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Curvature properties of (α, β)-metrics

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Abstract.

(α, β)-metrics form a rich class of computable Finsler metrics. Many (α, β)-metrics with special curvature properties have been found and discussed. They play an important role in Finsler geometry. In this article, we introduce the recent developments in the study of (α, β)-metrics.

§1. Introduction

In the past several years, we witness a rapid development in Finsler geometry. Various curvatures have been studied and their geometric meanings are better understood. This is partially due to the study of a special class of Finsler metrics. The special Finsler metrics we are going to discuss are expressed in terms of a Riemannian metric $\alpha = \sqrt{a_{ij}y^i y^j}$ and a 1-form $\beta = b_i y^i$. They are called (α, β)-metrics. The simplest (α, β)-metrics are the Randers metrics $F = \alpha + \beta$. Thus more intensive study has done on Randers metrics than other metrics. For example, a complete list of local structures of Randers metrics of constant flag curvature has been given in [13] recently. This motivates people to study more general (α, β)-metrics. In this article, we will introduce the recent development of (α, β)-metrics with special curvature properties.

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§2. Preliminaries

There are two important volume forms in Finsler geometry. One is the Busemann-Hausdorff volume form and the other is the Holmes-Thompson volume form.

For a Finsler metric $F = F(x, y)$ on an n-dimensional manifold M, the Holmes-Thompson volume form $dV_{HT} = \sigma_{HT}(x)dx$ is given by

$$\sigma_{HT}(x) = \frac{1}{\omega_n} \int_{\{F(x,y)<1\}} \det\Big(g_{ij}(x, y)\Big)dy,$$

and the Busemann-Hausdorff volume form $dV_B = \sigma_{BH}(x)dx$ is given by

$$\sigma_{BH}(x) = \frac{\omega_n}{\mathrm{Vol}\{(y^i) \in R^n | F(x, y) < 1\}}.$$

Here

$$\begin{aligned} \omega_n : &= \mathrm{Vol}(\mathrm{B}^n(1)) = \frac{1}{n}\mathrm{Vol}(\mathrm{S}^{n-1}) \\ &= \frac{1}{n}\mathrm{Vol}(\mathrm{S}^{n-2}) \int_0^\pi \sin^{n-2}(t)dt. \end{aligned}$$

When $F = \sqrt{g_{ij}(x)y^i y^j}$ is a Riemannian metric, both volume forms reduce to the same Riemannian volume form

$$dV_{BH} = dV_{HT} = \sqrt{\det(g_{ij})}dx.$$

For a Finsler metric, the geodesics are characterized by a system of 2nd ODEs:

$$\ddot{x}^i + 2G^i(x, \dot{x}) = 0,$$

where

$$G^i = \frac{1}{4}g^{il}\Big\{[F^2]_{x^m y^l} y^m - [F^2]_{x^l}\Big\}.$$

G^i define a global vector field $G := y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}$ on TM. G is called the spray of F and the local functions G^i are called the spray coefficients of F.

For a Finsler metric F and a volume form $dV = \sigma(x)dx$ on an n-dimensional manifold M, the S-curvature $\mathbf{S}$ is given by

$$\mathbf{S} = \frac{\partial G^m}{\partial x^m} - y^m \frac{\partial \ln \sigma}{\partial x^m}. \tag{1}$$

The volume form can be the Busemann-Hausdorff volume form $dV_{BH} = \sigma_{BH}dx$ or the Holmes-Thompson volume form $dV_{TH} = \sigma_{TH}(x)dx$. Unless specified, the S-curvature usually is defined with respect to the Busemann-Hausdorff volume form.

Definition 2.1. Let F be a Finsler metric on an n-dimensional manifold M.

(a) F is of weakly isotropic S-curvature if there exist a scalar function $c = c(x)$ and a 1-form η on M such that the S-curvature is in the following form,

$$\mathrm{S} = (n+1)c(x)F + \eta.$$

(b) F is of almost isotropic S-curvature if $c = c(x)$ is a scalar function and η is a closed 1-form on M;

(c) F is of isotropic S-curvature if $c = c(x)$ is a scalar function and $\eta = 0$;

(d) F is of constant S-curvature if c is a constant and $\eta = 0$.

A Finsler metric F is called a *Berwald metric* if its spray coefficients

$$G^i = \frac{1}{2}\Gamma^i_{jk}(x)y^j y^k$$

are quadratic in $y = y^i \frac{\partial}{\partial x^i}|_x \in T_x M$ for any $x \in M$. Thus Riemannian metrics are special Berwald metrics. The local structure of Berwald metrics has been completely determined [49].

It is known that for a Berwald metric, the S-curvature (with respect to the Busemann-Hausdorff volume form) vanishes, $\mathbf{S} = 0$. Thus Finsler metrics with vanishing S-curvature can be regarded as generalized Berwald metrics.

There is another important quantity—the Landsberg tensor defined by

$$L_{jkl} := -\frac{1}{2}FF_{y^i}\frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}. \tag{2}$$

Finsler metrics with $L_{jkl} = 0$ are called *Landsberg metrics.* It is obvious that if $G^i = \frac{1}{2}\Gamma^i_{jk}(x)y^j y^k$ are quadratic in $y = y^i \frac{\partial}{\partial x^i}|_x \in T_x M$ for any $x \in M$, then $L_{jkl} = 0$. Thus every Berwald metric is a Landsberg metric. Landsberg metrics can be regarded as generalized Berwald metrics. It is a long existing open problem whether or not any Landsberg metric is a Berwald metric.

In projective geometry of Finsler manifolds, there is an important projectively invariant quantity—the Douglas tensor defined by

$$D_j{}^i{}_{kl} = \frac{\partial^3 \Pi^i}{\partial y^j \partial y^k \partial y^l},$$

where

$$\Pi^i := G^i - \frac{1}{n+1}\frac{\partial G^m}{\partial y^m}y^i.$$

In local coordinates, the following three conditions are equivalent

$$D_j{}^i{}_{kl} = 0,$$

$$G^i = \frac{1}{2}\Gamma^i{}_{jk}(x)y^j y^k + P(x,y)y^i,$$

$$D^{ij} := G^i y^j - G^j y^i = A^{ij}_{klm}(x)y^k y^l y^m.$$

A Finsler metric is called a *Douglas metric* if $D_j{}^i{}_{kl} = 0$. The notion of Douglas metrics is first introduced in [6]. Douglas metrics are regarded as generalized Berwald metrics.

Finally, we come to the most important quantity — the Riemann curvature defined by

$$R^i{}_k := 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^m \partial y^k}y^m + 2G^m\frac{\partial^2 G^i}{\partial y^m \partial y^k} - \frac{\partial G^i}{\partial y^m}\frac{\partial G^m}{\partial y^k}.$$

The flag curvature $\mathbf{K} = \mathbf{K}(P,y)$ of a flag (P,y), where $P = \mathrm{span}\{y,u\} \subset T_xM$ is defined by

$$\mathbf{K} = \frac{g_{ij}R^i{}_k(x,y)u^j u^k}{F(x,y)^2 g_{ij}(x,y)u^i u^j - [g_{ij}(x,y)y^i u^j]^2}.$$

It is a natural problem to investigate Finsler metrics with special flag curvature properties.

Definition 2.2. Let $F = F(x,y)$ be a Finsler metric on a manifold M.

(a) F is of scalar flag curvature if $\mathbf{K} = \mathbf{K}(x,y)$ is independent of P containing $y \in T_xM$;
(b) F is of weakly isotropic flag curvature if $\mathbf{K} = 3\eta/F + \sigma$, where η is a 1-form and $\sigma = \sigma(x)$ is a scalar function on M;
(c) F is of almost isotropic flag curvature if $\mathbf{K} = 3c_{x^m}y^m/F + \sigma$, where $c = c(x)$ and $\sigma = \sigma(x)$ are scalar functions on M;
(d) F is of isotropic flag curvature if $\mathbf{K} = \sigma$ where $\sigma = \sigma(x)$ is a scalar function on M;
(e) F is of constant flag curvature if $\mathbf{K} = \sigma = constant$.

By Schur Lemma, in dimension $n \geq 3$, if F is of isotropic flag curvature, then it is of constant flag curvature.

The S-curvature is closely related to the flag curvature.

Theorem 2.3. ([16]) *Let F be a Finsler metric of scalar flag curvature on a manifold M. Suppose that the S-curvature is almost isotropic, $\mathbf{S} = (n+1)cF + \eta$, where $c = c(x)$ is a scalar function and $\eta = \eta_i y^i$ is a closed 1-form, then the flag curvature is almost isotropic in the following form*

$$\mathbf{K} = \frac{3c_{x^m}y^m}{F} + \sigma,$$

where $\sigma = \sigma(x)$ is a scalar function on M.

Let F be a Finsler metric of scalar flag curvature on a manifold M. In [33], we find a sufficient and necessary condition on a non-Riemannian quantity for the flag curvature to be weakly isotropic.

§3. (α, β)-metrics

In Finsler geometry, it is in general very difficult to compute the curvatures of a Finsler metric. Some Finsler metrics are defined by some elementary functions, but their expressions of curvatures are extremely complicated so that one can not easily determine their values.

There is a class of Finsler metrics defined by a Riemannian metric and a 1-form on a manifold, which is relatively simple with interesting curvature properties. More important, these metrics are "computable". Thus they first deserve our attention.

Let $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ be a Riemannian metric and $\beta = b_i(x)y^i$ be a 1-form on an n-dimensional manifold M. Using α and β one can define a function on TM as follows

$$F = \alpha\phi(s), \qquad s = \frac{\beta}{\alpha}. \tag{3}$$

where $\phi = \phi(s)$ is a C^∞ positive function on an open interval $(-b_o, b_o)$. The norm $\|\beta_x\|_\alpha$ of β with respect to α is defined by

$$\|\beta_x\|_\alpha := \sup_{y\in T_xM} \frac{\beta(x,y)}{\alpha(x,y)} = \sqrt{a^{ij}(x)b_i(x)b_j(x)}.$$

In order to define F, β must satisfy the condition $\|\beta_x\|_\alpha < b_o$ for all $x \in M$.

To find a condition on the function ϕ such that F in (3) is a Finsler metric, one computes the Hessian $g_{ij} := \frac{1}{2}[F^2]_{y^iy^j}$ as follows,

$$g_{ij} = \rho a_{ij} + \rho_0 b_i b_j + \rho_1(b_i\alpha_j + b_j\alpha_i) - s\rho_1\alpha_i\alpha_j,$$

where $\alpha_i := \alpha_{y^i}$ and

$$\rho = \phi^2 - s\phi\phi', \qquad \rho_0 = \phi\phi'' + \phi'\phi',$$

$$\rho_1 = -s(\phi\phi'' + \phi'\phi') + \phi\phi',$$

where the functions are evaluated on $s := \beta/\alpha$. By linear algebra, one gets

$$\det(g_{ij}) = \phi^{n+1}(\phi - s\phi')^{n-2}\left[(\phi - s\phi') + (b^2 - s^2)\phi''\right] det(a_{ij}).$$

Using the above formula, one can easily get the following

Lemma 3.1. ([20]) *The function $F = \alpha\phi(\beta/\alpha)$ is a Finsler metric for any $\alpha = \sqrt{a_{ij}y^iy^j}$ and any $\beta = b_iy^i$ with $\|\beta_x\|_\alpha < b_o$ if and only if $\phi = \phi(s)$ is a positive C^∞ function on $(-b_o, b_o)$ satisfying the following condition:*

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \qquad |s| \le b < b_o. \tag{4}$$

¿From (4), one can see that ϕ must satisfy

$$\phi(s) - s\phi'(s) > 0, \qquad |s| < b_o.$$

Some computations on the Hessian g_{ij} of (α, β)-metrics are done in [37].

