Advanced Studies in Pure Mathematics 48, 2007
Finsler Geometry, Sapporo 2005 – In Memory of Makoto Matsumoto
pp. 19–71

On two curvature-driven problems in Riemann–Finsler geometry

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Dedicated to the memory of Professor S.-S. Chern and Professor Makoto Matsumoto

Abstract.

This article uses the Berwald connection exclusively, together with its two curvatures, to cut an efficient path across the landscape of Finsler geometry. Its goal is to initiate differential geometers into two key research areas in the field: the search for unblemished "unicorns" and the study of Ricci flow. The exposition is almost self-contained.

Contents

§1. What are Finsler metrics?

1.1. All it takes is a change of perspective

Even for a diehard Finsler geometer such as myself, the physical distance between any two points P and Q on a manifold M should still be measured by a *Riemannian* metric μ:

$$\operatorname{dist}(P,Q) = \inf_{\sigma} \int_{\sigma} \sqrt{\mu_{\sigma}(\dot\sigma,\dot\sigma)}\, ds\,, \quad \text{where } \dot\sigma := \frac{d\sigma}{ds}\,.$$

Received January 6, 2007.
Revised March 14, 2007.
*Supported in part by the S.S. Chern Foundation for Mathematical Research, and by the University of Houston.

Here, it's understood that the infimum is taken over all piecewise smooth curves from P to Q. Finsler metrics become relevant only when one asks for the least travel time from P to Q, rather than mere physical distance.

The integrand $\sqrt{\mu_\sigma(\dot\sigma,\dot\sigma)}\,ds$ is the length of a tiny segment of the curve σ, and the parameter s may have nothing to do with actual travel time. To get the least amount of time it takes to traverse this segment, we divide that length by the fastest travel speed we could muster, say, c. In general, c depends on our location $x = \sigma(s) \in M$, the direction of our instantaneous tangent $y = \dot\sigma(s) \in T_xM$, and most likely the time of the day as well; but let us not insist on this last bit of reality. Since only the direction of y matters, c should satisfy $c(x, \lambda y) = c(x, y)$ for all $\lambda > 0$. Then the least travel time from P to Q is:

$$\mathrm{time}(P,Q) = \inf_\sigma \int_\sigma \frac{1}{c(\sigma,\dot\sigma)} \sqrt{\mu_\sigma(\dot\sigma,\dot\sigma)}\,ds\,, \quad \text{where } \dot\sigma := \frac{d\sigma}{ds}\,.$$

The new integrand is of the type

$$F(x,y) = \frac{1}{c(x,y)} \sqrt{\mu_x(y,y)} = \sqrt{\frac{\mu_x(y,y)}{c^2(x,y)}}\,.$$

It represents the shortest time required for traveling along y, from its basepoint x to the tip. The quantity inside the radical is typically not even rational in y; it is a quadratic function of y if and only if c has no y dependence, and in that case F is said to be **Riemannian**.

Formally, a **Finsler metric** is a continuous function $F : TM \to [0,\infty)$ with the following properties.

(1) **Regularity**: F is smooth on $TM \setminus 0 := \{(x,y) \in TM \mid y \neq 0\}$.
(2) **Positive homogeneity**: $F(x,\lambda y) = \lambda F(x,y)$ for all $\lambda > 0$.
(3) **Strong convexity**: the fundamental tensor $g_{ij} := \partial^2_{y^i y^j}(\frac{1}{2}F^2)$ is positive definite at all $(x,y) \in TM \setminus 0$.

Regularity is needed in an essential way when establishing certain global results in Finsler geometry. These include a Gauss–Bonnet–Chern theorem ([Bao–Chern 1996], [Lackey 2002]) and a Hodge decomposition theorem [Bao–Lackey 1996]; the former because arbitrary vector fields are involved (hence there can not be any forbidden directions y), and the latter because one has to average over *all* directions. The less stringent criterion of **y-locality**, namely, F being not even C^2 at some nonzero y, or is strongly convex only on some proper open cone in $TM \setminus 0$, sometimes opens the door to phenomena that are either impossible or not-yet-witnessed in the stricter regime. More on this in Section 4, where we discuss Landsberg metrics.

Strong convexity implies that the vectors $\{y \mid F(x,y) \leqslant 1\}$ comprise a strictly convex set in T_xM, though the converse does not hold; see [Bao et al. 2000]. Its inclusion here in our definition of a Finsler metric is to enable technology transfer from Riemannian geometry, and to narrow our scope. For applications to relativistic physics, this hypothesis has to be weakened to non-degeneracy, namely, $\det(g_{ij}) \neq 0$; see [Asanov 1985] for background and [Rutz 1996] for a concrete example. Thus, for the sake of generality, some authors (for example [Antonelli et al. 1993]) prefer non-degeneracy over strong convexity, and this point is articulated distinctly in [Matsumoto 1986].

The property of positive homogeneity that has been motivated by our discussion is of degree 1 (in y). Functions satisfying $\Phi(\lambda y) = \lambda^r\, \Phi(y)$ for all $\lambda > 0$ are said to be positively homogeneous of degree r. For them, **Euler's theorem** assures us that the following two statements are equivalent:

- $\Phi(\lambda y) = \lambda^r\, \Phi(y)$ for all $\lambda > 0$.
- $y^i \partial_{y^i} \Phi(y) = r\, \Phi(y)$.

Many computations in Finsler geometry would have been intractable without this basic fact.

Let us demystify the notation with examples from 2d. To reduce clutter, abbreviate y^1 as p and y^2 as q.

$r = 3$: Try $\Phi = p^5/q^2$. Then $p\partial_p\Phi + q\partial_q\Phi = p(5p^4/q^2) + q(-2p^5/q^3) = 3\Phi$, as expected.

$r = 0$: Try $\Phi = (p^2 - q^2)/(p^2 + q^2)$. Then the quantity $p\partial_p\Phi + q\partial_q\Phi = p(4pq^2/[p^2+q^2]^2) + q(-4qp^2/[p^2+q^2]^2) = 0$.

As a more serious illustration, two successive applications of Euler's theorem let us "invert" the defining relation

$$g_{ij}(x,y) := \partial_{y^i}\partial_{y^j}\left(\tfrac{1}{2}F^2\right)$$

of the **fundamental tensor** g_{ij} to recover F:

$$F^2(x,y) = g_{ij}(x,y)\, y^i y^j\,.$$

Consequently, strong convexity implies that F must be positive at all $y \neq 0$. The converse is false because, while $g_{ij}(x,y)\, y^i y^j = F^2(x,y)$ may be positive for $y \neq 0$, the quadratic $g_{ij}(x,y)\, v^i v^j$ could still be $\leqslant 0$ for some nonzero v. Given this, it is rather surprising to find that positivity *and* non-degeneracy together do imply strong convexity [Lovas 2005]!

1.2. A practical example

Imagine living on a Riemannian manifold (M, h), where the h-unit tangent vectors u represent the displacements we can make in 1 second, with our engine at full throttle. According to the mindset presented in the previous section, we have $\sqrt{h_x(y,y)} = \sqrt{\mu_x(y,y)}/\xi(x)$, where μ is some underlying Riemannian metric which measures physical distances, and $\xi(x)$ is a speed function which just happens to be *in*dependent of the direction of travel. This ξ is determined by the performance characteristics of our engine at various locations x on M.

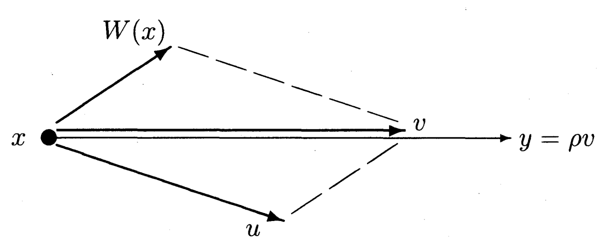
Now a wind starts blowing across this landscape, with its velocity at location x given by a tangent vector $W(x)$. For simplicity, let us assume that W does not vary with time. With the wind blowing, and with our engine still at full throttle, it is clear that in 1 second we can travel farther along those directions to which the wind lends a helping component. Thus, the new speed function c should depend on both x and y. The resulting $F(x,y) = \sqrt{\mu_x(y,y)}/c(x,y)$, which measures the shortest travel time from the base of y to its tip, is expected to be non-Riemannian.

The purpose of this section is to show that in practice, it is sometimes more efficient to derive the expression of F through first principles, from which we can then determine c, rather than the other way around.

Suppose our goal is to navigate along a tangent vector $y \in T_xM$, starting from its basepoint x. Within a split second Δt:

* The wind would have displaced us from x to the tip of $W(x)\Delta t$, had we turned off our engine.
* On the other hand, had the wind been absent, traveling at full power along any h-unit vector u would have taken us to $u\Delta t$.

So, in order to stay the course along y, we need to direct our engine at full throttle in the direction of a particular h-unit vector u, such that the resultant $(u+W)\Delta t$, denoted $v\Delta t$, has the same direction as y. This trend of thought is represented by the picture below.



Since $v\Delta t$ is the displacement achieved during the split second Δt, we have $F(x, v\Delta t) = \Delta t$ according to the meaning intended for F.

Invoking the positive homogeneity of F, we can cancel off Δt to get $F(x, v) = 1$. Expressing y as ρv for some $\rho > 0$, homogeneity effects $F(x, y) = \rho$. It remains to determine ρ explicitly. To that end:

$$\begin{aligned}
& h(u,u) = 1 \\
\Rightarrow\quad & h(v-W, v-W) = 1 \\
\Rightarrow\quad & \zeta + 2h(v,w) - h(v,v) = 0, \quad \text{with} \quad \zeta := 1 - h(W,W) \\
\Rightarrow\quad & \zeta\,\rho^2 + 2h(y,W)\,\rho - h(y,y) = 0 \\
\Rightarrow\quad & \rho = \frac{-h(y,W) + \sqrt{[h(y,W)]^2 + \zeta\,h(y,y)}}{\zeta}\,.
\end{aligned}$$

In the above derivation, we have assumed $h(W,W) < 1$ to reduce the myriad of possible cases down to one, and have reminded ourselves that the ρ we are solving for must be positive. Thus

$$F(x,y) = \frac{\sqrt{[h(y,W)]^2 + h(y,y)\{1 - h(W,W)\}} - h(y,W)}{1 - h(W,W)}\,.$$

Since $F(x,-y) \neq F(x,y)$, the function F can not possibly be Riemannian. This is also manifest from the fact that in the maximal travel speed $c = \sqrt{\mu_x(y,y)}/F(x,y)$, the y-dependence can not be eliminated.

The Finsler metric we obtained has the structure of a Riemannian part (the first term) plus a 1-form part (the second term), and is said to be of **Randers type** [Randers 1941]. The pair (h, W) is called the **navigation data** for this Randers metric. The condition $h(W,W) < 1$ is precisely what one needs to ensure that F is strongly convex; details can be found in [Bao–Robles 2004].

The idea that structures our discussion here is due to Zermelo; see [Zermelo 1931] and [Caratheodory 1999]. The discovery of the above remarkable formula in modern settings is due to Z. Shen; see [Shen 2003].

1.3. Chern's Riemannian vector bundles

Recall that the fundamental tensor is defined as the y-Hessian of $\frac{1}{2}F^2$. Using subscripts to signify partial differentiation, we have

$$g_{ij}(x,y) := \left(\tfrac{1}{2}F^2\right)_{y^i y^j} = F\,F_{y^i y^j} + F_{y^i}\,F_{y^j}\,.$$

In the Riemannian case, $F(x,y) := \sqrt{a_{rs}(x)\,y^r y^s}$, so g_{ij} is simply $a_{ij}(x)$, thereby recovering the Riemannian metric we started with.

