Advanced Studies in Pure Mathematics 48, 2007
Finsler Geometry, Sapporo 2005 – In Memory of Makoto Matsumoto
pp. 263–308

Ehresmann connections, metrics and good metric derivatives

Rezső L. Lovas, Johanna Pék and József Szilasi

Abstract.

In this survey we approach some aspects of tangent bundle geometry from a new viewpoint. After an outline of our main tools, i.e., the pull-back bundle formalism, we give an overview of Ehresmann connections and covariant derivatives in the pull-back bundle of a tangent bundle over itself. Then we define and characterize some special classes of generalized metrics. By a generalized metric we shall mean a pseudo-Riemannian metric tensor in our pull-back bundle. The main new results are contained in Section 5. We shall say, informally, that a metric covariant derivative is 'good' if it is related in a natural way to an Ehresmann connection determined by the metric alone. We shall find a family of good metric derivatives for the so-called weakly normal Moór – Vanstone metrics and a distinguished good metric derivative for a certain class of Miron metrics.

CONTENTS

Received February 3, 2006.
Revised May 27, 2006.

§1. Introduction

The background philosophy behind this paper is very simple: *Finsler metrics are special pseudo-Riemannian metrics in a special vector bundle.* The special character of the metric means *variationality* above all: it is the Hessian of a Lagrangian which is defined on an appropriate open submanifold $\widetilde{TM}$ of the tangent manifold TM of a base manifold M. If τ is the natural projection of TM onto M, we require that $\tau(\widetilde{TM}) = M$, and we formulate the theory in the pull-back bundle $\pi^*\tau$, $\pi := \tau \restriction \widetilde{TM}$. We arrive at the most important special case when $\widetilde{TM}$ is the split manifold $\overset{\circ}{T}M := \underset{p\in M}{\cup}(T_pM \setminus \{0_p\})$ and $\overset{\circ}{\tau} := \tau \restriction \overset{\circ}{T}M$.

As to the scene of the theory, there is no consensus among geometers: at least three other approaches are also in current use. In them we find the following basic geometric setups:

- the tangent bundle $\tau_{TM} : TTM \to TM$,
- the vertical subbundle τ^v_{TM} of τ_{TM},
- the pull-back of the frame bundle associated with $\tau : TM \to M$ over τ.

The first approach was initiated by Joseph Grifone [16, 17], for later use see e.g. [39, 41]. Compared with the other formulations, the world is duplicated in a non-canonical way: T^vTM occurs as a fibrewise $n(= \dim M)$-dimensional direct summand in TTM, and its complementary subbundle, called a *horizontal subbundle*, depends on another structure. However, in the presence of a Finsler structure a horizontal subbundle may be specified in a natural manner. In all certainty Grifone's main motivation in the extension of τ^v_{TM} was the intention of using the Frölicher – Nijenhuis calculus of vector-valued forms. In τ_{TM} this formalism is applicable without any difficulty, and provides an extremely concise and transparent formulation of the underlying geometric ideas. At the same time it serves as a powerful tool for calculations.

The second approach (scene the vertical bundle) is followed e.g. in the books [1, 5]. However, when the torsion of a covariant derivative operator in τ^v_{TM} is treated, the operator in both books is extended to the whole τ_{TM} using a horizontal structure.

The third approach first appeared in a paper of Louis Auslander [3], but it was elaborated in full detail by Makoto Matsumoto [28, 29]. Here the basic geometric setup is a (special) principle bundle, and for the treatment of *Finsler connections* the whole machinery of the theory of principal connections is available.

Why have we chosen the pull-back bundle framework? In our practice we have gained experience about all the three other formulations of the theory. We extremely enjoyed the elegance and efficiency of the Frölicher – Nijenhuis formalism in τ_{TM}. However, Mike Crampin's paper *'Connections of Berwald type'* [10] made a strong impression on us. When we realized that a convenient intrinsic calculus is also available in this framework, and found that the advantages of the Frölicher – Nijenhuis calculus may be preserved as well, we made our choice in favour of the $\pi^*\tau$-formalism. The third author's comprehensive study [40] has already originated in the spirit of this philosophy.

Our decision is rather typical than exceptional. The pull-back bundle $\overset{\circ}{\tau}{}^*\tau$ is the main scene of the theory in the delightful textbook of D. Bao, S.-S. Chern and Z. Shen [4]. This approach was consequently used by H. Akbar-Zadeh in his works (see e.g. [2]), as well as in B. T. Hassan's excellent Thesis [20]. Mention should be made of Peter Dombrowski's brilliant review [13], in which he translated a paper of M. Matsumoto from the principal bundle formalism into the pullback formalism. Dombrowski's review inspired a remarkable paper of Z. I. Szabó [38].

In this work we need only a modest algebraic-analytical formalism. We shall apply from beginning the *canonical v-covariant derivative in* $\pi^*\tau$. This is a possible intrinsic formulation of 'the well known fact that the partial differentiation of components of a tensor field by y^i gives rise to a new tensor field, as it has been noticed since the early period of Finsler geometry' ([29], p. 59; the y^i's mean fibre coordinates).

As in any vector bundle, covariant derivative operators are at our disposal also in $\pi^*\tau$. Their torsion(s) may be defined without any artificial extension process, using only canonical bundle maps arising in the short exact sequence (called the *basic exact sequence*) made from $\pi^*\tau$ and $\tau_{\widetilde{TM}}$.

Ehresmann connections (called also *nonlinear connections*) are further indispensable tools in the formulation of the theory, but they also deserve attention in their own right. The importance of 'nonlinear connections' for the foundation of Finsler geometry was first emphasized by A. Kawaguchi [23], but their systematic use in this context is due to M. Matsumoto. Recently Ehresmann connections have been applied to model the constraints in the mechanics of nonholonomic systems [6].

There are several equivalent possibilities to introduce the concept of Ehresmann connections. We define an Ehresmann connection as a splitting $\mathcal{H}$ of our basic exact sequence; then $\operatorname{Im}\mathcal{H}$ is a complementary subbundle to T^vTM, i.e., a horizontal subbundle. Strictly speaking, an Ehresmann connection is also required to be complete in the sense that horizontal lifts of complete vector fields on M are always complete vector fields on TM (see e.g. [14, 32]), but we shall not need this additional assumption.

Via linearization, an Ehresmann connection induces a covariant derivative operator in $\pi^*\tau$, the *Berwald derivative*, which acts in vertical directions as the canonical v-covariant derivative. In terms of the Berwald derivative, the basic geometric data (tension, torsions, curvature) may be defined or/and characterized.

Of course, all the material collected here concerning Ehresmann connections can be found (possibly sporadically) in the literature, or is well-established folklore. However, our presentation may provide some new insight, and it is adapted to the special demands of the treatment of (generalized) Finsler stuctures.

Having clarified the scene and the basic tools, we have to face a further question: *why generalized Finsler structures at all*? Since the age of Poincaré, there has been a standard answer to this type of question: in a more general setting it becomes clearer and more transparent what some facts depend on. Although nowadays it is a *cliché*, it remains true. We found again and again that some results concerning generalized metrics can be proved without any extra effort, while to obtain others we have to impose fine conditions, which, being satisfied automatically, are imperceptible in the standard Finslerian setup. To our best knowledge Masao Hashiguchi was the first who identified Finsler metrics among generalized metrics by the elegant condition of *normality* [19].

There is also a more practical reason of the investigation of generalized metrics. It turned out that the requirement of variationality is too restrictive for some important physical theories. As it was noticed first by J. Horváth and A. Moór [21], generalized metrics provide a natural framework for the so-called bilocal field theory initiated by H. Yukawa in the 1940s. Generalized metrics proved to be useful also in relativistic optics [35].

Riemannian (and pseudo-Riemannian) geometry enjoys the miracle of the existence of a unique distinguished metric derivative, the *Levi-Civita derivative. Cartan's derivative* in Finsler geometry also realizes such a miracle. However, this 'deus ex machina' does not work for the whole class of generalized metrics. There is quite a universal construction, discovered by Radu Miron for a special class of metrics [33], which produces a 'nice' metric derivative starting from a metric and an Ehresmann connection. In the Finslerian case one can obtain in this way Cartan's derivative. In general, however, it is very difficult to find (even if it exists) an Ehresmann connection which depends only on the metric and is 'nicely related' to the covariant derivative operator. In this paper we investigate and solve this problem for two special classes of generalized metrics: for the class of so-called *weakly normal Moór – Vanstone metrics* and the class of *positive definite Miron metrics.*

Acknowledgement

We are indebted to Tom Mestdag for his contribution to the mini-seminar organized by the third author a couple of years ago in Debrecen. This proved to be fruitful for the formulation of subsections 3.6 and 3.8 of the paper.

§2. Framework and tools

2.1. Notation

Throughout this paper 1_H denotes the identity map of a set H. If $\varphi : H \to \mathcal{U}$ and $\psi : H \to \mathcal{V}$ are two maps, we denote by (φ, ψ) the map $H \to \mathcal{U} \times \mathcal{V}$ given by

$$a \in H \mapsto (\varphi, \psi)(a) := (\varphi(a), \psi(a)) \in \mathcal{U} \times \mathcal{V}.$$

The product $\varphi_1 \times \varphi_2$ of two maps $\varphi_1 : H_1 \to \mathcal{U}_1$, $\varphi_2 : H_2 \to \mathcal{U}_2$ is given by $(\varphi_1 \times \varphi_2)(a_1, a_2) := (\varphi_1(a_1), \varphi_2(a_2))$; it maps $H_1 \times H_2$ into $\mathcal{U}_1 \times \mathcal{U}_2$.

The symbols $\mathbb{N}^*$ and $\mathbb{R}$ denote the positive integers and the reals.

2.2. Conventions and basic technicalities

(1) M will stand once and for all for an n-dimensional ($n \in \mathbb{N}^*$) smooth manifold whose topology is Hausdorff, second countable and connected. $C^\infty(M)$ is the $\mathbb{R}$-algebra of smooth functions on M. A smooth map $\varphi : M \to N$ between manifolds M and N is called a *diffeomorphism* if it has a smooth inverse.

(2) By an *(r-)vector bundle* over a manifold M we mean a smooth map $\pi : E \to M$ such that

(i) each $\pi^{-1}(p)$, $p \in M$, is an r-dimensional real vector space;

(ii) for each $p \in M$ there is a neighbourhood $\mathcal{U}$ of p in M and a diffeomorphism $\varphi : \mathcal{U} \times \mathbb{R}^r \to \pi^{-1}(\mathcal{U}) \subset E$ such that for each $q \in \mathcal{U}$, the map

$$v \in \mathbb{R}^r \mapsto \varphi(q, v) \in \pi^{-1}(q)$$

is a linear isomorphism.

Terminology: M is the *base manifold*, E the *total manifold*, π the *projection*, $E_p := \pi^{-1}(p)$ the *fibre over* p, $\mathbb{R}^r$ is the *standard fibre*, and φ is a *bundle chart* of the vector bundle.

If $\pi' : E' \to M'$ is a second vector bundle, then a *bundle map* from π to π' is a smooth map $\varphi : E \to E'$ that restricts to linear maps

$$\varphi_p : E_p \to E'_{\psi(p)}, \ p \in M.$$

The correspondence $p \in M \mapsto \psi(p) \in M'$ defines a smooth map $\psi : M \to M'$. If $M' = M$ and $\psi = 1_M$, then φ is called a *strong bundle map*.

A *section* of $\pi : E \to M$ is a smooth map $\sigma : M \to E$ such that $\pi \circ \sigma = 1_M$. The set $\Gamma(\pi)$ of all sections of π is a module over $C^\infty(M)$ under the pointwise addition and multiplication by functions in $C^\infty(M)$.

(3) The *tangent bundle* of M is the n-vector bundle $\tau : TM \to M$ whose fibre at $p \in M$ is the tangent space T_pM. We recall that T_pM is the space of linear maps $v : C^\infty(M) \to \mathbb{R}$ which satisfy $v(fg) = v(f)g(p) + f(p)v(g)$; $f, g \in C^\infty(M)$. The *vertical lift* of a function $f \in C^\infty(M)$ to TM is $f^v := f \circ \tau \in C^\infty(TM)$, and the *complete lift* of f to TM is the smooth function

$$f^c : TM \to \mathbb{R},\ v \mapsto f^c(v) := v(f).$$

The tangent bundle of TM is $\tau_{TM} : TTM \to TM$. We denote by $\overset{\circ}{T}M$ the open subset of the nonzero tangent vectors to M. $\overset{\circ}{\tau} := \tau \restriction \overset{\circ}{T}M : \overset{\circ}{T}M \to M$ is the *deleted bundle* for τ, and its tangent bundle is $\tau_{\overset{\circ}{T}M} : T\overset{\circ}{T}M \to \overset{\circ}{T}M$.

The *tangent map* of a smooth map $\varphi : M \to N$ is the bundle map $\varphi_* : TM \to TN$ whose restriction to T_pM is given by

$$(\varphi_*)_p(v)(h) := v(h \circ \varphi),\ h \in C^\infty(N).$$

(4) A *vector field* on M is a section of $\tau : TM \to M$; the module of all vector fields on M will be denoted by $\mathfrak{X}(M)$. Any vector field $X \in \mathfrak{X}(M)$ may be interpreted as a differential operator on $C^\infty(M)$ by the rule

$$f \in C^\infty(M) \mapsto Xf \in C^\infty(M);\ Xf(p) := X(p)(f),\ p \in M.$$

This interpretation makes X a *derivation* on the ring $C^\infty(M)$ and leads to a very convenient definition of the *bracket* $[X, Y]$ of two vector fields $X, Y \in \mathfrak{X}(M)$: this is the unique vector field sending each $f \in C^\infty(M)$ to $X(Yf) - Y(Xf)$. Under the bracket operation, $\mathfrak{X}(M)$ becomes a real Lie algebra.

The *complete lift* of a vector field $X \in \mathfrak{X}(M)$ to TM is the vector field $X^c \in \mathfrak{X}(TM)$ characterized by

$$X^c f^c := (Xf)^c,\ f \in C^\infty(M).$$

Typical vector fields on TM will be denoted by Greek letters $\xi, \eta, \ldots$.

(5) By definition, a *covariant* and a *vectorvariant tensor field* (briefly *tensor*) *of degree* $k \in \mathbb{N}^*$ on M is, respectively,

a $C^\infty(M)$-multilinear map $A : (\mathfrak{X}(M))^k \to C^\infty(M)$ and $B : (\mathfrak{X}(M))^k \to \mathfrak{X}(M)$. Then A is also called a type $(0,k)$, while B a type $(1,k)$ tensor field. If T is a (covariant or vectorvariant) tensor field of degree k on M, then its *contraction* $i_X T$ by $X \in \mathfrak{X}(M)$ is the tensor field of degree $k-1$ given by $(i_X T)(X_2, \dots, X_k) := T(X, X_2, \dots, X_k)$.

(6) *One-forms* on M are the objects dual to vector fields; their $C^\infty(M)$-module is denoted by $\mathfrak{X}^*(M)$. A one-form $\vartheta \in \mathfrak{X}^*(M)$ and a vector field $X \in \mathfrak{X}(M)$ combine naturally to give the smooth function

$$\vartheta(X) : M \to \mathbb{R},\ p \in M \mapsto \vartheta(X)(p) := \vartheta_p(X_p).$$

Each function $f \in C^\infty(M)$ gives rise to the one-form df, its *differential*, given by

$$df(X) := Xf,\ X \in \mathfrak{X}(M).$$

We shall need the *exterior derivative* $d\vartheta$ of a one-form ϑ on M. It is a skew-symmetric covariant tensor field of degree 2 on M which may practically be given by the formula

$$d\vartheta(X,Y) := X(\vartheta Y) - Y(\vartheta X) - \vartheta[X,Y];\ X, Y \in \mathfrak{X}(M).$$

(7) By a *covariant derivative operator* or simply a *covariant derivative* in a vector bundle $\pi : E \to M$ we mean a map

$$(X, \sigma) \in \mathfrak{X}(M) \times \Gamma(\pi) \mapsto D_X\sigma \in \Gamma(\pi)$$

satisfying the following conditions:

(i) $D_{fX+hY}\sigma = fD_X\sigma + hD_Y\sigma$,
(ii) $D_X(\sigma_1 + \sigma_2) = D_X\sigma_1 + D_X\sigma_2$,
(iii) $D_X(f\sigma) = (Xf)\sigma + fD_X\sigma$.

Here $X, Y \in \mathfrak{X}(M)$; $\sigma, \sigma_1, \sigma_2 \in \Gamma(\pi)$; $f, h \in C^\infty(M)$. The *covariant differential* of a section $\sigma \in \Gamma(\pi)$ is the map

$$D\sigma : \mathfrak{X}(M) \to \Gamma(\pi),\ X \mapsto D\sigma(X) := D_X\sigma,$$

which collects all the covariant derivatives of σ. If we set

$$R(X,Y)\sigma := D_X D_Y\sigma - D_Y D_X\sigma - D_{[X,Y]}\sigma;\ X, Y \in \mathfrak{X}(M),\ \sigma \in \Gamma(\pi),$$

then $R(X,Y)\sigma \in \Gamma(\pi)$, and the map

$$(X, Y, \sigma) \in \mathfrak{X}(M) \times \mathfrak{X}(M) \times \Gamma(\pi) \mapsto R(X, Y, \sigma) \in \Gamma(\pi)$$

is $C^\infty(M)$-trilinear; it is said to be the *curvature* of D.