Definition 3.2. A Finsler metric F on a manifold M is called an (α, β)-*metric* if it is expressed as $F = \alpha\phi(\beta/\alpha)$ with $\|\beta_x\|_\alpha < b_o$, where $\phi = \phi(s)$ is a positive C^∞ on $(-b_o, b_o)$ satisfying (4).

Let $\phi = 1 + s$. The (α, β)-metric defined by ϕ is given by

$$F = \alpha + \beta.$$

It is easy to verify that F is a Finsler metric if and only if $\|\beta_x\|_\alpha < 1$ for all $x \in M$. Such metric is called a *Randers metric*. General (α, β)-metrics were first studied by M. Matsumoto [28] in 1972 as a direct generalization of Randers metrics. They have many applications in physics and biology(ecology) ([3][9][36]). The study of (α, β)-metrics no doubt leads us to a better understanding on the geometric properties of Finsler metrics.

In order to study the geometric properties of (α, β)-metrics, one needs a formula for the spray coefficients of an (α, β)-metric. Let

$$r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i}), \qquad s_{ij} := \frac{1}{2}(b_{i|j} - b_{j|i}),$$

$$s^i{}_j := a^{ih}s_{hj}, \qquad s_j := b_is^i{}_j = b^ms_{mj}, \qquad e_{ij} := r_{ij} + b_is_j + b_js_i,$$

where "|" denotes the covariant derivative with respect to the Levi-Civita connection of α. We will denote $r_{00} := r_{ij}y^iy^j$, $s_0 := s_jy^j$, etc. Let G^i and $\bar{G}^i$ denote the spray coefficients of F and α, respectively, given by

$$G^i = \frac{g^{il}}{4}\Big\{[F^2]_{x^ky^l}y^k - [F^2]_{x^l}\Big\}, \qquad \bar{G}^i = \frac{a^{il}}{4}\Big\{[\alpha^2]_{x^ky^l}y^k - [\alpha^2]_{x^l}\Big\},$$

where $(g^{ij}) := (\frac{1}{2}[F^2]_{y^iy^j})$ and $(a^{ij}) := (a_{ij})^{-1}$. By a direct computation, one gets the following formula:

$$\begin{aligned} G^i &= \bar{G}^i + \alpha Q s^i{}_0 + \Theta\Big\{-2Q\alpha s_0 + r_{00}\Big\}\frac{y^i}{\alpha} \\ &\quad + \Psi\Big\{-2Q\alpha s_0 + r_{00}\Big\}b^i, \end{aligned} \tag{5}$$

where

$$\begin{aligned} Q: &= \frac{\phi'}{\phi - s\phi'}, \\ \Theta: &= \frac{\phi - s\phi'}{2\Big((\phi - s\phi') + (b^2 - s^2)\phi''\Big)}\cdot\frac{\phi'}{\phi} - s\Psi, \\ \Psi: &= \frac{\phi''}{2\Big((\phi - s\phi') + (b^2 - s^2)\phi''\Big)}, \end{aligned}$$

where $s := \beta/\alpha$ and $b := \|\beta_x\|_\alpha$. The formula (5) is given in [20] and [40]. A different version of (5) is given in [27].

The above formula (5) is very useful in computing curvatures of an (α, β)-metric $F = \alpha\phi(\beta/\alpha)$. However, it is still difficult to simplify a curvature equation expressed in terms of α and β, because the complexity of ϕ. A useful technique is to take a local coordinate system at x such that

$$\alpha = \sqrt{\sum_{i=1}^{n}(y^i)^2}, \qquad \beta = by^1. \tag{6}$$

Let $s = \beta/\alpha$. Then

$$y^1 = \frac{s}{\sqrt{b^2 - s^2}}\bar{\alpha},$$

where

$$\bar{\alpha} := \sqrt{\sum_{\alpha=2}^{n}(y^\alpha)^2}.$$

One obtains a coordinate transformation $(s, u^a) \to (y^i)$ given by

$$y^1 = \frac{s}{\sqrt{b^2 - s^2}}\bar{\alpha}, \qquad y^a = u^a. \tag{7}$$

Then

$$\alpha = \frac{b}{\sqrt{b^2 - s^2}}\bar{\alpha}, \qquad \beta = \frac{bs}{\sqrt{b^2 - s^2}}\bar{\alpha}.$$

If the curvature equation involves r_{ij}, s_{ij} or their covariant derivatives, one needs the following expressions:

$$r_1 = br_{11}, \quad r_\alpha = br_{1\alpha}, \quad s_1 = 0, \quad s_\alpha = bs_{1\alpha}.$$

$$r_{00} = \frac{s^2\bar{\alpha}^2}{b^2 - s^2}r_{11} + 2\frac{s\bar{\alpha}}{\sqrt{b^2 - s^2}}\bar{r}_{10} + \bar{r}_{00},$$

$$r_{10} = \frac{s\bar{\alpha}}{\sqrt{b^2 - s^2}}r_{11} + \bar{r}_{10}, \qquad s_{10} = \bar{s}_{10},$$

where $\bar{r}_{00} := r_{ab}u^a u^b$, $\bar{r}_{10} = r_{1a}u^a$ and $\bar{s}_{10} = s_{1a}u^a$. Then the curvature equation can be reduced to the following form

$$\Phi_1 + \Phi_2\bar{\alpha} = 0,$$

where $\Phi_1 = \Phi_1(s, u)$ and $\Phi_2 = \Phi_2(s, u)$ are polynomials in (u^a). Thus

$$\Phi_1 = 0, \qquad \Phi_2 = 0.$$

This technique is first used in [42].

Below are some important examples (cf. [43]).

(i) $\phi = 1 + s$. The metric defined by ϕ is a Randers metric given by

$$F = \alpha + \beta.$$

We have

$$Q = 1, \qquad \Theta = \frac{1}{2(1+s)}, \qquad \Psi = 0.$$

Thus the spray coefficients are given by

$$G^i = \bar{G}^i + \alpha s^i{}_0 + \frac{1}{2F}\Big\{ - 2\alpha s_0 + r_{00}\Big\}y^i. \tag{8}$$

(ii) $\phi = \phi(s)$ satisfies

$$\phi(s) - s\phi'(s) = (p + rs^2)\phi''(s). \tag{9}$$

In this case, both Θ and Ψ take the following simple forms

$$\begin{aligned} \Theta &= \frac{p + rs^2}{2(p + rs^2 + (b^2 - s^2))} \cdot \frac{\phi'}{\phi} - s\Psi \\ \Psi &= \frac{1}{2(p + rs^2 + (b^2 - s^2))}. \end{aligned}$$

Note that the only unpleasant term in Θ is the quotient ϕ'/ϕ.

For certain values of p and r, the solutions of (9) can be expressed in terms of elementary functions (cf. [43]).

(a) If $r = -1$ and $p = \pm 1$, then

$$\phi = \begin{cases} \sqrt{1 - s^2} + s \arctan\left(\frac{s}{\sqrt{1-s^2}}\right) + \varepsilon s, & if \text{ p=1} \\ \sqrt{1 + s^2} - s \ln(s + \sqrt{1 + s^2}) + \varepsilon s, & if \text{ p =-1.} \end{cases}$$

(b) If $r = 1$ and $p = \pm 1$, then

$$\phi = \begin{cases} \sqrt{1 + s^2} + \varepsilon s, & if \text{ p=1} \\ \sqrt{1 - s^2} + \varepsilon s, & if \text{ p=-1.} \end{cases}$$

(c) If $r = -1/2$, $p = \pm 1/2$, then

$$\phi = \begin{cases} 1 + s^2 + \varepsilon s, & if \text{p =1/2} \\ 1 - s^2 + \varepsilon s, & if \text{p =-1/2.} \end{cases}$$

(d) If $r = 1/2$ and $p = \pm 1/2$, then

$$\phi = \begin{cases} 1 + s \arctan(s) + \varepsilon s, & if \text{ p =1/2} \\ 1 + s \ln \sqrt{\frac{1-s}{1+s}} + \varepsilon s, & if \text{ p =-1/2.} \end{cases}$$

(e) If $r = -1/3$ and $p = \pm 1/3$, then

$$\phi = \begin{cases} \left(1 + \frac{1}{2}s^2\right)\sqrt{1 - s^2} + \frac{3}{2}s \arctan\left(\frac{s}{\sqrt{1-s^2}}\right) + \varepsilon s, & if \text{ p =1/3} \\ \left(1 - \frac{1}{2}s^2\right)\sqrt{1 + s^2} - \frac{3}{2}s \ln\left(s + \sqrt{1 + s^2}\right) + \varepsilon s, & if \text{ p=-1/3.} \end{cases}$$

(f) If $r = 1/3$ and $p = \pm 1/3$, then

$$\phi = \begin{cases} \sqrt{1+s^2} + \frac{s^2}{\sqrt{1+s^2}} + \varepsilon s, & if \text{ p =1/3} \\ \\ \sqrt{1-s^2} - \frac{s^2}{\sqrt{1-s^2}} + \varepsilon s, & if \text{ p =-1/3.} \end{cases}$$

(g) If $r = -1/4$ and $p = \pm 1/4$, then

$$\phi = \begin{cases} 1 + 2s^2 - \frac{1}{3}s^4 + \varepsilon s, & if \text{ p =1/4} \\ \\ 1 - 2s^2 - \frac{1}{3}s^4 + \varepsilon s, & if \text{ p =-1/4.} \end{cases}$$

(h) If $r = 1/4$ and $p = \pm 1/4$, then

$$\phi = \begin{cases} \frac{2+3s^2}{2(1+s^2)} + \frac{3}{2}s \arctan(s) + \varepsilon s, & if \text{ p=1/4} \\ \\ \frac{2-3s^2}{2(1-s^2)} + \frac{3}{2}s \ln\sqrt{\frac{1-s}{1+s}} + \varepsilon s, & if \text{ p =-1/4.} \end{cases}$$

One can easily write down a formula for the (α, β)-metric defined by any of the above functions ϕ. For example,

$$\begin{aligned}
F &= \alpha \pm \frac{\beta^2}{\alpha} + \varepsilon\beta, \\
F &= \alpha \pm 2\frac{\beta^2}{\alpha} - \frac{1}{3}\frac{\beta^4}{\alpha^3} + \varepsilon\beta, \\
F &= \alpha + \beta \arctan\left(\frac{\beta}{\alpha}\right) + \varepsilon\beta, \\
F &= \sqrt{\alpha^2+\beta^2} - \beta \ln\left[\frac{\sqrt{\alpha^2+\beta^2}+\beta}{\alpha}\right] + \varepsilon\beta, \\
F &= \alpha + \beta \ln\sqrt{\frac{\alpha-\beta}{\alpha+\beta}} + \varepsilon\beta, \\
F &= \left(1 + \frac{1}{2}\frac{\beta^2}{\alpha^2}\right)\sqrt{\alpha^2-\beta^2} + \frac{3}{2}\beta \arctan\left(\frac{\beta}{\sqrt{\alpha^2-\beta^2}}\right) + \varepsilon\beta, \\
F &= \left(1 - \frac{1}{2}\frac{\beta^2}{\alpha^2}\right)\sqrt{\alpha^2+\beta^2} - \frac{3}{2}\beta \ln\left[\frac{\sqrt{\alpha^2+\beta^2}+\beta}{\alpha}\right] + \varepsilon\beta, \\
F &= \frac{2\alpha^2+3\beta^2}{2(\alpha^2+\beta^2)}\alpha + \frac{3}{2}\beta \arctan\left(\frac{\beta}{\alpha}\right) + \varepsilon\beta, \\
F &= \frac{2\alpha^2-3\beta^2}{2(\alpha^2-\beta^2)}\alpha + \frac{3}{2}\beta \ln\sqrt{\frac{\alpha-\beta}{\alpha+\beta}} + \varepsilon\beta.
\end{aligned}$$

For any of the above functions ϕ, one can find α and β such that $F = \alpha\phi(\beta/\alpha)$ is of scalar flag curvature.