Outside the Riemannian realm, the fundamental tensor will always have a y-dependence. Let us illustrate with an example in 2d, where we write y^1 as p and y^2 as q.

$$F(x^1, x^2; p, q) := \sqrt{\sqrt{p^4 + q^4} + \psi(x^1, x^2)(p^2 + q^2)} \, ;$$

the term proportional to ψ is added to effect strong-convexity. For this F, the 2-by-2 matrix (g_{ij}) is

$$\begin{pmatrix} \psi(x^1, x^2) + \frac{p^2(p^4+3q^4)}{(p^4+q^4)^{3/2}} & \frac{-2p^3q^3}{(p^4+q^4)^{3/2}} \\ \frac{-2p^3q^3}{(p^4+q^4)^{3/2}} & \psi(x^1, x^2) + \frac{q^2(q^4+3p^4)}{(p^4+q^4)^{3/2}} \end{pmatrix} ,$$

whose y-dependence reaffirms that F is non-Riemannian.

The **Cartan tensor** quantifies the deviation of F from being Riemannian, and is defined as

$$A_{ijk} := \frac{F}{2}(g_{ij})_{y^k} = \frac{F}{2}\left(\tfrac{1}{2}F^2\right)_{y^i y^j y^k} .$$

This tensor is totally symmetric in all three indices. And its contraction with y vanishes by Euler's theorem, because g_{ij} is homogeneous of degree 0 in y. Indices on A are manipulated by g_{ij} and its inverse g^{ij}. With that in mind, one can show (for later use) that

$$F(g^{ij})_{y^k} = -2\, A^{ij}{}_k \, .$$

On account of the y-dependence of g_{ij}, the tensor

$$g := g_{ij}(x, y)\, dx^i \otimes dx^j$$

carries with it not one but a sphere's worth of inner products in every tangent space T_xM. Indeed, since $g_{ij}(x, y)$ is constant along each ray emanating from the origin of T_xM, there is exactly one inner product corresponding to any given direction. Each such inner product deserves its own copy of T_xM to act on. This realisation led Chern to the following "blow-up" type construction.

- Replace each $x \in M$ by the collection of rays which emanate from the origin of T_xM. Each element in this collection is a ray $(x, [y_o]) := \{(x, \lambda y_o) \mid \lambda > 0\}$, with $y_o \neq 0$. And, $(x, [z]) = (x, [y])$ if and only if $z = \lambda y$ for some $\lambda > 0$.
- Over each ray $(x, [y])$, erect a copy of T_xM and equip the latter with the inner product $g_{ij}(x, y)\, dx^i \otimes dx^j$. The inner product is well defined because $g_{ij}(x, \lambda y) = g_{ij}(x, y)$ for all $\lambda > 0$.

In this way, the original n-dimensional manifold M is replaced by the $(2n-1)$-dimensional **manifold of rays** SM (also called the *projective* sphere bundle). It is to be emphasised that SM is independent of F. A Riemannian vector bundle of fibre dimension n is built over SM, with fibre metric given by the g defined above. This is called the pulled-back bundle over SM. Note that g is Riemannian precisely because F is presumed strongly convex.

However, computations on SM are more easily done using the affine coordinates provided by $TM \setminus 0$, the $2n$-dimensional **manifold of nonzero tangent vectors** (also known as the *slit* tangent bundle). Elements of $TM \setminus 0$ are of the form (x,y), with $y \neq 0$. Over each (x,y), erect a copy of T_xM, and equip it with the inner product $g_{ij}(x,y)dx^i \otimes dx^j$. The resulting Riemannian vector bundle of fibre dimension n is called the pulled-back bundle over $TM \setminus 0$, and is denoted by π^*TM.

There is a global section of π^*TM, defined by

$$\ell(x,y) := \frac{y^i}{F(x,y)} \frac{\partial}{\partial x^i} .$$

We shall call it the **canonical section**. Euler's theorem implies that ℓ has unit length with respect to the bundle metric g.

The dual of π^*TM is the pulled-back cotangent bundle, π^*T^*M. It too, has an important global section

$$\omega := \partial_{y^i}[F(x,y)]\ dx^i = F_{y^i}\ dx^i ,$$

called the **Hilbert form**. Euler's theorem implies that

$$\ell_i := g_{ij}\,\ell^j = F_{y^i} , \quad \text{namely,} \quad \omega = g(\ell,\ ) .$$

Consequently, ω is also of g-unit length, and is naturally dual to ℓ.

Simple calculations give the following useful statements:

$$\begin{aligned} F\,(\ell^i)_{y^j} &= \delta^i{}_j - \ell^i\,\ell_j , \\ F\,(\ell_i)_{y^j} &= \delta_{ij} - \ell_i\,\ell_j . \end{aligned}$$

We shall need these in later computations.

§2. Geodesic sprays and nonlinear parallel transport

2.1. The manifold of nonzero tangent vectors

The manifold $TM \setminus 0$ is the base space of Chern's pulled-back bundle π^*TM. Elements of TM are of the form (x,y), with $y = y^i\,\partial_{x^i} \in T_xM$. Thus we have natural coordinates $(x^1,\ldots,x^n;y^1,\ldots,y^n)$ on TM

that are local in x but global in y. The associated coordinate basis is $\{\partial_{x^i}; \partial_{y^i}\}$, with dual basis $\{dx^i; dy^i\}$.

A local coordinate change $x^i = x^i(\tilde{x}^1, \ldots, \tilde{x}^n)$ induces the transformation

$$\frac{\partial}{\partial \tilde{x}^p} = \frac{\partial x^i}{\partial \tilde{x}^p} \frac{\partial}{\partial x^i} \quad \text{on } M\,.$$

Since y can be expanded as either $y^i \partial_{x^i}$ or $\tilde{y}^p \partial_{\tilde{x}^p}$, we must have

$$y^i = \frac{\partial x^i}{\partial \tilde{x}^p}\, \tilde{y}^p\,.$$

When used as a differential operator on TM, $\partial_{\tilde{x}^p}$ encounters functions Φ with *both* x and y dependences. The chain rule

$$\frac{\partial \Phi}{\partial \tilde{x}^p} = \frac{\partial \Phi}{\partial x^i} \frac{\partial x^i}{\partial \tilde{x}^p} + \frac{\partial \Phi}{\partial y^i} \frac{\partial y^i}{\partial \tilde{x}^p}$$

then implies that

$$\frac{\partial}{\partial \tilde{x}^p} = \frac{\partial x^i}{\partial \tilde{x}^p} \frac{\partial}{\partial x^i} + \frac{\partial^2 x^i}{\partial \tilde{x}^p \partial \tilde{x}^q}\, \tilde{y}^q\, \frac{\partial}{\partial y^i} \quad \text{on } TM\,.$$

A similar situation holds for $d\tilde{y}^p$, namely

$$d\tilde{y}^p = \frac{\partial \tilde{x}^p}{\partial x^i}\, dy^i + \frac{\partial^2 \tilde{x}^p}{\partial x^i \partial x^j}\, y^j\, dx^i\,,$$

though the "problem" does not extend to $d\tilde{x}^p$ and $\partial_{\tilde{y}^p}$.

Surprisingly, the remedy comes from a term in the differential equation for geodesics $\sigma(s)$ with constant Finslerian speed $F(\sigma, \dot{\sigma})$. That equation reads

$$\ddot{\sigma}^i + \dot{\sigma}^j\, \dot{\sigma}^k\, \gamma^i{}_{jk}(\sigma, \dot{\sigma}) = 0\,,$$

where

$$\gamma^i{}_{jk} := g^{is}\, \tfrac{1}{2} \left(g_{sj,x^k} - g_{jk,x^s} + g_{ks,x^j} \right)$$

are the fundamental tensor's formal Christoffel symbols of the second kind. Motivated by the second term in the geodesic equation, we define the **geodesic spray coefficients**

$$G^i(x,y) := \tfrac{1}{2}\, \gamma^i{}_{jk}(x,y)\, y^j y^k\,,$$

while hastening to point out that the definition in [Bao et al. 2000] lacks that factor of $\frac{1}{2}$.

In practice, one first computes

$$\begin{aligned} G_i &:= g_{ij}\, G^j \\ &= \tfrac{1}{4}\{g_{ij,x^k} - g_{jk,x^i} + g_{ki,x^j}\}\, y^j y^k \\ &= \tfrac{1}{4}\{(F\ell_i)_{x^k}\, y^k - (F^2)_{x^i} + (F\ell_i)_{x^j}\, y^j\}\,, \end{aligned}$$

that is,

$$G_i = \tfrac{1}{2}\{(F\ell_i)_{x^j}\, y^j - F F_{x^i}\}\,,$$

and then raise the index to get G^i. As an example, consider the Numata-type metric

$$F(x,y) = \sqrt{\delta_{pq}\, y^p y^q} \;+\; f_{x^p}\, y^p\,,$$

where f is some function of x only. Then the components of the Hilbert form are

$$\ell_i = F_{y^i} = \frac{\delta_{ip}\, y^p}{\sqrt{\delta_{pq}\, y^p y^q}} + f_{x^i}\,,$$

from which we find that

$$G_i = \tfrac{1}{2}\, \ell_i\, f_{x^p x^q}\, y^p y^q\,.$$

Hence

$$G^i = g^{ij}\, G_j = \tfrac{1}{2}\, \ell^i\, f_{x^p x^q}\, y^p y^q\,.$$

Using the geodesic spray coefficients, we generate the **nonlinear connection**

$$N^i{}_j := (G^i)_{y^j}\,.$$

For instance, in the Numata-type example considered above, we have

$$N^i{}_j = \frac{1}{2F}(\delta^i{}_j - \ell^i\, \ell_j)\, f_{x^p x^q}\, y^p y^q + \ell^i\, f_{x^j x^p}\, y^p\,.$$

With the nonlinear connection in hand, let us modify ∂_{x^i} and dy^i on $TM \smallsetminus 0$, as follows:

$$\begin{aligned} \frac{\delta}{\delta x^i} &:= \frac{\partial}{\partial x^i} - N^s{}_i\, \frac{\partial}{\partial y^s}\,, \\ \delta y^i &:= dy^i + N^i{}_s\, dx^s\,. \end{aligned}$$

Amazingly, local coordinate changes on M now induce the simple transformations:

$$\begin{aligned} \frac{\delta}{\delta \tilde{x}^p} &= \frac{\partial x^i}{\partial \tilde{x}^p}\, \frac{\delta}{\delta x^i}\,, \\ \delta \tilde{y}^p &= \frac{\partial \tilde{x}^p}{\partial x^i}\, \delta y^i\,. \end{aligned}$$

This is why, on $TM \setminus 0$, it is preferable to use the basis

$$\left\{ \frac{\delta}{\delta x^i}; F\frac{\partial}{\partial y^i} \right\} \quad \text{and its dual} \quad \left\{ dx^i; \frac{\delta y^i}{F} \right\},$$

rather than

$$\left\{ \frac{\partial}{\partial x^i}; F\frac{\partial}{\partial y^i} \right\} \quad \text{and} \quad \left\{ dx^i; \frac{dy^i}{F} \right\}.$$

Here, factors of F and $1/F$ are introduced to render all objects invariant under positive rescaling $y \mapsto \lambda y$ in y, so that they also make sense on the manifold of rays SM.