2.3. The pull-back bundle $\pi^*\tau$

Let $\widetilde{TM} \subset TM$ be an open set such that $\tau(\widetilde{TM}) = M$, and let $\pi := \tau \restriction \widetilde{TM}$. Consider the fibre product

$$\pi^*TM := \widetilde{TM} \times_M TM := \left\{(v,w) =: w_v \in \widetilde{TM} \times TM \middle| \pi(v) = \tau(w)\right\},$$

and define the projection $\pi^*\tau : \pi^*TM \to \widetilde{TM}$ by $\pi^*\tau(w_v) := v$. Then the fibres $(\pi^*TM)_v := (\pi^*\tau)^{-1}(v) = \{v\} \times T_{\pi(v)}M$ carry a natural linear structure given by

$$\alpha(w_1)_v + \beta(w_2)_v := (\alpha w_1 + \beta w_2)_v;\ w_1, w_2 \in T_{\pi(v)}M;\ \alpha, \beta \in \mathbb{R}.$$

Thus $\pi^*\tau : \widetilde{TM} \times_M TM \to TM$ becomes an n-vector bundle, called the *pull-back of* τ *over* π. Note that the fibres of $\pi^*\tau$ may canonically be identified with the real vector spaces $T_{\pi(v)}M$'s by the isomorphisms $w_v \in (\pi^*TM)_v \mapsto w \in T_{\pi(v)}M$.

The most important special cases arise when

$$\widetilde{TM} = TM,\ \pi = \tau, \quad \text{and} \quad \widetilde{TM} = \overset{\circ}{T}M,\ \pi = \overset{\circ}{\tau};$$

then we get the *pull-back* $\tau^*\tau$ *of* τ *over itself* and the *pull-back* $\overset{\circ}{\tau}{}^*\tau$ *of* τ *over* $\overset{\circ}{\tau}$, respectively.

Comparing with Matsumoto's theory of Finsler connections, in our approach the bundle $\pi^*\tau$ plays the same role as his 'Finsler bundle' in [29] or the 'vectorial frame bundle' in [30]; see also [13].

2.4. Tensors along π

Instead of $\Gamma(\pi^*\tau)$ we denote by $\mathfrak{X}(\pi)$ the $C^\infty(\widetilde{TM})$-module of sections of $\pi^*\tau$. The elements of $\mathfrak{X}(\pi)$ are of the form

$$\begin{cases} \tilde{X} = \left(1_{\widetilde{TM}}, \underline{X}\right) : \widetilde{TM} \to \widetilde{TM} \times_M TM, \\ \underline{X} : \widetilde{TM} \to TM \text{ is a smooth map such that } \tau \circ \underline{X} = \pi. \end{cases}$$

$\tilde{X}$ and $\underline{X}$ may be identified. These sections will also be mentioned as *vector fields along* π. We have the *canonical section*

$$\delta := \left(1_{\widetilde{TM}}, 1_{\widetilde{TM}}\right) : v \in \widetilde{TM} \mapsto \delta(v) = (v,v) \in \widetilde{TM} \times_M TM.$$

If $X \in \mathfrak{X}(M)$, then $\hat{X} := \left(1_{\widetilde{TM}}, X \circ \pi\right)$ is a vector field along π, called a *basic vector field.* Obviously, the $C^\infty(\widetilde{TM})$-module $\mathfrak{X}(\pi)$ is generated by basic vector fields. Analogously, if $\alpha \in \mathfrak{X}^*(M)$, then $\hat{\alpha} := \left(1_{\widetilde{TM}}, \alpha \circ \pi\right) \in \mathfrak{X}^*(\pi)$; $\hat{\alpha}$ is said to be a *basic one-form along* π.

As in the classical case (2.2. (5)), by a covariant or a vectorvariant *tensor (field)* of degree $k \in \mathbb{N}^*$ *along* π, resp., we mean a $C^\infty(\widetilde{TM})$-multilinear map $\tilde{A} : (\mathfrak{X}(\pi))^k \to C^\infty(\widetilde{TM})$ or $\tilde{B} : (\mathfrak{X}(\pi))^k \to \mathfrak{X}(\pi)$. For details we refer to [40], 2.22.

2.5. The basic short exact sequence

Starting from the vector bundles $\pi^*\tau$ and $\tau_{\widetilde{TM}} : T\widetilde{TM} \to \widetilde{TM}$ we may construct the short exact sequence of strong bundle maps

$$(2.5.1) \qquad 0 \to \widetilde{TM} \times_M TM \xrightarrow{\mathbf{i}} T\widetilde{TM} \xrightarrow{\mathbf{j}} \widetilde{TM} \times_M TM \to 0,$$

where $\mathbf{i}$ identifies the fibre $\{v\} \times T_{\pi(v)}M$ with the tangent vector space $T_v T_{\pi(v)}M$ for all $v \in \widetilde{TM}$, while $\mathbf{j} := (\tau_{\widetilde{TM}}, \pi_*)$. The property that (2.5.1) is an exact sequence means that $\mathbf{i}$ is injective, $\mathbf{j}$ is surjective, and $\operatorname{Im}\mathbf{i} = \operatorname{Ker}\mathbf{j}$. The bundle maps $\mathbf{i}$ and $\mathbf{j}$ induce the $C^\infty(\widetilde{TM})$-homomorphisms

$$\tilde{X} \in \mathfrak{X}(\pi) \mapsto \mathbf{i}\tilde{X} := \mathbf{i} \circ \tilde{X} \in \mathfrak{X}(\widetilde{TM}),\ \xi \in \mathfrak{X}(\widetilde{TM}) \mapsto \mathbf{j}\xi := \mathbf{j} \circ \xi \in \mathfrak{X}(\pi)$$

at the level of modules of sections, and we denote these maps also by $\mathbf{i}$ and $\mathbf{j}$. Thus we also have the exact sequence of $C^\infty(\widetilde{TM})$-homomorphisms

$$(2.5.2) \qquad 0 \to \mathfrak{X}(\pi) \xrightarrow{\mathbf{i}} \mathfrak{X}(\widetilde{TM}) \xrightarrow{\mathbf{j}} \mathfrak{X}(\pi) \to 0.$$

Both (2.5.1) and (2.5.2) will be referred to as the *basic (short) exact sequence.*

$$\mathfrak{X}^v(\widetilde{TM}) := \left\{ \mathbf{i}\tilde{X} \in \mathfrak{X}(\widetilde{TM}) \middle| \tilde{X} \in \mathfrak{X}(\pi) \right\}$$

is a subalgebra of the Lie algebra $\mathfrak{X}(\widetilde{TM})$, called the algebra of *vertical vector fields* on $\widetilde{TM}$. In particular, for any vector field X on M, $X^v := \mathbf{i}\hat{X}$ is a vertical vector field, the *vertical lift* of X. The map

$$\ell^v : \mathfrak{X}(M) \to \mathfrak{X}(TM),\ X \mapsto \ell^v(X) := X^v$$

is said to be the *vertical lifting* from $\mathfrak{X}(M)$ into $\mathfrak{X}(TM)$.

The basic short exact sequence makes it possible to define the following canonical objects:

$C := \mathbf{i}\delta$ – the *Liouville vector field* on $\widetilde{TM}$,

$J := \mathbf{i} \circ \mathbf{j}$ – the *vertical endomorphism* of $\mathfrak{X}(\widetilde{TM})$.

Then we have immediately that

$$\operatorname{Im} J = \operatorname{Ker} J = \mathfrak{X}^v(\widetilde{TM}), \ J^2 = 0; \tag{2.5.3}$$

$$[C, X^v] = -X^v, \ [C, X^c] = 0 \quad (X \in \mathfrak{X}(M)). \tag{2.5.4}$$

2.6. Brackets

For any vector field ξ on $\widetilde{TM}$ and type (1,1) tensors A, B on $\widetilde{TM}$ we define the *Frölicher – Nijenhuis brackets* $[\xi, A]$ and $[A, B]$ by the following rules for calculation:

$$[\xi, A]\eta := [\xi, A\eta] - A[\xi, \eta], \ \eta \in \mathfrak{X}(\widetilde{TM}); \tag{2.6.1}$$

$$\begin{aligned}[A, B](\xi, \eta) := {} & [A\xi, B\eta] + [B\xi, A\eta] + (A \circ B + B \circ A)[\xi, \eta] \\ & - A[B\xi, \eta] - A[\xi, B\eta] - B[A\xi, \eta] - B[\xi, A\eta].\end{aligned} \tag{2.6.2}$$

In particular, $N_A := \frac{1}{2}[A, A]$ is said to be the *Nijenhuis torsion* of A. Then we have

$$[X^v, J] = [X^c, J] = 0 \ (X \in \mathfrak{X}(M)), \ [C, J] = -J, \ N_J = 0. \tag{2.6.3}$$

For a systematic treatment and applications of *Frölicher – Nijenhuis theory* of vector valued differential forms we refer to [18, 32, 40].

2.7. The canonical v-covariant derivative

Let $\tilde{X}$ be a section along π, and set

$$\nabla^v_{\tilde{X}} F := (\mathbf{i}\tilde{X})F = (dF \circ \mathbf{i})(\tilde{X}) \text{ if } F \in C^\infty(\widetilde{TM}); \tag{2.7.1}$$

$$\begin{aligned}\nabla^v_{\tilde{X}} \tilde{Y} := \mathbf{j}\left[\mathbf{i}\tilde{X}, \eta\right] & \text{ if } \tilde{Y} \in \mathfrak{X}(\pi) \text{ and } \eta \in \mathfrak{X}(\widetilde{TM}) \\ & \text{ such that } \mathbf{j}\eta = \tilde{Y}.\end{aligned} \tag{2.7.2}$$

Then $\nabla^v_{\tilde{X}} \tilde{Y}$ is well-defined: it does not depend on the choice of η. The map

$$\nabla^v : \mathfrak{X}(\pi) \times \mathfrak{X}(\pi) \to \mathfrak{X}(\pi), \ (\tilde{X}, \tilde{Y}) \mapsto \nabla^v_{\tilde{X}} \tilde{Y}$$

has the properties 2.2. (7) (i), (ii) of a covariant derivative operator, but the product rule takes the slightly different form

$$\nabla^v_{\tilde{X}} F\tilde{Y} = \left(\nabla^v_{\tilde{X}} F\right) \tilde{Y} + F\nabla^v_{\tilde{X}} \tilde{Y}.$$

∇^v is said to be the *canonical v-covariant derivative* in $\pi^*\tau$. It may be extended to be a tensor derivation of the tensor algebra of $\mathfrak{X}(\pi)$ by the standard pattern. For any tensor field $\tilde{A}$ along π, the v-covariant *differential* $\nabla^v \tilde{A}$ collects all the v-covariant derivatives of $\tilde{A}$, cf. 2.2. (7).

2.8. Lagrangian forms and Hessians

Let $F : \widetilde{TM} \to \mathbb{R}$ be a smooth function, mentioned also as a *Lagrangian* in this context. Then

$$\vartheta_F := dF \circ J \text{ and } \omega_F := d\vartheta_F = d(dF \circ J)$$

are one- and two-forms on $\widetilde{TM}$, called the *Lagrangian 1-form* and the *Lagrangian 2-form* associated to F. The second v-covariant differential

$$g_F := \nabla^v \nabla^v F$$

of F is a symmetric covariant tensor field of degree 2 along π, called the *Hessian* of F. ω_F and g_F are related by

$$\omega_F(J\xi, \eta) = g_F(\mathbf{j}\xi, \mathbf{j}\eta);\ \ \xi, \eta \in \mathfrak{X}(\widetilde{TM}). \tag{2.8.1}$$

It now follows immediately that *ω_F and g_F are non-degenerate at the same time.* In case of non-degeneracy F is said to be a *regular Lagrangian.*

Warning. Throughout the paper, *non-degeneracy should be meant pointwise* and not at the level of sections; the latter would be a weaker condition on g_F.

2.9. Semisprays and Lagrangian vector fields

We recall that a vector field S on $\widetilde{TM}$ is said to be a *second-order vector field* or a *semispray* if $JS = C$ or, equivalently, if $\mathbf{j}S = \delta$. If, in addition, $\overset{\circ}{T}M \subset \widetilde{TM}$ and $[C, S] = S$, then S is called a *spray.* The condition $[C, S] = S$ implies that the fibre components of a spray are positively homogeneous functions of degree 2. If S is a semispray, then we have

$$J[J\xi, S] = J\xi \quad \text{for all } \xi \in \mathfrak{X}(\widetilde{TM}). \tag{2.9.1}$$

This useful relation was established by J. Grifone [16].

Now suppose that $F \in C^\infty(\widetilde{TM})$ is a regular Lagrangian, and let $L_F := CF - F$. It is a well-known but fundamental fact that there is a unique semispray S on $\widetilde{TM}$ such that

$$i_S \omega_F = -dL_F; \tag{2.9.2}$$

for a proof see e.g. [25] 7.1. The semispray S given by (2.9.2) is said to be the *Lagrangian vector field* for F.

§3. Ehresmann connections. Covariant derivatives in $\pi^*\tau$

3.1. Ehresmann connections and associated maps

By an *Ehresmann connection* (or a *nonlinear connection*) $\mathcal{H}$ on $\widetilde{TM}$ we mean a right splitting of the basic exact sequence (2.5.1), i.e., a strong bundle map $\mathcal{H} : \widetilde{TM} \times_M TM \to T\widetilde{TM}$ such that $\mathbf{j} \circ \mathcal{H} = 1_{\widetilde{TM} \times_M TM}$. The type (1,1) tensor field $\mathbf{h} := \mathcal{H} \circ \mathbf{j}$ on $\widetilde{TM}$ is the *horizontal projector* belonging to $\mathcal{H}$, and $\mathbf{v} := 1_{T\widetilde{TM}} - \mathbf{h}$ is the complementary *vertical projector.* To every Ehresmann connection $\mathcal{H}$ corresponds a unique strong bundle map $\mathcal{V} : T\widetilde{TM} \to \widetilde{TM} \times_M TM$ such that

$$\mathcal{V} \circ \mathbf{i} = 1_{\widetilde{TM} \times_M TM} \text{ and } \operatorname{Ker} \mathcal{V} = \operatorname{Im} \mathcal{H};$$

$\mathcal{V}$ is called the *vertical map* associated to $\mathcal{H}$. $\mathcal{H}$ and $\mathcal{V}$ induce $C^\infty(\widetilde{TM})$-homomorphisms at the level of sections by the rules

$$\tilde{X} \in \mathfrak{X}(\pi) \mapsto \mathcal{H} \circ \tilde{X} =: \mathcal{H}\tilde{X} \in \mathfrak{X}(\widetilde{TM}),$$
$$\xi \in \mathfrak{X}(\widetilde{TM}) \mapsto \mathcal{V} \circ \xi =: \mathcal{V}\xi \in \mathfrak{X}(\pi).$$

Thus $\mathcal{H}$ and $\mathcal{V}$ give rise to a right and a left splitting also of (2.5.2), respectively, denoted by the same symbols.

$$\mathfrak{X}^h(\widetilde{TM}) := \left\{ \mathcal{H}\tilde{X} \in \mathfrak{X}(\widetilde{TM}) \middle| \tilde{X} \in \mathfrak{X}(\pi) \right\}$$

is the $C^\infty(\widetilde{TM})$-module of the *horizontal vector fields* on $\widetilde{TM}$. The horizontal vector fields do not form, in general, a subalgebra of the Lie algebra $\mathfrak{X}(\widetilde{TM})$; this failure will be measured by the curvature of the Ehresmann connection.

3.2. Horizontal lifts

Let an Ehresmann connection $\mathcal{H}$ be specified on $\widetilde{TM}$. For any vector field X on M, $X^h := \mathcal{H} \circ \hat{X} =: \mathcal{H}\hat{X}$ is a horizontal vector field, called the *horizontal lift* of X. The map $\ell^h : X \in \mathfrak{X}(M) \mapsto \ell^h(X) := X^h$ is said to be the *horizontal lifting* with respect to $\mathcal{H}$.

The following useful observation can easily be proved.

Lemma 1. *If ℓ^h is the horizontal lifting with respect to the Ehresmann connection $\mathcal{H}$, then*

(i) $\ell^h(fX) = f^v \ell^h(X)$ *for any vector field X on M and function $f \in C^\infty(M)$,*

(ii) $J \circ \ell^h = \ell^v$.

Conversely, given a map $\ell^h : \mathfrak{X}(M) \to \mathfrak{X}(\widetilde{TM})$ *satisfying* (i) *and* (ii), *there is a unique Ehresmann connection* $\mathcal{H}$ *on* $\widetilde{TM}$ *such that* ℓ^h *is the horizontal lifting with respect to* $\mathcal{H}$. *Setting*

$$\mathcal{H}\hat{X} := \ell^h(X) \text{ if } X \in (M),$$

$\mathcal{H}$ *can be given via* $C^\infty(\widetilde{TM})$*-linear extension.* □

3.3. The difference tensor

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Ehresmann connections on $\widetilde{TM}$. Since

$$J \circ (\mathcal{H}_1 - \mathcal{H}_2) = \mathbf{i} \circ (\mathbf{j} \circ \mathcal{H}_1 - \mathbf{j} \circ \mathcal{H}_2) = \mathbf{i} \circ \left(1_{\widetilde{TM} \times_M TM} - 1_{\widetilde{TM} \times_M TM}\right) = 0,$$

$\mathcal{H}_1 - \mathcal{H}_2$ is vertical-valued. So there is a unique endomorphism P along π such that

$$\mathcal{H}_1 - \mathcal{H}_2 = \mathbf{i} \circ P; \tag{3.3.1}$$

P is said to be the *difference tensor* of $\mathcal{H}_1$ and $\mathcal{H}_2$. It is easy to check that

$$\mathbf{i} \circ P = \mathbf{v}_2 \circ \mathcal{H}_1, \tag{3.3.2}$$

where $\mathbf{v}_2$ is the vertical projector belonging to $\mathcal{H}_2$.