§4. Volume forms of (α, β)-metrics

To compute the S-curvature, one should first find a formula for the Busemann-Hausdorff volume forms dV_{BH} and the Holmes-Thompson dV_{HT}.

Proposition 4.1. ([19]) *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α, β)-metric on an n-dimensional manifold M.*

Let

$$f(b) := \begin{cases} \dfrac{\int_0^\pi \sin^{n-2}(t)dt}{\int_0^\pi \frac{\sin^{n-2}(t)}{\phi(b\cos(t))^n}dt} & \text{if } dV = dV_{BH}, \\ \dfrac{\int_0^\pi \sin^{n-2}(t)T(b\cos t)dt}{\int_0^\pi \sin^{n-2}(t)dt} & \text{if } dV = dV_{HT}. \end{cases}$$

Then the volume form dV is given by

$$dV = f(b)dV_\alpha,$$

where $dV_\alpha = \sqrt{\det(a_{ij})}dx$ denotes the Riemannian volume form of α.

Proof. In a coordinate system, the determinant of $g_{ij} := \frac{1}{2}[F^2]_{y^iy^j}$, is given by

$$\det(g_{ij}) = \phi^{n+1}(\phi - s\phi')^{n-2}[(\phi - s\phi') + (b^2 - s^2)\phi'']\det(a_{ij}).$$

First we take an orthonormal basis at a point x with respect to α so that

$$\alpha = \sqrt{\sum (y^i)^2}, \qquad \beta = by^1,$$

where $b = \|\beta_x\|_\alpha$. Then the volume form $dV_\alpha = \sigma_\alpha dx$ at x is given by

$$\sigma_\alpha = \sqrt{\det(a_{ij})} = 1.$$

In order to evaluate the integrals

$$\mathrm{Vol}\Big\{(y^i) \in R^n | F(x,y) < 1\Big\} = \int_{\{F(x,y)<1\}} dy = \int_{\{\alpha\phi(\beta/\alpha)<1\}} dy,$$

and

$$\int_{\{F(x,y)<1\}} \det(g_{ij})dy = \int_{\{\alpha\phi(\beta/\alpha)<1\}} \det(g_{ij})dy,$$

we take the coordinate transformation, $\psi : (s, u^a) \to (y^i)$ given by (7):

$$y^1 = \frac{s}{\sqrt{b^2 - s^2}}\bar{\alpha}, \qquad y^a = u^a,$$

where $\bar{\alpha} = \sqrt{\sum_{a=2}^{n}(y^a)^2}$. Then

$$F = \alpha\phi(\beta/\alpha) = \frac{b\phi(s)}{\sqrt{b^2 - s^2}}\bar{\alpha}.$$

and the Jacobian of the transformation ψ is given by $b^2(b^2 - s^2)^{-3/2}\bar{\alpha}$. Then

$$\begin{aligned}
&\mathrm{Vol}\Big\{(y^i) \in R^n | F(x, y) < 1\Big\} \\
= &\int_{\frac{b\phi(s)}{\sqrt{b^2-s^2}}\bar{\alpha}<1} \frac{b^2}{(b^2 - s^2)^{3/2}}\bar{\alpha} ds du \\
= &\int_{-b}^{b} \frac{b^2}{(b^2 - s^2)^{3/2}} \Big[\int_{\bar{\alpha}<\frac{\sqrt{b^2-s^2}}{b\phi(s)}} \bar{\alpha} du\Big] ds \\
= &\frac{1}{n}\mathrm{Vol}(\mathrm{S}^{n-2}) \int_{-b}^{b} \frac{b^2}{(b^2 - s^2)^{3/2}} \Big(\frac{\sqrt{b^2 - s^2}}{b\phi(s)}\Big)^n ds \\
= &\frac{1}{n}\mathrm{Vol}(\mathrm{S}^{n-2}) \int_{-b}^{b} \frac{(b^2 - s^2)^{(n-3)/2}}{b^{n-2}\phi(s)^n} ds \\
= &\frac{1}{n}\mathrm{Vol}(\mathrm{S}^{n-2}) \int_{0}^{\pi} \frac{\sin^{n-2}(t)}{\phi(b\cos(t))^n} dt.
\end{aligned}$$

Therefore

$$\sigma_{BH} = \frac{\int_0^\pi \sin^{n-2}(t) dt}{\int_0^\pi \frac{\sin^{n-2}(t)}{\phi(b\cos(t))^n} dt}\, \sigma_\alpha.$$

Let

$$T(s) := \phi(\phi - s\phi')^{n-2}[(\phi - s\phi') + (b^2 - s^2)\phi''] \tag{10}$$

Then

$$\det(g_{ij}) = \phi(s)^n T(s) \det(a_{ij}).$$

By a similar argument, we get

$$\begin{aligned}\sigma_{HT} &= \frac{1}{\omega_n}\int_{\{F(x,y)<1\}} \phi(s)^n T(s) dy^1 \cdots dy^n \\ &= \frac{1}{n\omega_n}\mathrm{Vol}(\mathrm{S}^{n-2})\int_{-b}^{b} \frac{b^2}{(b^2-s^2)^{3/2}}\Big(\frac{\sqrt{b^2-s^2}}{b}\Big)^n T(s) ds \\ &= \frac{\int_0^\pi \sin^{n-2}(t) T(b\cos t) dt}{\int_0^\pi \sin^{n-2}(t) dt}.\end{aligned}$$

Thus

$$\sigma_{HT} = \frac{\int_0^\pi \sin^{n-2}(t) T(b\cos t) dt}{\int_0^\pi \sin^{n-2}(t) dt}\, \sigma_\alpha.$$

This proves the proposition. Q.E.D.

It is surprised to see that for certain ϕ, $dV_{TH} = dV_\alpha$.

Corollary 4.2. *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α, β)-metric on an n-dimensional manifold M. Let $T = T(s)$ be defined in (10). Suppose that $T - 1$ is an odd function of s. Then $dV_{TH} = dV_\alpha$.*

Proof. Let $\varphi(s) = T(s) - 1$. By assumption, $\varphi(-s) = -\varphi(s)$. It is easy to see that

$$\int_0^\pi \sin^{n-2}(t)\varphi(b\cos(t)) dt = 0.$$

Thus

$$\int_0^\pi \sin^{n-2}(t) T(b\cos(t)) dt = \int_0^\pi \sin^{n-1}(t) dt.$$

This implies that $\sigma_{HT} = 1$ in the above special coordinate system at x. Then in a general coordinate system $\sigma_{HT} = \sigma_\alpha$. Q.E.D.

§5. Randers metrics of scalar flag curvature

Randers metrics are the simplest (α, β)-metrics. The spray coefficients of a Randers metric are given by (8). Then one can use (8) to compute the Ricci curvature and the Riemann curvature. On the other hand, it is one of important problems in Finsler geometry to study and characterize Finsler metrics of constant (or scalar) flag curvature. Thus it is natural to investigate Randers metrics first.

Bao-Robles [11] [12] first observe that for a Randers metric $F = \alpha + \beta$ on an n-dimensional manifold M, if the Ricci curvature is in the following form

$$\mathbf{Ric} = (n-1)\sigma F^2, \tag{11}$$

where $\sigma = \sigma(x)$ is a scalar function, then the 1-form $\beta = b_i(x)y^i$ satisfies the following PDE:

$$r_{00} + 2s_0\beta = 2c(\alpha^2 - \beta^2), \tag{12}$$

where c is a constant. The equation (12) is equivalent to a condition on the S-curvature,

$$\mathbf{S} = (n+1)cF, \tag{13}$$

where c is a constant (cf. [17]).

In [11], Bao-Robles obtain another PDE on β which together with (12) characterizes Randers metrics of constant flag curvature (see also [12]). Independently, Matsumoto-Shimada obtain the same result ([31]). However, it is very difficult to solve these PDEs for α and β to classify such metrics.

Using Zermelo's navigation idea ([39][40]), one can obtain Randers metrics of constant flag curvature. The crucial idea is to express a Randers metric $F = \alpha + \beta$ in terms of a Riemannian metric $h = \sqrt{h_{ij}(x)y^iy^j}$ and a vector field $W = W^i\frac{\partial}{\partial x^i}$ by

$$F = \frac{\sqrt{\lambda h^2 + W_0^2}}{\lambda} - \frac{W_0}{\lambda}, \quad W_0 := W_i y^i, \tag{14}$$

where $W_i := h_{ij}W^j$ and

$$\lambda := 1 - W_iW^i = 1 - h(x, W)^2.$$

Theorem 5.1. ([13]) *For a Randers metric F expressed in the form (14), it has constant flag curvature, $\mathbf{K} = k$ if and only if h has constant sectional curvature $\bar{\mathbf{K}} = k + c^2$ and W satisfies*

$$W_{0;0} = -2ch^2, \tag{15}$$

where c is a constant, $W_{0;0} := W_{i;j}y^iy^j$ and the covariant derivatives $DW = W_{i;j}dx^i \otimes dx^j$ are taken with respect to h.

When h has constant sectional curvature $\bar{\mathbf{K}} = \mu$, it is easy to solve (15) for W to obtain a complete list of local structure of Randers metrics of constant flag curvature.

Theorem 5.2. ([13]) *Let $F = \alpha + \beta$ be a Randers metric on a manifold M which is expressed in terms of a Riemannian metric h and a vector field W by (14). F has constant flag curvature if and only if at*

any point, there is a local coordinate system in which h and W are given by

$$h = \frac{\sqrt{|y|^2 + \mu(|x|^2|y|^2 - \langle x, y\rangle^2)}}{1 + \mu|x|^2}, \tag{16}$$

$$W = -2c\sqrt{1 + \mu|x|^2}\, x + xQ + b + \mu\langle b, x\rangle x, \tag{17}$$

where c and μ are constants with $c\mu = 0$, $Q = (q_j{}^i)$ is an anti-symmetric matrix and $a, b \in \mathrm{R}^n$ are constant vectors. In this case, the flag curvature is given by $\mathbf{K} = \mu - c^2$.

