i,n}t^i) \mid a_{i,j} \in \mathbb{F}\}$$

$$J^\infty X = \{(\sum_{i=0}^\infty a_{i,1}t^i, ..., \sum_{i=0}^\infty a_{i,n}t^i) \mid a_{i,j} \in \mathbb{F}, \text{ the series are convergent}\},$$

$$J^\infty[[X]] = \{(\sum_{i=0}^\infty a_{i,1}t^i, ..., \sum_{i=0}^\infty a_{i,n}t^i) \mid a_{i,j} \in \mathbb{F}\}.$$

□

We introduce formally the following definitions.

Definition 6.10. Let R be one of the rings $\mathbb{F}[t], \mathbb{F}\{t\}, \mathbb{F}[[z]]$. An algebraic scheme X defined over R is said to be *R-hyperbolic* if there does not exists any non-trivial morphism $f : \text{Spec } R \to X$. □

For $\mathbb{F} = \mathbb{C}$, $\mathbb{C}[t]$-hyperbolicity is usually known as *algebraic hyperbolicity* and $\mathbb{C}\{t\}$-hyperbolicity is known as *analytic hyperbolicity* (= Brody hyperbolicity). We shall also refer to $\mathbb{C}[t]-$hyperbolicity as *formal hyperbolicity.* The following implications are immediate:

formal hyperbolicity
$\Longrightarrow$ analytic hyperbolicity
$\Longrightarrow$ algebraic hyperbolicity.

It is also well-known that an elliptic curve (over $\mathbb{C}$) is algebraically hyperbolic but not analytically hyperbolic.

§7. Construction of Jet Metrics via Global Jet Differentials

7.1. Riemann-Roch Theorem for Jet Differentials

The main references of this section are [28] and [11]. In the case of a *compact* complex manifold an obvious construction of jet metrics can be carried out as follows (the obvious generalization to varieties defined over a valuated field will be left to the readers). Take a basis $\omega_1, ..., \omega_N$ of global holomorphic k-jet differentials of weight m (provided that these exist) and define:

$$\rho_k(j^k f) = (\sum_{i=1}^{N} |\omega_i(j^k f)|^2)^{1/2m}. \tag{7.1}$$

Since a k-jet differential of weight m is a weighted symmetric homogeneous polynomial of degree m on the k-jet bundle (i.e. $\omega(\lambda \cdot j^k f) = \lambda^m \omega(j^k f)$), we see readily that $\rho_k(\lambda \cdot j^k f) = |\lambda| \rho_k(j^k f)$. It is clear from the definition that ρ_k is continuous on $J^k X$, real analytic on $J^k X \setminus \{\text{zero-section}\}$; indeed, ρ_k^{2m} is real analytic on $J^k X$. If $\mathcal{J}_k^m X$ is ample then the result of section 2 can be extended to show the existence of a Finsler k-jet metric with strongly negative holomorphic bisectional curvature (the details will appear in a forthcoming article) which implies that X is Kobayashi-hyperbolic. In other words, we have:

Theorem 7.1. *Let X a be a compact complex manifold such that $\mathcal{J}_k^m X$ is ample then X is Kobayashi-hyperbolic.*

For $k = m = 1$ the condition reduces to the ampleness of T^*X. In fact the analogue of Theorem 5.1 remains true:

Theorem 7.2. *Let X be a compact complex manifold admitting a continuous Finsler k-jet metric with strongly negative hsc. Then X is Kobayashi-hyperbolic.*

At this point the only proof that I know of of requires Nevanlinna Theory and shall be discussed thoroughly in a forth coming article. For concrete applications Theorem 7.1 is not enough as many important examples of interest do not have ample jet differentials. For example, the k-jet differentials of hypersurfaces of $\mathbb{P}^n$ are never ample for any k if $n \geq 3$. On the other hand, we do not have any good techniques of producing Finsler metric with strongly negative hsc except when $\mathcal{J}_k^m X$

is ample (in which case we get actually a k-jet metric with strongly negative bhsc). For these reasons we shall deal with k-jet metric of the type as given by (7.1). The idea is simply to look for conditions guaranteeing the existence of lots of global holomorphic k-jet differentials. This leads us back to the concept of big bundles in section 3. The main tool here is the Riemann-Roch Theorem for coherent sheaves (the k-jet differentials $\mathcal{J}_k^m X$, though not a vector bundle, is a coherent sheaf) and the conditions will be expressed in terms of Chern numbers which is, of course, intrinsically defined:

Theorem 7.3. *Let X be a compact complex manifold of dimension n then*

$$\chi(\mathcal{J}_k^m X) = \deg\ \mathrm{ch}\,(\mathcal{J}_k^m X) \cdot \mathrm{td}\, X$$

where $\chi(\mathcal{J}_k^m X)$ is the Euler characteristic, ch *is the Chern character and* td *is the Todd class.*

As is clear from the discussions in the previous sections, it is more convenient to work on the projectivization. The projectivization $\mathbb{P}(J^k X)$ is defined via the $\mathbb{C}^*$-action given in (6.2). Each fiber of the bundle

$$p : \mathbb{P}(J^k X) \to X$$

is isomorphic to the *weighted projective space* $\mathbb{P}(Q_{k,n})$ where

$$Q_{k,n} = (\underbrace{1, ..., 1}_{n}, \underbrace{2, ..., 2}_{n}, ..., \underbrace{k, ..., k}_{n})$$

is the weight (see [12] and [15] for details). The standard projective space $\mathbb{P}^n$ corresponds to $\mathbb{P}(Q)$ where the weight is taken to be

$$Q = (\underbrace{1, ..., 1}_{n}).$$

The dimension of the weighted projective space $\mathbb{P}(Q_{k,n}) = \mathbb{C}^{nk}/Q_{k,n}$ is $nk-1$. The equivalence class, in $\mathbb{P}(Q_{k,n})$, of a point $(z_{11}, ..., z_{1n}, \cdots, z_{n1}, ..., z_{nn}) \in \mathbb{C}^{nk}$ shall be denoted by $[z_{11}, ..., z_{1n}, \cdots, z_{n1}, ..., z_{nn}]_{Q_{k,n}}$.

Example 7.4. The weighted projective space $\mathbb{P}(Q_{k,2})$ is of dimension $2k-1$. It is non-singular if and only if $k = 1$. For $k \geq 2$, $\dim \mathbb{P}(Q_{k,2})_{\text{sing}} =$

1 and the singular points of $\mathbb{P}(Q_{k,2})$ are of the form:

$$[0,0;*,*;0,0;0,0;...;0,0]_{Q_{k,2}}$$
$$[0,0;0,0;*,*;0,0;...;0,0]_{Q_{k,2}}$$
$$\cdots$$
$$\cdots$$
$$[0,0;0,0;0,0;...;0,0;*,*]_{Q_{k,2}}$$

where $*$ represents non-zero complex numbers. Let X be a complex surface then the each of the fibers of the projectivized jet bundle $\mathbb{P}(J^kX)$ over X is isomorphic to $\mathbb{P}(Q_{k,2})$; hence $\dim_{\mathbb{C}} \mathbb{P}(J^kX) = 2k+1$. It is non-singular if and only if $k = 1$ and for $k \geq 2$

$$\dim \mathbb{P}(J^kX)_{\text{sing}} = 3.$$

In general, the weighted projective space $\mathbb{P}(Q_{k,n})$ is of dimension $nk-1$. It is non-singular if and only if $k = 1$ and

$$\dim \mathbb{P}(Q_{k,2})_{\text{sing}} = n - 1.$$

The projectivized jet bundle $\mathbb{P}(J^kX)$ over a manifold X of dimension n is of dimension $n(k+1)-1$. It is non-singular if and only if $k = 1$ and

$$\dim \mathbb{P}(J^kX)_{\text{sing}} = 2n - 1.$$

□

Just as in the case of the projective space, there is a line sheaf $\mathcal{L}$, the Serre line sheaf $\mathcal{L}$, over $\mathbb{P}(J^kX)$ defined to be the line sheaf whose restriction to each fiber is $\mathcal{O}_{\mathbb{P}(Q_{k,n})}(1)$. Thus the situation is essentially the same as the situation in sections 2 and 3. The isomorphism theorem of Grothendieck/Serre is still valid:

Theorem 7.5. *Let X be a complex manifold of dimension n and $p : \mathbb{P}(J^kX) \to X$ be the k-jet bundle. Then*

$$H^q(X, \mathcal{J}_k^n X) \cong H^q(\mathbb{P}(J^kX) \otimes \mathcal{S}, \mathcal{L}^m \otimes p^*\mathcal{S}), \quad q \geq 0$$

where $\mathcal{S}$ is a sheaf on X and $\mathcal{L} = \mathcal{L}_{\mathbb{P}(J^kX)}$ is the Serre line sheaf on $\mathbb{P}(J^kX)$.