3.4. The Crampin – Grifone construction

Note first that if $\mathcal{H}$ is an Ehresmann connection on $\widetilde{TM}$, then

$$S_{\mathcal{H}} := \mathcal{H}\delta$$

is a semispray, called the *associated semispray* to $\mathcal{H}$. Now, conversely, we sketch a process discovered (independently) by M. Crampin [7, 8, 9] and J. Grifone [16] that results in an Ehresmann connection if we start from a semispray S on $\widetilde{TM}$. Consider the map

$$\ell^h : X \in \mathfrak{X}(M) \mapsto \ell^h(X) := \frac{1}{2}(X^c - [S, X^v]). \tag{3.4.1}$$

A straightforward calculation shows that ℓ^h satisfies the conditions of Lemma 1, therefore gives rise to an Ehresmann connection $\mathcal{H}_S$ with horizontal lifting ℓ^h. The horizontal projector belonging to $\mathcal{H}_S$ is given by the formula

$$\mathbf{h}_S = \frac{1}{2}\left(1_{\mathfrak{X}(\widetilde{TM})} - [S, J]\right). \tag{3.4.2}$$

The semispray associated to $\mathcal{H}_S$ is $\frac{1}{2}(S + [C, S])$, which coincides with the original semispray S if and only if it is a spray.

3.5. Torsions

Let a covariant derivative operator

$$D : \mathfrak{X}(\widetilde{TM}) \times \mathfrak{X}(\pi) \to \mathfrak{X}(\pi),\ (\xi, \tilde{Y}) \mapsto D_\xi \tilde{Y}$$

be given in $\pi^*\tau$. By the *torsion* of D we mean the map

$$(3.5.1)\quad T(D) : (\xi, \eta) \in \mathfrak{X}(\widetilde{TM}) \times \mathfrak{X}(\widetilde{TM}) \mapsto T(D)(\xi, \eta) := D_\xi \mathbf{j}\eta - D_\eta \mathbf{j}\xi - \mathbf{j}[\xi, \eta] \in \mathfrak{X}(\pi).$$

Then $T(D)$ is an $\mathfrak{X}(\pi)$-valued tensor field of degree 2 on $\widetilde{TM}$. We may also form a type (1,2) tensor field $T^v(D)$ along π given by

$$(3.5.2)\ T^v(D)(\tilde{X}, \tilde{Y}) := D_{\mathbf{i}\tilde{X}} \tilde{Y} - D_{\mathbf{i}\tilde{Y}} \tilde{X} - \mathbf{i}^{-1}\left[\mathbf{i}\tilde{X}, \mathbf{i}\tilde{Y}\right];\ \tilde{X}, \tilde{Y} \in \mathfrak{X}(\pi);$$

$T^v(D)$ is called the *vertical torsion* of D. Following J.-G. Diaz and G. Grangier [12], we call the map

$$(3.5.3)\qquad \mathcal{S} : (\tilde{X}, \tilde{Y}) \in \mathfrak{X}(\pi) \times \mathfrak{X}(\pi) \mapsto \mathcal{S}(\tilde{X}, \tilde{Y}) := \nabla^v_{\tilde{Y}} \tilde{X} - D_{\mathbf{i}\tilde{Y}} \tilde{X}$$

the *Finsler torsion* of D; it is again a type (1,2) tensor field along π. If $\mathcal{S} = 0$, then we say that D is *vertically natural.* In the presence of an Ehresmann connection $\mathcal{H}$ in $\pi^*\tau$ we may define the *horizontal torsion*

$$(3.5.4)\qquad \mathcal{T} := T(D) \circ (\mathcal{H} \times \mathcal{H})$$

of D. It is a type (1,2) tensor field along π such that

$$\mathcal{T}(\tilde{X}, \tilde{Y}) := D_{\mathcal{H}\tilde{X}} \tilde{Y} - D_{\mathcal{H}\tilde{Y}} \tilde{X} - \mathbf{j}\left[\mathcal{H}\tilde{X}, \mathcal{H}\tilde{Y}\right];\ \tilde{X}, \tilde{Y} \in \mathfrak{X}(\pi).$$

The torsion of D is completely determined by the horizontal torsion and the Finsler torsion, namely for any vector fields ξ, η on $\widetilde{TM}$ we have

$$T(D)(\xi, \eta) = \mathcal{T}(\mathbf{j}\xi, \mathbf{j}\eta) + \mathcal{S}(\mathbf{j}\xi, \mathcal{V}\eta) - \mathcal{S}(\mathbf{j}\eta, \mathcal{V}\xi),$$

where $\mathcal{V}$ is the vertical map associated to $\mathcal{H}$.

3.6. Deflections and regularities

By the *deflection* of a covariant derivative operator D in $\pi^*\tau$ we mean the covariant differential $\mu := D\delta$ of the canonical section. The type (1,1) tensor field

$$\tilde{\mu} := \mu \circ \mathbf{i} : \tilde{X} \in \mathfrak{X}(\pi) \mapsto \tilde{\mu}\tilde{X} := \mu(\mathbf{i}\tilde{X}) = D_{\mathbf{i}\tilde{X}} \delta$$

along π is said to be the *v-deflection* of D. The v-deflection and the Finsler torsion of D are related by

$$\tilde{\mu} = 1_{\mathfrak{X}(\pi)} - i_\delta \mathcal{S}. \tag{3.6.1}$$

Indeed, let $\tilde{X}$ be a vector field along π. It can be represented in the form $\tilde{X} = \mathbf{j}\xi$, $\xi \in \mathfrak{X}(\widetilde{TM})$. Using an arbitrary 'auxiliary' Ehresmann connection $\mathcal{H}$ on $\widetilde{TM}$, and taking into account (2.7.2) and (2.9.1) we have

$$\begin{aligned}(i_\delta \mathcal{S})(\tilde{X}) &= \mathcal{S}(\delta, \tilde{X}) = \nabla^v_{\tilde{X}}\delta - D_{\mathrm{i}\tilde{X}}\delta = \mathbf{j}\big[\mathbf{i}\tilde{X}, \mathcal{H}\delta\big] - \tilde{\mu}\tilde{X} \\ &= \mathbf{j}[J\xi, S_{\mathcal{H}}] - \tilde{\mu}\tilde{X} = \mathbf{j}\xi - \tilde{\mu}\tilde{X} = \big(1_{\mathfrak{X}(\pi)} - \tilde{\mu}\big)\big(\tilde{X}\big).\end{aligned}$$

The covariant derivative operator is said to be

regular if $\tilde{\mu}$ is (pointwise) invertible,
strongly regular if $\tilde{\mu} = 1_{\mathfrak{X}(\pi)}$,
Moór – Vanstone regular if $\big(\tilde{\mu} - 1_{\mathfrak{X}(\pi)}\big)^2 = 0$.

In view of (3.6.1) these properties can be expressed in terms of the Finsler torsion of D as follows:

D is regular $\iff$ $1_{\mathfrak{X}(\pi)} - i_\delta\mathcal{S}$ is bijective;
D is strongly regular $\iff$ $i_\delta\mathcal{S} = 0$;
D is Moór – Vanstone regular $\iff$ $(i_\delta\mathcal{S})^2 = 0$.

As a consequence, we have the following implications for a covariant derivative D in $\pi^*\tau$:

D is vertically natural $\implies$ D is strongly regular
$\implies$ D is Moór – Vanstone regular $\implies$ D is regular.

The concept of Moór – Vanstone regularity appeared in the papers [36, 42] in the form of quite a strange coordinate expression. As a matter of fact, it is not difficult to produce Moór – Vanstone regular (and hence regular) covariant derivatives. This assertion may be made more vivid by an example. Consider a type (1,1) tensor field $\tilde{A}$ along π such that $\tilde{A}\delta = 0$, and let

$$\mathcal{S}(\tilde{X}, \tilde{Y}) := \big(\tilde{A}\tilde{Y}\big)\tilde{X};\ \tilde{X}, \tilde{Y} \in \mathfrak{X}(\pi).$$

If D is a covariant derivative operator such that

$$D_{\mathrm{i}\tilde{X}}\tilde{Y} := \nabla^v_{\tilde{X}}\tilde{Y} - \mathcal{S}\big(\tilde{Y}, \tilde{X}\big),$$

then the Finsler torsion of D is just the given tensor $\mathcal{S}$, and we have for any section $\tilde{X} \in \mathfrak{X}(\pi)$

$$(i_\delta \mathcal{S})^2(\tilde{X}) = i_\delta \mathcal{S}\big((\tilde{A}\tilde{X})\delta\big) = (\tilde{A}\tilde{X})\mathcal{S}(\delta,\delta) = (\tilde{A}\tilde{X})(\tilde{A}\delta)\delta = 0,$$

therefore D is Moór – Vanstone regular. This observation is due to Tom Mestdag.

3.7. Associatedness

Let a covariant derivative operator D and an Ehresmann connection $\mathcal{H}$ be given in $\pi^*\tau$. As above, we denote by μ and $\mathcal{V}$ the deflection of D and the vertical map belonging to $\mathcal{H}$, resp. By the *h-deflection* of D (with respect to $\mathcal{H}$) we mean the map

$$\mu^h := \mu \circ \mathcal{H} : \tilde{X} \in \mathfrak{X}(\pi) \mapsto \mu^h \tilde{X} := \mu(\mathcal{H}\tilde{X}) = D_{\mathcal{H}\tilde{X}}\delta \in \mathfrak{X}(\pi).$$

D is said to be

associated to $\mathcal{H}$ if $\operatorname{Ker}\mu = \operatorname{Im}\mathcal{H}$,
strongly associated to $\mathcal{H}$ if $\mu = \mathcal{V}$.

We have the simple

Lemma 2. *A covariant derivative operator D in $\pi^*\tau$ is*

(i) *associated to an Ehresmann connection $\mathcal{H}$ if and only if it is regular and has vanishing h-deflection;*
(ii) *strongly associated to $\mathcal{H}$ if and only if it is strongly regular and has vanishing h-deflection.* □

3.8. Ehresmann connections induced by covariant derivatives

Proposition 3. *Let D be a regular covariant derivative in $\pi^*\tau$, with deflection μ and v-deflection $\tilde{\mu}$. Then the map*

$$\ell_D : X \in \mathfrak{X}(M) \mapsto \ell_D(X) := X^c - \mathbf{i}\tilde{\mu}^{-1}D_{X^c}\delta$$

is a horizontal lifting in the sense that it satisfies conditions (i) *and* (ii) *of Lemma 1. Thus ℓ_D determines a unique Ehresmann connection $\mathcal{H}_D$ such that $\mathcal{H}_D\hat{X} = \ell_D(X)$ for any vector field X on M. The h-deflection of D with respect to $\mathcal{H}_D$ vanishes, therefore* $\operatorname{Ker}\mu = \operatorname{Im}\mathcal{H}_D$. *$\mathcal{H}_D$ is the unique Ehresmann connection with this property.*

Proof. Let $X \in \mathfrak{X}(M)$, $f \in C^\infty(M)$. Then

$$\begin{aligned}\ell_D(fX) &:= (fX)^c - \mathbf{i}\tilde{\mu}^{-1}D_{(fX)^c}\delta = f^vX^c + f^cX^v - \mathbf{i}\tilde{\mu}^{-1}D_{f^vX^c+f^cX^v}\delta \\ &= f^v\left(X^c - \mathbf{i}\tilde{\mu}^{-1}D_{X^c}\delta\right) + f^c\left(X^v - \mathbf{i}\tilde{\mu}^{-1}D_{\mathbf{i}\hat{X}}\delta\right) \\ &= f^v\ell_D(X) + f^c\left(X^v - \mathbf{i}\tilde{\mu}^{-1}\tilde{\mu}\hat{X}\right) = f^v\ell_D(X),\end{aligned}$$

$$J \circ \ell_D(X) = JX^c - \mathbf{i} \circ \mathbf{j} \circ \mathbf{i}\tilde{\mu}^{-1} D_{X^c}\delta = X^v = \ell^v(X),$$

so ℓ_D is indeed a horizontal lifting. Since

$$\begin{aligned} D_{\mathcal{H}_D \hat{X}}\delta &= D_{X^c}\delta - D_{\mathbf{i}\circ\tilde{\mu}^{-1}D_{X^c}\delta}\delta = D_{X^c}\delta - \tilde{\mu}\left(\tilde{\mu}^{-1}D_{X^c}\delta\right) \\ &= D_{X^c}\delta - D_{X^c}\delta = 0, \end{aligned}$$

and the basic vector fields generate $\mathfrak{X}(\pi)$, it follows that the h-deflection of D with respect to $\mathcal{H}_D$ vanishes.

Suppose, finally, that $\mathcal{H}$ is another Ehresmann connection with this property, i.e.,

$$D_{\mathcal{H}\tilde{X}}\delta = 0, \ \tilde{X} \in \mathfrak{X}(\pi).$$

Then, by (3.3.1) and (3.3.2),

$$\mathcal{H} - \mathcal{H}_D = \mathbf{v}_D \circ \mathcal{H} = \mathbf{i} \circ \mathcal{V}_D \circ \mathcal{H}$$

($\mathbf{v}_D$ and $\mathcal{V}_D$ are the vertical projector and vertical map belonging to $\mathcal{H}_D$), and for any section $\tilde{X} \in \mathfrak{X}(\pi)$ we have

$$\begin{aligned} \tilde{\mu}\big((\mathcal{H} - \mathcal{H}_D)(\tilde{X})\big) &= \mu\big(\mathbf{i} \circ \mathcal{V}_D(\mathcal{H}\tilde{X})\big) = D_{\mathbf{i}\mathcal{V}_D(\mathcal{H}\tilde{X})}\delta \\ &= D_{(\mathcal{H}-\mathcal{H}_D)(\tilde{X})}\delta = D_{\mathcal{H}\tilde{X}}\delta - D_{\mathcal{H}_D\tilde{X}}\delta = 0. \end{aligned}$$

This implies that $\mathcal{H} = \mathcal{H}_D$, since $\tilde{\mu}$ is injective by the regularity of D. □

The Ehresmann connection characterized by Proposition 3 is said to be the *Ehresmann connection induced by D*. If, in particular, D is strongly regular, then

$$\mathcal{H}_D\hat{X} = X^c - \mathbf{i}D_{X^c}\delta, \ X \in \mathfrak{X}(M). \tag{3.8.1}$$

3.9. The Berwald derivative. Tension

Let an Ehresmann connection $\mathcal{H}$ be given in $\pi^*\tau$, and let $\mathcal{V}$ be the vertical map associated to $\mathcal{H}$. Define the mapping $\nabla : \mathfrak{X}(\widetilde{TM}) \times \mathfrak{X}(\pi) \to \mathfrak{X}(\pi)$ by

$$\nabla_{\mathcal{H}\tilde{X}}\tilde{Y} := \mathcal{V}\left[\mathcal{H}\tilde{X}, \mathbf{i}\tilde{Y}\right], \ \nabla_{\mathbf{i}\tilde{X}}\tilde{Y} := \nabla^v_{\tilde{X}}\tilde{Y}; \ \tilde{X}, \tilde{Y} \in \mathfrak{X}(\pi). \tag{3.9.1}$$

Then ∇ is a covariant derivative operator in $\pi^*\tau$, called the *Berwald derivative* induced by $\mathcal{H}$. Thus ∇ is vertically natural *ab ovo*, and hence it is strongly regular. Sometimes it is convenient to separate the *horizontal part* ∇^h of ∇ given by

$$\nabla^h_{\tilde{X}}\tilde{Y} := \nabla_{\mathcal{H}\tilde{X}}\tilde{Y}; \ \tilde{X}, \tilde{Y} \in \mathfrak{X}(\pi).$$

The h-deflection

$$\mathbf{t} := \nabla^h \delta$$

of ∇ (with respect to $\mathcal{H}$) is said to be the *tension of the Ehresmann connection* $\mathcal{H}$. If $\overset{\circ}{T}M \subset \widetilde{TM}$ and $\mathbf{t} = 0$, then $\mathcal{H}$ is said to be *homogeneous.* Since

$$\mathbf{i}(\mathbf{t}\hat{X}) = \mathbf{i}\nabla_{X^h}\delta = \mathbf{i} \circ \mathcal{V}\left[X^h, C\right]$$

for all $X \in \mathfrak{X}(M)$, it follows that in the homogeneous case the fibre components of the horizontally lifted vector fields are positively homogeneous of degree 1. If $\overset{\circ}{T}M \subset \widetilde{TM}$, from Lemma 2 we may also conclude that *the Berwald derivative induced by an Ehresmann connection* $\mathcal{H}$ *is strongly associated to* $\mathcal{H}$ *if and only if* $\mathcal{H}$ *is homogeneous.*

The tension has another slightly different interpretation. Let $\mathcal{H}_\nabla$ be the Ehresmann connection induced by the Berwald derivative determined by $\mathcal{H}$. Then *the difference tensor of* $\mathcal{H}$ *and* $\mathcal{H}_\nabla$ *is just the tension of* $\mathcal{H}$*:*

$$\mathcal{H} - \mathcal{H}_\nabla = \mathbf{i} \circ \mathbf{t}. \tag{3.9.2}$$

Indeed, since ∇ is strongly regular, we may apply (3.8.1) to get

$$\begin{aligned}\mathcal{H}_\nabla\hat{X} &= X^c - \mathbf{i}\nabla_{X^c}\delta = X^c - \mathbf{i}\nabla_{\mathcal{H}(\mathbf{j}X^c)}\delta - \mathbf{i}\nabla_{\mathbf{i}(\mathcal{V}X^c)}\delta \\ &= X^c - \mathbf{i}(\mathbf{t}\hat{X}) - \mathbf{i}(\mathcal{V}X^c) = X^c - \mathbf{v}X^c - \mathbf{i}\circ\mathbf{t}(\hat{X}) = (\mathcal{H} - \mathbf{i}\circ\mathbf{t})(\hat{X}).\end{aligned}$$

3.10. Curvature and torsions of Ehresmann connections

Let an Ehresmann connection $\mathcal{H}$ be given in $\pi^*\tau$, with associated vertical map $\mathcal{V}$, horizontal and vertical projectors $\mathbf{h}$ and $\mathbf{v}$ and induced Berwald derivative ∇.