In fact, without the condition on the flag curvature, (15) is equivalent to (12) for any scalar function $c = c(x)$ ([50]). Therefore (15) is equivalent to (13) for any scalar function $c = c(x)$ by [17].

The expression (14) is a key to classify Randers metrics of constant flag curvature. It is a natural idea to use (14) to study Randers metrics of scalar flag curvature.

Let $F = \alpha + \beta$ be defined by (14). Let

$$\mathcal{R}_{ij} := \frac{1}{2}(W_{i;j} + W_{j;i}), \quad \mathcal{S}_{ij} := \frac{1}{2}(W_{i;j} - W_{j;i}),$$

$$\mathcal{R}_j := W^i\mathcal{R}_{ij}, \quad \mathcal{R} := W^j\mathcal{R}_j, \quad \mathcal{S}_j := W^i\mathcal{S}_{ij}.$$

Let G^i and $\tilde{G}^i$ denote the spray coefficients of F and h, respectively. We have the following

(18)
$$G^i = \tilde{G}^i - \frac{1}{2}F^2(\mathcal{S}^i + \mathcal{R}^i) - F\mathcal{S}^i{}_0 + \frac{1}{2}(y^i/F - W^i)(2\mathcal{R}_0 F - \mathcal{R}_{00} - \mathcal{R}F^2),$$

where $\mathcal{S}^i := h^{ij}\mathcal{S}_j$, $\mathcal{R}^i := h^{ij}\mathcal{R}_j$, $\mathcal{R}_0 := \mathcal{R}_i y^i$ and $\mathcal{R}_{00} := \mathcal{R}_{ij}y^i y^j$. Formula (18) is due to C. Robles [35].

By a direct computation, one can obtain from (18) that

$$\frac{\partial G^m}{\partial y^m} = \frac{\partial \tilde{G}^m}{\partial y^m} + \frac{n+1}{2F}\Big\{2F\mathcal{R}_0 - \mathcal{R}_{00} - F^2\mathcal{R}\Big\}. \tag{19}$$

Let $dV_F = \sigma_F dx^1 \cdots dx^n$ and $dV_h = \sigma_h dx^1 \cdots dx^n$ denote the volume form of F and h respectively. An important fact is that $dV_F = dV_h$, i.e., $\sigma_F = \sigma_h$. Since h is a Riemannian metric, we have

$$\frac{\partial \tilde{G}^m}{\partial y^m} = y^m \frac{\partial}{\partial x^m}\Big(\ln \sigma_h\Big). \tag{20}$$

Then it follows from (19) and (20) that

$$\mathbf{S} = \frac{\partial G^m}{\partial y^m} - y^m \frac{\partial}{\partial x^m}\Big(\ln \sigma_F\Big) \tag{21}$$

$$= \frac{n+1}{2F}\Big\{2F\mathcal{R}_0 - \mathcal{R}_{00} - F^2\mathcal{R}\Big\}. \tag{22}$$

Let

$$\xi^i := y^i - F(x,y)W^i. \tag{23}$$

Since $\|W_x\|_h < 1$, the vector $\xi := \xi^i \frac{\partial}{\partial x^i}|_x \in T_xM$ can be arbitrary. Moreover, it is easy to verify that

$$h_{ij}(x)\xi^i\xi^j = F(x,y)^2.$$

Then it follows from (22) that

$$\frac{\mathbf{S}}{F} = -\frac{n+1}{2}\frac{\mathcal{R}_{ij}\xi^i\xi^j}{h_{ij}\xi^i\xi^j}. \tag{24}$$

By (24), one gets the following

Lemma 5.3. ([50] [20]) *Let F be a Randers metric F defined by (14) and $c = c(x)$ be a scalar function on an n-dimensional manifold.* $\mathbf{S} = (n+1)cF$ *if and only if*

$$\mathcal{R}_{00} = -2ch^2. \tag{25}$$

Note that (25) is equivalent to (15).

In the following, we are going to discuss Randers metrics of scalar flag curvature and isotropic S-curvature.

First we assume that a Randers metric F expressed in (14) has isotropic S-curvature, $\mathbf{S} = (n+1)cF$. By Lemma 5.3, W satisfies (25). Then the spray coefficients G^i in (18) are reduced to the following expression:

$$G^i = \tilde{G}^i - F\mathcal{S}^i{}_0 - \frac{1}{2}F^2\mathcal{S}^i + cFy^i. \tag{26}$$

By the simplified expression (26), one can express the Riemann curvature in terms of h and W. Rewrite (26) as follows

$$G^i = \tilde{G}^i + Q^i,$$

where

$$Q^i := -F\mathcal{S}^i{}_0 - \frac{1}{2}F^2\mathcal{S}^i + cFy^i.$$

The Riemann curvature $R^i{}_k = R_j{}^i{}_{kl}(x,y)y^jy^l$ of F and the Riemann curvature $\bar{R}^i{}_k = \bar{R}_j{}^i{}_{kl}(x)y^jy^l$ of h are related by

$$R^i{}_k = \bar{R}^i{}_k + 2Q^i{}_{;k} - [Q^i{}_{;m}]_{y^k}y^m + 2Q^m[Q^i]_{y^my^k} - [Q^i]_{y^m}[Q^m]_{y^k}. \tag{27}$$

where ";" denotes the horizontal covariant differentiation with respect to h (cf.[41]). By a direct and lengthy argument, one can get

$$\begin{aligned} R^i{}_k &= \bar{R}_p{}^i{}_{kq}(y^p - FW^p)(y^q - FW^q) \\ &\quad -F_{y^k}\bar{R}_p{}^i{}_{mq}(y^p - FW^p)(y^q - FW^q)W^m \\ &\quad +\Big(\frac{3c_{x^m}y^m}{F} - c^2 - 2c_{x^m}W^m\Big)\Big\{F^2\delta^i_k - FF_{y^k}y^i\Big\}. \end{aligned} \tag{28}$$

Let

$$\tilde{h} := \sqrt{h_{ij}(x)\xi^i\xi^j}, \qquad \tilde{R}^i{}_k := \bar{R}_p{}^i{}_{kq}\xi^p\xi^q.$$

It follows from (28) that for any scalar function $\mu = \mu(x)$ on M,

$$\begin{aligned} &R^i{}_k - \Big(\frac{3c_{x^m}y^m}{F} + \mu - c^2 - 2c_{x^m}W^m\Big)\Big\{F^2\delta^i_k - FF_{y^k}y^i\Big\} \\ &= \tilde{R}^i{}_k - \mu\Big(\tilde{h}^2\delta^i_k - \xi_k\xi^i\Big) - \frac{\xi_k}{\tilde{h} + \tilde{W}_0}\Big\{\tilde{R}^i{}_p - \mu\Big(\tilde{h}^2\delta^i_p - \xi_p\xi^i\Big)\Big\}W^p, \end{aligned} \tag{29}$$

where $\xi_i := h_{ij}\xi^j$.

From (29), one can easily prove the following

Theorem 5.4. ([18]) *Let F be a Randers metric on n-dimensional manifold M defined by (14). Suppose that $\mathbf{S} = (n+1)cF$ where $c = c(x)$ is a scalar function. Then F is of scalar flag curvature if and only if h is of sectional curvature $\bar{\mathbf{K}} = \mu$, where $\mu = \mu(x)$ is a scalar function (=constant if $n \geq 3$). In this case, the flag curvature of F is given by*

$$\mathbf{K} = \frac{3c_{x^m}y^m}{F} + \sigma, \tag{30}$$

where $\sigma := \mu - c^2 - 2c_{x^m}W^m$.

Proof. Assume that F is of scalar curvature, then by Theorem 2.3 above, the flag curvature of F is given by

$$\mathbf{K} = 3c_{x^m}y^m/F + \sigma,$$

where $\sigma = \sigma(x)$ is a scalar function on M. This is equivalent to the following equation:

$$R^i{}_k = \Big(\frac{3c_{x^m}y^m}{F} + \sigma\Big)\Big\{F^2\delta^i_k - FF_{y^k}y^i\Big\}. \tag{31}$$

Plugging (31) into (29) yields

$$\tilde{R}^i{}_k - \mu\Big(\tilde{h}^2\delta^i_k - \xi_k\xi^i\Big) - \frac{1}{\tilde{h} + \tilde{W}_0}\xi_k\Big\{\tilde{R}^i{}_p - \mu\Big(\tilde{h}^2\delta^i_p - \xi_p\xi^i\Big)\Big\}W^p = 0,$$

where $\mu := \sigma + c^2 + 2c_{x^m}W^m$. Immediately, one obtains

$$\tilde{R}^i{}_k = \mu\Big(\tilde{h}^2\delta^i_k - \xi_k\xi^i\Big). \tag{32}$$

Thus h has sectional curvature $\bar{\mathbf{K}} = \mu(x)$. By the Schur lemma, $\mu =$ *constant* in dimension $n \geq 3$.

Conversely, if h has sectional curvature $\bar{\mathbf{K}} = \mu(x)$, then (32) holds. By (29) again, we get (31) with $\sigma = \mu - c^2 - 2c_{x^m}W^m$. Thus F is of scalar curvature and its flag curvature is given by (30). Q.E.D.

If a Riemannian metric h has constant curvature $\bar{\mathbf{K}} = \mu$, then one can easily solve (25) for W and obtain the list of local structures of Randers metrics of scalar flag curvature and isotropic S-curvature.

Theorem 5.5. ([18]) *Let $F = \alpha + \beta$ be a Randers metric on a manifold M of dimension $n \geq 3$, which is expressed in terms of a Riemannian metric h and a vector field W by (14). Suppose that F is of isotropic S-curvature $\mathbf{S} = (n+1)cF$. Then it is of scalar flag curvature, $\mathbf{K} = K(x,y)$, if and only if at any point, there is a local coordinate system in which h, c and W are given by*

$$h = \frac{\sqrt{|y|^2 + \mu(|x|^2|y|^2 - \langle x,y\rangle^2)}}{1+\mu|x|^2}, \tag{33}$$

$$c = \frac{\delta + \langle a,x\rangle}{\sqrt{1+\mu|x|^2}} \tag{34}$$

(35)

$$W = -2\Big\{\Big(\delta\sqrt{1+\mu|x|^2} + \langle a,x\rangle\Big)x - \frac{|x|^2 a}{\sqrt{1+\mu|x|^2}+1}\Big\} + xQ + b + \mu\langle b,x\rangle x,$$

where δ, μ are constants, $Q = (q_j{}^i)$ is an anti-symmetric matrix and $a, b \in \mathrm{R}^n$ are constant vectors. In this case, the flag curvature is given by (30).

Since locally projectively flat Randers metrics are always of scalar flag curvature, Theorem 5.5 generalizes the main result in [16]. Since every Randers metric of constant flag curvature must have constant S-curvature, the class of Randers metrics with isotropic S-curvature and scalar flag curvature contains all Randers metrics of constant flag curvature.