The fundamental tensor g_{ij} is an object that wears several hats. We have already seen that its first job is to provide a Riemannian metric for Chern's pulled-back bundle π^*TM and the associated tensor products. Now we describe its second job: to provide, via a Sasaki type lift, a Riemannian metric on the manifold $TM \setminus 0$ of nonzero tangent vectors. That metric is

$$\hat{g}(x,y) \ := \ g_{ij}(x,y)\, dx^i \otimes dx^j + g_{ij}(x,y)\, \frac{\delta y^i}{F} \otimes \frac{\delta y^j}{F}.$$

With respect to $\hat{g}$, the span of $\{\delta/\delta x^i\}$ is orthogonal to that of $\{F\partial_{y^i}\}$, and is therefore said to define a **horizontal distribution** on $TM \setminus 0$. Since the $\delta/\delta x^i$ are constructed directly from the $N^s{}_i$, the latter then acquires the status of an Ehresmann connection; this is why it is called the nonlinear connection.

2.2. Horizontal constancy of Finsler metrics

We have seen that by naively lifting ∂_{x^i} from M to $TM \setminus 0$, one gets a vector field with less-than-satisfactory transformation properties

$$\frac{\partial}{\partial \tilde{x}^p} = \frac{\partial x^i}{\partial \tilde{x}^p}\frac{\partial}{\partial x^i} + \frac{\partial^2 x^i}{\partial \tilde{x}^p \partial \tilde{x}^q}\, \tilde{y}^q\, \frac{\partial}{\partial y^i}$$

induced by local coordinate changes $x^i = x^i(\tilde{x}^1, \ldots, \tilde{x}^n)$ of M. On the other hand, the horizontal lift $\delta/\delta x^i$ of ∂_{x^i} effects the transformation law

$$\frac{\delta}{\delta \tilde{x}^p} = \frac{\partial x^i}{\partial \tilde{x}^p}\frac{\delta}{\delta x^i}$$

we had in mind, somewhat surprisingly.

What is even more surprising is the following fact:

$$\frac{\delta}{\delta x^j} F \ = \ 0\,.$$

A moment's thought gives the equivalent statement:

F is constant along any curve with horizontal velocity.

The computation which establishes the above fact illustrates well the notation and formalism developed up to this point, so we give the details here.

$$\begin{aligned}
&\frac{\delta}{\delta x^j} F \\
= \;& F_{x^j} - N^i{}_j \, F_{y^i} \\
= \;& F_{x^j} - (G^i)_{y^j} \, \ell_i \\
= \;& F_{x^j} - (g^{is} \, G_s)_{y^j} \, \ell_i \\
= \;& F_{x^j} - (g^{is})_{y^j} \, G_s \, \ell_i - g^{is} \, (G_s)_{y^j} \, \ell_i \\
= \;& F_{x^j} + \frac{2}{F} A^{is}{}_j \, G_s \, \ell_i - \ell^s \, \tfrac{1}{2} \{ (F\ell_s)_{x^r} \, y^r - F F_{x^s} \}_{y^j} \\
= \;& F_{x^j} + 0 - \tfrac{1}{2} \ell^s \{ (F\ell_s)_{x^r y^j} \, y^r + (F\ell_s)_{x^r} \, \delta^r{}_j - \ell_j \, F_{x^s} - F \, F_{x^s y^j} \} \\
= \;& F_{x^j} - \tfrac{1}{2} \ell^s \{ [\ell_j \ell_s + F \, (\ell_s)_{y^j}]_{x^r} \, y^r + (F\ell_s)_{x^j} - \ell_j \, F_{x^s} - F \, (\ell_j)_{x^s} \} \\
= \;& F_{x^j} - \tfrac{1}{2} \ell^s \{ [\ell_j \ell_s + g_{sj} - \ell_s \ell_j]_{x^r} \, y^r + F_{x^j} \, \ell_s + F \, (\ell_s)_{x^j} - (F\ell_j)_{x^s} \} \\
= \;& F_{x^j} - \tfrac{1}{2} \ell^s \{ [F F_{y^s y^j} + \ell_s \ell_j]_{x^r} \, y^r + F_{x^j} \, \ell_s + F \, (\ell_s)_{x^j} - (F\ell_j)_{x^s} \} \\
= \;& F_{x^j} - \tfrac{1}{2} \{ [F F_{y^j y^s} \, y^s + y^s \ell_s \ell_j]_{x^r} \, \ell^r + F_{x^j} + (y^s \ell_s)_{x^j} - (F\ell_j)_{x^s} \, \ell^s \} \\
= \;& F_{x^j} - \tfrac{1}{2} \{ [0 + F\ell_j]_{x^r} \, \ell^r + F_{x^j} + F_{x^j} - (F\ell_j)_{x^s} \, \ell^s \} \\
= \;& 0 \, .
\end{aligned}$$

On account of the horizontal constancy of F, we have

$$\begin{aligned}
dF \;&=\; (dF) \left(\frac{\delta}{\delta x^i} \right) dx^i + (dF) \left(F \frac{\partial}{\partial y^i} \right) \frac{\delta y^i}{F} \\
&=\; F F_{y^i} \frac{\delta y^i}{F} \, .
\end{aligned}$$

That is,

$$dF = \ell_i \, \delta y^i \, ,$$

an identity that will be useful later.

2.3. A canonical nonlinear parallel transport

Let $\sigma(t)$ be any injective smooth curve in M which emanates from x at $t = 0$. Its velocity field is

$$\dot\sigma(t) := \frac{d\sigma^i}{dt} \, \frac{\partial}{\partial x^i}_{|\sigma(t)} \, .$$

At each location $\sigma(t)$, we horizontally lift $\dot\sigma(t)$ to *every* point $(\sigma(t), y)$ on the fibre of $TM \setminus 0$ over $\sigma(t)$:

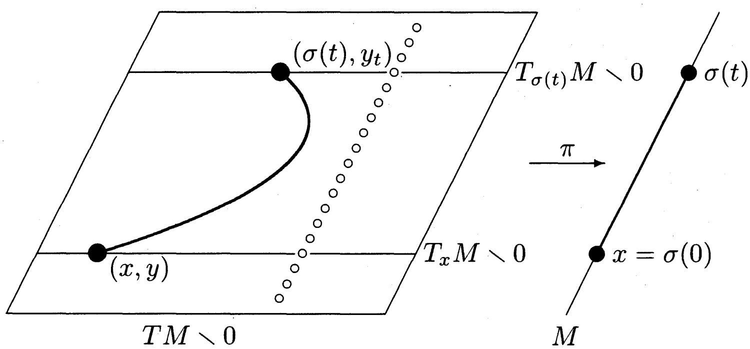
$$\hat{\dot\sigma}(\sigma(t), y) := \frac{d\sigma^i}{dt} \, \frac{\delta}{\delta x^i}_{|(\sigma(t),y)} .$$

This defines a vector field on a subset of $TM \setminus 0$, namely, the subset comprised of all fibres of $TM \setminus 0$ over the curve σ.

We now describe how to parallel translate any fixed nonzero $y \in T_xM$ along the curve σ, from x to $\sigma(t)$.

- Go to the point (x, y) in $TM \setminus 0$. Find the unique integral curve of $\hat{\dot\sigma}$ which emanates from (x, y) at $t = 0$.
- Denote the time t location along this integral curve as $(\sigma(t), y_t)$, and declare that to be the time t **canonical parallel translate** of (x, y).

This construction originates from [Ichijyo 1978].



The instantaneous velocity of $(\sigma(t), y_t)$ is

$$\begin{aligned} & \dot\sigma^i(t) \, \frac{\partial}{\partial x^i} + \dot y_t^i \, \frac{\partial}{\partial y^i} \\ = \; & \dot\sigma^i(t) \left\{ \frac{\delta}{\delta x^i} + N^j{}_i(\sigma(t), y_t) \, \frac{\partial}{\partial y^j} \right\} + \dot y_t^i \, \frac{\partial}{\partial y^i} \\ = \; & \dot\sigma^i(t) \, \frac{\delta}{\delta x^i} + \{ \dot y_t^i + N^i{}_j(\sigma(t), y_t) \, \dot\sigma^j \} \, \frac{\partial}{\partial y^i} . \end{aligned}$$

Since $(\sigma(t), y_t)$ is an integral curve of $\hat{\dot\sigma}$, that velocity equals $\dot\sigma^i(t) \, \delta/\delta x^i$ as well. Hence we must have

$$\dot y_t^i + N^i{}_j(\sigma(t), y_t) \, \dot\sigma^j = 0 .$$

Let us re-write this differential equation into a more familiar form. To that end, introduce the **Berwald connection coefficients**

$$ {}^b\Gamma^i{}_{jk} := (N^i{}_j)_{y^k} = (G^i)_{y^j y^k} \,, $$

which are manifestly symmetric in j and k.

- In the Riemannian case, $G^i = \frac{1}{2}\{{}^i{}_{jk}\}\, y^j y^k$, hence the ${}^b\Gamma^i{}_{jk}$ are simply the usual Christoffel symbols $\{{}^i{}_{jk}\}$.
- In the Finsler setting, the Berwald connection coefficients typically depend on y. Take for example the Numata type metric $F(x,y) = \sqrt{\delta_{pq}\, y^p y^q} + f_{x^p}\, y^p$ that we've been working with. Differentiating the $N^i{}_j$ obtained earlier, we get

$$ \begin{aligned} {}^b\Gamma^i{}_{jk} \quad = \quad & \ell^i\, f_{x^j x^k} \\ & + \{h^i{}_j\, f_{x^k x^p} + h^i{}_k\, f_{x^j x^p}\}\, \ell^p \\ & - \tfrac{1}{2}\, f_{x^p x^q}\, \ell^p \ell^q\, \{h^i{}_j\, \ell_k + h_{jk}\, \ell^i + h_k{}^i\, \ell_j\} \,, \end{aligned} $$

  where $h_{ij} := g_{ij} - \ell_i \ell_j$ is the so-called angular metric.

By Euler's theorem, ${}^b\Gamma^i{}_{jk}\, y^k = N^i{}_j$. The second term in our differential equation is then $y^k_t\; {}^b\Gamma^i{}_{jk}\, \dot\sigma^j$. After invoking the symmetry of ${}^b\Gamma^i{}_{jk}$, and relabeling, that becomes $y^j_t\; {}^b\Gamma^i{}_{jk}\, \dot\sigma^k$. The differential equation for the canonical parallel transport of (x,y) along σ now reads

$$ \dot y^i_t + y^j_t\; {}^b\Gamma^i{}_{jk}(\sigma(t), y_t)\, \dot\sigma^k \;=\; 0 \,. $$

This is almost identical to that for the Riemannian case, with one exception: the connection coefficients here typically depend on the vector being transported!

So, in the general Finsler setting, canonical parallel transport defines an *a priori* nonlinear map ϕ_t from $T_x M \smallsetminus 0$ to $T_{\sigma(t)} M$. Note that F must remain constant along $(\sigma(t), y_t)$ because this curve has horizontal velocity. That is,

$$ F(\sigma(t), y_t) = F(x,y) \,. $$

This, together with the fact that F is nonzero (in fact, positive) away from the origin of each tangent space, implies that:

* y_t can not be zero, because $y \neq 0$; hence the range of ϕ_t is contained in $T_{\sigma(t)} M \smallsetminus 0$.