The obstruction to integrability of $\operatorname{Im}\mathcal{H}$ is measured by the *curvature* Ω of $\mathcal{H}$ defined by

$$\Omega(\tilde{X}, \tilde{Y}) := -\mathcal{V}\left[\mathcal{H}\tilde{X}, \mathcal{H}\tilde{Y}\right]; \ \tilde{X}, \tilde{Y} \in \mathfrak{X}(\pi).$$

A somewhat simpler formula:

$$\mathbf{i}\Omega(\hat{X}, \hat{Y}) = -\mathbf{v}\left[X^h, Y^h\right]; \ X, Y \in \mathfrak{X}(M).$$

In this paper the curvature will not be applied.

By the *torsion* $\mathbf{T}$ of $\mathcal{H}$ we mean the horizontal torsion of ∇: $\mathbf{T} := T(\nabla)\circ(\mathcal{H}\times\mathcal{H})$. (It is called *weak torsion* by J. Grifone.) Then we have

$$\mathbf{T}(\tilde{X}, \tilde{Y}) = \mathcal{V}\left[\mathcal{H}\tilde{X}, \mathbf{i}\tilde{Y}\right] - \mathcal{V}\left[\mathcal{H}\tilde{Y}, \mathbf{i}\tilde{X}\right] - \mathbf{j}\left[\mathcal{H}\tilde{X}, \mathcal{H}\tilde{Y}\right]; \ \tilde{X}, \tilde{Y} \in \mathfrak{X}(\pi).$$

A more impressive formula:

$$\mathbf{i}\mathbf{T}(\hat{X},\hat{Y}) = \left[X^h, Y^v\right] - \left[Y^h, X^v\right] - [X,Y]^v;\ X,Y \in \mathfrak{X}(M).$$

It can be shown by a pleasant calculation that $\mathcal{H}$ has vanishing torsion if it is generated by a semispray according to the Crampin – Grifone construction. The converse is also true, but much more difficult: *if $T_pM \cap \widetilde{TM}$ is simply connected for any $p \in M$, and an Ehresmann connection on $\widetilde{TM}$ has vanishing torsion, then it arises from a semispray by* (3.4.1) *(a theorem of M. Crampin)* For a fascinating proof we refer to [7], see also [40].

The type (1,1) tensor field $\mathbf{T}^s$ along π given by

$$\mathbf{T}^s := \mathbf{t} + i_\delta \mathbf{T}$$

is said to be the *strong torsion* of the Ehresmann connection $\mathcal{H}$. Its meaning is clarified by

Proposition 4. *If $S := \mathcal{H}\delta$ is the semispray associated to $\mathcal{H}$, and $\mathcal{H}_S$ is the Ehresmann connection arising from S by the Crampin – Grifone construction, then*

$$\mathcal{H} - \mathcal{H}_S = \mathbf{i} \circ \frac{1}{2}\mathbf{T}^s,$$

i.e., $\frac{1}{2}\mathbf{T}^s$ is the difference tensor of $\mathcal{H}$ and $\mathcal{H}_S$.

Proof. For any vector field X on M we have

$$\begin{aligned}\mathbf{i}(\mathbf{T}^s\hat{X}) &= \mathbf{i}(\mathbf{t}\hat{X}) + \mathbf{i}\mathbf{T}(\delta,\hat{X}) = \mathbf{v}\left[X^h, C\right] + \mathbf{v}[S, X^v] - \mathbf{v}\left[X^h, C\right] \\ &\quad - J\left[S, X^h\right] = \mathbf{v}[S, X^v] + J\left[X^h, S\right],\end{aligned}$$

where the vertical projector $\mathbf{v}$ and the horizontal lift X^h belong to $\mathcal{H}$. By the definition of $\mathcal{H}_S$, and taking into account $\mathcal{H}_S - \mathcal{H} = \mathbf{v} \circ \mathcal{H}_S$ (see (3.3.1) and (3.3.2)), we get

$$\begin{aligned}\mathbf{v}[S, X^v] &= \mathbf{v}\left(X^c - 2\mathcal{H}_S\hat{X}\right) = X^c - \mathbf{h}X^c - 2\mathbf{v}(\mathcal{H}_S\hat{X}) \\ &= X^c - X^h - 2\left(\mathcal{H}_S\hat{X} - X^h\right) = X^h + X^c - 2\mathcal{H}_S\hat{X}.\end{aligned}$$

The term $J\left[X^h, S\right]$ can be formed as follows:

$$J\left[X^h, S\right] = J[\mathbf{h}X^c, S] = J[X^c, S] - J[\mathbf{v}X^c, S].$$

Relations $[X^c, J] = 0$ and $[C, X^c] = 0$ imply $J[X^c, S] = 0$, while $J[\mathbf{v}X^c, S] = \mathbf{v}X^c = X^c - X^h$ by (2.9.1). Thus

$$\mathbf{i}(\mathbf{T}^s\hat{X}) = X^h + X^c - 2\mathcal{H}_S\hat{X} + X^h - X^c = 2(\mathcal{H} - \mathcal{H}_S)(\hat{X}),$$

which proves the Proposition. □

Now we conclude J. Grifone's following important result as an immediate consequence.

Corollary 5. *The strong torsion of an Ehresmann connection vanishes if and only if its torsion and tension vanish. An Ehresmann connection is uniquely determined by its associated semispray and strong torsion.* □

§4. Generalized metrics

4.1. Basic concepts

By a *generalized metric* (called also a *generalized Finsler* or a *generalized Lagrange* metric), briefly a *metric* we mean a pseudo-Riemannian metric in $\pi^*\tau$. So g is a metric in this sense if it is a map that sends a non-degenerate symmetric bilinear form

$$g_v : T_{\pi(v)}M \times T_{\pi(v)}M \to \mathbb{R}$$

to any vector $v \in \widetilde{TM}$. It is assumed that g_v varies smoothly. This means that for any two (smooth) vector fields $\tilde{X}, \tilde{Y}$ along π the function

$$g(\tilde{X}, \tilde{Y}) : v \in \widetilde{TM} \mapsto g_v(\tilde{X}(v), \tilde{Y}(v)) \in \mathbb{R}$$

should be a smooth function. Due to the non-degeneracy we can define the *canonical musical dualities.* If $\tilde{\alpha}$ is a one-form along π, then there is a unique vector field $\tilde{\alpha}^\sharp$ along π such that

$$\tilde{\alpha}(\tilde{Y}) = g(\tilde{\alpha}^\sharp, \tilde{Y})$$

for any vector field $\tilde{Y}$ along π. Conversely, if $\tilde{X}$ is a vector field along π, then a one-form $\tilde{X}^\flat \in \mathfrak{X}^*(\pi)$ is determined by the condition

$$\tilde{X}^\flat(\tilde{Y}) = g(\tilde{X}, \tilde{Y})$$

for every vector field $\tilde{Y}$ along π.

We associate to any metric g in $\pi^*\tau$ the *Lagrange one-form*

$$\vartheta_g : \tilde{X} \in \mathfrak{X}(\pi) \mapsto \vartheta_g\tilde{X} := g(\tilde{X}, \delta) \in C^\infty(\widetilde{TM})$$

and the *Lagrange two-form*

$$\omega_g := d(\vartheta_g \circ \mathbf{j}),$$

cf. 2.8. Note that the Lagrange one-form lives along π, while ω_g is a usual two-form on $\widetilde{TM}$.

By the *absolute energy* of a metric g we mean the smooth function $L := \frac{1}{2}g(\delta, \delta)$ on $\widetilde{TM}$. If L is a regular Lagrangian, i.e., its Hessian $g_L := \nabla^v\nabla^v L$ is non-degenerate, then g is said to be an *energy-regular* metric. If

$$\nabla^v_\delta g = \nabla_C g = 0,$$

then g is called *homogeneous.* In this case the components of g with respect to any induced chart on $\widetilde{TM}$ are positively homogeneous functions of degree 0. Now it turns out why it is crucial that g need not be defined on the zero section: otherwise, under homogeneity, it should be a basic tensor field along τ arising from a pseudo-Riemannian metric on M, and the velocity-dependent character of the theory is lost.

If a metric is energy-regular and homogeneous, then we speak of a *Moór – Vanstone metric.*

4.2. Finsler energies and Finsler metrics

By a *Finsler energy* on $\overset{\circ}{T}M$ we mean a smooth function L on $\overset{\circ}{T}M$ which is positively homogeneous of degree 2 and whose Hessian $g_L := \nabla^v\nabla^v L$ is non-degenerate. Then g_L is a metric in $\overset{\circ}{\tau}{}^*\tau$ in the above sense, called a *Finsler metric* on $\overset{\circ}{T}M$. We note that this concept of a Finsler metric is more general than the usual one since the positiveness of L is not required. It can be shown that if L is, in addition, everywhere positive, then the metric tensor g_L is positive definite [26]. It may be seen immediately that *the absolute energy of a Finsler metric g_L is just the given Finsler energy L, and g_L is homogeneous.* Thus *any Finsler metric is a Moór – Vanstone metric.* The converse is, of course, definitely false in general, although the absolute energy of a Moór – Vanstone metric is, by the definition, a Finsler energy.

Due to its homogeneity property, any Finsler energy $L : \overset{\circ}{T}M \to \mathbb{R}$ can uniquely be extended to a C^1 function $\tilde{L} : TM \to \mathbb{R}$ such that $\tilde{L}(v) = 0$ if $v = 0$. Since $CL - L = L$, the Lagrangian vector field S for L is determined by the relation

$$i_S d(dL \circ J) = -dL,$$

(cf. (2.9.2)), and it is, in fact, a *spray.* S can also be prolonged to a C^1 map $\tilde{S} : TM \to TTM$ such that $\tilde{S} \restriction TM \setminus \overset{\circ}{T}M = 0$. S is said to be the *canonical spray* for the Finsler energy L. According to the Crampin–Grifone construction, S generates an Ehresmann connection $\mathcal{H}_L$ in $\overset{\circ}{\tau}{}^*\tau$, called the *Barthel connection* of the *Finsler manifold* (M, L).

4.3. The Cartan tensors and the Miron tensor

Let g be a metric in $\pi^*\tau$. The *covariant Cartan tensor* of g is the v-covariant differential $\mathcal{C}_\flat := \nabla^v g$ of the metric. The *vectorvariant Cartan tensor* $\mathcal{C}$ is defined by the musical duality given by

$$\mathcal{C}_\flat(\tilde{X}, \tilde{Y}, \tilde{Z}) = g(\mathcal{C}(\tilde{X}, \tilde{Y}), \tilde{Z});\ \tilde{X}, \tilde{Y}, \tilde{Z} \in \mathfrak{X}(\pi).$$

Both of these metrically equivalent tensors will be mentioned as *the Cartan tensor* of g. In terms of the Cartan tensor, the homogeneity of g can be expressed as follows:

$$\text{(4.3.1)} \qquad \nabla^v_\delta g = 0 \iff \forall \tilde{X} \in \mathfrak{X}(\pi) : \mathcal{C}(\delta, \tilde{X}) = 0.$$

If, in particular, g is a Finsler metric, then its Cartan tensor $\mathcal{C}_\flat$ is totally symmetric. In general, $\mathcal{C}_\flat$ is symmetric only in its last two variables. This lack of the symmetries of the Cartan tensors is perhaps the main source of the difficulties one has to face studying generalized metrics.

By the *Miron tensor* of g we mean the type (1,1) tensor

$$A : \tilde{X} \in \mathfrak{X}(\pi) \mapsto A\tilde{X} := \tilde{X} + \mathcal{C}(\tilde{X}, \delta)$$

along π. It is related to the Lagrange two-form ω_g of g by

$$\text{(4.3.2)} \qquad \omega_g(J\xi, \eta) = g(A(\mathbf{j}\xi), \mathbf{j}\eta);\ \xi, \eta \in \mathfrak{X}(\widetilde{TM}).$$

From this it may easily be concluded that ω_g *is non-degenerate if and only if the Miron tensor A is pointwise injective* (and hence invertible). In this case we call the metric *Miron-regular*.

4.4. Some special classes of metrics

A metric g in $\pi^*\tau$ is said to be

variational if its vectorvariant Cartan tensor is symmetric (or, equivalently, its covariant Cartan tensor is totally symmetric);
weakly variational if $\mathcal{C}_\flat(\tilde{X}, \tilde{Y}, \delta) = \mathcal{C}_\flat(\tilde{Y}, \tilde{X}, \delta)$ for any vector fields $\tilde{X}, \tilde{Y}$ along π;
normal if $\mathcal{C}(\tilde{X}, \delta) = 0$ for every $\tilde{X} \in \mathfrak{X}(\pi)$;
weakly normal if $\mathcal{C}_\flat(\tilde{X}, \delta, \delta) = 0$ for any vector field $\tilde{X}$ along π.

By a *Miron metric* we mean a weakly normal Miron-regular metric.

We now have

Theorem 6. *Let g be a metric in $\pi^*\tau$.*

(i) *Suppose that $T_pM \cap \widetilde{TM}$ is simply connected for every point $p \in M$. Then g is variational if and only if it is the Hessian of a smooth function on $\widetilde{TM}$, i.e.,*

$$\exists F \in C^\infty(\widetilde{TM}) : g = \nabla^v\nabla^v F.$$

Setting $L_F := CF - F$, the absolute energy of g is $\frac{1}{2}CL_F$. The Lagrangian forms associated to g coincide with the Lagrangian forms associated to L_F, more precisely

$$\vartheta_g \circ \mathbf{j} = \vartheta_{L_F},\ \omega_g = \omega_{L_F}.$$

(ii) *We continue to assume that $T_pM \cap \widetilde{TM}$ is simply connected for every $p \in M$. g is weakly variational if and only if the Lagrangian one-form ϑ_g associated to g is ∇^v-exact, i.e.,*

$$\exists F \in C^\infty(\widetilde{TM}) : \vartheta_g = \nabla^v F.$$

(iii) (M. Hashiguchi) *A metric on $\overset{\circ}{T}M$ is normal if and only if it is a Finsler metric. More precisely, if g is a normal metric on $\overset{\circ}{T}M$, then the absolute energy L of g is a Finsler energy, and $g = g_L = \nabla^v\nabla^v L$. Conversely, any Finsler metric is normal.*

(iv) *If $\widetilde{TM} = \overset{\circ}{T}M$, then g is weakly normal if and only if its Lagrange one-form can be identified with the Lagrange one-form associated to its absolute energy L:*

$$\vartheta_g \circ \mathbf{j} = \vartheta_L, \text{ or (equivalently) } \vartheta_g = dL \circ \mathbf{i} = \nabla^v L.$$

A complete proof of this theorem can be found in [31]. For the convenience of the reader, and for its own interest, we present here a self-contained proof of part (iii), Hashiguchi's theorem.

Suppose first g is a normal metric on $\overset{\circ}{T}M$, i.e., it satisfies the condition

(4.4.1) $$\mathcal{C}(\tilde{X}, \delta) = 0 \text{ for any vector field } \tilde{X} \text{ along } \overset{\circ}{\tau}.$$

We check that the absolute energy $L = \frac{1}{2}g(\delta, \delta)$ of g is positively homogeneous of degree 2. This is a routine verification:

$$\begin{aligned} CL &= \frac{1}{2}C(g(\delta, \delta)) = \frac{1}{2}((\nabla_C g)(\delta, \delta) + 2g(\nabla_C \delta, \delta)) \\ &= \frac{1}{2}\mathcal{C}_\flat(\delta, \delta, \delta) + g(\delta, \delta) \overset{(4.4.1)}{=} 2L. \end{aligned}$$

Our condition (4.4.1) also implies that the Miron tensor of g is the identity endomorphism along $\overset{\circ}{\tau}$, i.e., $A = 1_{\mathfrak{X}(\overset{\circ}{\tau})}$. Thus relation (4.3.2) reduces to

$$\omega_g(J\xi, \eta) = g(\mathbf{j}\xi, \mathbf{j}\eta); \ \xi, \eta \in \mathfrak{X}\left(\overset{\circ}{T}M\right).$$

By the choice $\xi := X^c$, $\eta := Y^c$; $X, Y \in \mathfrak{X}(M)$ this takes the form

$$\omega_g(X^v, Y^c) = g\big(\hat{X}, \hat{Y}\big).$$

Since

$$\begin{aligned} \omega_g(X^v, Y^c) &= d(\vartheta_g \circ \mathbf{j})(X^v, Y^c) = X^v\big(\vartheta_g \hat{Y}\big) - Y^c(\vartheta_g \mathbf{j} X^v) \\ &\quad - \vartheta_g(\mathbf{j}[X^v, Y^c]) = X^v\big(\vartheta_g \hat{Y}\big) - \vartheta_g(\mathbf{j}[X, Y]^v) = X^v g\big(\hat{Y}, \delta\big), \end{aligned}$$

and

$$0 = \mathcal{C}\big(\hat{Y}, \delta, \delta\big) = (\nabla_{Y^v} g)(\delta, \delta) = 2Y^v L - 2g\big(\hat{Y}, \delta\big),$$

it follows that

$$g\big(\hat{X}, \hat{Y}\big) = X^v(Y^v L) = g_L\big(\hat{X}, \hat{Y}\big),$$

which concludes the proof of (iii).