This and the Riemann-Roch Theorem imply that

Corollary 7.6. *Let X be a compact complex manifold of dimension n. Then*

$$\chi(\mathcal{J}_k^m X) = \chi(\mathcal{L}^m_{\mathbb{P}(J^kX)}) = \frac{c_1(\mathcal{L}_{\mathbb{P}(J^kX)})^{(n+1)k-1}}{((n+1)k-1)!} m^{(n+1)k-1} + O(m^{(n+1)k-2}).$$

The Chern number $c_1(\mathcal{L}^m_{\mathbb{P}(J^kX)})$ on the right above can be computed via the Filtration Theorem of Green-Griffiths (see [28] and [11]) in terms of the invariants of the Chern numbers of X. In the case of compact complex surfaces we have (the details can be found in [11]):

Theorem 7.7. *Let X be a non-singular complex surface. Then, for $m >> k$,*

$$\chi(\mathcal{J}_k^{k!m}X) = \frac{(k!)^{2k+1}}{2}\big(\alpha_k c_1^2(TX) - \beta_k c_2(TX)\big)\, m^{2k+1} + O(m^{2k})$$

where

$$\alpha_k = \frac{2}{(k!)^2(2k+1)!}\Big(\sum_{i=1}^{k}\frac{1}{i}\sum_{j=1}^{i-1}\frac{1}{j} - \sum_{i=1}^{k}\frac{1}{i^2}\Big), \quad \beta_k = \frac{2}{(k!)^2(2k+1)!}\sum_{i=1}^{k}\frac{1}{i^2}.$$

The Poincaré characteristic is the alternating sum of the dimension of the cohomologies. Thus the invariant $\dim H^0(\mathcal{J}_k^m X)$ can be computed or at least estimated if the dimension of the higher cohomolgies can be estimated. For X a compact complex surface the following result was obtained in [11] by combining all the ideas mentioned above:

Theorem 7.8. *Let X be a non-singular minimal surface of general type with* $\operatorname{Pic} X \cong \mathbb{Z}$. *Then the sheaf of k-jet differentials, $\mathcal{J}_kX$, is big. In particular, for a generic hypersurface X of degree $d \geq 5$ in $\mathbb{P}^3$, $\mathcal{J}_kX$ is big.*

A few remarks about the theorem above is in order. First we recall that a surface is said to be *minimal* if it does not contain any rational curve C with self intersection number $C^2 = C \cdot C = -1$. The *Picard group* is the group of isomorphism classes of holomorphic line bundles. A surface is said to be of *general type* if the canonical bundle $\mathcal{K}_X$ is big. This concept is, in many ways, the analogue of curves (Riemann surfaces) of genus $g \geq 2$. The term general type suggests that "most" surfaces belong to this class. A *smooth* hypersurface of degree $d \geq 5$ is of general type but the Picard group may not be isomorphic to the group of integers but most of them do. This accounts for the "generic" condition in the theorem. More precisely, the space $S_{d,3}$ of surfaces of degree d in $\mathbb{P}^3$ is isomorphic to $\mathbb{P}^N$ where

$$N = C_d^{d+3} = \frac{(d+3)!}{d!3!}.$$

A subset of $S_{d,3}$ is said to be generic if it is the complement of a countable union of subvarieties of strictly lower dimensions. It was shown in [11] that the preceding theorem implies that

Theorem 7.9. *Let X be a smooth minimal surface of general type with positive geometric genus and* $\operatorname{Pic} X \cong \mathbb{Z}$. *Then every holomorphic map $f : \mathbb{C} \to X$ is algebraically degenerate. If, in addition, the surface X contains no rational nor elliptic curve then X is hyperbolic. In particular, a generic hypersurface of degree $d \geq 5$ in $\mathbb{P}^3$ is hyperbolic.*