* The only continuous extension of ϕ_t to the origin is $\phi_t(x,0) := (\sigma(t), 0)$; but it may not be differentiable there.

In this way, we have defined a differentiable map

$$ \phi_t : T_x M \smallsetminus 0 \to T_{\sigma(t)} M \smallsetminus 0 $$

that is F-preserving and injective, the latter because integral curves of $\hat{\dot\sigma}$ can not cross each other. The map ϕ_t is also surjective by the following reasoning:

* Denote the reverse of σ by $-\sigma$. Horizontally lifting the velocity of $-\sigma$, we get $-\hat{\dot\sigma}$. This implies that the integral curves needed for parallel translation along $-\sigma$ are merely the reverse of those for parallel translation along σ.
* Thus, given any element $(\sigma(t), z) \in T_{\sigma(t)}M$, its pre-image under ϕ_t can be recovered by parallel translating it along $-\sigma$, from $\sigma(t)$ back to $\sigma(0) = x$.

Canonical parallel transport, $\phi_t : T_xM \smallsetminus 0 \to T_{\sigma(t)}M \smallsetminus 0$, is therefore an *a priori* nonlinear F-preserving diffeomorphism.

There is also a linear version of parallel transport, and its differential equation reads:

$$\dot y_t^i + y_t^j \, {}^b\Gamma^i{}_{jk}(\sigma(t), \dot\sigma(t))\, \dot\sigma^k = 0 .$$

This notion, together with its underlying covariant derivative operator, plays a key role in establishing comparison theorems for Finsler metrics. See [Bao et al. 2000] for an expository account, and references.

§3. The Berwald connection and its two curvatures

3.1. Structural equations

The Berwald connection coefficients ${}^b\Gamma^i{}_{jk} := (G^i)_{y^j y^k}$ were introduced above, when we worked out the differential equation for canonical parallel transport. We now show that these are indeed the coefficients of a bona-fide connection, with which one covariantly differentiates sections of the pulled-back bundle π^*TM and its tensor products. This connection is called the Berwald connection; its associated connection forms are 1-forms $\omega_j{}^i$ which live on the base space of π^*TM, namely, the manifold $TM \smallsetminus 0$ of nonzero tangent vectors of M.

The **first structural equation** for the Berwald connection states that it is torsion-free:

$$d(dx^i) - dx^j \wedge \omega_j{}^i = 0 , \quad \text{that is,} \;\; dx^j \wedge \omega_j{}^i = 0 .$$

Being 1-forms on $TM \smallsetminus 0$, $\omega_j{}^i$ have the expansion $\Gamma^i{}_{jk}\, dx^k + Z^i{}_{jk}\, \delta y^k / F$. Substituting this into the above equation, one concludes that $Z^i{}_{jk}$ must vanish and $\Gamma^i{}_{jk}$ must be symmetric in j, k. Thus, torsion-freeness implies that

$$\omega_j{}^i = \Gamma^i{}_{jk}\, dx^k , \quad \text{where} \;\; \Gamma^i{}_{kj} = \Gamma^i{}_{jk} .$$

The **second structural equation** for the Berwald connection measures its lack of compatibility with $g := g_{ij}(x,y)\, dx^i \otimes dx^j$, the Riemannian metric of the vector bundle π^*TM:

$$dg_{ij} - g_{kj}\,\omega_i{}^k - g_{ik}\,\omega_j{}^k = -\,2\,\dot{A}_{ijk}\,dx^k + 2\,A_{ijk}\,\frac{\delta y^k}{F}\,.$$

Here, $\dot{A}$ is the so-called **Landsberg tensor**, defined as:

$$\dot{A}_{jkl} := -\tfrac{1}{2}\,(g_{is}\,y^s)(G^i)_{y^j y^k y^l} = -\tfrac{1}{2}\,F\ell_i\,(G^i)_{y^j y^k y^l}\,.$$

Note that

$$\begin{aligned} dg_{ij} &= \frac{\delta g_{ij}}{\delta x^k}\,dx^k + F\frac{\partial g_{ij}}{\partial y^k}\,\frac{\delta y^k}{F} \\ &= \left(g_{ij,x^k} - 2\,A_{ijs}\,\frac{N^s{}_k}{F}\right) dx^k + 2\,A_{ijk}\,\frac{\delta y^k}{F}\,. \end{aligned}$$

Substituting this and the refined expansion $\omega_j{}^i = \Gamma^i{}_{jk}\,dx^k$ into the second structural equation, we get

$$\Gamma_{ijk} + \Gamma_{jik} = g_{ij,x^k} - 2\,A_{ijs}\,\frac{N^s{}_k}{F} + 2\,\dot{A}_{ijk}\,.$$

Implementing Christoffel's trick, namely,

$$(\Gamma_{ijk} + \Gamma_{jik}) - (\Gamma_{jki} + \Gamma_{kji}) + (\Gamma_{kij} + \Gamma_{ikj})\,,$$

produces a formula for Γ_{ijk}. Raising the index i with g^{li} gives

$$\Gamma^l{}_{jk} = \gamma^l{}_{jk} - g^{li}\left(A_{ijs}\frac{N^s{}_k}{F} - A_{jks}\frac{N^s{}_i}{F} + A_{kis}\frac{N^s{}_j}{F}\right) + \dot{A}^l{}_{jk}\,.$$

A comparison with [Bao et al. 2000] shows that the righthand side consists of the Chern connection coefficients plus the components of the Landsberg tensor. For the purpose at hand, this specific relationship with the Chern connection is not relevant. What is important is that we have established that there is exactly one solution to the two structural equations.

In that computation, we have been writing $\Gamma^l{}_{jk}$ without the superscript ^{b} because we do not yet know whether it coincides with the ${}^b\Gamma^l{}_{jk}$ introduced earlier. Directly showing that ${}^b\Gamma^l{}_{jk} := (G^l)_{y^j y^k}$ equals the above righthand side turns out to be daunting. Instead, we shall check that ${}^b\Gamma^l{}_{jk}\,dx^k$ satisfies the two structural equations, for then by uniqueness it must agree with *the* only solution that $\Gamma^l{}_{jk}\,dx^k$ is describing.

Since ${}^b\Gamma^l{}_{kj} = {}^b\Gamma^l{}_{jk}$, torsion-freeness is automatic. Next, replacing $\omega_q{}^p$ by ${}^b\Gamma^p{}_{qr}\,dx^r$ and using $dg_{ij} = (g_{ij,x^k} - 2\,A_{ijs}\,N^s{}_k/F)\,dx^k + 2\,A_{ijk}\,\delta y^k/F$, we get

$$\begin{aligned} & dg_{ij} - g_{kj}\,\omega_i{}^k - g_{ik}\,\omega_j{}^k \\ = \ & \left\{(F F_{x^k})_{y^i y^j} - \frac{2}{F}\,A_{ijs}\,N^s{}_k - ({}^b\Gamma_{jik} + {}^b\Gamma_{ijk})\right\} dx^k + 2\,A_{ijk}\,\frac{\delta y^k}{F}\,, \end{aligned}$$

which we need to simplify to $-2\dot{A}_{ijk}\,dx^k + 2A_{ijk}\,\delta y^k/F$. The key ingredient for that purpose is $\delta F/\delta x^i = 0$, equivalently $F_{x^i} = N^j{}_i\,\ell_j$. Indeed,

$$\begin{aligned} & (F F_{x^k})_{y^i y^j} \\ = \ & (F\,N^p{}_k\,\ell_p)_{y^i y^j} \\ = \ & (N^p{}_k\,g_{pq}\,y^q)_{y^i y^j} \\ = \ & \left\{(N^p{}_k)_{y^i}\,g_{pq}\,y^q + N^p{}_k\,\frac{2}{F}\,A_{pqi}\,y^q + N^p{}_k\,g_{pq}\,\delta^q{}_i\right\}_{y^j} \\ = \ & \{(N^p{}_k)_{y^i}\,g_{pq}\,y^q + 0 + N^p{}_k\,g_{pi}\}_{y^j} \\ = \ & (N^p{}_k)_{y^i y^j}\,g_{pq}\,y^q + (N^p{}_k)_{y^i}\,\frac{2}{F}\,A_{pqj}\,y^q + (N^p{}_k)_{y^i}\,g_{pq}\,\delta^q{}_j \\ & + (N^p{}_k)_{y^j}\,g_{pi} + N^p{}_k\,\frac{2}{F}\,A_{pij} \\ = \ & F\ell_p\,(G^p)_{y^i y^j y^k} + 0 + {}^b\Gamma_{jki} + {}^b\Gamma_{ikj} + \frac{2}{F}\,A_{ijp}\,N^p{}_k \\ = \ & -2\,\dot{A}_{ijk} + \frac{2}{F}\,A_{ijp}\,N^p{}_k + ({}^b\Gamma_{jik} + {}^b\Gamma_{ijk})\,, \end{aligned}$$

which, upon substituting into the above formula for $dg_{ij} - g_{kj}\,\omega_i{}^k - g_{ik}\,\omega_j{}^k$, gives the second structural equation. This completes the proof that the **Berwald connection**

$${}^b\omega_j{}^i := {}^b\Gamma^i{}_{jk}\,dx^k = (G^i)_{y^j y^k}\,dx^k$$

is the unique solution of our two structural equations.

With this connection, we can covariantly differentiate any section of the tensor products of π^*TM. A basic example is the covariant differential

$${}^b\nabla\ell = ({}^b\nabla\ell)^i \otimes \partial_{x^i} := \{d\ell^i + \ell^r\ {}^b\omega_r{}^i\} \otimes \partial_{x^i}$$

of the canonical section ℓ. This gives a π^*TM-valued (that is, vector-valued) 1-form on the manifold $TM \setminus 0$. Since

$$d\ell^i = \frac{\delta\ell^i}{\delta x^s}\,dx^s + F\frac{\partial\ell^i}{\partial y^s}\,\frac{\delta y^s}{F}\,,$$

we have

$$ {}^b\nabla\ell = \left\{ \frac{\delta \ell^i}{\delta x^s} + \ell^r \, {}^b\Gamma^i{}_{rs} \right\} dx^s \otimes \partial_{x^i} + \left\{ F \frac{\partial \ell^i}{\partial y^s} \right\} \frac{\delta y^s}{F} \otimes \partial_{x^i} \, . $$

But

$$ \begin{aligned} \frac{\delta \ell^i}{\delta x^s} + \ell^r \, {}^b\Gamma^i{}_{rs} &= \frac{-1}{F^2} \frac{\delta F}{\delta x^s} y^i + \frac{1}{F} \frac{\delta y^i}{\delta x^s} + \frac{1}{F} (G^i)_{y^r y^s} \, y^r \\ &= 0 + \frac{1}{F} (0 - N^p{}_s \, \delta^i{}_p) + \frac{1}{F} N^i{}_s \;=\; 0 \end{aligned} $$

and

$$ F \frac{\partial \ell^i}{\partial y^s} = \delta^i{}_s - \ell^i \, \ell_s \, . $$

Therefore

$$ {}^b\nabla\ell = (\delta^i{}_s - \ell^i \, \ell_s) \, \frac{\delta y^s}{F} \otimes \partial_{x^i} \, . $$

In other words:

$$ {}^b\nabla_{\frac{\delta}{\delta x^s}} \ell = 0 \, ; \qquad {}^b\nabla_{F \partial_{y^s}} \ell = \partial_{x^s} - \ell_s \, \ell \, . $$

The first statement says that the canonical section ℓ, just like F, is covariantly constant along the horizontal directions of $TM \setminus 0$.