As an immediate consequence, we have (under the hypotheses of the theorem) the following implications:

$$\begin{array}{ccc} & g \text{ is a Finsler metric} & \\ & \Updownarrow & \\ & \boxed{g \text{ is normal}} & \\ \swarrow & & \searrow \\ \text{g is weakly normal} & & \text{g is variational} \\ \searrow & & \swarrow \\ & g \text{ is weakly variational} & \end{array}$$

To conclude this section, we collect here some further useful corollaries.

Corollary 7. *A metric in $\overset{\circ}{\tau}{}^*\tau$ is a Finsler metric if and only if its Miron tensor is the unit tensor along $\overset{\circ}{\tau}$.*

This is just a reformulation of Hashiguchi's theorem.

Corollary 8. *If g is a weakly variational metric in $\overset{\circ}{\tau}{}^*\tau$, then its Miron tensor yields a selfadjoint linear transformation in every fibre.*

Proof. For any $v \in \overset{\circ}{T}M$; $w_1, w_2 \in T_{\tau(v)}M$ we have

$$\begin{aligned} g_v(\mathcal{C}_v(w_1, v), w_2) &= (\mathcal{C}_\flat)_v(w_1, v, w_2) = (\mathcal{C}_\flat)_v(w_2, v, w_1) \\ &= g_v(\mathcal{C}_v(w_2, v), w_1), \end{aligned}$$

therefore

$$\begin{aligned} g_v(A_v(w_1), w_2) &= g_v(w_1, w_2) + g_v(\mathcal{C}_v(w_1, v), w_2) \\ &= g_v(w_1, w_2) + g_v(\mathcal{C}_v(w_2, v), w_1) = g_v(w_1, A_v(w_2)). \end{aligned}$$

□

Corollary 9. *The absolute energy of a Miron metric is a Finsler energy.*

Proof. Let g be a Miron metric in $\overset{\circ}{\tau}{}^*\tau$. First we show that *weak normality of g implies*

$$g(A\hat{X}, \hat{Y}) = g_L(\hat{X}, \hat{Y});\ X, Y \in \mathfrak{X}(M). \tag{4.4.2}$$

Indeed, by part (iv) of Theorem 6 we have $\vartheta_g = \nabla^v L$. Hence, applying (4.3.2), we get

$$\begin{aligned} g(A\hat{X}, \hat{Y}) &= g(A(\mathbf{j}X^c), \mathbf{j}Y^c) = \omega_g(JX^c, Y^c) = d(\nabla^v L \circ \mathbf{j})(X^v, Y^c) \\ &= X^v(\nabla^v L(\hat{Y})) - Y^c(\nabla^v L(\mathbf{j}X^v)) - \nabla^v L(\mathbf{j}[X^v, Y^c]) \\ &= X^v(Y^v L) = g_L(\hat{X}, \hat{Y}). \end{aligned}$$

Now by the Miron-regularity of g it follows from (4.4.2) that g_L is non-degenerate. The homogeneity property $CL = 2L$ can be obtained by the same calculation as in the proof of Hashiguchi's theorem. □

§5. Good metric derivatives

5.1. Metric derivatives in a pseudo-Riemannian vector bundle

We start with some remarks on a very general situation, so in this subsection $\pi : E \to M$ will be an r-vector bundle.

(1) It can be shown by a standard partition of unity argument that there exist covariant derivative operators in π. (For two other types of proof we refer to [15], 7.11.) If D^1 and D^2 are covariant derivatives in π, then the map

$$\psi : \mathfrak{X}(M) \times \Gamma(\pi) \to \Gamma(\pi),\ (X, \sigma) \mapsto \psi(X, \sigma) := D^1_X\sigma - D^2_X\sigma$$

is $C^\infty(M)$-bilinear. It is called the *difference tensor* of D^1 and D^2. ψ may be considered as an $\mathrm{End}(\Gamma(\pi))$-valued tensor on M by the interpretation

$$\begin{cases} X \in \mathfrak{X}(M) \mapsto \psi_X \in \mathrm{End}(\Gamma(\pi)), \\ \psi_X(\sigma) := \psi(X, \sigma),\ \sigma \in \Gamma(\pi). \end{cases}$$

We denote by $L_{C^\infty(M)}(\mathfrak{X}(M), \mathrm{End}(\Gamma(\pi)))$ the module of $C^\infty(M)$-linear maps $\mathfrak{X}(M) \to \mathrm{End}(\Gamma(\pi))$. If $\overset{\circ}{D}$ is a covariant derivative in π, $\psi \in L_{C^\infty(M)}(\mathfrak{X}(M), \mathrm{End}(\Gamma(\pi)))$, and

$$D_X\sigma := \overset{\circ}{D}_X\sigma + \psi_X\sigma;\ X \in \mathfrak{X}(M),\ \sigma \in \Gamma(\pi),$$

then D is also a covariant derivative operator in π, and every covariant derivative can be obtained in this way.

(2) Now suppose that π is endowed with a pseudo-Riemannian metric $g : \Gamma(\pi) \times \Gamma(\pi) \to C^\infty(M)$; then the pair (π, g) is said to be a pseudo-Riemannian vector bundle. In the pointwise interpretation (cf. 4.1) g sends to each point $p \in M$ a non-degenerate, symmetric bilinear form $g_p : E_p \times E_p \to \mathbb{R}$ such that the function

$$g(\sigma_1, \sigma_2) : p \in M \mapsto g_p(\sigma_1(p), \sigma_2(p)) \in \mathbb{R}$$

is smooth for any sections $\sigma_1, \sigma_2 \in \Gamma(\pi)$. We define the *Cartan tensors* $\mathcal{C}$ and $\mathcal{C}_\flat$ of g with respect to a covariant derivative D in π on the analogy of 4.3 as follows:

$$\mathcal{C}_\flat := Dg,\ \mathcal{C}_\flat(X, \sigma_1, \sigma_2) =: g(\mathcal{C}(X, \sigma_1), \sigma_2)$$

for all $X \in \mathfrak{X}(M)$; $\sigma_1, \sigma_2 \in \Gamma(\pi)$. Then

$$\mathcal{C} \in L_{C^\infty(M)}(\mathfrak{X}(M), \mathrm{End}(\Gamma(\pi))).$$

(3) A covariant derivative D in (π, g) is said to be *metric* if

$$Xg(\sigma_1, \sigma_2) = g(D_X\sigma_1, \sigma_2) + g(\sigma_1, D_X\sigma_2)$$

for every $X \in \mathfrak{X}(M)$; $\sigma_1, \sigma_2 \in \Gamma(\pi)$. *Any pseudo-Riemannian vector bundle admits a metric covariant derivative.* Indeed, let $\overset{\circ}{D}$ be any covariant derivative in (π, g), and let $\mathcal{C}_0$ be the vectorvariant Cartan tensor of g with respect to $\overset{\circ}{D}$. Then, as it may immediately be seen, the map

$$D : (X, \sigma) \in \mathfrak{X}(M) \times \Gamma(\pi) \mapsto D_X\sigma := \overset{\circ}{D}_X\sigma + \frac{1}{2}\mathcal{C}_0(X, \sigma)$$

is a metric derivative in (π, g). *The difference tensor of two metric derivatives is skew-symmetric with respect to g.* Conversely, if $\psi \in L_{C^\infty(M)}(\mathfrak{X}(M), \mathrm{End}(\Gamma(\pi)))$ is skew-symmetric, i.e., for all $X \in \mathfrak{X}(M)$; $\sigma_1, \sigma_2 \in \Gamma(\pi)$ we have

$$g(\psi_X(\sigma_1), \sigma_2) + g(\sigma_1, \psi_X(\sigma_2)) = 0,$$

and D is a metric derivative in (π, g), then

$$\tilde{D} : (X, \sigma) \mapsto \tilde{D}_X\sigma := D_X\sigma + \psi_X(\sigma)$$

is also a metric derivative.

(4) Let

$$L^{\mathrm{sym}}_{C^\infty(M)}(\mathfrak{X}(M), \mathrm{End}(\Gamma(\pi))) \text{ and } L^{\mathrm{skew}}_{C^\infty(M)}(\mathfrak{X}(M), \mathrm{End}(\Gamma(\pi)))$$

denote the submodule of symmetric and skew-symmetric $\mathrm{End}(\Gamma(\pi))$-valued 1-tensors on M, respectively. Then $L_{C^\infty(M)}(\mathfrak{X}(M), \mathrm{End}(\Gamma(\pi)))$ is the direct sum of these submodules, so there exists a unique endomorphism Ob of $L_{C^\infty(M)}(\mathfrak{X}(M), \mathrm{End}(\Gamma(\pi)))$ such that

$$\mathrm{Ob} \circ \mathrm{Ob} = \mathrm{Ob}, \text{ i.e., Ob is a projector;}$$
$$\mathrm{Im}\,\mathrm{Ob} = L^{\mathrm{skew}}_{C^\infty(M)}(\mathfrak{X}(M), \mathrm{End}(\Gamma(\pi)));$$
$$\mathrm{Ker}\,\mathrm{Ob} = L^{\mathrm{sym}}_{C^\infty(M)}(\mathfrak{X}(M), \mathrm{End}(\Gamma(\pi))).$$

We call this projection operator the *Obata operator* of skew-symmetrization. It can explicitly be given as follows: for every $\Phi \in L_{C^\infty(M)}(\mathfrak{X}(M), \mathrm{End}(\Gamma(\pi)))$,

$$g((\mathrm{Ob}\,\Phi)_X(\sigma_1), \sigma_2) = \frac{1}{2}(g(\Phi_X(\sigma_1), \sigma_2) - g(\sigma_1, \Phi_X(\sigma_2)))$$

$(X \in \mathfrak{X}(M)$; $\sigma_1, \sigma_2 \in \Gamma(\pi))$.

To sum up, we have the following

Proposition 10. *Any metric covariant derivative in a vector bundle (π, g) can be represented in the form*

$$D_X\sigma = \overset{\circ}{D}_X\sigma + \frac{1}{2}\mathcal{C}_0(X,\sigma) + (\mathrm{Ob}\,\Phi)_X(\sigma);\ X \in \mathfrak{X}(M),\ \sigma \in \Gamma(\pi),$$

where $\overset{\circ}{D}$ is an arbitrary covariant derivative in π, $\mathcal{C}_0$ is the vectorvariant Cartan tensor of g with respect to $\overset{\circ}{D}$, Ob *is the Obata operator of skew-symmetrization, and Φ is an arbitrary* $\mathrm{End}(\Gamma(\pi))$*-valued tensor on M.* □

5.2. The Miron construction

Now we go back to the main scene of our considerations, to the pull-back bundle $\pi^*\tau$.

Lemma 11. *Let a metric g and an Ehresmann connection $\mathcal{H}$ be given in $\pi^*\tau$. There is a unique metric covariant derivative in $\pi^*\tau$ which has vanishing vertical torsion and whose horizontal torsion coincides with the torsion of $\mathcal{H}$.*

For a complete proof we refer to [40] 2.51 or [31], we sketch here only the main steps.

Step 1. We consider the Berwald derivative ∇ induced by $\mathcal{H}$ according to (3.9.1).

Step 2. We introduce the (vectorvariant) h-Cartan tensor $\mathcal{C}^h$ of g (with respect to $\mathcal{H}$) by means of the musical relation

$$g(\mathcal{C}^h(\tilde{X},\tilde{Y}),\tilde{Z}) = (\nabla_{\mathcal{H}\tilde{X}}g)(\tilde{Y},\tilde{Z});\ \tilde{X},\tilde{Y},\tilde{Z} \in \mathfrak{X}(\pi).$$

Step 3. Using Christoffel's trick, we define two further vectorvariant tensors $\overset{\circ}{\mathcal{C}}$ and $\overset{\circ}{\mathcal{C}}{}^h$ along π determined by the conditions

(5.2.1)
$$g\Big(\overset{\circ}{\mathcal{C}}(\tilde{X},\tilde{Y}),\tilde{Z}\Big) = g(\mathcal{C}(\tilde{X},\tilde{Y}),\tilde{Z}) + g(\mathcal{C}(\tilde{Y},\tilde{Z}),\tilde{X}) - g(\mathcal{C}(\tilde{Z},\tilde{X}),\tilde{Y})$$

and

(5.2.2)
$$g\Big(\overset{\circ}{\mathcal{C}}{}^h(\tilde{X},\tilde{Y}),\tilde{Z}\Big) = g(\mathcal{C}^h(\tilde{X},\tilde{Y}),\tilde{Z}) + g(\mathcal{C}^h(\tilde{Y},\tilde{Z}),\tilde{X}) - g(\mathcal{C}^h(\tilde{Z},\tilde{X}),\tilde{Y}),$$

where $\tilde{X},\tilde{Y},\tilde{Z}$ are vector fields along π, and $\mathcal{C}$ is the Cartan tensor of g (4.3). It can be seen at once that $\overset{\circ}{\mathcal{C}}$ and $\overset{\circ}{\mathcal{C}}{}^h$ are both symmetric.

Step 4. We define the desired covariant derivative operator D by the following rules for calculation:

$$(5.2.3)\quad D_{\mathrm{i}\tilde{X}}\tilde{Y} := \nabla_{\mathrm{i}\tilde{X}}\tilde{Y} + \frac{1}{2}\overset{\circ}{\mathcal{C}}(\tilde{X},\tilde{Y}),\ D_{\mathcal{H}\tilde{X}}\tilde{Y} := \nabla_{\mathcal{H}\tilde{X}}\tilde{Y} + \frac{1}{2}\overset{\circ}{\mathcal{C}}{}^h(\tilde{X},\tilde{Y}) \quad (\tilde{X},\tilde{Y} \in \mathfrak{X}(\pi)).$$

It can be checked immediately that D is indeed a metric derivative, and has the properties

$$T^v(D) = 0,\ T(D) \circ (\mathcal{H} \times \mathcal{H}) = T(\nabla) \circ (\mathcal{H} \times \mathcal{H}).$$

We call the covariant derivative obtained in this way the *Miron derivative* arising from g and $\mathcal{H}$. The idea, formulated here intrinsically, is due to Radu Miron. In his paper [33] he started from a Miron metric g and the Barthel connection of the associated Finsler manifold (M, L), $L := \frac{1}{2}g(\delta,\delta)$. Using an induced chart on $\overset{\circ}{T}M$, he determined the coordinate expression of the Barthel connection, defined the tensors $\overset{\circ}{\mathcal{C}}$ and $\overset{\circ}{\mathcal{C}}{}^h$ in terms of the components of g, and, finally, he deduced the Christoffel symbols of D.

If, in particular, (M, L) is a Finsler manifold, $\mathcal{H}_L$ is the Barthel connection of (M, L), then the Miron derivative arising from $g_L := \nabla^v\nabla^v L$ is called the *Cartan derivative* in (M, L). In this case (5.2.3) reduces to

$$(5.2.4)\quad D_{\mathrm{i}\tilde{X}}\tilde{Y} = \nabla_{\mathrm{i}\tilde{X}}\tilde{Y}+\frac{1}{2}\mathcal{C}(\tilde{X},\tilde{Y}),\ D_{\mathcal{H}_L\tilde{X}}\tilde{Y} := \nabla_{\mathcal{H}_L\tilde{X}}\tilde{Y}+\frac{1}{2}\mathcal{C}^h(\tilde{X},\tilde{Y}),$$

where $\mathcal{C}$ is the Cartan tensor of g_L (and hence it is symmetric), and $\mathcal{C}^h_\flat(\tilde{X},\tilde{Y},\tilde{Z}) := \left(\nabla_{\mathcal{H}_L\tilde{X}}g_L\right)(\tilde{Y},\tilde{Z})$. Then $\mathcal{C}^h$ is also (totally) symmetric. In particular, for any basic vector fields $\hat{X},\hat{Y}$ along $\overset{\circ}{\tau}$ we have

$$D_{X^v}\hat{Y} = \frac{1}{2}\mathcal{C}(\hat{X},\hat{Y}),\ D_{X^{h_L}}\hat{Y} = \mathcal{V}_L\left[X^{h_L}, Y^v\right] + \frac{1}{2}\mathcal{C}^h(\hat{X},\hat{Y})$$

(subscript L refers to objects derived from $\mathcal{H}_L$).

5.3. An algebraic lemma

Lemma 12. *Let K be a commutative ring with a unit element and with characteristic different to 2. Let V be a K-module and V^* its dual module. Suppose that $f : V \to V^*$ is an isomorphism, and the bilinear function*

$$\langle\ ,\ \rangle : V \times V \to K,\ (v,w) \mapsto \langle v,w\rangle := f(v)(w)$$

is symmetric. Then for every skew-symmetric K-bilinear map $\omega : V \times V \to V$ there is a unique K-bilinear map $\psi : V \times V \to V$ such that

(i) $\forall u, v, w \in V : \langle \psi(u,v), w\rangle + \langle v, \psi(u,w)\rangle = 0,$
(ii) $\forall v, w \in V : \omega(v,w) = \psi(v,w) - \psi(w,v).$

This lemma is taken from the monograph [15], where (in section 7.26) it plays a crucial role in the proof of the following fundamental result, the so-called *Ricci lemma: on every pseudo-Riemannian manifold there is a unique metric covariant derivative, the* Levi-Civita derivative, *with vanishing torsion.* In what follows, we shall present several generalizations of Ricci's lemma, applying also Lemma 12.

5.4. Metric v-covariant derivatives

We begin with a reasonable axiomatization of the basic properties of the canonical v-covariant derivative ∇^v (see 2.7).