Let us take a look at a special example. In (33)-(35), let $\mu = 0, \delta = 0, Q = 0$ and $b = 0$. We get

$$h = |y|, \quad c = \langle a, x\rangle, \quad W = -2\langle a, x\rangle x + |x|^2 a.$$

The Randers metric $F = \alpha + \beta$ is given by

$$\begin{aligned} F &= \frac{\sqrt{(1-|a|^2|x|^4)|y|^2 + (|x|^2\langle a, y\rangle - 2\langle a, x\rangle\langle x, y\rangle)^2}}{1-|a|^2|x|^4} \\ &\quad - \frac{|x|^2\langle a, y\rangle - 2\langle a, x\rangle\langle x, y\rangle}{1-|a|^2|x|^4}. \end{aligned}$$

The S-curvature and the flag curvature are given by

$$\mathbf{S} = (n+1)\langle a, x\rangle\, F, \qquad \mathbf{K} = \frac{3\langle a, y\rangle}{F} + 3\langle a, x\rangle^2 - 2|a|^2|x|^2.$$

Clearly F is not locally projectively flat because β is not closed. This example is constructed in [40].

According to Theorem 2.3, for a Finsler metric F of scalar flag curvature on an n-dimensional manifold M, if the S-curvature is isotropic, $\mathbf{S} = (n+1)cF$, where $c = c(x)$ is a scalar function on M, then the flag curvature must take the form

$$\mathbf{K} = \frac{3c_m y^m}{F} + \sigma. \tag{36}$$

A natural question arises: does (36) imply that the S-curvature is isotropic? The answer is affirmative for Randers metrics.

Let $F = \alpha + \beta$ and let

$$t_{ij} := s_i{}^m s_{mj}, \qquad q_{ij} := r_i{}^m s_{mj}, \qquad t_j := b^m t_{mj},$$

where $b^i := a^{ij} b_j$. By a direct and lengthy computation, we get the following formula for the Ricci curvature:

$$\begin{aligned} \mathbf{Ric} &= \overline{\mathbf{Ric}} + \Big\{2\alpha s^m_{0|m} - 2t_{00} - \alpha^2 t^m_{\ m}\Big\} \\ &\quad + (n-1)\Big\{\frac{3(r_{00} - 2s_0\alpha)^2}{4(\alpha+\beta)^2} + \frac{4\alpha[q_{00} - t_0\alpha] - [r_{00|0} - 2s_{0|0}\alpha]}{2(\alpha+\beta)}\Big\}. \end{aligned} \tag{37}$$

where $\overline{\mathbf{Ric}}$ denotes the Ricci curvature of α. The formula (37) is due to Bao-Robles ([12], p 220).

Lemma 5.6. ([46]) *Let $F = \alpha+\beta$ be a Randers metric on a manifold M. Then the Ricci curvature is in the form*

$$\mathbf{Ric} = (n-1)\Big\{\frac{3c_{x^m}y^m}{F} + \sigma\Big\}F^2, \tag{38}$$

where $c = c(x)$ and $\sigma = \sigma(x)$ are scalar functions, if and only if

$$\begin{aligned}\overline{\mathbf{Ric}} &= (n-1)\Big\{(\sigma - 3\tilde{c}^2)\alpha^2 + (\sigma+\tilde{c}^2)\beta^2 + (3c_0 - \tilde{c}_0)\beta - s_{0|0} - s_0^2\Big\} \\ &\quad +2t_{00} + \alpha^2 t^m{}_m \end{aligned} \tag{39}$$

$$s^m{}_{0|m} = \frac{n-1}{2}\Big\{3c_0 + \tilde{c}_0 + 2(\sigma+\tilde{c}^2)\beta + 4\tilde{c}s_0 + 2t_0\Big\} \tag{40}$$

$$r_{00} + 2s_0\beta = 2\tilde{c}(\alpha^2 - \beta^2), \tag{41}$$

where $\tilde{c} = \tilde{c}(x)$ is a scalar function on M.

Note that (41) is equivalent to that $\mathbf{S} = (n+1)\tilde{c}F$. Then one can easily prove the following

Theorem 5.7. ([46]) *Let $F = \alpha + \beta$ be a Randers metric on a manifold M and (h, W) be its navigation representation (14). Then the flag curvature $\mathbf{K}$ of F is in the form (36) if and only if the sectional curvature $\tilde{\mathbf{K}}$ of h and the vector field W satisfy*

$$\tilde{\mathbf{K}} = \mu \tag{42}$$

$$W_{0;0} = -2\tilde{c}h^2. \tag{43}$$

where $\mu = \mu(x)$ and $\tilde{c} = \tilde{c}(x)$ are scalar functions on M. In either case, the scalar functions are related by

$$c - \tilde{c} = constant, \qquad \sigma = \mu - \tilde{c}^2 - 2\tilde{c}_{x^m}W^m.$$

Q.E.D.

By Theorem 5.7 and Theorem 5.5, we can classify n-dimensional Randers metrics ($n \geq 3$) with flag curvature in the form (36).

For the Ricci curvature, we have the following

Theorem 5.8. ([46]) *Let $F = \alpha + \beta$ be a Randers metric on a manifold M and (h, W) be its navigation representation. Then the Ricci curvature of F is in the form*

$$\mathbf{Ric} = (n-1)\Big\{\frac{3c_{y^m}y^m}{F} + \sigma\Big\}F^2, \tag{44}$$

where $c = c(x)$ and $\sigma = \sigma(x)$ are scalar functions, if and only if the Ricci curvature $\widetilde{\mathbf{Ric}}$ of h and the vector field W satisfy

$$\widetilde{\mathbf{Ric}} = (n-1)\mu h^2, \tag{45}$$

$$W_{0;0} = -2\tilde{c}h^2. \tag{46}$$

where $\mu = \mu(x)$ and $\tilde{c} = \tilde{c}(x)$ are scalar functions on M. In this case, the scalar functions are related by

$$c - \tilde{c} = constant, \qquad \sigma = \mu - \tilde{c}^2 - 2\tilde{c}_{x^m}W^m.$$

§6. (α, β)-metrics of Landsberg type

It is a long existing open problem in Finsler geometry whether or not every Landsberg metric is of Berwald type. Since (α, β)-metrics are "computable" metrics, it is natural to investigate this problem on (α, β)-metrics.

Let $F = \alpha\phi(\beta/\alpha)$ be an (α, β)-metric. If β is parallel with respect to α ($r_{ij} = 0$ and $s_{ij} = 0$), then by (5), $G^i = \bar{G}^i$ are quadratic in y. Thus F is a Berwald metric. The converse is true too ([23][29]). In fact, one can show that every Landsberg (α, β)-metric must satisfies that $s_{ij} = 0$ and $r_{ij} = 0$. Thus it is a Berwald metric.

By a simple computation, one gets

$$FF_{y^i} = g_{ij}y^j = (\phi^2 - s\phi\phi')y_i + \phi\phi'\alpha b_i. \tag{47}$$

By (2), (5) and (47), one gets the following formula for L_{jkl}.

$$L_{jkl} = -\frac{\rho}{6\alpha^5}\Big\{h_jh_kC_l + h_jh_lC_k + h_kh_lC_j + 3E_jh_{kl} + 3E_kh_{jl} + 3E_lh_{kl}\Big\}, \tag{48}$$

where

$$
\begin{aligned}
h_j: &= \alpha b_j - s y_j \\
h_{jk}: &= \alpha^2 a_{jk} - y_j y_k, \\
C_j: &= (A_1 r_{00} + A_2 \alpha s_0) h_j + 3\Lambda J_j \\
E_j: &= (B_1 r_{00} + B_2 \alpha s_0) h_j + 3\mu J_j \\
J_j: &= \alpha^2 (s_{j0} + \Gamma r_{j0} + \Pi \alpha s_j) - (\Gamma r_{00} + \Pi \alpha s_0) y_j \\
A_1: &= \frac{1}{2\Delta^2}\Big\{ -2\Delta Q''' + 3(Q - sQ')Q'' + 3(b^2 - s^2)(Q'')^2 \Big\}, \\
A_2: &= \frac{1}{\Delta^2}\Big\{ 2\Delta Q Q''' + 3\Delta Q' Q'' - 3QQ''\Big(Q - sQ' + (b^2 - s^2)Q''\Big) \Big\}, \\
B_1: &= \frac{1}{2\Delta^2}\Big\{ (Q - sQ')^2 + \Big(2(s + b^2 Q) - (b^2 - s^2)(Q - sQ')\Big)Q'' \Big\}, \\
B_2: &= \frac{1}{\Delta^2}\Big\{ -\Big(s\Delta + (s + b^2 Q)\Big)QQ'' \\
&\qquad\qquad + (Q - sQ')\Big(\Delta Q' - (Q - sQ')Q\Big) \Big\}, \\
\Lambda: &= -Q'', \\
\mu: &= -\frac{1}{3}(Q - sQ'), \\
\Gamma: &= \frac{1}{\Delta} \\
\Pi: &= -\frac{Q}{\Delta}, \\
Q: &= \frac{\phi'}{\phi - s\phi'} \\
\rho: &= \phi(\phi - s\phi') \\
\Delta: &= 1 + sQ + (b^2 - s^2)Q'.
\end{aligned}
$$

Let

$$
\begin{aligned}
J: &= b^j J_j = \alpha\Big\{\alpha(s_0 + \Gamma r_0) - (\Gamma r_{00} + \Pi \alpha s_0)s\Big\}. \\
C: &= b^j C_j = (A_1 r_{00} + A_2 \alpha s_0)\alpha(b^2 - s^2) + 3\Lambda J, \\
E: &= b^j E_j = (B_1 r_{00} + B_2 \alpha s_0)\alpha(b^2 - s^2) + 3\mu J.
\end{aligned}
$$

It follows from the definitions of C_j and E_j that

$$
\alpha(b^2 - s^2)C_j - Ch_j = 3\Lambda\Big\{\alpha(b^2 - s^2)J_j - Jh_j\Big\}, \tag{49}
$$

$$
\alpha(b^2 - s^2)E_j - Eh_j = 3\mu\Big\{\alpha(b^2 - s^2)J_j - Jh_j\Big\}. \tag{50}
$$

One can easily prove the following

Lemma 6.1. ([42]) *Assume that the dimension* $n \geq 3$. *Then the following two conditions are equivalent.*

(i) $L_{jkl} = 0$,
(ii) $C_j = 0$ *and* $E_j = 0$.