7.2. Oka Principle and The Moving Lemma

The preceding theorem works well for surfaces but the combinatoric of computing Chern classes become quite involved in higher dimension. For this reason we introduce below another approach which turns out to be quite effective even in higher dimension. Let X be a projective manifold of dimension n and $f : \mathbb{C} \to X$ be a holomorphic map such that the k-jet $j^k f : \mathbb{C} \to J^k X$ is non-trivial in the sense that the image $j^k f(\mathbb{C})$ is not entirely contained in the zero-section of $J^k X$ then

$$[j^k f] : \mathbb{C} \to \mathbb{P}(J^k X)$$

(where $[\ ] : J^k X \to \mathbb{P}(J^k X)$ is the quotient map) is a well-defined holomorphic map. This map shall be referred to as the canonical lifting of f. Clearly we have $p \circ [j^k f] = f$ where $p : \mathbb{P}(J^k X) \to X$ is the projection. *Thus the problem of investigating the "size" of the image of f is reduced to the study of the "size" of the image of* $[j^k f]$. For example the map f is constant (hyperbolicity) is equivalent to the condition that the image of $[j^k f]$ is contained in a single fiber. This idea, though simple, turns out to be quite effective as we may now exploit the geometric structure of $J^k X$ or $\mathbb{P}(J^k X)$. For example we do not know if we can move, holomorphically, the image of f but it may be possible to move the image of $[j^k f]$. To do this we make use of one of the most important tool in complex analysis: the Oka Principle (see section 4 above and [72], [73]).

We have a commutative diagram:

$$\begin{array}{ccc}
f^*(J^k X) & \xrightarrow{f_*} & J^k X \\
[\,]\downarrow & & \downarrow[\,] \\
f^*\mathbb{P}(J^k X) & \xrightarrow{f_*} & \mathbb{P}(J^k X) \\
\rho\downarrow & [j^k f]\nearrow & \downarrow p \\
\mathbb{C} & \xrightarrow{f} & X
\end{array}$$

Since $\mathbb{C}$ is contractible and Stein, Oka Principle implies that the pull-back bundle $f^*(J^kX) \to \mathbb{C}$ is trivial. The same is then also true for $f^*\mathbb{P}(J^kX) \to \mathbb{C}$. Thus we have a commutative diagram

$$\begin{array}{ccccc} \mathbb{C}\times\mathbb{P}(Q_{k,n}) & \xrightarrow{\Phi} & f^*\mathbb{P}(J^kX) & \xrightarrow{f_*} & \mathbb{P}(J^kX) \\ p_1\downarrow & & \rho\downarrow\ \ [j^kf]\nearrow & & \downarrow p \\ \mathbb{C} & \xrightarrow{\text{id}} & \mathbb{C} & \xrightarrow{f} & X \end{array}$$

where $\mathbb{P}(Q_{k,n})$ is the weighted projective space which is isomorphic to any of the fibers of $\mathbb{P}(J^kX)$, Φ is the trivialization isomorphism and p_1 is the projection onto the first factor.

Observe that there is a section $\phi : \mathbb{C} \to \mathbb{C}\times\mathbb{P}(Q_{k,n})$ $(p_1 \circ \phi = \text{id})$ such that

$$f_* \circ \Phi \circ \phi = [j^kf].$$

The main idea is to move the canonical lifting $[j^kf]$ by moving the section ϕ. To move ϕ we simply take a dominating rational self-map $\gamma : \mathbb{P}(Q_{k,n}) \to \mathbb{P}(Q_{k,n})$ (for example, in the case $k = 1, \mathbb{P}(Q_{1,n})$ is just the usual projective space $\mathbb{P}^{n-1}$ which is a homogeneous manifold and we can take γ to be an automorphism) inducing a rational map

$$G : \mathbb{C}\times\mathbb{P}(Q_{k,n}) \to \mathbb{C}\times\mathbb{P}(Q_{k,n}),\ \ G(\zeta,\xi) = (\zeta,\gamma(\xi)).$$

With this we get a section

$$\psi = G\circ\phi : \mathbb{C} \to \mathbb{C}\times\mathbb{P}(Q_{k,n})$$

which is actually *holomorphic.* The map

$$[g_k] = f_* \circ \Phi \circ \psi$$

is holomorphic and is a lifting of f, i.e., $p\circ[g_k] = f$. We have thus successively move $[j_kf]$ to $[g_k]$. Moreover, by Runge approximation theorem it is easy to see that the trivialization Φ can be chosen to have the same "growth" as f. The best way of measuring growth is via Nevanlinna Theory (see [11], [72], [73]) and what we obtain out of this construction is that

$$T_{[g_k]}(r) = O(T_{j^kf}(r))$$

where, for a holomorphic map F, $T_F(r)$ is the characteristic function (for a holomorphic function the characteristic function is essentially the logarithmic maximum modulus). We summarize the construction above in the following lemma:

Lemma 7.10. (Moving Lemma) *Let $f : \mathbb{C} \to X$ be a holomorphic map into a projective manifold X. Assume that the image of $j^k f$ is not contained entirely in the zero section of $J^k X$. Let ζ_0 be an arbitrary point of $\mathbb{C}$ and $\xi_0 \in \mathbb{P}(J^k X)_{f(\zeta_0)} \cong \mathbb{P}(Q_{k,n})$ such that $\gamma(f'(\zeta_0), \cdots, f^{(k)}(\zeta_0)) = \xi_0$. Then there exists a holomorphic map*

$$[g_k] : \mathbb{C} \to \mathbb{P}(J^k X)$$

such that (i) $p \circ [g_k] = f$, (ii) $[g_k](\zeta_0) = \xi_0$ and (iii) $T_{[g_k]}(r) = O(T_{[j^k f]}(r))$.