Earlier, we had shown that $dF = \ell_s \, \delta y^s$. Combining this with the above formula for ${}^b\nabla\ell$, we have

$$ {}^b\nabla(F\ell) = (dF) \otimes \ell + F \, {}^b\nabla\ell = \delta y^i \otimes \partial_{x^i} \, , $$

an intriguing identity that we shall later find useful, because it allows us to *computationally* convert $\hat{v} = v^i \, \partial_{y^i}$ into $v = v^i \, \partial_{x^i}$:

$$ {}^b\nabla_{\hat{v}}(F\ell) = \delta y^i(\hat{v}) \, \partial_{x^i} = (dy^i + N^i{}_s \, dx^s)(v^j \, \partial_{y^j}) \, \partial_{x^i} = v^i \, \partial_{x^i} = v \, ! $$

Let's close this section with an observation just for curiosity's sake. In terms of ${}^b\nabla$, the differential equation for the canonical nonlinear parallel transport (along σ) of $y \in T_xM$ can be restated as

$$ {}^b\nabla_{\hat{\dot\sigma}} \, y_t \;=\; 0 \quad \text{along the 'unknown' horizontal curve } (\sigma(t), y_t) \, . $$

After the equation is solved and y_t becomes known, the curve in question is the horizontal lift (of σ) which emanates from (x, y). On the other hand, the differential equation for the linear parallel transport of y along σ can be restated as

$$ {}^b\nabla_{\partial_t(\sigma, \dot\sigma)} \, y_t \;=\; 0 \quad \text{along the known curve } (\sigma(t), \dot\sigma(t)). $$

Here, $(\sigma, \dot\sigma)$ is the canonical lift of σ, and $\partial_t(\sigma, \dot\sigma)$ is its velocity field.

3.2. Curvatures of Finsler metrics

The curvature 2-forms ${}^b\Omega_j{}^i$ of the Berwald connection ${}^b\omega_j{}^i$ are defined as

$$ {}^b\Omega_j{}^i := d\,{}^b\omega_j{}^i - {}^b\omega_j{}^k \wedge {}^b\omega_k{}^i \,. $$

Being 2-forms on the manifold of nonzero tangent vectors $TM \smallsetminus 0$, they are *a priori* the sum of three types of terms: $\frac{1}{2}\,{}^bR_j{}^i{}_{kl}\,dx^k \wedge dx^l$, ${}^bP_j{}^i{}_{kl}\,dx^k \wedge \frac{\delta y^l}{F}$, and $\frac{1}{2}\,{}^bQ_j{}^i{}_{kl}\,\frac{\delta y^k}{F} \wedge \frac{\delta y^l}{F}$. Here, both ${}^bR$ and ${}^bQ$ are, *without loss of generality*, skew-symmetric in their last two indices k, l. However, taking the exterior differential of the torsion-freeness criterion $dx^j \wedge {}^b\omega_j{}^i = 0$, we get

$$ \begin{aligned} & dx^j \wedge d\,{}^b\omega_j{}^i = 0 \\ \Rightarrow\quad & dx^j \wedge \{{}^b\Omega_j{}^i + {}^b\omega_j{}^k \wedge {}^b\omega_k{}^i\} = 0 \\ \Rightarrow\quad & dx^j \wedge {}^b\Omega_j{}^i + 0 \wedge {}^b\omega_k{}^i = 0 \\ \Rightarrow\quad & dx^j \wedge {}^b\Omega_j{}^i = 0 \\ \Rightarrow\quad & \tfrac{1}{2}\,{}^bR_j{}^i{}_{kl}\; dx^j \wedge dx^k \wedge dx^l + {}^bP_j{}^i{}_{kl}\; dx^j \wedge dx^k \wedge \frac{\delta y^l}{F} \\ & + \tfrac{1}{2}\,{}^bQ_j{}^i{}_{kl}\; dx^j \wedge \frac{\delta y^k}{F} \wedge \frac{\delta y^l}{F} \;=\; 0\,. \end{aligned} $$

In particular, since ${}^bQ$ is already presumed skew-symmetric in k and l, it must vanish. Thus

$$ d\,{}^b\omega_j{}^i - {}^b\omega_j{}^k \wedge {}^b\omega_k{}^i = \tfrac{1}{2}\,{}^bR_j{}^i{}_{kl}\; dx^k \wedge dx^l + {}^bP_j{}^i{}_{kl}\; dx^k \wedge \frac{\delta y^l}{F} $$

which, upon the substitution of ${}^b\omega_j{}^i := {}^b\Gamma^i{}_{jk}\,dx^k$, leads to

$$ \begin{aligned} {}^bR_j{}^i{}_{kl} &= \frac{\delta\,{}^b\Gamma^i{}_{jl}}{\delta x^k} + {}^b\Gamma^i{}_{hk}\,{}^b\Gamma^h{}_{jl} - (\text{terms with } k,l \text{ interchanged})\,, \\ {}^bP_j{}^i{}_{kl} &= -F\frac{\partial\,{}^b\Gamma^i{}_{jk}}{\partial y^l} \;=\; -F\,(G^i)_{y^j y^k y^l}\,. \end{aligned} $$

For ease of exposition, we shall call ${}^bR$ the **Riemann curvature**, and ${}^bP$ the **Berwald curvature**. These are the only two curvature tensors for Finsler metrics, associated to the Berwald connection. In general, the use of any torsion-free connection (for instance the Chern connection) effects only two associated curvature tensors.

In the Riemannian setting, the Berwald connection coefficients are simply the usual Christoffel symbols $\{{}^i{}_{jk}\}$. These are independent of the direction variable y. Hence the Riemann curvature reduces to the familiar formula $\partial_{x^k}\{{}^i{}_{jl}\} + \{{}^i{}_{hk}\}\{{}^h{}_{jl}\}$ – (terms with k, l interchanged), and the Berwald curvature is zero.

Finsler metrics whose F have no x dependence may at first glance be said to be of Minkowski type. However, we have seen earlier that every local coordinate change $x^i = x^i(\tilde{x}^1, \ldots, \tilde{x}^n)$ on M induces a transformation $y^i = \tilde{y}^p\,(\partial x^i/\partial \tilde{x}^p)$ on y, through which a x-dependence will reassert itself in such F. Thus the concept of F having no x-dependence is not a coordinate-invariant statement. Instead, we say that a Finsler metric is **locally Minkowskian** if all its x dependence can be suppressed by a transformation of the type just described. With x absent from the formula for F in a special coordinate system, the fundamental tensor g_{ij} depends only on y; hence its formal Christoffel symbols must vanish, and so must the geodesic spray coefficients G^i. Consequently, the Berwald connection coefficients vanish, leading to the conclusion that the Riemann and Berwald curvatures are both zero, which is happily a tensorial and hence coordinate-invariant statement. It turns out that the converse holds as well. Namely, the vanishing of ${}^bR$ and ${}^bP$ implies that F must be locally Minkowskian. For a leisurely exposition of this fact, see [Bao et al. 2000], while keeping in mind that this reference uses exclusively the Chern connection (which differs from the Berwald connection by the Landsberg tensor $\dot{A}$).

We summarise here the sequence of definitions that leads us from F to ${}^bP$ and $\dot{A}$, as it highlights the economy of the Berwald connection.

Object	Formula
g_{ij}	$(\frac{1}{2}F^2)_{y^i y^j}$
$\gamma^i{}_{jk}$	$g^{is}\,\frac{1}{2}\left(g_{sj,x^k} - g_{jk,x^s} + g_{ks,x^j}\right)$
G^i	$\frac{1}{2}\,\gamma^i{}_{jk}\,y^j y^k$
$N^i{}_j$	$(G^i)_{y^j}$
${}^b\Gamma^i{}_{jk}$	$(N^i{}_j)_{y^k} = (G^i)_{y^j y^k}$
${}^bP_j{}^i{}_{kl}$	$-F\,({}^b\Gamma^i{}_{jk})_{y^l} = -F\,(G^i)_{y^j y^k y^l}$
$\dot{A}_{jkl}$	$\frac{1}{2}\,\ell_i\,{}^bP_j{}^i{}_{kl} = -\frac{1}{2}\,F\ell_i\,(G^i)_{y^j y^k y^l}$

3.3. Berwald spaces

Recall that the canonical parallel transport of $y \in T_xM$ along any curve $\sigma(t)$ which emanates from x at $t = 0$ is governed by the differential equation

$$\dot{y}_t^i + y_t^j \, {}^b\Gamma^i{}_{jk}(\sigma(t), y_t) \, \dot{\sigma}^k \; = \; 0 \, .$$

In the general Finsler setting, ${}^b\Gamma^i{}_{jk}$ typically depends on the evolution y_t of y along σ, hence the process is *a priori* nonlinear in y. It seems reasonable to expect that:

Canonical parallel transport is linear in y if and only if ${}^b\Gamma^i{}_{jk}$ does not depend on y.

Indeed, if the above differential equation is linear in y, then

$$\begin{aligned}
& y^j \; {}^b\Gamma^i{}_{jk}(\sigma, y) \text{ is linear in } y \\
\Rightarrow \quad & \{ y^j \; {}^b\Gamma^i{}_{jk} \}_{y^r y^s} = 0 \\
\Rightarrow \quad & \{ \delta^j{}_r \, {}^b\Gamma^i{}_{jk} + y^j \, ({}^b\Gamma^i{}_{jk})_{y^r} \}_{y^s} = 0 \\
\Rightarrow \quad & \{ {}^b\Gamma^i{}_{rk} + (G^i)_{y^r y^k y^j} \, y^j \}_{y^s} = 0 \\
\Rightarrow \quad & \{ {}^b\Gamma^i{}_{rk} + 0 \}_{y^s} = 0 \\
\Rightarrow \quad & {}^b\Gamma^i{}_{rk} \text{ is independent of } y.
\end{aligned}$$

Compare with the treatment in [Aikou 2001].

Finsler spaces for which canonical parallel transport is a linear process are said to be of **Berwald type**. Thus, on Berwald spaces, the F-preserving diffeomorphisms $\phi_t : T_x M \setminus 0 \to T_{\sigma(t)} M \setminus 0$ generated by canonical parallel transport become linear isometries between "normed" tangent spaces [Ichijyo 1976]. For example, Riemannian spaces and locally Minkowskian spaces belong to this family. If, as Z. Shen suggests, we agree to assign a unique colour to each Minkowskian norm, then the above description may be paraphrased to read: Berwald spaces are monochromatic creatures, while Finsler spaces are in general multicoloured.

Since the Berwald curvature is given by ${}^bP_j{}^i{}_{kl} \; = \; -F \, ({}^b\Gamma^i{}_{jk})_{y^l}$, we see that: F is of Berwald type if and only if ${}^bP_j{}^i{}_{kl} \; = \; 0$. In view of this curvature criterion for Berwald metrics, we may characterise locally Minkowskian spaces as Berwald spaces with zero Riemann curvature ${}^bR$.