By the above lemma, one can show the following

Theorem 6.2. ([42]) *Let* $\phi = \phi(s)$ *be a* C^∞ *positive function on an interval* $I = (-b_o, b_o)$. *Suppose that* ϕ *satisfies (4), but* $\phi \neq k_2\sqrt{1 + k_1 s^2}$ *on* I *for any constants* k_1 *and* $k_2 > 0$. *For an* (α, β)*-metric* $F = \alpha\phi(\beta/\alpha)$ *with* $0 < b(x) := \|\beta_x\|_\alpha < b_o$ *on a manifold of dimension* $n \geq 3$, F *is a Landsberg metric if and only if* β *is parallel with respect to* α *(i.e.* F *is a Berwald metric).*

One can use (ii) in Lemma 6.1 to prove Theorem 6.2. The problem is how to deal with the terms involving $\phi(\beta/\alpha)$ in simplifying the equation $E_j = 0$. To overcome this difficulty, we change the y-coordinates (y^i) at a point to "polar" coordinates (s, y^α), where $i = 1, \cdots, n$ and $\alpha = 2, \cdots, n$. Fix an arbitrary point $x \in M$. Take an orthonormal basis $\{e_i\}$ at x such that

$$\alpha = \sqrt{\sum_{i=1}^{n}(y^i)^2}, \qquad \beta = by^1.$$

Here $b = \|\beta_x\|_\alpha$. Take a coordinate transformation $(s, u^a) \to (y^i)$ given by (7). By a direct computation $E_1 = 0$ is equivalent to the following two equations:

$$\Big\{(b^2 - s^2)B_1 - 3s\mu\Gamma\Big\}\bar{r}_{00} + s\Big\{sB_1 + 3\mu\Gamma\Big\}r_{11}\bar{\alpha}^2 = 0, \tag{51}$$

$$\begin{aligned}&(b^2 - s^2)\Big\{\Big(2sB_1 + 3\mu\Gamma\Big)r_{10} + \Big(b^2B_2 + 3\mu\Big)s_{10}\Big\}\\ &-3s\mu\Big\{s\Gamma r_{10} + (b^2\Pi - s)s_{10}\Big\} = 0.\end{aligned} \tag{52}$$

$E_a = 0$ is equivalent to the following two equations:

$$\begin{aligned}&s\Big\{\Big((b^2 - s^2)B_1 - 3s\mu\Gamma\Big)\bar{r}_{00} + s\Big(sB_1 + 3\mu\Gamma\Big)r_{11}\bar{\alpha}^2\Big\}u^a\\ &-3b^2\mu\Big\{\Gamma\Big(\bar{r}_{0a}\bar{\alpha}^2 - \bar{r}_{00}u^a\Big) - \bar{s}_{0a}\bar{\alpha}^2\Big\} = 0.\end{aligned} \tag{53}$$

$$
\begin{aligned}
&s\Big\{(b^2-s^2)\Big[\Big(2sB_1+3\mu\Gamma\Big)r_{10}+\Big(b^2B_2+3\mu\Big)s_{10}\Big]\\
&-3\mu s\Big[\Gamma s r_{10}+(b^2\Pi-s)s_{10}\Big]\Big\}u^a\\
&=3\mu b^2\Big\{\Big[\Gamma s r_{1a}+(b^2\Pi-s)s_{1a}\Big]\bar{\alpha}^2-\Big[\Gamma s r_{10}+(b^2\Pi-s)s_{10}\Big]u^a\Big\}. \qquad (54)
\end{aligned}
$$

Equation (53) implies (51) and (54) implies (52). But we need (51) and (52) to simplify (53) and (54). Similarly, we get four equations for $C_j=0$ by replacing B_i and μ by A_i and Λ respectively.

By the above equations, one can show the following

Proposition 6.3. *($n\geq 3$) Assume that $\phi\neq k_1\sqrt{1+k_2s^2}$ for any constants k_1 and k_2. Assume that β is parallel with respect to α. Then $F=\alpha\phi(\beta/\alpha)$ is a Landsberg metric if and only if β satisfies*

$$s_{ij}=\frac{1}{b^2}\Big(b_is_j-b_js_i\Big), \qquad (55)$$

$$r_{ij}=k(b^2a_{ij}-b_ib_j)+cb_ib_j+d(b_is_j+b_js_i). \qquad (56)$$

where $k=k(x)$, $c=c(x)$ and $d=d(x)$ are scalar functions, and ϕ satisfies

$$\Big\{(b^2-s^2)A_1-3s\Lambda\Gamma\Big\}k+\Big\{s^2A_1+3\Lambda\Gamma s\Big\}c=0, \qquad (57)$$

$$\Big\{(b^2-s^2)B_1-3\mu s\Gamma\Big\}k+\Big\{s^2B_1+3\mu\Gamma s\Big\}c=0. \qquad (58)$$

If $s_0\neq 0$, then $\phi=\phi(s)$ satisfies three additional ODEs:

$$db^2s\Gamma+b^2\Pi-s=0, \qquad (59)$$

$$\Big\{2sA_1+3\Lambda\Gamma\Big\}db^2+\Big\{b^2A_2+3\Lambda\Big\}=0. \qquad (60)$$

$$\Big\{2sB_1+3\mu\Gamma\Big\}db^2+\Big\{b^2B_2+3\mu\Big\}=0, \qquad (61)$$

One can actually solve the above ODE's for ϕ. The key idea is as follows. Since β is not parallel, (57) and (58) hold for $(k,c)\neq(0,0)$. Then

$$\mu A_1-\Lambda B_1=\frac{(Q-sQ')Q'''+3s(Q'')^2}{3(1+sQ+(b^2-s^2)Q')}=0. \qquad (62)$$

The numerator of the equation (62) is independent of the norm of β! More surprisingly, it is solvable. If we impose the regularity on Q at $s = 0$, we obtain the following general solution

$$Q = c_1\sqrt{1 + c_2 s^2} + c_3 s,$$

where c_1, c_2 and c_3 are constants with $c_1 \neq 0$. Then plugging it into the equations in Proposition 6.3, one can completely determine the coefficients c_i, that is, $c_2 = -1/b^2$. Thus $b = constant$. Since $\phi = \phi(s)$ is C^∞ on $(-b_o, b_o)$, we conclude that $b = b_o$. Then

$$\phi(s) = c_4 \exp\Big[\int_0^s \frac{c_1\sqrt{1 - (t/b_o)^2} + c_3 t}{1 + c_3 t^2 + c_1 t\sqrt{1 - (t/b_o)^2}} dt\Big]. \tag{63}$$

However, the function $F = \alpha\phi(\beta/\alpha)$ is always singular in the directions $y \in T_x M$ with $|\beta(x, y)| = b_o\alpha(x, y)$. Therefore a regular (α, β)-metric $F = \alpha\phi(\beta/\alpha)$ is a Landsberg metric if and only if β is parallel with respect to α. In this case, it is a Berwald metric. This proves Theorem 6.2.

The above argument also gives us singular Landsberg (α, β)-metrics. Let $\phi = \phi(s)$ be a positive C^∞ on $(-b_o, b_o)$ satisfying (4). For any Riemannian metric $\alpha = \sqrt{a_{ij}y^i y^j}$ and any 1-form $\beta = b_i y^i$ on an n-dimensional manifold M, the function $F := \alpha\phi(\beta/\alpha)$ has the following properties: (i) $F(x, y) > 0$ and (ii) $g_{ij} = \frac{1}{2}[F^2]_{y^i y^j}(x, y) > 0$ for any $y \in T_x M$ with $|\beta(x, y)| < b_o\alpha(x, y)$. But F might be singular or even not defined for $y \in T_x M$ with $|\beta(x, y)| \geq b_o\alpha(x, y)$. Such function is called an *almost regular* (α, β)-metric.

Theorem 6.4. ([42]) *$(n \geq 3)$ Let $F = \alpha\phi(\beta/\alpha)$ be an almost regular (α, β)-metric where $\phi = \phi(s)$ is a function on $(-b_o, b_o)$ such that $\phi \neq k_2\sqrt{1 + k_1 s^2}$ for any constants k_1 and $k_2 > 0$. Then F is a Landsberg metric if and only if β is parallel with respect to α (hence F is a Berwald metric) or*

$$b(x) \equiv b_o. \tag{64}$$

$$\phi(s) = c_4 \exp\Big[\int_0^s \frac{c_1\sqrt{1 - (t/b_o)^2} + c_3 t}{1 + c_3 t^2 + c_1 t\sqrt{1 - (t/b_o)^2}} dt\Big]. \tag{65}$$

$$s_{ij} = 0, \tag{66}$$

$$r_{ij} = k(b^2 a_{ij} - b_i b_j), \tag{67}$$

where c_1, c_3, c_4 are constants with $c_1 \neq 0$ and $c_4 > 0$ such that

$$1 + c_3 s^2 + c_1 s\sqrt{1-(s/b_o)^2} > 0, \qquad (|s| < b_o),$$

and $k = k(x)$ is a scalar function. Moreover, F is not a Berwald metric if and only if $k \neq 0$.

It follows from Theorem 6.4 that a regular (α, β)-metric is a Landsberg metric if and only if it is a Berwald metric. The third author claimed this in the first version of [42] in May 2004. But there is a computational mistake in an expression for E_j. After two years, G.S. Asanov discovered that his metrics arising from Physics actually are Landsberg metric but not Berwaldian [1][2]. Then the third author corrected the expression for E_j and proved the claim for regular (α, β)-metrics. Meanwhile he characterizes almost regular Landsberg (α, β)-metrics and obtains a two-parameter family of Landsberg (α, β)-metrics including Asanov's examples.

Below is a simple example.

Example 6.5. *At a point $\mathbf{x} = (x, y, z) \in R^3$ and in the direction $\mathbf{y} = (u, v, w) \in T_{\mathbf{x}}R^3$, define $\alpha = \alpha(\mathbf{x}, \mathbf{y})$ and $\beta = \beta(\mathbf{x}, \mathbf{y})$ by*

$$\begin{aligned} \alpha: &= \sqrt{u^2 + e^{2kx}(v^2 + w^2)} \\ \beta: &= u \end{aligned}$$

where $k \neq 0$ is an arbitrary constant. Then α and β satisfy (66) and (67) with $b = \|\beta_{\mathbf{x}}\|_\alpha \equiv 1$. Let

$$F = \alpha \exp\Big[\int_0^{\beta/\alpha} \frac{c_1\sqrt{1-t^2} + c_3 t}{1 + c_3 t^2 + c_1 t\sqrt{1-t^2}} dt\Big]$$

Then F is a Landsberg metric but not a Berwald metric. This metric is singular in two directions $\mathbf{y} = (\pm 1, 0, 0) \in T_{\mathbf{x}}R^3$ at any point $\mathbf{x}$.

The reader is referred to [22] for the early discussion on Landsberg (α, β)-metrics in dimension two. Recently, Li and the third author have given a complete characterization of almost regular Landsberg (α, β)-metrics and shown that they are all Berwaldian. Further, they have characterized weakly Landsberg (α, β)-metrics and shown that there are almost regular (α, β)-metrics which are weakly Landsbergian but not Berwaldian ([25]).

§7. (α, β)-metrics of Douglas type

Douglas metrics can be viewed as generalized Berwald metrics. What makes them special is that being a Douglas metric is a projective property, that is, if a Finsler metric F has the same geodesics as a Douglas metric, then F is a Douglas metric.

In 1997, the first author and M. Matsumoto proved the following

Theorem 7.1. ([6]) *A Randers metric $F = \alpha + \beta$ is a Douglas metric if and only if β is closed.*

Proof. From (8), we have

$$\begin{aligned} G^i y^j - G^j y^i &= (\bar{G}^i y^j - \bar{G}^j y^i) + (Q^i y^j - Q^j y^i) \\ &= (\bar{G}^i y^j - \bar{G}^j y^i) + \alpha(s^i{}_0 y^j - s^j{}_0 y^i), \end{aligned}$$

where $\bar{G}^i$ denote the spray coefficients of α and $Q^i = \alpha s^i{}_0$. Thus $\bar{G}^i y^j - \bar{G}^j y^i$ are homogeneous polynomials in (y^i) of degree three.