By a *v-covariant derivative operator* in $\pi^*\tau$ we mean a map

$$D^v : \mathfrak{X}(\pi) \times \mathfrak{X}(\pi) \to \mathfrak{X}(\pi),\ \left(\tilde{X}, \tilde{Y}\right) \mapsto D^v_{\tilde{X}}\tilde{Y}$$

which is $C^\infty\left(\widetilde{TM}\right)$-linear in its first variable, additive in its second variable, and satisfies

$$D^v_{\tilde{X}} F\tilde{Y} = \left((\mathbf{i}\tilde{X})F\right)\tilde{Y} + FD^v_{\tilde{X}}\tilde{Y};\ \tilde{X}, \tilde{Y} \in \mathfrak{X}(\pi),\ F \in C^\infty\left(\widetilde{TM}\right).$$

Then the maps $D^v_{\tilde{X}} : \mathfrak{X}(\pi) \to \mathfrak{X}(\pi)$, $\tilde{Y} \mapsto D^v_{\tilde{X}}\tilde{Y}$ can be uniquely extended to any tensor along π, to be a tensor derivation ([40], 2.42). Obviously, any covariant derivative operator D in $\pi^*\tau$ leads to a v-covariant derivative given by

$$D^v_{\tilde{X}}\tilde{Y} := D_{\mathbf{i}\tilde{X}}\tilde{Y};\ \tilde{X}, \tilde{Y} \in \mathfrak{X}(\pi).$$

This suggests to define the *torsion* $T(D^v)$ of D^v on the analogy of (3.5.2):

$$T(D^v)\left(\tilde{X}, \tilde{Y}\right) := D^v_{\tilde{X}}\tilde{Y} - D^v_{\tilde{Y}}\tilde{X} - \mathbf{i}^{-1}\left[\mathbf{i}\tilde{X}, \mathbf{i}\tilde{Y}\right];\ \tilde{X}, \tilde{Y} \in \mathfrak{X}(\pi).$$

Given a metric g in $\pi^*\tau$, D^v is said to be *metric* if $D^v g = 0$, i.e.,

$$(\mathbf{i}\tilde{X})g\left(\tilde{Y}, \tilde{Z}\right) = g\left(D^v_{\tilde{X}}\tilde{Y}, \tilde{Z}\right) + g\left(\tilde{Y}, D^v_{\tilde{X}}\tilde{Z}\right)$$

for all $\tilde{X}, \tilde{Y}, \tilde{Z} \in \mathfrak{X}(\pi)$.

Lemma 13. *Let g be a metric in $\pi^*\tau$. There is a unique metric v-covariant derivative in $\pi^*\tau$ whose torsion vanishes.*

Proof. The existence is clear from the Miron construction: if $\overset{\circ}{\mathcal{C}}$ is the vectorvariant tensor given by (5.2.1), and

$$D^v_{\tilde{X}}\tilde{Y} := \nabla^v_{\tilde{X}}\tilde{Y} + \frac{1}{2}\overset{\circ}{\mathcal{C}}(\tilde{X},\tilde{Y}),$$

then D^v is a metric v-covariant derivative.

To show the uniqueness, suppose that $\tilde{D}^v$ is another metric v-covariant derivative in $\pi^*\tau$ such that $T(\tilde{D}^v) = 0$. Let ψ^v be the *difference tensor* of D^v and $\tilde{D}^v$ defined by

$$\psi(\tilde{X},\tilde{Y}) := D^v_{\tilde{X}}\tilde{Y} - \tilde{D}^v_{\tilde{X}}\tilde{Y};\ \tilde{X},\tilde{Y} \in \mathfrak{X}(\pi).$$

Since D^v and $\tilde{D}^v$ are both metric derivatives, for any vector fields $\tilde{X},\tilde{Y},\tilde{Z}$ along π we have

$$\begin{aligned}
0 &= \left(D^v_{\tilde{X}}g\right)\left(\tilde{Y},\tilde{Z}\right) = \left(\mathbf{i}\tilde{X}\right)g\left(\tilde{Y},\tilde{Z}\right) - g\left(D^v_{\tilde{X}}\tilde{Y},\tilde{Z}\right) - g\left(\tilde{Y},D^v_{\tilde{X}}\tilde{Z}\right)\\
&= g\left(\tilde{D}^v_{\tilde{X}}\tilde{Y},\tilde{Z}\right) + g\left(\tilde{Y},\tilde{D}^v_{\tilde{X}}\tilde{Z}\right) - g\left(D^v_{\tilde{X}}\tilde{Y},\tilde{Z}\right) - g\left(\tilde{Y},D^v_{\tilde{X}}\tilde{Z}\right)\\
&= -g\left(\psi^v(\tilde{X},\tilde{Y}),\tilde{Z}\right) - g\left(\tilde{Y},\psi^v(\tilde{X},\tilde{Z})\right).
\end{aligned}$$

On the other hand, the vanishing of $T(D^v)$ and $T(\tilde{D}^v)$ yields

$$\begin{aligned}
0 &= D^v_{\hat{X}}\hat{Y} - D^v_{\hat{Y}}\hat{X} = \tilde{D}^v_{\hat{X}}\hat{Y} + \psi^v\left(\hat{X},\hat{Y}\right) - \tilde{D}^v_{\hat{Y}}\hat{X} - \psi^v\left(\hat{Y},\hat{X}\right)\\
&= \psi^v\left(\hat{X},\hat{Y}\right) - \psi^v\left(\hat{Y},\hat{X}\right)
\end{aligned}$$

for any vector fields X,Y on M. Thus ψ^v satisfies the conditions of Lemma 12 with the choice $\omega := 0$, therefore $\psi^v = 0$ and hence $\tilde{D}^v = D^v$. □

5.5. A characterization of the Cartan derivative

Now we present a deduction of the existence and uniqueness of Cartan's covariant derivative without a prior specification of an Ehresmann connection. This approach to Cartan's derivative can be found e.g. in Abate and Patrizio's book [1], but our proof is strongly different and completely coordinate-free.

In this subsection we shall work on a Finsler manifold (M,L). The vectorvariant Cartan tensor, the Barthel connection and and the Cartan derivative of (M,L) will be denoted by $\mathcal{C}$, $\mathcal{H}_L$ and D, respectively. Some objects determined by (M,L) alone will be distinguished by the subscript L.

Lemma 14. *Let $\mathcal{H}$ be an Ehresmann connection and $\tilde{D}$ be a metric covariant derivative in $\overset{\circ}{\tau}{}^*\tau$. Suppose that the vertical torsion $T^v(\tilde{D})$ and* *the horizontal torsion* $\tilde{\mathcal{T}} := T(\tilde{D}) \circ (\mathcal{H} \times \mathcal{H})$ *of* $\tilde{D}$ *vanish. Then we have for all* $\tilde{X}, \tilde{Y} \in \mathfrak{X}(\overset{\circ}{\tau})$

$$\tilde{D}_{\mathcal{H}\tilde{X}}\tilde{Y} = D_{\mathcal{H}\tilde{X}}\tilde{Y} + \frac{1}{2}\left(\mathcal{C}(\tilde{X}, P\tilde{Y}) - P^*\mathcal{C}(\tilde{X}, \tilde{Y})\right), \tag{5.5.1}$$

where P *is the difference tensor of* $\mathcal{H}$ *and* $\mathcal{H}_L$*, and* P^* *is the adjoint of* P *with respect to the Finsler metric* g_L*.*

Proof. Let ψ^h be the difference tensor of the horizontal part of $\tilde{D}$ and D, i.e., let

$$\psi^h(\tilde{X}, \tilde{Y}) := \tilde{D}_{\mathcal{H}\tilde{X}}\tilde{Y} - D_{\mathcal{H}\tilde{X}}\tilde{Y};\ \tilde{X}, \tilde{Y} \in \mathfrak{X}(\overset{\circ}{\tau}).$$

First we show that ψ^h satisfies conditions (i), (ii) of Lemma 12 with ω given by

$$\omega(\tilde{X}, \tilde{Y}) := \frac{1}{2}\left(\mathcal{C}(\tilde{X}, P\tilde{Y}) - \mathcal{C}(\tilde{Y}, P\tilde{X})\right);\ \tilde{X}, \tilde{Y} \in \mathfrak{X}(\overset{\circ}{\tau}).$$

Since both $\tilde{D}$ and D are metric, for any sections $\tilde{X}, \tilde{Y}, \tilde{Z} \in \mathfrak{X}(\overset{\circ}{\tau})$ we have

$$\begin{aligned}
0 &= (\tilde{D}_{\mathcal{H}_L\tilde{X}}g)(\tilde{Y}, \tilde{Z}) \\
&= (\mathcal{H}_L\tilde{X})g(\tilde{Y}, \tilde{Z}) - g(\tilde{D}_{\mathcal{H}_L\tilde{X}}\tilde{Y}, \tilde{Z}) - g(\tilde{Y}, \tilde{D}_{\mathcal{H}_L\tilde{X}}\tilde{Z}) \\
&= g(D_{\mathcal{H}_L\tilde{X}}\tilde{Y} - \tilde{D}_{\mathcal{H}_L\tilde{X}}\tilde{Y}, \tilde{Z}) + g(\tilde{Y}, D_{\mathcal{H}_L\tilde{X}}\tilde{Z} - \tilde{D}_{\mathcal{H}_L\tilde{X}}\tilde{Z}) \\
&\overset{(*)}{=} g(D_{\mathcal{H}\tilde{X}}\tilde{Y} - \tilde{D}_{\mathcal{H}\tilde{X}}\tilde{Y}, \tilde{Z}) + g(\tilde{Y}, D_{\mathcal{H}\tilde{X}}\tilde{Z} - \tilde{D}_{\mathcal{H}\tilde{X}}\tilde{Z}) \\
&= -g\left(\psi^h(\tilde{X}, \tilde{Y}), \tilde{Z}\right) - g\left(\tilde{Y}, \psi^h(\tilde{X}, \tilde{Z})\right),
\end{aligned}$$

using at the step denoted by $(*)$ the coincidence of D^v and $\tilde{D}^v$ assured by Lemma 13. Thus condition (i) is satisfied. Next we apply the vanishing of the horizontal torsions

$$\mathcal{T} = T(D) \circ (\mathcal{H}_L \times \mathcal{H}_L) \text{ and } \tilde{\mathcal{T}} = T(\tilde{D}) \circ (\mathcal{H} \times \mathcal{H}).$$

For any vector fields X, Y on M we have

$$\begin{aligned}
0 &= \tilde{\mathcal{T}}(\hat{X},\hat{Y}) = \tilde{D}_{X^h}\hat{Y} - \tilde{D}_{Y^h}\hat{X} - \mathbf{j}\left[X^h, Y^h\right] \\
&= \tilde{D}_{X^{h_L}}\hat{Y} - \tilde{D}_{Y^{h_L}}\hat{X} - \widehat{[X,Y]} + \tilde{D}_{\mathbf{i}P\hat{X}}\hat{Y} - \tilde{D}_{\mathbf{i}P\hat{Y}}\hat{X} \\
&= D_{X^{h_L}}\hat{Y} - D_{Y^{h_L}}\hat{X} - \widehat{[X,Y]} + \psi^h(\hat{X},\hat{Y}) - \psi^h(\hat{Y},\hat{X}) \\
&\quad + \nabla_{\mathbf{i}P\hat{X}}\hat{Y} + \frac{1}{2}\mathcal{C}(P\hat{X},\hat{Y}) - \nabla_{\mathbf{i}P\hat{Y}}\hat{X} - \frac{1}{2}\mathcal{C}(P\hat{Y},\hat{X}) \\
&= \mathcal{T}(\hat{X},\hat{Y}) + \psi^h(\hat{X},\hat{Y}) - \psi^h(\hat{Y},\hat{X}) + \omega(\hat{X},\hat{Y}) \\
&= \psi^h(\hat{X},\hat{Y}) - \psi^h(\hat{Y},\hat{X}) + \omega(\hat{X},\hat{Y}),
\end{aligned}$$

so condition (ii) is also satisfied. (In our calculation we used Lemma 13 and Miron's construction again, as well as the fact that the canonical v-covariant derivatives of basic vector fields vanish.) Thus Greub, Halperin and Vanstone's algebraic lemma implies that ψ^h is uniquely determined.

It remains to show that the only possible choice for ψ^h is the map

$$(\tilde{X},\tilde{Y}) \in \mathfrak{X}(\overset{\circ}{\tau}) \times \mathfrak{X}(\overset{\circ}{\tau}) \mapsto \frac{1}{2}\left(\mathcal{C}(\tilde{X},P\tilde{Y}) - P^*\mathcal{C}(\tilde{X},\tilde{Y})\right). \tag{5.5.2}$$

This is quite an immediate verification again. For any vector fields $\tilde{X},\tilde{Y},\tilde{Z}$ along $\overset{\circ}{\tau}$ we have

$$\begin{aligned}
&g\left(\mathcal{C}(\tilde{X},P\tilde{Y}) - P^*\mathcal{C}(\tilde{X},\tilde{Y}),\tilde{Z}\right) + g\left(\tilde{Y},\mathcal{C}(\tilde{X},P\tilde{Z}) - P^*\mathcal{C}(\tilde{X},\tilde{Z})\right) \\
&\quad = g\left(\mathcal{C}(\tilde{X},P\tilde{Y}),\tilde{Z}\right) - g\left(\mathcal{C}(\tilde{X},\tilde{Y}),P\tilde{Z}\right) \\
&\qquad + g\left(\tilde{Y},\mathcal{C}(\tilde{X},P\tilde{Z})\right) - g\left(P\tilde{Y},\mathcal{C}(\tilde{X},\tilde{Z})\right) = \mathcal{C}_\flat(\tilde{X},P\tilde{Y},\tilde{Z}) \\
&\qquad - \mathcal{C}_\flat(\tilde{X},\tilde{Y},P\tilde{Z}) + \mathcal{C}_\flat(\tilde{X},P\tilde{Z},\tilde{Y}) - \mathcal{C}_\flat(\tilde{X},\tilde{Z},P\tilde{Y}) = 0,
\end{aligned}$$

due to the symmetry of $\mathcal{C}_\flat$. Thus the map (5.5.2) satisfies condition (i) of Lemma 12. It also satisfies the second condition of the Lemma, since

$$\begin{aligned}
&\frac{1}{2}\left(\mathcal{C}(\tilde{X},P\tilde{Y}) - P^*\mathcal{C}(\tilde{X},\tilde{Y}) - \mathcal{C}(\tilde{Y},P\tilde{X}) + P^*\mathcal{C}(\tilde{Y},\tilde{X})\right) \\
&\qquad\qquad = \frac{1}{2}\left(\mathcal{C}(\tilde{X},P\tilde{Y}) - \mathcal{C}(\tilde{Y},P\tilde{X})\right) = \omega(\tilde{X},\tilde{Y}).
\end{aligned}$$

□

Proposition 15. *Let (M,L) be a Finsler manifold. If $\mathcal{H}$ is an Ehresmann connection and $\tilde{D}$ is a covariant derivative operator in $\overset{\circ}{\tau}{}^*\tau$ such that*

(i) *$\tilde{D}$ is metric;*

(ii) *the vertical torsion of $\tilde{D}$ vanishes;*
(iii) *the horizontal torsion $\mathcal{T} = T(\tilde{D}) \circ (\mathcal{H} \times \mathcal{H})$ of $\tilde{D}$ vanishes;*
(iv) *$\tilde{D}$ is strongly associated to $\mathcal{H}$,*

then $\tilde{D}$ is the Cartan derivative and $\mathcal{H}$ is the Barthel connection of (M, L).

Proof. $\tilde{D}^v = D^v$ by (i), (ii) and Lemma 13. We show that $\tilde{D}^h = D^h$ is also true. First we note that conditions (i)–(iii) imply by the previous lemma that $D_{\mathcal{H}\tilde{X}}\tilde{Y}$ is given by (5.5.1). Since $\tilde{D}$ is strongly associated to $\mathcal{H}$, its h-deflection vanishes by Lemma 2, therefore we have for all $\tilde{X} \in \mathfrak{X}(\mathring{\tau})$

$$\begin{aligned} 0 &= 2\tilde{D}_{\mathcal{H}\tilde{X}}\delta = 2D_{\mathcal{H}\tilde{X}}\delta + \mathcal{C}(\tilde{X}, P\delta) - P^*\mathcal{C}(\tilde{X}, \delta) \\ &= 2D_{\mathcal{H}_L\tilde{X}}\delta + 2D_{\mathrm{i}P\tilde{X}}\delta + \mathcal{C}(\tilde{X}, P\delta) \\ &= 2\nabla_{\mathrm{i}P\tilde{X}}\delta + \mathcal{C}(P\tilde{X}, \delta) + \mathcal{C}(\tilde{X}, P\delta) = 2P\tilde{X} + \mathcal{C}(\tilde{X}, P\delta); \end{aligned}$$

applying (5.2.3), (5.2.4), the normality of g_L and the fact that D is strongly associated to $\mathcal{H}_L$. By the choice $\tilde{X} := \delta$ it follows that $P\delta = 0$, whence $P\tilde{X} = 0$ for all $\tilde{X} \in \mathfrak{X}(\pi)$. This means that $P = 0$, therefore $\mathcal{H} = \mathcal{H}_L$ and $\tilde{D} = D$. □

5.6. When is a metric derivative good?

This is, of course, only a heuristic question, and the possible answers are context-dependent. Our answer is strongly motivated by our experience in Finsler geometry.

In the more general context of the pull-back bundle $\pi^*\tau$, a covariant derivative operator is said to be 'good' (or it may be called good) if it has some of the regularity properties formulated in 3.6. For example, in the book of Abate and Patrizio [1], the regular covariant derivative operators are called good covariant derivatives. The attribute refers to the fact that the covariant derivative induces an Ehresmann connection to which it is associated. In the metric case all geometric data have to be determined by the metric alone, and one expects a 'harmony' between the metric, the covariant derivative and the induced Ehresmann connection. From this viewpoint, Cartan's covariant derivative and, in particular, the Levi-Civita derivative in Riemannian geometry are ideal. It is instructive to observe how important it is to require associatedness in Proposition 15 in the identification of the Barthel connection.