On the other hand, *ad hoc* constructions of explicit Berwald metrics rely on the following characterisation in terms of $G^i := \frac{1}{2} \gamma^i{}_{pq} \, y^p y^q$, where $\gamma^i{}_{pq}$ are the formal Christoffel symbols of the fundamental tensor.

$${}^b\Gamma^i{}_{jk} \text{ is independent of } y \;\Leftrightarrow\; G^i \text{ is quadratic in } y.$$

Indeed, if G^i is quadratic in y, then ${}^b\Gamma^i{}_{jk} \; = \; (G^i)_{y^j y^k}$ won't have any y dependence (even though $\gamma^i{}_{jk}$ may). Conversely, if ${}^b\Gamma^i{}_{jk}$ is independent of y, then $\frac{1}{2} {}^b\Gamma^i{}_{jk} \, y^j y^k \; = \; \frac{1}{2} (G^i)_{y^j y^k} \, y^j y^k$ is quadratic in y; but by Euler's theorem, the latter is simply G^i.

To bring out the utility of this last viewpoint, consider Finsler metrics of Randers type:

$$F(x,y) = \sqrt{a_{ij}(x)\, y^i y^j} + b_i(x)\, y^i =: \alpha + \beta\,.$$

A straightforward computation gives

$$G^i = \tfrac{1}{2}(\{{}^i{}_{jk}\} + \ell^i\, b_{j|k})\, y^j y^k + \tfrac{1}{2}\,\alpha\,(a^{ij} - \ell^i b^j)(b_{j|k} - b_{k|j})\, y^k\,,$$

where $\{{}^i{}_{jk}\}$ are the Christoffel symbols of the Riemannian metric a_{ij}, and $b_{j|k}$ denotes the covariant derivative of the 1-form b with respect to a. Observe that if b is parallel, then G^i reduces to $\frac{1}{2}\{{}^i{}_{jk}\}\, y^j y^k$, which is quadratic in y, hence ${}^b\Gamma^i{}_{jk} = (G^i)_{y^j y^k}$ is independent of y and F is of Berwald type [Hashiguchi–Ichijyo 1975]. The converse also holds, namely, if a Randers metric is of Berwald type, then b must be parallel with respect to a. A *direct* proof is given in [Kikuchi 1979], though indirect arguments exist in [Matsumoto 1974] and [Shibata et al. 1977]; see also M. Crampin's proof in the Errata for [Bao et al. 2000].

When a Randers metric is of Berwald type:

- ${}^b\Gamma^i{}_{jk}$ reduce to the Christoffel symbols $\{{}^i{}_{jk}\}$ of a. The Riemann curvature ${}^bR_j{}^i{}_{kl}$ of the Berwald connection then becomes the usual curvature tensor of a.
- Since b is parallel, it must have constant length with respect to a, so it is either identically zero or nowhere zero. To support the latter, the Euler number $\chi(M)$ needs to vanish if M is compact boundaryless, or if $b^\sharp$ is transversal to ∂M.

In order to construct a Berwald metric of Randers type which is neither Riemannian nor locally Minkowskian, we take a non-flat Riemannian metric a, and a nonzero parallel 1-form b. Since a is non-flat, the Riemann curvature ${}^bR$ of the Berwald connection is nonzero, hence F can not be locally Minkowskian; since b is nonzero, F is not Riemannian.

For an explicit example, we choose M to be $S^1 \times S^2$, with coordinates (t, φ, θ), the latter two being the usual spherical coordinates. Since M is compact boundaryless and 3-dimensional, it automatically satisfies the topological constraint $\chi(M) = 0$. Let a be the product metric $dt \otimes dt + d\varphi \otimes d\varphi + (\sin^2 \varphi)\, d\theta \otimes d\theta$. Set $b := \epsilon\, dt$, where ϵ is any constant in $(0,1)$, and is needed to ensure the strong convexity of F. The geometry of Cartesian products tells us that a is non-flat, and b is parallel with respect to a. Expanding tangent vectors y as $y^t\, \partial_t + y^\varphi\, \partial_\varphi + y^\theta\, \partial_\theta$, the formula for our Berwald metric F reads:

$$F(x,y) = \sqrt{(y^t)^2 + (y^\varphi)^2 + (\sin^2 \varphi)(y^\theta)^2} + \epsilon\, y^t\,.$$

Berwald spaces have been classified by Z.I. Szabó in [Szabó 1981] and explicitly constructed in [Szabó 2006]. His key insight is the realisation that for a Berwald space (M, F), the Berwald connection coefficients ${}^b\Gamma^i{}_{jk}$ always coincide with the Christoffel symbols of a corresponding non-unique Riemannian metric on M. Thus, in order to account for all Berwald metrics, it suffices to consider the set of linear connections generated by all Riemannian metrics and, for each such connection, determine all the Finsler metrics that can claim this connection as their Berwald connection. However, for this purpose, the second structural equation that we used to characterise the Berwald connection is impractical. Instead, Szabó relies on the following characterisation:

> Let F be any fixed Finsler metric and denote its Berwald connection by ${}^b\Gamma^i{}_{jk}\, dx^k$. Suppose we are given a torsion-free connection $\tilde{\Gamma}^i{}_{jk}\, dx^k$, where $\tilde{\Gamma}$ is invariant under $y \mapsto \lambda y$, $\lambda > 0$. Set $\tilde{G}^i := \frac{1}{2}\, \tilde{\Gamma}^i{}_{jk}\, y^j y^k$ and $\tilde{N}^i{}_j := (\tilde{G}^i)_{y^j}$. Then, $\tilde{\Gamma}^i{}_{jk} = {}^b\Gamma^i{}_{jk}$ if and only if the following two criteria are satisfied:
>
> $$(1)\ \partial_{x^i} F - \tilde{N}^j{}_i\, \partial_{y^j} F = 0\,; \qquad (2)\ (\tilde{G}^i)_{y^j y^k} = \tilde{\Gamma}^i{}_{jk}\,.$$

One direction is simple: if $\tilde{\Gamma}^i{}_{jk} = {}^b\Gamma^i{}_{jk} := (G^i)_{y^j y^k}$, then Euler's theorem gives $\tilde{G}^i = G^i$; hence (2) is a tautology and (1) is immediate because of the horizontal constancy of F. Let us give a self-contained argument for the converse. Criterion (1) says that $F_{x^i} = \tilde{N}^j{}_i\, \ell_j$. Likewise, $\delta F/\delta x^i = 0$ is equivalent to $F_{x^i} = N^j{}_i\, \ell_j$. Hence $(\tilde{N}^j{}_i - N^j{}_i)\, \ell_j = 0$. Upon differentiation, this basic statement

$$
\begin{aligned}
\Rightarrow\quad & (\tilde{N}^j{}_i - N^j{}_i)_{y^k}\, \ell_j + (\tilde{N}^j{}_i - N^j{}_i)\, (\ell_j)_{y^k} \;=\; 0 \\
\Rightarrow\quad & (\tilde{\Gamma}^j{}_{ik} - {}^b\Gamma^j{}_{ik})\, \ell_j + (\tilde{N}^j{}_i - N^j{}_i)\, \frac{1}{F}(g_{jk} - \ell_j \ell_k) \;=\; 0 \\
\Rightarrow\quad & (\tilde{\Gamma}^j{}_{ik} - {}^b\Gamma^j{}_{ik})\, \ell_j + (\tilde{N}^j{}_i - N^j{}_i)\, \frac{1}{F}\, g_{jk} - 0 \;=\; 0 \\
\Rightarrow\quad & (\tilde{N}^j{}_i - N^j{}_i)\, g_{jk}\, g^{ks} = ({}^b\Gamma^j{}_{ik} - \tilde{\Gamma}^j{}_{ik})\, F \ell_j\, g^{ks} \\
\Rightarrow\quad & \tilde{N}^s{}_i - N^s{}_i = ({}^b\Gamma^j{}_{ki} - \tilde{\Gamma}^j{}_{ki})\, F \ell_j\, g^{ks} \\
\Rightarrow\quad & (\tilde{G}^s - G^s)_{y^i} = (N^j{}_k - \tilde{N}^j{}_k)_{y^i}\, F \ell_j\, g^{ks} \\
\Rightarrow\quad & (\tilde{G}^s - G^s)_{y^i}\, y^i = (N^j{}_k - \tilde{N}^j{}_k)_{y^i}\, y^i\, F \ell_j\, g^{ks} \\
\Rightarrow\quad & 2\, (\tilde{G}^s - G^s) = (N^j{}_k - \tilde{N}^j{}_k)\, F \ell_j\, g^{ks} \\
\Rightarrow\quad & 2\, (\tilde{G}^s - G^s) = 0\ !
\end{aligned}
$$

Consequently, $\tilde{N}^i{}_j = N^i{}_j$ and $\tilde{\Gamma}^i{}_{jk} = {}^b\Gamma^i{}_{jk}$. In the above steps, that basic statement has been invoked twice, criterion (2) thrice, and Euler's theorem twice (in which the hypothesised homogeneity of $\tilde{\Gamma}$ is needed).

Here's how Szabó uses the established characterisation to avoid unwarranted computations.

- Imagine that we have a *Riemannian* connection $\tilde{\Gamma}^i{}_{jk}\, dx^k$, from which we construct $\tilde{G}^i$ and $\tilde{N}^i{}_j$ as we did in the above discussion, then criterion (2) is automatically satisfied because $\tilde{\Gamma}$ has no y dependence. Also, the parallel transport defined by $\tilde{\Gamma}$ can be realised by lifting any curve $\sigma(t)$ on M to an appropriate integral curve of $\dot{\sigma}^i(t)\{\partial_{x^i} - \tilde{N}^j{}_i\, \partial_{y^j}\}_{|(\sigma(t),y)}$, a vector field on a subset of $TM \setminus 0$.
- Suppose, in some fixed tangent space T_xM, we have a Minkowski norm F_o which is invariant under the holonomy group of $\tilde{\Gamma}$ at x. Then F_o can be unambiguously extended to a F on TM by imposing constancy along all said integral curves involved in the parallel transport. In doing so we will have ensured that $\dot{\sigma}^i(t)\{\partial_{x^i}F - \tilde{N}^j{}_i\, \partial_{y^j}F\}_{|(\sigma(t),y)} = 0$ for *all* σ, from which we can extract criterion (1).
- The characterisation now tells us that $\tilde{\Gamma}$ is in fact the Berwald connection of F. Since $\tilde{\Gamma}$ has no y dependence, F must be a Berwald metric.

§4. The unicorn problem

4.1. Punctured tangent spaces as Riemannian manifolds

The fundamental tensor g_{ij} has been put to work in two different contexts so far.

* First, it provides a Riemannian metric for Chern's pulled-back vector bundle π^*TM.
* Second, its Sasaki-type lift gives a Riemannian metric on the manifold $TM \setminus 0$ of nonzero tangent vectors.

Let us describe a third role played by this remarkable tensor. Fix any tangent space T_xM. The coordinate basis $\{\partial_{x^i}\}$ allows us to expand $y \in T_xM$ as $y^i\, \partial_{x^i}$, thereby obtaining the global linear coordinates (y^i). At each "point" y of T_xM, we then have a coordinate basis $\{\partial_{y^i}\}$ and its dual $\{dy^i\}$. Since, in the general Finsler setting, $g_{ij}(x,y)$ are defined at all $y \neq 0$ (and typically not at the origin), the punctured tangent space $T_xM \setminus 0$ can be endowed with a Riemannian metric

$$\tilde{g}_x := g_{ij}(x,y)\, dy^i \otimes dy^j\,.$$

The point we are making here is: every Banach space, with the origin deleted, is automatically a curved Hilbert manifold. For an exposition on the Christoffel symbols and the Riemann curvature tensor of $\tilde{g}$, see [Bao et al. 2000].