Armed with this we now have the following important result ([11], [72], [73]:

Theorem 7.11. (Moving Schwarz Lemma) *Let X be a compact complex manifold and assume that the sheaf of k-jet differentials is big. Let $Y \subset \mathbf{P}(J^k X)$ be a subvariety and suppose that there exists a non-trivial section*

$$\sigma \in H^0(Y, \mathcal{L}^m_{\mathbb{P}(J^k X)}|_Y \otimes p|_Y^*[-D])$$

where $\mathcal{L}_{\mathbb{P}(J^k X)}$ is the Serre line bundle, D is an ample divisor in X and $p : \mathbb{P}(J^k X) \to X$ is the projection map. Let $[g_k] : \mathbb{C} \to \mathbb{P}(J^k X)$ be a lifting as constructed via the Moving Lemma. Then $\sigma([g_k]) \equiv 0$, i.e., the image $[g_k](\mathbb{C})$ is contained in Y.

The Moving Schwarz Lemma can be interpreted in terms of Finsler geometry as follows. Recall that we use global k-jet differentials to construct k-jet metric (see (7.1)). The k-jet differentials are identified with global sections of the Serre line bundle $\mathcal{L}$ on $\mathbb{P}(J^k X)$. The construction of k-jet metric relies on the existence of global sections. This accounts for the condition that the bundle $\mathcal{J}_k X$ or, equivalently, $\mathcal{L}$ is big. Unlike the case of an ample bundle the common zeros (called the *base locus*) of global sections of a big bundle may not be empty; in other words ρ_k as constructed in (7.1) may not be positive definite. The base locus of $\mathcal{L}$ is a subvariety in $\mathbb{P}(J^k X)$. Theorem 7.11 provides the precise conditions, under which, the images of the liftings obtained via the Moving Lemma are contained in the base locus. The proof of Theorem 7.11 is based on Nevanlinna Theory. The geometric meaning being that, outside of the base locus, ρ_k is positive definite. Moreover, the condition "big" implies that the holomorphic sectional curvature is strongly negative in the horizontal direction (compare the situation of ample bundles in sections 2 and 3). This implies that the image of a lifting, being an image of $\mathbb{C}$ cannot be entirely contained outside of the base locus. The more difficult part of the proof is to eliminate the case where the image intersects, but not entirely contained in the base locus. In this case we have to deal with curvature current of pseudo-metrics. Fortunately, Nevanlinna

Theory is well-designed to deal with such situation. The main reason being that Nevanlinna Theory is designed to deal with integral estimates rather than pointwise estimates. For example, the condition that ρ_k being a positive definite Finsler metric with strongly negative horizontal holomorphic sectional curvature implies the pointwise estimate

$$dd^c \log \rho_k(g_k) \geq c\rho_k(g_k).$$

But Nevanlinna Theory requires only that

$$\int_0^r \frac{dt}{t} \int_{|\zeta|<t} dd^c \log \rho_k(g_k(\zeta)) \geq c \int_0^{2\pi} \rho_k(g_k(re^{\sqrt{-1}\theta}))\frac{d\theta}{2\pi}$$

where ρ_k is required only to be semi-definite so long as the integrals make sense as distributions.