To make a long story short, we formulate our concept of 'goodness' as follows: *a metric derivative in $\pi^*\tau$ is good if it is associated to an Ehresmann connection determined by the metric alone.*

5.7. Good metric derivatives for weakly normal Moór–Vanstone metrics

Let g be a Moór–Vanstone metric in $\overset{\circ}{\tau}{}^*\tau$. Then (M, L), $L := \frac{1}{2}g(\delta, \delta)$, is a Finsler manifold, so the Barthel connection $\mathcal{H}_L$ induced by the canonical spray S_L of (M, L) is available to apply Miron's construction to get a candidate for a good metric derivative. It turns out, however, that the covariant derivative so obtained has no satisfactory relation to $\mathcal{H}_L$. Thus to find a good metric derivative for g one has to search a more suitable Ehresmann connection than $\mathcal{H}_L$. The problem can be formulated more efficiently as follows: *find a type (1,1) tensor field P along $\overset{\circ}{\tau}$ such that the Miron derivative arising from g and $\mathcal{H} := \mathcal{H}_L - \mathbf{i} \circ P$ be good.* This reformulation can essentially be found in the unpublished manuscript [22], where the authors could not find the correct solution. In the sequel we shall only sketch our solution, for a detailed account we refer to [27].

Keeping the notation just introduced, we shall also use the symbol $\sharp$ for the musical isomorphism described in 4.1. We have to prescribe an additional condition on g assuming that it is *weakly normal* at the same time.

The first step towards the solution of the problem is the following observation:

Lemma 16. *Assume that g is a weakly normal Moór–Vanstone metric in $\overset{\circ}{\tau}{}^*\tau$. We write $\overset{L}{\nabla}$ for the Berwald derivative induced by the Ehresmann connection $\mathcal{H}_L$. Let P be a type (1,1) tensor field along $\overset{\circ}{\tau}$ such that the Ehresmann connection $\mathcal{H} := \mathcal{H}_L - \mathbf{i} \circ P$ has the properties*

$$\mathcal{H}\delta = S_L \text{ and } \operatorname{Im} \mathcal{H} \subset \operatorname{Ker}(dL).$$

The Miron derivative arising from g and $\mathcal{H}$ is strongly associated to $\mathcal{H}$ if and only if for any two sections $\tilde{X}, \tilde{Y}$ in $\mathfrak{X}(\overset{\circ}{\tau})$ we have

$$g\big((\nabla^v_\delta P)(\tilde{X}), \tilde{Y}\big) + g\big(\tilde{X}, (\nabla^v_\delta P)(\tilde{Y})\big) = -\Big(\overset{L}{\nabla}_{S_L} g\Big)(\tilde{X}, \tilde{Y}). \tag{5.7.1}$$

Sketch of proof. By Lemma 2 (ii), we have to show that (5.7.1) holds if and only if the Miron derivative D arising from g and $\mathcal{H}$ is strongly regular and has vanishing h-deflection. In the first step, we show that D is automatically strongly regular. If $\tilde{X}, \tilde{Y} \in \mathfrak{X}(\overset{\circ}{\tau})$, we have

$$\tilde{\mu}\tilde{X} = D_{\mathrm{i}\tilde{X}}\delta = \nabla_{\mathrm{i}\tilde{X}}\delta + \frac{1}{2}\overset{\circ}{\mathcal{C}}(\tilde{X}, \delta),$$

$$g\left(\overset{\circ}{\mathcal{C}}(\tilde{X},\delta),\tilde{Y}\right) = g\big(\mathcal{C}(\tilde{X},\delta),\tilde{Y}\big) + g\big(\mathcal{C}(\delta,\tilde{Y}),\tilde{X}\big) - g\big(\mathcal{C}(\tilde{Y},\tilde{X}),\delta\big)$$
$$= g\big(\mathcal{C}(\tilde{X},\tilde{Y}),\delta\big) + (\nabla^v_\delta g)(\tilde{Y},\tilde{X}) - g\big(\mathcal{C}(\tilde{X},\tilde{Y}),\delta\big) = 0,$$

thus $\tilde{\mu}$ is indeed the identity map. In the second and longer step one has to show that the h-deflection of D with respect to $\mathcal{H}$ vanishes if and only if (5.7.1) holds. Since g is non-degenerate, it is enough to consider the expression $g\big(D_{\mathcal{H}\tilde{X}}\delta,\tilde{Y}\big)$ with $\tilde{X},\tilde{Y} \in \mathfrak{X}(\overset{\circ}{\tau})$ arbitrary. By a rather cumbersome calculation, which we omit here, one obtains

$$2g\big(D_{\mathcal{H}\tilde{X}}\delta,\tilde{Y}\big) = g\big((\nabla^v_\delta P)(\tilde{X}),\tilde{Y}\big) + g\big(\tilde{X},(\nabla^v_\delta P)(\tilde{Y})\big) + \left(\overset{L}{\nabla}_{S_L} g\right)(\tilde{X},\tilde{Y}),$$

which implies the desired equivalence.

For a detailed proof see [27] or the Thesis [26] of the first author.

Theorem 17. *Notation and assumption as in the previous Lemma. Define a type (1,1) tensor field P along $\overset{\circ}{\tau}$ by*

(i) $P\tilde{X} := -\frac{1}{2}\left(i_{\tilde{X}}\overset{L}{\nabla}_{S_L} g\right)^\sharp + P_s\tilde{X} + P_a\tilde{X},\ \tilde{X} \in \mathfrak{X}(\overset{\circ}{\tau}),$

where

- (ii) *P_s is a symmetric, P_a is a skew-symmetric type (1,1) tensor along $\overset{\circ}{\tau}$ with respect to g;*
- (iii) *P_s is homogeneous of degree 0, i.e., $\nabla^v_\delta P_s = 0$;*
- (iv) $\operatorname{Im} P_s$ *and* $\operatorname{Im} P_a$ *are contained in the g-orthogonal complement of the canonical section.*

Then the Ehresmann connection $\mathcal{H} := \mathcal{H}_L - \mathbf{i} \circ P$ has the properties

- (v) $\mathcal{H}\delta = S_L$;
- (vi) $\operatorname{Im}\mathcal{H} \subset \operatorname{Ker}(dL)$,

and the Miron derivative D arising from g and $\mathcal{H}$ is strongly associated to $\mathcal{H}$, i.e.,

- (vii) $D\delta = \mathcal{V}$.

Conversely, if $\mathcal{H} = \mathcal{H}_L - \mathbf{i} \circ P$ is an Ehresmann connection with the properties (v) *and* (vi), *and the Miron derivative arising from g and $\mathcal{H}$ is strongly associated to $\mathcal{H}$, then P is of the form* (i), *satisfying relations* (ii), (iii) *and* (iv).

Idea of the proof. In view of Lemma 16, taking into account the fact that $\mathcal{H}_L$ also satisfies (v) and (vi), we have to solve the following mixed system of algebraic equations and a partial differential equation for P:

$$\text{(viii)} \begin{cases} P\delta = 0, \\ (\mathbf{i}P\tilde{X})L = 0, \\ g\left((\nabla^v_\delta P)(\tilde{X}), \tilde{Y}\right) + g\left(\tilde{X}, (\nabla^v_\delta P)(\tilde{Y})\right) = -\left(\overset{L}{\nabla}_{S_L} g\right)(\tilde{X}, \tilde{Y}); \end{cases}$$

$\left(\tilde{X}, \tilde{Y} \in \mathfrak{X}(\overset{\circ}{\tau})\right)$. As the system is linear, its general solution may be searched as the sum of a particular solution and the general solution of the associated homogeneous system. Fortunately, a particular solution may be 'found out' quite easily: it is, roughly speaking, the map given by the first term in the right-hand side of (i). However, the verification of this observation needs a lengthy and troublesome calculation.

The associated homogeneous system differs to (viii) only in the third equation, which takes the form

$$\text{(ix)} \quad g\left((\nabla^v_\delta P)(\tilde{X}), \tilde{Y}\right) + g\left(\tilde{X}, (\nabla^v_\delta P)(\tilde{Y})\right) = 0; \ \tilde{X}, \tilde{Y} \in \mathfrak{X}(\overset{\circ}{\tau}).$$

Decomposing P into its symmetric part P_s and skew-symmetric part P_a, one can show that (ix) holds if and only if P_s is homogeneous of degree 0, while the algebraic relations in (viii) are valid if and only if both the sum and the difference of P_s and P_a are contained in the g-orthogonal complement of δ. This concludes our sketchy proof of the Theorem.

Remark. Theorem 17 provides, in fact, a family of good metric derivatives for a weakly normal Moór – Vanstone metric, since there is a considerable freedom in the choice of P_s and P_a in (i). Conditions (v) and (vi) have a clear geometric meaning. (v) assures that the geodesics of $\mathcal{H}$ (see e.g. [40]) coincide with the geodesics of (M, L), while (vi) expresses the requirement that the energy L is a first integral of the $\mathcal{H}$-horizontal vector fields.

5.8. The adjoint tensor $\mathcal{C}^*$

Let first g be any metric along $\pi^*\tau$ and $\mathcal{C}$ its vectorvariant Cartan tensor. Choose a fixed vector field $\tilde{Y}$ along π, and consider the $C^\infty(\widetilde{TM})$-linear map

$$\mathcal{C}_{\tilde{Y}} : \tilde{X} \in \mathfrak{X}(\pi) \mapsto \mathcal{C}_{\tilde{Y}}(\tilde{X}) := \mathcal{C}(\tilde{X}, \tilde{Y}) \in \mathfrak{X}(\pi).$$

To $\mathcal{C}_{\tilde{Y}}$ there corresponds a unique $C^\infty(\widetilde{TM})$-linear endomorphism $\mathcal{C}^*_{\tilde{Y}}$ along π such that

$$g(\mathcal{C}^*_{\tilde{Y}}(\tilde{X}), \tilde{Z}) = g(\tilde{X}, \mathcal{C}_{\tilde{Y}}(\tilde{Z})); \ \tilde{X}, \tilde{Z} \in \mathfrak{X}(\pi).$$

$\mathcal{C}^*_{\tilde{Y}}$ is called the *adjoint* of $\mathcal{C}_{\tilde{Y}}$. (Fixing the first variable of $\mathcal{C}$, we would obtain in this way a self-adjoint endomorphism of $\mathfrak{X}(\pi)$.) Keeping in mind that in forming adjoints the *second* variable of $\mathcal{C}$ is fixed, occasionally we shall write simply $\mathcal{C}^*(\tilde{X},\tilde{Y})$ instead of $\mathcal{C}^*_{\tilde{Y}}(\tilde{X})$.

Now suppose that g is *weakly variational.* Then $\mathcal{C}^*$ satisfies

$$\mathcal{C}^*(\delta,\tilde{X}) = \mathcal{C}(\tilde{X},\delta),\ \ \tilde{X} \in \mathfrak{X}(\pi). \tag{5.8.1}$$

Indeed, for any sections $\tilde{X},\tilde{Y}$ in $\mathfrak{X}(\pi)$ we have

$$\begin{aligned} g(\mathcal{C}^*(\delta,\tilde{X}),\tilde{Y}) &= g(\mathcal{C}^*_{\tilde{X}}(\delta),\tilde{Y}) = g(\delta,\mathcal{C}_{\tilde{X}}(\tilde{Y})) \\ &= g(\mathcal{C}(\tilde{Y},\tilde{X}),\delta) \overset{(*)}{=} g(\mathcal{C}(\tilde{X},\tilde{Y}),\delta) = g(\mathcal{C}(\tilde{X},\delta),\tilde{Y}), \end{aligned}$$

which implies by the non-degeneracy of g the desired relation. Weak variationality was applied at the step denoted by $(*)$.

5.9. A good metric derivative for a class of Miron metrics

In this subsection we shall frequently use the Miron tensor A introduced in 4.3. Suppose that g is a Miron metric in $\mathring{\tau}^*\tau$. Then its Miron tensor yields a self-adjoint linear transformation $A_v \in \operatorname{End}(T_{\mathring{\tau}(v)}M)$ at each $v \in \mathring{T}M$ by Corollary 8 because Miron metrics are weakly normal, and weak normality implies weak variationality. For the same reason, the adjoint tensor $\mathcal{C}^*$ satisfies (5.8.1). Note finally that the canonical section δ is an eigenvector-field of A with corresponding eigenvalue 1, since by the weak normality of g

$$A\delta := \delta + \mathcal{C}(\delta,\delta) = \delta.$$

The next important observation, as well as its proof, is a friend of Lemma 14.

Lemma 18. *Assume that g is a Miron metric in $\mathring{\tau}^*\tau$. Let $\mathcal{H}$ and $\tilde{\mathcal{H}}$ be Ehresmann connections, D and $\tilde{D}$ be metric derivatives in $\mathring{\tau}^*\tau$ such that*

(i) $T^v(D) = T^v(\tilde{D}) = 0$,
(ii) *the torsion* $\tilde{\mathbf{T}}$ *of* $\tilde{\mathcal{H}}$ *vanishes,*
(iii) *the horizontal torsions* $\mathcal{T} := T(D)\circ(\mathcal{H}\times\mathcal{H})$ *and* $\tilde{\mathcal{T}} := T(\tilde{D})\circ(\tilde{\mathcal{H}}\times\tilde{\mathcal{H}})$ *vanish.*

If P is the difference tensor of $\mathcal{H}$ and $\tilde{\mathcal{H}}$, and P^* is its adjoint with respect to g, then for any two sections $\tilde{X},\tilde{Y}$ in $\mathfrak{X}(\pi)$ we have

$$\begin{aligned} D_{\mathcal{H}\tilde{X}}\tilde{Y} = \tilde{D}_{\mathcal{H}\tilde{X}}\tilde{Y} + \frac{1}{2}\Big(&\mathcal{C}(P\tilde{Y},\tilde{X}) - \mathcal{C}(\tilde{Y},P\tilde{X}) \\ &+\mathcal{C}^*(\tilde{Y},P\tilde{X}) - P^*\mathcal{C}^*(\tilde{Y},\tilde{X})\Big). \end{aligned} \tag{5.9.1}$$

Sketch of proof. Conditions $T^v(\tilde{D}) = 0$, $\tilde{\mathcal{T}} = 0$ and $\tilde{\mathbf{T}} = 0$ imply that $\tilde{D}$ is just the Miron derivative arising from g and $\tilde{\mathcal{H}}$. Let ψ^h be the difference tensor of the $\mathcal{H}$-horizontal part of D and $\tilde{D}$, i.e., let

$$\psi^h(\tilde{X}, \tilde{Y}) := D_{\mathcal{H}\tilde{X}}\tilde{Y} - \tilde{D}_{\mathcal{H}\tilde{X}}\tilde{Y};\ \tilde{X}, \tilde{Y} \in \mathfrak{X}(\overset{\circ}{\tau}).$$

Since $D^v = \tilde{D}^v$ by (i) and Lemma 13, this relation can also be written in the form

$$\psi^h(\tilde{X}, \tilde{Y}) := D_{\tilde{\mathcal{H}}\tilde{X}}\tilde{Y} - \tilde{D}_{\tilde{\mathcal{H}}\tilde{X}}\tilde{Y};\ \tilde{X}, \tilde{Y} \in \mathfrak{X}(\overset{\circ}{\tau}).$$

Using the condition that both D and $\tilde{D}$ are metric, a routine calculation similar to that in the proof of Lemma 14 yields

$$g(\psi^h(\tilde{X}, \tilde{Y}), \tilde{Z}) + g(\tilde{Y}, \psi^h(\tilde{X}, \tilde{Z})) = 0;\ \tilde{X}, \tilde{Y}, \tilde{Z} \in \mathfrak{X}(\overset{\circ}{\tau}). \tag{5.9.2}$$

Next applying our condition $\mathcal{T} = \tilde{\mathcal{T}} = 0$, after a lengthy but quite straightforward calculation we get for any vector fields X, Y on M

$$\psi^h(\hat{X}, \hat{Y}) - \psi^h(\hat{Y}, \hat{X}) = -\frac{1}{2}\Big(\overset{\circ}{\mathcal{C}}(P\hat{X}, \hat{Y}) - \overset{\circ}{\mathcal{C}}(P\hat{Y}, \hat{X})\Big), \tag{5.9.3}$$

where $\overset{\circ}{\mathcal{C}}$ was defined by (5.2.1). Thus the map ω given by

$$\omega(\tilde{X}, \tilde{Y}) := -\frac{1}{2}\Big(\overset{\circ}{\mathcal{C}}(P\tilde{X}, \tilde{Y}) - \overset{\circ}{\mathcal{C}}(P\tilde{Y}, \tilde{X})\Big);\ \tilde{X}, \tilde{Y} \in \mathfrak{X}(\overset{\circ}{\tau})$$

satisfies conditions (i), (ii) of Lemma 12, therefore it is uniquely determined by (5.9.2) and (5.9.3). Finally, an immediate (but also lengthy) calculation shows that if ψ^h is given by the second term on the right-hand side of (5.9.1), then it satisfies (5.9.2) and (5.9.3). This concludes the proof.