4.2. Geometrical significance of the Landsberg tensor

Recall that the map $\phi_t : T_xM \setminus 0 \to T_{\sigma(t)}M \setminus 0$ which defines canonical parallel transport, namely

$$\phi_t(x, y) := (\sigma(t), y_t)\,,$$

is a F-preserving diffeomorphism. Now that we have realised these punctured spaces as Riemannian manifolds, it is natural to wonder whether ϕ_t is a Riemannian local isometry, namely, $\phi_t^*\, \tilde{g}_{\sigma(t)} = \tilde{g}_x$? If not, can we identify the obstruction?

Toward that end, we compute the quantity

$$\frac{d}{dt}(\phi_t^*\, \tilde{g}_{\sigma(t)})(\hat{v}, \hat{w}) = \frac{d}{dt}\tilde{g}_{\sigma(t)}(\phi_{t*}\hat{v}, \phi_{t*}\hat{w}) = \frac{d}{dt}\tilde{g}_{\sigma(t)}(\hat{v}_t, \hat{w}_t)\,.$$

Here, $\hat{v}$, $\hat{w}$ are two arbitrary tangent vectors in $T_y(T_xM \setminus 0)$, and we have introduced abbreviations

$$\hat{v}_t := \phi_{t*}\hat{v} \quad \text{and} \quad \hat{w}_t := \phi_{t*}\hat{w}$$

for their push-forwards, which are tangent vectors based at the point $(\sigma(t), y_t)$ of $T_{\sigma(t)}M \setminus 0$.

In order to express that rate of change using the Berwald connection, we need to get the structural equations involved. But the latter concerns the Riemannian metric $g := g_{ij}\, dx^i \otimes dx^j$ of the vector bundle π^*TM, not $\tilde{g}$. To overcome this obstacle, we expand

$$\hat{v}_t = \hat{v}_t^i\, \partial_{y^i}\,, \quad \hat{w}_t = \hat{w}_t^i\, \partial_{y^i}$$

and, using the resulting coefficients, define

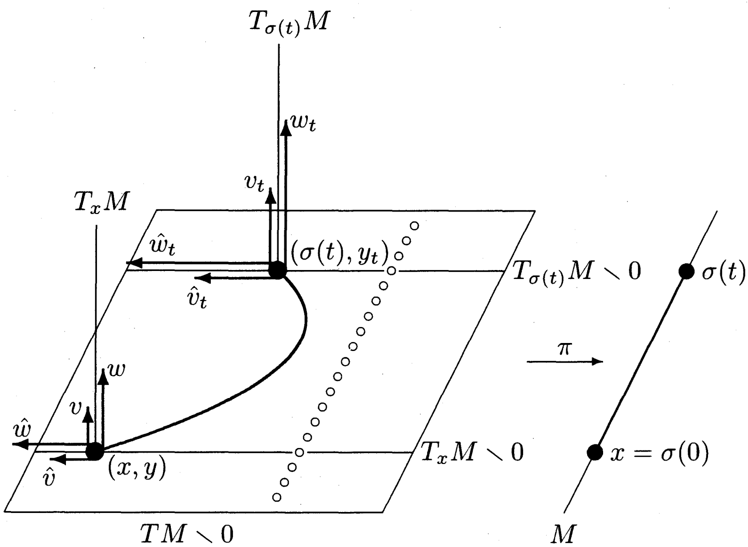
$$v_t := \hat{v}_t^i\, \partial_{x^i}\,, \quad w_t := \hat{w}_t^i\, \partial_{x^i}\,.$$

Consequently,

$$\tilde{g}_{\sigma(t)}(\hat{v}_t, \hat{w}_t) = g_{(\sigma(t), y_t)}(v_t, w_t)$$

and the original rate of change we wanted to compute now becomes

$$\frac{d}{dt}(\phi_t^*\, \tilde{g}_{\sigma(t)})(\hat{v}, \hat{w}) = \frac{d}{dt}g_{(\sigma(t), y_t)}(v_t, w_t)\,.$$



Let $c(t)$ be a curve in $TM \smallsetminus 0$ with velocity $\dot{c}$, and let $V(t)$, $W(t)$ be sections of π^*TM over c. In view of the second structural equation

$$dg_{ij} - g_{kj}\,\omega_i{}^k - g_{ik}\,\omega_j{}^k = -2\,\dot{A}_{ijk}\,dx^k + 2\,A_{ijk}\,\frac{\delta y^k}{F}$$

for the Berwald connection, it is straightforward to show that

$$\begin{aligned}\frac{d}{dt}g(V,W) &= g\left({}^b\nabla_{\dot{c}}V, W\right) + g\left(V, {}^b\nabla_{\dot{c}}W\right) \\ &\quad + 2\,A_{ijk}\,V^i\,W^j\,\frac{\delta y^k}{F}(\dot{c}) - 2\,\dot{A}_{ijk}\,V^i\,W^j\,dx^k(\dot{c})\,.\end{aligned}$$

For the case at hand, the curve in question is $(\sigma(t), y_t)$, whose velocity field is the horizontal lift $\hat{\dot{\sigma}} = \dot{\sigma}^s\,\delta/\delta x^s$ of $\dot{\sigma} = \dot{\sigma}^s\,\partial_{x^s}$. So

$$\begin{aligned}\frac{d}{dt}g_{(\sigma(t),y_t)}(v_t, w_t) &= g\left({}^b\nabla_{\hat{\dot{\sigma}}}\,v_t, w_t\right) + g\left(v_t, {}^b\nabla_{\hat{\dot{\sigma}}}\,w_t\right) \\ &\quad - 2\,\dot{A}_{ijk}\,v_t^i\,w_t^j\,\dot{\sigma}^k(t)\,.\end{aligned}$$

We digress to explain why

$${}^b\nabla_{\hat{\dot{\sigma}}}\,v_t = 0\,, \quad {}^b\nabla_{\hat{\dot{\sigma}}}\,w_t = 0\,.$$

We had exhibited near the end of §3.1 a trick that transforms $\hat{v}_t$ to v_t, namely, $v_t = {}^b\nabla_{\hat{v}_t}(F\ell)$. Hence

$$\begin{aligned} & {}^b\nabla_{\hat{\dot{\sigma}}}\, v_t \\ = \; & {}^b\nabla_{\hat{\dot{\sigma}}}\, {}^b\nabla_{\hat{v}_t}(F\ell) \\ = \; & {}^b\nabla_{\hat{v}_t}\, {}^b\nabla_{\hat{\dot{\sigma}}}(F\ell) + {}^b\nabla_{[\hat{\dot{\sigma}}, \hat{v}_t]}(F\ell) + {}^b\Omega_j{}^i(\hat{\dot{\sigma}}, \hat{v}_t)\,(F\ell)^j\,\partial_{x^i}\,. \end{aligned}$$

There are, however, some subtleties concerning the second step! Observe that ${}^b\nabla_{\hat{\dot{\sigma}}}(F\ell)$, being a section over the horizontal curve $(\sigma(t), y_t)$, is undefined along any curve tangent to $\hat{v}_t$, so we can not apply the operator ${}^b\nabla_{\hat{v}_t}$ to it. Likewise, the Lie bracket $[\hat{\dot{\sigma}}, \hat{v}_t]$ does not make sense because the velocity field $\hat{\dot{\sigma}}$ of $(\sigma(t), y_t)$ is undefined along curves tangent to $\hat{v}_t$.

Happily, the remedy is standard. We first generate a family of horizontal curves by the canonical parallel transport of vectors $y + s\hat{v}$ near y, so that $(\sigma(t), y_t)$ is the "base curve" and $\hat{v}_t$ is the "variational vector field". Explicitly,

$$\xi(s,t) := \phi_t(x, y + s\hat{v})\,.$$

Note that $s = 0$ corresponds to the base curve, and we have

$$\begin{aligned} \left(\xi_* \frac{\partial}{\partial t}\right)_{|s=0} &= \frac{\partial}{\partial t}_{|s=0} \phi_t(x, y+s\hat{v}) = \frac{\partial}{\partial t}(\sigma(t), y_t) = \hat{\dot{\sigma}}(t)\,, \\ \left(\xi_* \frac{\partial}{\partial s}\right)_{|s=0} &= \frac{\partial}{\partial s}_{|s=0} \phi_t(x, y+s\hat{v}) = \phi_{t*}\left(\frac{\partial}{\partial s}_{|s=0}(y + s\hat{v})\right) = \hat{v}_t\,. \end{aligned}$$

Next, in the troublesome quantities described above, we replace $\hat{\dot{\sigma}}$ by $\xi_*\partial_t$ and $\hat{v}_t$ by $\xi_*\partial_s$, carry out the stipulated computations, and then set $s = 0$ in order to restrict back to the base curve $(\sigma(t), y_t)$. Through this corrected perspective, we have:

$$\begin{aligned} & {}^b\nabla_{\hat{\dot{\sigma}}}\, v_t = {}^b\nabla_{\hat{\dot{\sigma}}}\, {}^b\nabla_{\hat{v}_t}(F\ell) \\ = \; & {}^b\nabla_{\xi_*\partial_t}\, {}^b\nabla_{\xi_*\partial_s}(F\ell) \\ = \; & {}^b\nabla_{\xi_*\partial_s}\, {}^b\nabla_{\xi_*\partial_t}(F\ell) + {}^b\nabla_{[\xi_*\partial_t, \xi_*\partial_s]}(F\ell) + {}^b\Omega_j{}^i(\hat{\dot{\sigma}}, \hat{v}_t)\,(F\ell)^j\,\partial_{x^i}\,. \end{aligned}$$

Immediately, we see that in the last line:

- The first term is zero because F and ℓ are both covariantly constant along horizontal curves.
- The second term vanishes because $[\xi_*\partial_t, \xi_*\partial_s] = \xi_*[\partial_t, \partial_s] = \xi_* 0 = 0$.

Though the third term needs no re-interpretation, it is zero nonetheless:

$$\begin{aligned}
&{}^b\Omega_j{}^i(\hat{\dot\sigma}, \hat v_t)\,(F\ell)^j\,\partial_{x^i}\\
= &\left\{\tfrac{1}{2}\,{}^bR_j{}^i{}_{kl}\,dx^k\wedge dx^l + {}^bP_j{}^i{}_{kl}\,dx^k\wedge\frac{\delta y^l}{F}\right\}(\hat{\dot\sigma}, \hat v_t)\,y^j\,\partial_{x^i}\\
= &\ 0 + {}^bP_j{}^i{}_{kl}\,\dot\sigma^k\,\frac{\hat v_t^l}{F}\,y^j\,\partial_{x^i}\\
= &\ -F\,(G^i)_{y^jy^ky^l}\,y^j\,\dot\sigma^k\,\frac{\hat v_t^l}{F}\,\partial_{x^i}\\
= &\ 0 \ \text{ by Euler's theorem.}
\end{aligned}$$

Thus ${}^b\nabla_{\hat{\dot\sigma}}\, v_t = 0$ and likewise ${}^b\nabla_{\hat{\dot\sigma}}\, w_t = 0$, as claimed. This ends our digression.