Assume that F is a Douglas metric. Then the terms on the left side are homogeneous polynomials in (y^i) of degree three. Note that α is irrational in (y^i), one concludes that the coefficients of α must be zero, namely,

$$(s^i{}_0 y^j - s^j{}_0 y^i) = 0.$$

Then it follows that $s^i{}_0 = 0$. That is, β is closed.

Conversely, if β is closed, then $Q^i = 0$ in (8). Therefore, from (8),

$$G^i y^j - G^j y^i = \bar{G}^i y^j - \bar{G}^j y^i$$

are homogeneous polynomials in (y^i) of degree three. Hence, F is a Douglas metric. Q.E.D.

For a general (α, β)-metric $F = \alpha\phi(\beta/\alpha)$, it follows from (5) that

$$\begin{aligned} G^i y^j - G^j y^i &= \bar{G}^i y^j - \bar{G}^j y^i + \alpha Q(s^i{}_0 y^j - s^j{}_0 y^i) \\ &\quad + \Psi\Big\{ - 2Qs_0 + r_{00}\Big\}(b^i y^j - b^j y^i). \end{aligned} \tag{68}$$

In 1989, M. Matsumoto introduced an (α, β)-metric $F = \alpha^2/(\alpha - \beta)$ as a realization of P. Finsler's idea " a slope measure of a mountain with respect to a time measure". This metric is called Matsumoto metric. By (68), one can show the following

Proposition 7.2. ([7][27]) *A Matsumoto metric $F = \alpha^2/(\alpha - \beta)$ is Douglas metric if and only if β is parallel with respect to α. In this case, F is a Berwald metric.*

Proposition 7.3. ([51]) *The exponential metric* $F = \alpha \exp(\beta/\alpha) + \varepsilon\beta$ *is a Douglas metric if and only if* β *is parallel with respect to* α. *In this case,* F *is a Berwald metric.*

The above two propositions show that the Matsumoto metric and the exponential metric are very "projectively hard". These metrics might have no good flag curvature properties.

It is natural to consider more general (α, β)-metrics $F = \alpha\phi(s)$, where $\phi = \phi(s)$ satisfies (9), i.e.,

$$\phi - s\phi' = (p + rs^2)\phi'', \tag{69}$$

It is easy to verify that if β satisfies

$$b_{i|j} = 2\tau\Big\{(p + b^2)a_{ij} + (r - 1)b_i b_j\Big\}, \tag{70}$$

where $\tau = \tau(x)$ is a scalar function, then F is a Douglas metric. Several people have shown that the converse is true too for some specific (α, β)-metrics. For example, $F = \alpha \pm \frac{\beta^2}{\alpha} + \varepsilon\beta$ is a Douglas metric if and only if

$$b_{i|j} = \tau\Big\{(\pm 1 + 2b^2)a_{ij} - 3b_i b_j\Big\}, \tag{71}$$

where $\tau = \tau(x)$ is a scalar function. This generalizes a theorem in ([7][27]). The Finsler metric $F = \alpha + \beta^2/\alpha$ was first proposed in [27]. Another example is that $F = \alpha \pm 2\frac{\beta^2}{\alpha} - \frac{1}{3}\frac{\beta^4}{\alpha^3} + \varepsilon\beta$ is a Douglas metric if and only if

$$b_{i|j} = \frac{\tau}{2}\Big\{(\pm 1 + 4b^2)a_{ij} - 5b_i b_j\Big\}, \tag{72}$$

where $\tau = \tau(x)$ is a scalar function. The above two examples show that $F = \alpha \pm \frac{\beta^2}{\alpha} + \varepsilon\beta$ and $F = \alpha \pm 2\frac{\beta^2}{\alpha} - \frac{1}{3}\frac{\beta^4}{\alpha^3} + \varepsilon\beta$ are "projectively soft". Most recently, B. Li and the third author have just completely characterized all (α, β)-metrics of Douglas type using a result from [44]. See Theorem 8.4 below.

Theorem 7.4. ([26]) *Let* $F = \alpha\phi(s)$, $s = \beta/\alpha$, *be an* (α, β)*-metric on an open subset* $\mathcal{U}$ *in the n-dimensional Euclidean space* R^n $(n \geq 3)$, *where* $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ *and* $\beta = b_i(x)y^i \neq 0$. *Suppose that the following conditions: (a)* β *is not parallel with respect to* α, *(b)* $\phi \neq k_1\sqrt{1 + k_2 s^2} + k_3 s$ *for any constants* k_1, k_2 *and* k_3, *and (c)* $db \neq 0$ *everywhere or* $b =$ *constant on* $\mathcal{U}$. *Then* F *is a Douglas metric on* $\mathcal{U}$ *if and only if the function* $\phi = \phi(s)$ *satisfies*

$$\Big\{1 + (k_1 + k_2 s^2)s^2 + k_3 s^2\Big\}\phi''(s) = (k_1 + k_2 s^2)\Big\{\phi(s) - s\phi'(s)\Big\}, \tag{73}$$

and β *satisfies*

$$b_{i|j} = 2\tau\Big\{(1 + k_1 b^2)a_{ij} + (k_2 b^2 + k_3)b_i b_j\Big\}, \tag{74}$$

where $\tau = \tau(x)$ *is a scalar function on* $\mathcal{U}$ *and* k_1, k_2 *and* k_3 *are constants with* $(k_2, k_3) \neq (0, 0)$.

§8. Projectively flat (α, β)-metrics

It is Hilbert's Fourth Problem in the regular case to study and characterize Finsler metrics on an open domain $\mathcal{U} \subset \mathrm{R}^n$ whose geodesics are straight lines. Finsler metrics with this property are called *projectively flat* metrics.

It is easy to see that a Finsler metric $F = F(x, y)$ on an open subset $\mathcal{U} \subset \mathrm{R}^n$ is projectively flat if and only if the spray coefficients are in the following form

$$G^i = Py^i,$$

where $P = P(x, y)$ is a positively homogeneous function of degree one in y. In 1903, G. Hamel found a system of partial differential equations that characterize projectively flat metrics $F = F(x, y)$ on an open subset $\mathcal{U} \subset \mathrm{R}^n$. That is,

$$F_{x^m y^i} y^m = F_{x^i}. \tag{75}$$

A natural problem is to find projectively flat metrics by solving (75). According to the Beltrami Theorem, a Riemannian metric $F = \sqrt{g_{ij}(x)y^i y^j}$ is projectively flat if and only if it is of constant sectional curvature. Thus this problem has been solved in Riemannian geometry. However for Finsler metrics, this problem is far from being solved. In this section, we shall discuss some projectively flat (α, β)-metrics.

The Funk metric on the unit ball $\mathrm{B}^n \subset$ is given by

$$\Theta = \frac{\sqrt{(1 - |x|^2)|y|^2 + \langle x, y\rangle^2}}{1 - |x|^2} + \frac{\langle x, y\rangle}{1 - |x|^2}, \tag{76}$$

where $y \in T_x\mathrm{B}^n \approx \mathrm{R}^n$. Here $|\cdot|$ and $\langle\ ,\ \rangle$ denote the standard Euclidean norm and inner product. By a direct computation, one can verify that the Funk metric is projectively flat on B^n. The Funk metric has a very important curvature property: the flag curvature $\mathbf{K} = -1/4$.

Note that the Funk metric Θ on B^n is a special Randers metric expressed in the form

$$\Theta = \bar{\alpha} + \bar{\beta}, \tag{77}$$

where

$$\bar{\alpha} = \frac{\sqrt{(1-|x|^2)|y|^2 + \langle x,y\rangle^2}}{1-|x|^2}, \qquad \bar{\beta} = \frac{\langle x,y\rangle}{1-|x|^2}. \tag{78}$$

Thus it is natural to investigate projectively flat Randers metrics with constant flag curvature.

First we have the following

Theorem 8.1. *A Randers metric $F = \alpha + \beta$ on an open subset $\mathcal{U} \subset \mathrm{R}^n$ is projectively flat if and only if α is projectively flat and β is closed.*

The proof is straight forward using Theorem 7.1.

In [38], it is proved that a Randers metric on a manifold is locally projectively flat with constant flag curvature if and only if it is locally Minkowskian or up to a scaling and reversing, it is locally isometric to

$$\Theta_a = \bar{\alpha} + \bar{\beta}_a,$$

where $\bar{\alpha}$ is defined above and $\bar{\beta}_a$ is given by

$$\bar{\beta}_a := \frac{\langle x,y\rangle}{1-|x|^2} + \frac{\langle a,y\rangle}{1+\langle a,x\rangle},$$

where $a \in \mathrm{R}^n$ is a constant vector with $|a| < 1$. The metric Θ_a is projectively flat with $\mathbf{K} = -1/4$.

L. Berwald ([4]) constructed a projectively flat metric with zero flag curvature on the unit ball B^n, which is given by

$$B = \frac{(\sqrt{(1-|x|^2)|y|^2 + \langle x,y\rangle^2} + \langle x,y\rangle)^2}{(1-|x|^2)^2\sqrt{(1-|x|^2)|y|^2 + \langle x,y\rangle^2}},$$

where $y \in T_x\mathrm{B}^n \equiv \mathrm{R}^n$. Berwald's metric can be expressed in the form

$$B = \frac{(\lambda\bar{\alpha} + \lambda\bar{\beta})^2}{\lambda\bar{\alpha}} = \lambda\frac{(\bar{\alpha}+\bar{\beta})^2}{\bar{\alpha}}, \tag{79}$$

where $\bar{\alpha}$ and $\bar{\beta}$ are defined in (78) and $\lambda := 1/(1-|x|^2)$.

In [32], it is shown that the following metric B_a on $\mathrm{B}^n \subset \mathrm{R}^n$ is projectively flat with $\mathbf{K} = 0$.

$$B_a := \frac{(\lambda_a\bar{\alpha} + \lambda_a\bar{\beta}_a)^2}{\lambda_a\bar{\alpha}} = \lambda_a\frac{(\bar{\alpha}+\bar{\beta}_a)^2}{\bar{\alpha}}, \tag{80}$$

where $\lambda_a := (1 + \langle a, x\rangle)^2/(1 - |x|^2)$ and $a \in \mathrm{R}^n$ with $|a| < 1$.

The above discussion leads us to study the following metric on a manifold M,

$$F = \frac{(\alpha + \beta)^2}{\alpha}, \tag{81}$$

where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M. It is known that $F = (\alpha + \beta)^2/\alpha$ is a Finsler metric if and only if $b(x) := \|\beta_x\|_\alpha < 1$ at any point $x \in M$. A natural question arises: is there any other projectively flat metric in the form (81) with constant flag curvature?

Theorem 8.2. ([47]) *Let $F = (\alpha + \beta)^2/\alpha$ be a Finsler metric on a manifold M. F is projectively flat if and only if*

(i) $b_{i|j} = 2\tau\{(\frac{1}{2} + b^2)a_{ij} - \frac{3}{2}b_i b_j\}$,

(ii) *the spray coefficients $\bar{G}^i$ of α are in the form: $\bar{G}^i = \theta y^i - \tau\alpha^2 b^i$,*

where $b := \|\beta_x\|_\alpha$, $b_{i|j}$ denote the covariant derivatives of β with respect to α, $\tau = \tau(x)$ is a scalar function and $\theta = a_i(x)y^i$ is a 1-form on M.