Theorem 19. *Let g be a positive definite Miron metric in $\overset{\circ}{\tau}{}^*\tau$. Suppose that the self-adjoint linear transformations $A_v \in \operatorname{End}\big(T_{\overset{\circ}{\tau}(v)}M\big)$ $\big(v \in \overset{\circ}{T}M\big)$ have no eigenvalues λ_i, λ_j such that $\lambda_i + \lambda_j = 0$. Then there is a unique Ehresmann connection $\mathcal{H}$ and a unique metric derivative D in $\overset{\circ}{\tau}{}^*\tau$ such that*

(i) *the vertical torsion of D vanishes;*
(ii) *the horizontal torsion $\mathcal{T} := T(D) \circ (\mathcal{H} \times \mathcal{H})$ of D vanishes;*
(iii) $\operatorname{Im}\mathcal{H} \subset \operatorname{Ker}\mu$, *where μ is the deflection of D.*

Proof. Choose a fixed Ehresmann connection $\tilde{\mathcal{H}}$ with vanishing torsion (e.g., $\tilde{\mathcal{H}} := \mathcal{H}_L$, $L = \frac{1}{2}g(\delta, \delta)$), and consider the Miron derivative $\tilde{D}$ arising from g and $\tilde{\mathcal{H}}$. Then it satisfies the prescriptions imposed on $\tilde{D}$ in Lemma 18. As in the proof of Theorem 17, we look for the desired Ehresmann connection in the form

$$\mathcal{H} = \tilde{\mathcal{H}} + \mathbf{i} \circ P.$$

If D is a metric derivative satisfying (i) and (ii), then by the previous Lemma its $\mathcal{H}$-horizontal part acts by the rule (5.9.1), while $D^v = \tilde{D}^v$ as we have seen in the proof of the Lemma. Thus D is uniquely determined by (i) and (ii). Our only task is to show that the further condition (iii) forces the existence and uniqueness of the 'unknown' difference tensor P.

First we observe that *condition* (iii) *is equivalent to the relation*

$$\text{(iii)}^* \quad \begin{aligned} &2g(\tilde{\mu}^h \tilde{X}, \tilde{Y}) + 2g(P\tilde{X}, \tilde{Y}) + g(\mathcal{C}(P\tilde{X}, \delta), \tilde{Y}) \\ &+g(\mathcal{C}(P\delta, \tilde{X}), \tilde{Y}) - g(\mathcal{C}(P\tilde{Y}, \tilde{X}), \delta) = 0 \end{aligned}$$

for P, where $\tilde{\mu}^h$ is the h-deflection of $\tilde{D}$ with respect to $\tilde{\mathcal{H}}$; $\tilde{X}, \tilde{Y} \in \mathfrak{X}(\overset{\circ}{\tau})$. This can be verified by a routine, but lengthy calculation, which we omit. Substituting $\tilde{X} := \delta$ into (iii)*, the last term vanishes by the weak normality of g. Then the non-degeneracy of g yields

$$2\tilde{\mu}^h \delta + 2P\delta + 2\mathcal{C}(P\delta, \delta) = 0.$$

With the help of the Miron tensor of g this can be written in the form

$$A(P\delta) = -\tilde{\mu}^h \delta.$$

Due to the Miron regularity of g, A is invertible, and we get

$$P\delta = -A^{-1} \circ \tilde{\mu}^h(\delta).$$

Substituting this expression of $P\delta$ into (iii)* we find

$$\begin{aligned} 0 &= 2g(\tilde{\mu}^h \tilde{X}, \tilde{Y}) + 2g(P\tilde{X}, \tilde{Y}) + g(\mathcal{C}(P\tilde{X}, \delta), \tilde{Y}) \\ &\quad - g(\mathcal{C}(A^{-1} \circ \tilde{\mu}^h(\delta), \tilde{X}), \tilde{Y}) - g(\mathcal{C}_{\tilde{X}}(P\tilde{Y}), \delta) \\ &= g(2\tilde{\mu}^h \tilde{X} + 2P\tilde{X} + \mathcal{C}(P\tilde{X}, \delta) \\ &\quad - \mathcal{C}(A^{-1} \circ \tilde{\mu}^h(\delta), \tilde{X}) - P^*\mathcal{C}^*(\delta, \tilde{X}), \tilde{Y}). \end{aligned}$$

Again by the non-degeneracy of g, and taking into account (5.8.1), this relation is equivalent to

$$2P\tilde{X} + \mathcal{C}(P\tilde{X}, \delta) - P^*\mathcal{C}(\tilde{X}, \delta) = \mathcal{C}(A^{-1} \circ \tilde{\mu}^h(\delta), \tilde{X}) - 2\tilde{\mu}^h \tilde{X}.$$

Using the Miron tensor of g, our equality can also be written in the form

(iv) $\quad (P + P^* + A \circ P - P^* \circ A)(\tilde{X}) = \mathcal{C}(A^{-1} \circ \tilde{\mu}^h(\delta), \tilde{X}) - 2\tilde{\mu}^h\tilde{X}.$

Thus we have obtained that *under the condition* $P\delta = -A^{-1} \circ \tilde{\mu}^h(\delta)$ *relations* (iii)* *and* (iv), *and hence* (iii) *and* (iv), *are equivalent.*

Now we define an endomorphism B along $\overset{\circ}{\tau}$ by the rule

$$B\tilde{X} := \mathcal{C}(A^{-1} \circ \tilde{\mu}^h(\delta), \tilde{X}) - 2\tilde{\mu}^h\tilde{X}, \ \tilde{X} \in \mathfrak{X}(\overset{\circ}{\tau}),$$

and a further map $\Phi : \operatorname{End}(\mathfrak{X}(\overset{\circ}{\tau})) \to \operatorname{End}(\mathfrak{X}(\overset{\circ}{\tau}))$ by

$$\Phi(Q) := Q + Q^* + A \circ Q - Q^* \circ A, \ Q \in \operatorname{End}(\mathfrak{X}(\overset{\circ}{\tau})).$$

With these new ingredients our problem reduces to the solvability of the equation

$$\text{(v)} \qquad \Phi(\mathcal{X}) = B$$

in $\operatorname{End}(\mathfrak{X}(\overset{\circ}{\tau}))$. It has a unique solution if and only if

$$\operatorname{Ker}\Phi = \{0\} \subset \operatorname{End}(\mathfrak{X}(\overset{\circ}{\tau})),$$

meaning that $\operatorname{Ker}\Phi_v = \{0_v\} \subset \operatorname{End}(T_{\overset{\circ}{\tau}(v)}M)$ for all $v \in \overset{\circ}{T}M$.

We show that Φ satisfies this criterion. Let $Q \in \operatorname{Ker}\Phi$. Then for any section $\tilde{X}$ along $\overset{\circ}{\tau}$ we have

$$\begin{aligned} 0 &= g(Q\tilde{X}, \tilde{X}) + g(Q^*\tilde{X}, \tilde{X}) + g(A \circ Q(\tilde{X}), \tilde{X}) - g(Q^* \circ A(\tilde{X}), \tilde{X}) \\ &= 2g(Q\tilde{X}, \tilde{X}), \end{aligned}$$

showing that Q is skew-symmetric with respect to g, i.e., $Q^* = -Q$. Thus $\Phi(Q) = 0$ reduces to

$$\text{(vi)} \qquad A \circ Q + Q \circ A = 0.$$

Now let $v \in \overset{\circ}{T}M$ be arbitrary. As we have learnt, A_v is self-adjoint with respect to g_v. Hence, by the positive-definiteness of the metric, there exists a g_v-orthonormal basis $(e_i)_{i=1}^n$ of $T_{\overset{\circ}{\tau}(v)}M$ consisting of eigenvectors of A_v; i.e.,

$$A_v e_i = \lambda_i e_i; \ \lambda_i \in \mathbb{R}, \ i \in \{1, \dots, n\}.$$

Thus (vi) yields

$$\begin{aligned} 0 = g_v((A_v \circ Q_v + Q_v \circ A_v)(e_i), e_j) &= g_v(A_v \circ Q_v(e_i), e_j) \\ + \lambda_i g_v(Q_v(e_i), e_j) &= (\lambda_i + \lambda_j) g_v(Q_v(e_i), e_j); \end{aligned}$$

$1 \leqq i, j \leqq n$. This implies that $Q = 0$ because $\lambda_i + \lambda_j \neq 0$ by our condition on the eigenvalues of A_v. We conclude that $\operatorname{Ker}\Phi = 0$, hence equation (v) indeed has a unique solution P in $\operatorname{End}\big(\mathfrak{X}(\overset{\circ}{\tau})\big)$. To complete the proof of the theorem, we have to check that P satisfies $P\delta = -A^{-1} \circ \tilde{\mu}^h(\delta)$. Then, as we have shown, relations $\Phi P = B$ and (iii)* are equivalent, therefore $\mathcal{H} := \tilde{\mathcal{H}} + \mathbf{i} \circ P$ is the only Ehresmann connection which satisfies (iii).

Substitute $\tilde{X} := \delta$ into (iv). Since $A\delta = \delta$, the left-hand side yields

$$P\delta + A(P\delta) = \big(1_{\mathfrak{X}(\overset{\circ}{\tau})} + A\big)P\delta.$$

The right-hand side can be formed as follows:

$$\begin{aligned}
&\mathcal{C}\big(A^{-1} \circ \tilde{\mu}^h(\delta), \delta\big) - 2\tilde{\mu}^h\delta = \mathcal{C}\big(A^{-1} \circ \tilde{\mu}^h(\delta), \delta\big) - 2A\big(A^{-1} \circ \tilde{\mu}^h(\delta)\big) \\
&= \mathcal{C}\big(A^{-1} \circ \tilde{\mu}^h(\delta), \delta\big) - 2A^{-1} \circ \tilde{\mu}^h(\delta) - 2\mathcal{C}\big(A^{-1} \circ \tilde{\mu}^h(\delta), \delta\big) \\
&= -\Big(\mathcal{C}\big(A^{-1} \circ \tilde{\mu}^h(\delta), \delta\big) + A^{-1} \circ \tilde{\mu}^h(\delta)\Big) - A^{-1} \circ \tilde{\mu}^h(\delta) \\
&= -A\big(A^{-1} \circ \tilde{\mu}^h(\delta)\big) - A^{-1} \circ \tilde{\mu}^h(\delta) = \big(1_{\mathfrak{X}(\overset{\circ}{\tau})} + A\big)\big(-A^{-1} \circ \tilde{\mu}^h(\delta)\big),
\end{aligned}$$

therefore we get

$$\big(1_{\mathfrak{X}(\overset{\circ}{\tau})} + A\big)P\delta = \big(1_{\mathfrak{X}(\overset{\circ}{\tau})} + A\big)\big(-A^{-1} \circ \tilde{\mu}^h(\delta)\big).$$

Since δ is an eigenvector-field of A with corresponding eigenvalue 1, A has no eigenvalue -1 by our condition. Hence the endomorphism $1_{\mathfrak{X}(\overset{\circ}{\tau})} + A$ is (pointwise) invertible, and we obtain the desired relation $P\delta = -A^{-1} \circ \tilde{\mu}^h(\delta)$. □

Remark. Property (iii) in Theorem 19 may be named as the *weak associatedness* of D to $\mathcal{H}$. We see that the constructed covariant derivative operator is not 'good' in the sense of 5.6, but it is 'nearly good'. If, in particular, D proves to be regular, then it is associated to $\mathcal{H}$, and so it becomes a good metric derivative. It can be shown by an easy calculation that this occurs if and only if the metric satisfies the additional condition that the tensor

$$\tilde{X} \in \mathfrak{X}(\overset{\circ}{\tau}) \mapsto \frac{1}{2}\mathcal{C}(\delta, \tilde{X})$$

is (pointwise) invertible.

References

[1] M. Abate and G. Patrizio, Finsler metrics – A Global Approach, Lecture Notes in Math., **1591**, Springer-Verlag, Berlin, 1994.

[2] H. Akbar-Zadeh, Espaces de nullité en géométrie finslérienne, Tensor (N. S.), **26** (1972), 89–101.

[3] L. Auslander, On curvature in Finsler geometry, Trans. Amer. Math. Soc., **79** (1955), 378–388.

[4] D. Bao, S.-S. Chern and Z. Shen, An Introduction to Riemann – Finsler Geometry, Springer-Verlag, Berlin, 2000.

[5] A. Bejancu, Finsler Geometry and Applications, Ellis Horwood, Chichester, 1990.

[6] A. M. Bloch, P. S. Krishnaprasad, J. E. Marsden and R. M. Murray, Nonholonomic mechanical systems with symmetry, Arch. Rat. Mech. Anal., **136** (1996), 21–99.

[7] M. Crampin, On horizontal distributions on the tangent bundle of a differentiable manifold, J. London Math. Soc. (2), **3** (1971), 178–182.

[8] M. Crampin, On the differential geometry of the Euler – Lagrange equation and the inverse problem of Lagrangian dynamics, J. Phys. A, **14** (1981), 2567–2575.

[9] M. Crampin, Generalized Bianchi identities for horizontal distributions, Math. Proc. Camb. Phil. Soc., **94** (1983), 125–132.

[10] M. Crampin, Connections of Berwald type, Publ. Math. Debrecen, **57** (2000), 455–473.

[11] M. Crampin and L. A. Ibort, Graded Lie algebras of derivations and Ehresmann connections, J. Math. Pures Appl. (9), **66** (1987), 113–125.

[12] J.-G. Diaz and G. Grangier, Courbure et holonomie des variétés finslériennes, Tensor (N. S.), **30**, 95–109.

[13] P. Dombrowski, MR, **37** # 3469.

[14] W. Greub, S. Halperin and J. R. Vanstone, Connections, Curvature, and Cohomology Vol. 1, Academic Press, New York and London, 1972.

[15] W. Greub, S. Halperin and J. R. Vanstone, Connections, Curvature, and Cohomology Vol. 2, Academic Press, New York and London, 1973.

[16] J. Grifone, Structure presque tangente et connexions I., Ann. Inst. Fourier (Grenoble), **22** (1972) 1, 287–334.

[17] J. Grifone, Structure presque tangente et connexions II., Ann. Inst. Fourier (Grenoble), **22** (1972) 3, 291–338.

[18] J. Grifone and Z. Muzsnay, Variational principles for second-order differential equations: application of the Spencer theory to characterize variational sprays, World Scientific, Singapore, 2000.

[19] M. Hashiguchi, On parallel displacements in Finsler spaces, J. Math. Soc. Japan, **10** (1958), 365–379.

[20] B. T. M. Hassan, The theory of geodesics in Finsler spaces, thesis, Southampton, 1967.

[21] J. I. Horváth and A. Moór, Entwicklung einer Feldtheorie begründet auf einen allgemeinen metrischen Linienelementraum I, II, Indag. Math., **17** (1955), 421–429, 581–587.

[22] H. Izumi, T. Sakaguchi and M. Yoshida, Remarks on the assumptions of generalized metric $g_{ij}(x, y)$, unpublished manuscript.

[23] A. Kawaguchi, On the theory of non-linear connections II, Tensor (N. S.), **6** (1956), 165–199.

[24] O. Krupková, Variational metrics on $\mathbb{R} \times TM$ and the geometry of nonconservative mechanics, Math. Slovaca, **44** (1994), 315–335.

[25] M. de León and P. R. Rodrigues, Methods of differential geometry in analytical mechanics, North-Holland, Amsterdam, 1989.

[26] R. L. Lovas, Geometric vector fields of spray and metric structures, PhD thesis, Debrecen, 2005.

[27] R. L. Lovas and J. Szilasi, Good metric derivatives for a class of Moór – Vanstone metrics, submitted.

[28] M. Matsumoto, The Theory of Finsler Connections, Publ. of the Study Group of Geometry, **5**, Okayama, 1970.

[29] M. Matsumoto, Foundations of Finsler Geometry and Special Finsler Spaces, Kaiseisha Press, Otsu, 1986.

[30] M. Matsumoto, Finsler Geometry in the 20th Century, In: Handbook of Finsler Geometry, **1, 2**, (ed. P. L. Antonelli), Kluwer Academic Publishers, Dordrecht, 2003.

[31] T. Mestdag, J. Szilasi and V. Tóth, On the geometry of generalized metrics, Publ. Math. Debrecen, **62** (2003), 511–545.

[32] P. W. Michor, Topics in Differential Geometry, http://www.mat.univie.ac.at/~michor/dgbook.ps, draft from 15th September 2004.

[33] R. Miron, Metrical Finsler structures and metrical Finsler connections, J. Math. Kyōto Univ., **23** (1983), 219–224.

[34] R. Miron, M. Anastasiei and I. Bucataru, The Geometry of Lagrange Spaces, In: Handbook of Finsler Geometry, **2**, (ed. P. L. Antonelli), Kluwer Academic Publishers, Dordrecht, 2003.

[35] R. Miron and T. Kawaguchi, Relativistic geometrical optics, Internat. J. Theoret. Phys., **30** (1991), 1521–1543.

[36] A. Moór, Entwicklung einer Geometrie der allgemeinen metrischen Linienelementräume, Acta Sci. Math. (Szeged), **17** (1956), 85–120.

[37] A. Moór, Eine Verallgemeinerung der metrischen Übertragung in allgemeinen metrischen Räumen, Publ. Math. Debrecen, **10** (1963), 145–150.

[38] Z. I. Szabó, Über Zusammenhänge vom Finsler-Typ, Publ. Math. Debrecen, **27** (1980), 77–88.

[39] J. Szilasi, Notable Finsler connections on a Finsler manifold, Lect. Mat., **19** (1998), 7–34.

[40] J. Szilasi, A Setting for Spray and Finsler Geometry, In: Handbook of Finsler Geometry, **2**, (ed. P. L. Antonelli), Kluwer Academic Publishers, Dordrecht, 2003.

[41] J. Szilasi and Cs. Vincze, A new look at Finsler connections and special Finsler manifolds, Acta Math. Acad. Paedagog. Nyházi., **16** (2000), 33–63, http://www.emis.de/journals.

[42] J. R. Vanstone, A generalization of Finsler geometry, Canad. J. Math., **14** (1962), 87–112.

Institute of Mathematics
University of Debrecen
H-4010 Debrecen, P.O.B. 12
Hungary