The proof of the Moving Schwarz Lemma is based on the Lemma of Logarithmic Derivatives in Nevanlinna Theory:

Theorem 7.12. (Lemma of Logarithmic Derivatives) *Let X be a projective variety and let (i) D be an effective divisor with simple normal crossings, or (ii) D be the trivial divisor in X (i.e. the support of D is empty or equivalently, the line bundle associated to D is trivial). Let $f : \mathbb{C} \to X$ be an algebraically non-degenerate holomorphic map and $g_k : \mathbb{C} \to J^kX$ be a holomorphic lifting (i.e., $p(g_k) = f$ where $p : J^kX \to X$ is the projection) satisfying the estimate $T_{[g_k]}(r) = O(T_{j^kf}(r))$. Let $\omega \in H^0(X, \mathcal{J}_k^m X(\log D))$ (resp. $H^0(X, \mathcal{J}_k^m X)$ in case (ii)) a jet differential such that $\omega \circ g_k$ is not identically zero. Then*

$$T_{\omega\circ g_k}(r) = \int_0^{2\pi} \log^+ \left| \omega(g_k(re^{\sqrt{-1}\theta})) \right| \frac{d\theta}{2\pi} \leq O(\log T_f(r)) + O(\log r).$$

The proof of the preceding theorem can be found in [73] (Theorem 9.3), a special case was established in Theorem 6.1 in [11]. The proof of The Moving Schwarz Lemma now follows from Theorem 7.12.

Proof of The Moving Schwarz Lemma. The proof is the same as the proof of Corollary 6.2 in [11] except that we now use the stronger form of the Lemma of logarithmic derivatives above. By Grothendieck's isomorphism (Theorem 7.5) a section

$$\sigma \in H^0(\mathbb{P}(J^kX)^m), \mathcal{L}^m_{\mathbb{P}(J^kX)} \otimes p^*[-D])$$

is identified with a section

$$\omega \in H^0(X, \mathcal{J}_k^m X \otimes [-D])$$

and that $\sigma([g_k]) \equiv 0$ is equivalent to $\omega(g_k) \equiv 0$. Suppose that $\omega \circ g_k \not\equiv 0$. We may assume without loss of generality that $\log r = o(T_f(r))$. By Theorem 6.1, we have

$$\int_0^{2\pi} \log^+ |\omega \circ g_k| \frac{d\theta}{2\pi} = T_{\omega \circ g_k}(r) \ . \leq . \ O\big(\log(rT_f(r))\big).$$

On the other hand, since ω vanishes on D and D is apmle (hence can be chosen to be generic), we obtain via Jensen's Formula (see [60]):

$$\begin{aligned} T_f(r) &\leq N_f(D; r) + O\big(\log(rT_f(r))\big) \\ &= \int_0^{2\pi} \log^+ |\omega \circ g_k| \frac{d\theta}{2\pi} + O(\log\big(rT_f(r)\big)) \end{aligned}$$

which, together with the preceding estimate, implies that:

$$T_f(r) \ . \leq . \ O\big(\log(rT_f(r))\big).$$

This is impossible; hence we must have $\omega \circ g_k \equiv 0$. □

We get as Corollary of the Moving Schwarz's Lemma:

Theorem 7.13. *Let X be a projective manifold and assume that the sheaf of germs of k-jet differentials $\mathcal{J}_k X$ is big. Let $p : \mathbb{P}(J^k X) \to X$ be the projectivized k-jet bundle and let B be the base locus of the Serre line bundle $\mathcal{L}_{\mathbb{P}(J^k X)}$. Let $f : \mathbb{C} \to X$ be a holomorphic map and N be the Zariski closure of $f(\mathbb{C})$. Then $p^{-1}(N) \subset B$ where $p : \mathbb{P}(J^k X) \to X$ is the projection. In particular, if X is a surface then $p^{-1}(C) \subset B$ for every rational or elliptic curve C in X.*

Corollary 7.14. *Let X be a complex projective manifold and assume that the sheaf of germs of k-jet differentials is big. Then every holomorphic map $f : \mathbb{C} \to X$ is algebraically degenerate.*

However, it is not clear what sort of conditions guarantee hyperbolicity.

Open Problem 11. *Extend the preceding theory and find precise conditions for hyperbolicity in higher dimensions.*

The preceding theory requires that the manifold X be compact (actually the proof of the Moving Schwarz Lemma reuires that the variety be *projective*). However, we have seen in section 5 that Finsler geometry can still be useful in the case of $X = \mathbb{P}^2 \setminus C$ where C is a curve, by using the logarithmic bundle $T^*\mathbb{P}^2(\log C)$. Indeed this can be extended to the situation where X is the complement of a *divisor* D in a *projective* variety $\overline{X}$ (so $X = \overline{X} \setminus D$). In such cases the logarithmic bundle $T^*\overline{X}(\log D)$

(where X is the compactification of X) can still be defined. In fact the logarithmic k-jet differentials can be defined and the discussions in this section can be carried over to these bundles.

Open Problem 12. *Extend the theory of logarithmic vector bundle to logarithmic jet bundles.*

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