In [32], it has been shown that if α and β satisfy the conditions (i) and (ii), then $F = (\alpha + \beta)^2/\alpha$ is locally projectively flat. Theorem 8.2 asserts that the converse is true too.

By Theorem 8.2, we can completely determine the local structure of a projectively flat Finsler metric F in the form (81) which is of constant flag curvature.

Theorem 8.3. ([47]) *Let $F = (\alpha + \beta)^2/\alpha$ be an (α, β)-metric on a manifold M. Then F is locally projectively flat with constant flag curvature if and only if one of the following conditions holds*

(a) *α is flat and β is parallel with respect to α. In this case, F is locally Minkowskian;*

(b) *Up to a scaling on x and a scaling on F, F is locally isometric to B_a in (80).*

In either case (a) or (b), the flag curvature of F must be zero, $\mathbf{K} = 0$.

Below is an outline of the proof of Theorem 8.3. By imposing the curvature condition that the flag curvature be constant, one first shows that the flag curvature must be zero, $\mathbf{K} = 0$. If $\tau = 0$, then F is locally Minkowskian. In the case when $\tau \neq 0$, one gets that

$$d\tau + 2\tau^2\beta = 0, \qquad \theta_{x^k}y^k - \theta^2 = 3\tau^2(\alpha^2 - 2\beta^2). \tag{82}$$

Then we show that $\tau\beta$ is closed. Thus there is a local scalar function $\rho = \rho(x)$ such that $\tau\beta = \frac{1}{2}d\rho$ and $\tau = ce^{-\rho}$ for some constant c. Immediately, one can see that $\bar{\alpha} := e^{-\rho}\alpha$ is projectively flat, hence $\bar{\alpha}$ is of constant curvature $\bar{\mathbf{K}} = \mu$ by the Beltrami theorem. The constant μ must be nonpositive. By choosing the projective form of $\bar{\alpha}$, one can solve (82) for ρ. Then one can determine α and β. The detailed argument is given in [47].

The reader is referred to [14], [48],[52] and [51] for results on other special (α, β)-metrics.

Recently the third author has characterized all projectively flat (α, β)-metrics.

Theorem 8.4. ([44]) *Let $\phi := \phi(s)$, $-b_o < s < b_o$, be a positive C^∞ function satisfying (4). Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α, β)-metric on an open subset $\mathcal{U}$ in the n-dimensional Euclidean space R^n $(n \geq 3)$, where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ and $\beta = b_i(x)y^i \neq 0$. Suppose that the following conditions:*

(a) *β is not parallel with respect to α,*
(b) *$\phi \neq k_2\sqrt{1 + k_1s^2} + k_3s$ for some constants k_1, k_2 and k_3 with $k_2 > 0$, and*
(c) *the norm $b(x) := \|\beta_x\|_\alpha$ satisfies either $db \neq 0$ everywhere or $b =$ constant on $\mathcal{U}$.*

Then F is projectively flat on $\mathcal{U}$ if and only if

$$\Big\{1 + (c_1 + c_2s^2)s^2 + c_3s^2\Big\}\phi''(s) = (c_1 + c_2s^2)\Big\{\phi(s) - s\phi'(s)\Big\}, \tag{83}$$

$$b_{i|j} = 2\tau\Big\{(1 + c_1b^2)a_{ij} + (c_2b^2 + c_3)b_ib_j\Big\}, \tag{84}$$

$$G^i_\alpha = \xi y^i - \tau\Big(c_1\alpha^2 + c_2\beta^2\Big)b^i, \tag{85}$$

where $\tau = \tau(x)$ is a scalar function on $\mathcal{U}$ and c_1, c_2 and c_3 are constants with $(c_2, c_3) \neq (0, 0)$.

Let $c_1 = 1/p$, $c_2 = 0$ and $c_3 = (r-1)/p$, where p and r are constants with $p \neq 0$. Then (83) becomes

$$\phi(s) - s\phi'(s) = (p + rs^2)\phi''(s). \tag{86}$$

Projectively flat (α, β)-metrics $F = \alpha\phi(\beta/\alpha)$ with $\phi = \phi(s)$ satisfying (86) were first studied by the third author in [43]. He finds a sufficient

condition on β under which $F = \alpha\phi(\beta/\alpha)$ is projectively flat. That is, if ϕ satisfies (86) and α and β satisfy

$$b_{i|j} = 2\tau\Big\{(p + b^2)a_{ij} + (r-1)b_i b_j\Big\} \tag{87}$$

$$G^i_\alpha = \xi y^i - \tau\alpha^2 b^i, \tag{88}$$

where $\xi = \xi_i(x)y^i$ is a 1-form, then $F = \alpha\phi(\beta/\alpha)$ is projectively flat. Later on, the second author and Li prove that (87) and (88) are also necessary conditions for $F = \alpha\phi(\beta/\alpha)$ to be projectively flat provided that $\phi = \phi(s)$ is analytic in s.

Explicit examples can be constructed.

Example 8.5. ([43]) *Let $\phi = \phi(s)$ be a function satisfying (4) and (86) with $p \neq 0$. Let*

$$h := \frac{1}{\sqrt{1+\mu|x|^2}}\Big\{C_1 + \langle a, x\rangle + \frac{\eta|x|^2}{1+\sqrt{1+\mu|x|^2}}\Big\}, \tag{89}$$

and let $\rho = \rho(t)$ be given by

$$\rho(t) = \begin{cases} \frac{(C_2)^2}{p}\Big(C_3 + \eta t - \frac{1}{2}\mu t^2\Big) & \text{if } r = 0 \\ \ln\Big[-\frac{2r(C_2)^2}{p}\Big(C_3 + \eta t - \frac{1}{2}\mu t^2\Big)\Big]^{-\frac{1}{2r}} & \text{if } r \neq 0 \end{cases} \tag{90}$$

where η and C_i are constants ($C_2 > 0$) and $a \in \mathrm{R}^n$ is a constant vector. Define

$$\alpha := e^{\rho(h)}\alpha_\mu, \qquad \beta := C_2 e^{(r+1)\rho(h)}dh,$$

where

$$\alpha_\mu = \frac{\sqrt{|y|^2 + \mu(|x|^2|y|^2 - \langle x, y\rangle^2)}}{1+\mu|x|^2}.$$

Then α and β satisfy (87) and (88) with

$$\tau = \frac{\rho'(h)}{2C_2 e^{(r+1)\rho(h)}}.$$

Thus the Finsler metric $F = \alpha\phi(\beta/\alpha)$ is projectively flat.

Using Theorem 8.4, one can classify locally projectively flat (α, β)-metrics $F = \alpha\phi(\beta/\alpha)$ of constant flag curvature. Roughly speaking if F is not trivial, then $\phi = \sqrt{1+ks^2} + \varepsilon s$ or $\phi = (\sqrt{1+ks^2} + \varepsilon s)^2/\sqrt{1+ks^2}$. See [24] for more details.

§9. (α, β)-metrics with isotropic S-curvature

The S-curvature is an important geometric quantity. It interacts the flag curvature in a delicate way. This stimulates our interest in Finsler metrics with special S-curvature property. Our goal is to characterize (α, β)-metrics with isotropic S-curvature.

Let $\phi = \phi(s)$ be a positive C^∞ function on $(-b_o, b_o)$. For a number $b \in [0, b_o)$, let

$$\Phi := -(Q - sQ')\Big\{n\Delta + 1 + sQ\Big\} - (b^2 - s^2)(1 + sQ)Q'', \tag{91}$$

where $\Delta := 1 + sQ + (b^2 - s^2)Q'$ and $Q := \phi'/(\phi - s\phi')$.

Let $F = \alpha\phi(\beta/\alpha)$ be an (α, β)-metric on an n-dimensional manifold M, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form. Let G^i and $\bar{G}^i$ denote the spray coefficients of F and α respectively. G^i are related to $\bar{G}^i$ by (5). To compute the S-curvature of F, we need the following identities:

$$\frac{\partial s}{\partial y^m} = \frac{1}{\alpha}\Big\{b_m - s\frac{y_m}{\alpha}\Big\},$$

$$\frac{\partial \alpha}{\partial y^m} = \frac{y_m}{\alpha},$$

$$\frac{\partial \bar{G}^m}{\partial y^m} = y^m \frac{\partial}{\partial x^m}\Big(\ln \sigma_\alpha\Big).$$

Using the above identities, we obtain

$$\frac{\partial G^m}{\partial y^m} = y^m \frac{\partial}{\partial x^m}\Big(\ln \sigma_\alpha\Big) + 2\Psi(r_0 + s_0) - \alpha^{-1}\frac{\Phi}{2\Delta^2}(r_{00} - 2\alpha Q s_0),$$

where Φ is given in (91) with $b = \|\beta_x\|_\alpha$.

Let $dV = \bar{\sigma}dx$ denote the volume form of α. By Proposition 4.1, $dV = \sigma dx = f(b)\sigma_\alpha dx$. Thus

$$y^m \frac{\partial}{\partial x^m}\Big(\ln \sigma\Big) = \frac{f'(b)}{f(b)} y^m \frac{\partial b}{\partial x^m} + y^m \frac{\partial}{\partial x^m}\Big(\ln \sigma_\alpha\Big).$$

$$y^m \frac{\partial b}{\partial x^m} = \frac{b^i b_{i|m} y^m}{b} = \frac{r_0 + s_0}{b}. \tag{92}$$

Then the S-curvature is given by

$$\mathbf{S} = \Big\{2\Psi - \frac{f'(b)}{bf(b)}\Big\}(r_0 + s_0) - \alpha^{-1}\frac{\Phi}{2\Delta^2}(r_{00} - 2\alpha Q s_0). \tag{93}$$

Using (93), we can prove the following

Theorem 9.1. ([19]) *Let $F = \alpha\phi(s), s = \beta/\alpha$, be an (α, β)-metric on a manifold and $b := \|\beta_x\|_\alpha$. Suppose that $\phi \neq k_1\sqrt{1 + k_2 s^2} + k_3 s$ for any constants $k_1 > 0, k_2$ and k_3. Then F is of isotropic S-curvature, $\mathbf{S} = (n+1)cF$, if and only if one of the following holds*

(i) *β satisfies*

$$r_j + s_j = 0 \tag{94}$$

and $\phi = \phi(s)$ satisfies

$$\Phi = 0. \tag{95}$$

In this case, $\mathbf{S} = 0$.

(ii) *β satisfies*

$$r_{ij} = \varepsilon\Big\{b^2 a_{ij} - b_i b_j\Big\}, \qquad s_j = 0, \tag{96}$$

where $\varepsilon = \varepsilon(x)$ is a scalar function, and $\phi = \phi(s)$ satisfies

$$\Phi = -2(n+1)k\frac{\phi\Delta^2}{b^2 - s^2}, \tag{97}$$

where k is a constant. In this case, $\mathbf{S} = (n+1)cF$ with $c = k\epsilon$.

(iii) *β satisfies*

$$r_{ij} = 0, \qquad s_j = 0. \tag{98}$$

In this case, $\mathbf{S} = 0$, regardless of the choice of a particular ϕ.

It is easy to see that (98) implies (96), while (96) implies (94). The condition (94) is equivalent to that $b := \|\beta_x\|_\alpha = constant$. Thus (95) and (97) are independent of $x \in M$.

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