Returning to the question of whether $\phi_t^*\,\tilde g_{\sigma(t)} = \tilde g_x$, we now see that

$$\frac{d}{dt}(\phi_t^*\,\tilde g_{\sigma(t)})(\hat v, \hat w) = \frac{d}{dt}g_{(\sigma(t),y_t)}(v_t, w_t) = -\,2\,\dot A_{ijk}\,v_t^i\,w_t^j\,\dot\sigma^k(t)\,.$$

Since $\hat v$, $\hat w$, and $\dot\sigma$ are arbitrary, this allows us to conclude [Ichijyo 1978] that

$$\phi_t^*\,\tilde g_{\sigma(t)} = \tilde g_x \quad\text{if and only if}\quad \dot A = 0.$$

In other words, the Landsberg tensor $\dot A$ is the sole obstruction that prevents canonical parallel transport $\phi_t : T_xM \smallsetminus 0 \to T_{\sigma(t)}M \smallsetminus 0$ from being a local isometry between the Riemannian metrics $\tilde g_x$ and $\tilde g_{\sigma(t)}$. I learned the structure of this calculation from Z. Shen, around 1996.

Compare the exposition here with that in [Aikou 2001], which uses the viewpoint of fibred manifolds.

4.3. Landsberg spaces and Asanov's breakthrough

A Finsler metric F is said to be of **Landsberg type** if canonical parallel transport ϕ_t is a Riemannian local isometry between the punctured Riemannian manifolds $(T_xM \smallsetminus 0,\ \tilde g_x)$ and $(T_{\sigma(t)}M \smallsetminus 0,\ \tilde g_{\sigma(t)})$. The previous section tells us that an equivalent description of such metrics is $\dot A = 0$. Since the Landsberg tensor $\dot A$ is related to the Berwald curvature ${}^bP$ via

$$\dot A_{jkl} := -\tfrac{1}{2}\,F\ell_i\,(G^i)_{y^jy^ky^l} = \tfrac{1}{2}\,\ell_i\,{}^bP_j{}^i{}_{kl}\,,$$

and since Berwald metrics are characterised by ${}^bP = 0$, we see that every Berwald metric must be of Landsberg type. Thus we have

Riemannian & locally Minkowskian $\subset$ Berwald $\subset$ Landsberg

among four families of metrics.

The characterisations ${}^bP = 0$ and $\dot{A} = 0$ were originally the defining criteria of Berwald and Landsberg metrics, respectively. See the papers [Berwald 1929], [Berwald 1947], as well as [Landsberg 1907a], [Landsberg 1907b], [Landsberg 1908]. See also the books [Rund 1959], [Matsumoto 1986], [Antonelli et al. 1993], and especially [Shen 2001].

The Landsberg family encompasses the Berwald family, which, in view of Szabó's classification, is already a geometrically rich class of Finsler metrics. However, on Landsberg spaces that are not of Berwald type, canonical parallel transport is a local isometry between Riemannian metrics but not a linear isometry between Minkowski norms, so there is still an intellectual need for explicit examples of such spaces. This rather obvious distinction between the Landsberg and Berwald families, brought out by canonical parallel transport, serves to instill a sense of optimism into the search.

Furthermore, the $\dot{A} = 0$ description of Landsberg *surfaces* leads to a particularly elegant Gauss–Bonnet–Chern theorem for the compact boundaryless ones among such surfaces. See [Chern 1990] together with [Bao–Chern 1996]; a leisurely treatment is given in [Bao et al. 2000]. In this theorem, the Finsler metrics are required to be C^4 and strongly convex at all $y \neq 0$. It is a corollary of Szabó's classification that imposing **y-globality** (C^4 and strongly convex at all $y \neq 0$) and restricting to two dimensions dramatically cuts down on the richness of the Berwald family, for in that setting the latter consists only of Riemannian and locally Minkowskian metrics! This then, is yet another reason why one is driven to find Landsberg metrics which are not of Berwald type.

Research in the last few decades indicates that the said metrics are much more elusive than expected. For instance, in [Matsumoto 1996], one finds a list of rigidity results which almost suggest that such metrics do not exist. In 2003, Professor Matsumoto regained his sense of optimism and declared that the search for such metrics represents the next frontier of Finsler geometry.

For the sake of simpler prose, I shall from now on refer to Landsberg metrics that are not of Berwald type as **unicorns**, by analogy with those mythical single-horned horse-like creatures for which no confirmed sighting is available.

The picture has begun to change in the last few years. On several occasions since 2002, Robert Bryant has assured his audience that there is absolutely no doubt about the existence of *generalised* unicorns in two dimensions. For his exterior differential systems approach to the problem, and for the definition of a generalised Finsler metric in his framework, see [Bryant 1995]. In his announcements, Bryant states that in two dimensions, there is an abundance of such generalised metrics, depending on two families of functions of two variables; among those, there is a subclass with zero flag curvature (to be defined in the next section), depending on one family of functions of two variables.

Also, a perturbative approach has been advocated in [Bao 2006]. It is applicable to all dimensions, but is confronted at the outset by the issue of linearisation stability articulated in [Fischer–Marsden 1975a], [Fischer–Marsden 1975b].

The breakthrough came in March of 2006. In two remarkable manuscripts ([Asanov 2006a], [Asanov 2006b]), later consolidated into a single one [Asanov 2006c], Asanov produced y-local examples of unicorns in dimensions $\geqslant 3$. He was led to the prototype of such metrics in [Asanov 1995], by requiring that each **indicatrix** $I_x M := \{y \in T_x M : F(x,y) = 1\}$ has constant sectional curvature when equipped with the pull-back of the Riemannian metric $\tilde{g}_x$. This prototype was then developed in [Asanov 1998] for the absolutely homogeneous case $F(x, \lambda y) = |\lambda|\, F(x,y)$, and finally extended to the positively homogeneous setting in 2006.

Asanov's unicorns can be described in our notation as follows. The underlying manifold is the product

$$M := (0,1) \times N\,,$$

where N can be any smooth $(n-1)$-dimensional manifold with a Riemannian metric $h := h_{\mu\nu}\, dz^\mu \otimes dz^\nu$, $2 \leqslant \mu, \nu \leqslant n$. Let $\psi(t)$ be any positive function on $(0,1)$ with nowhere zero derivative ψ'. Then we have the Riemannian warped-product metric

$$a := dt \otimes dt + \psi^2(t)\; h$$

on M, together with a natural 1-form

$$b := dt\,.$$

Computing with the special coordinates $(t, z^2, \ldots, z^n)$, we find that:

- The 1-form b is closed, and has length 1 with respect to a; it is not parallel because ψ' is nowhere zero.
- The covariant derivative of b with respect to a satisfies the equation $b_{i|j} = (\psi'/\psi)\,(a_{ij} - b_i b_j)$.

Denote arbitrary local coordinates on M by $(x^i), i = 1, \ldots, n$. As usual, denote tangent vectors by y. Set

$$\begin{aligned}
\alpha &:= \sqrt{a_{ij}(x)\, y^i y^j}\,, \\
\beta &:= b_i(x)\, y^i\,, \\
q &:= \sqrt{\alpha^2 - \beta^2}\,.
\end{aligned}$$

Our α, β are, respectively, Asanov's "S" and "b". Note that $q = 0$ if and only if y is a multiple of $b^\sharp$.

Next, let

$$\begin{aligned} C_1 &:= \text{any constant in } (-2,2)\,, \\ C_2 &:= \sqrt{1-\frac{C_1^2}{4}}\,, \\ C_3 &:= \frac{C_1}{C_2}\,. \end{aligned}$$

These constants correspond, respectively, to Asanov's "g", "h", and "G". Define

$$\Phi := \begin{cases} +\frac{\pi}{2} + \arctan\left(\frac{C_3}{2}\right) - \arctan\left(\frac{q+C_1\,\beta/2}{C_2\,\beta}\right) & \text{if } \beta \geqslant 0\,, \\ \\ -\frac{\pi}{2} + \arctan\left(\frac{C_3}{2}\right) - \arctan\left(\frac{q+C_1\,\beta/2}{C_2\,\beta}\right) & \text{if } \beta \leqslant 0\,. \end{cases}$$

Then Asanov's unicorns are the Finsler metrics

$$\begin{aligned} F(x,y) &= e^{\{C_3\,\Phi/2\}}\,\sqrt{\tfrac{1}{2}[\beta + (\tfrac{1}{2}C_1 + C_2)q]^2 + \tfrac{1}{2}[\beta + (\tfrac{1}{2}C_1 - C_2)q]^2} \\ &= e^{\{C_3\,\Phi/2\}}\,\sqrt{\beta^2 + C_1\,\beta\,q + q^2}\,, \end{aligned}$$

whose indicatrices have [Asanov 2006a] constant positive sectional curvature C_2^2! Note that these F are y-local because the factor $1/\sqrt{\alpha^2 - \beta^2}$ is present in *all* derivatives of q, thereby causing a singularity whenever y is a multiple of $b^\sharp$. Our F here corresponds to Asanov's "K".

Being Landsberg metrics that are not of Berwald type, unicorns are characterised by the statement:

$$\dot{A}_{jkl} = -\,\tfrac{1}{2}\,F\ell_i\,(G^i)_{y^j y^k y^l} = 0\,, \quad \text{with } G^i \textit{ not} \text{ quadratic in } y.$$

In Asanov's metrics,

$$G^i = \tfrac{1}{2}\,\{{}^i{}_{jk}\}\,y^j y^k + \tfrac{1}{2}\,C_1\,(y^i - \beta\,b^i)\;q\;\frac{\psi'}{\psi}\,,$$

where $\{{}^i{}_{jk}\}$ are the Christoffel symbols of the Riemannian warped-product metric a, and ψ is the warping factor with nowhere zero derivative. Our convention for G^i here is *half* that in [Bao et al. 2000] and [Asanov 2006a], [Asanov 2006b].

- In $\dim M = 2$, we can always find another 1-form e such that $\{b, e\}$ is an orthonormal coframe field with respect to a. Since $a = b \otimes b + e \otimes e$, the quantity q loses its square root and becomes $|e_s(x)\, y^s|$, in which case the above G^i are quadratic in y, and F is of Berwald type.

- In $\dim M \geqslant 3$, the square root on q persists, so the G^i remain irrational, making it impossible for $(G^i)_{y^j y^k y^l}$ to vanish. Hence F can never be of Berwald type.

Direct computations show that

$$F\ell_i = \{a_{is}\, y^s + C_1\, q\, b_i\}\, \frac{F^2}{\beta^2 + C_1\, \beta\, q + q^2}\,,$$

together with

$$(a_{is}\, y^s)(G^i)_{y^j y^k y^l} \;=\; 0 \;=\; b_i\, (G^i)_{y^j y^k y^l}\,.$$

These effect the Landsberg criterion $\dot{A}_{jkl} = 0$; complete details are given in [Asanov 2006b]. Thus, Asanov's y-local metrics are unicorns in dimension $\geqslant 3$, but *not* in dimension 2.

Asanov's unicorns belong to the class of so-called (α, β) metrics, of which Matsumoto's "slope of a mountain" metric [Matsumoto 1989] is an inspiring example; see [Antonelli et al. 1993], [Shen 2004] for expositions, and [Bao–Robles 2004] for a synopsis of Matsumoto's account. Prompted by Asanov's discovery, Z. Shen debugged a project that had been in progress since 2004, eventually leading to [Shen 2006]. In this work, he proved with the help of Maple computations that in dimension at least three, there are only y-local (that is, F being either not even C^2 or not strongly convex at some nonzero y) unicorns of (α, β) type. He did so by showing that, within this class, there are exactly two families of unicorns, both y-local, with one consisting of Asanov's metrics. Shen also finds that the situation in dimension 2 is no better.

The search for y-global unicorns should continue, because they represent an unblemished ideal. In view of the discussion about the Gauss–Bonnet–Chern theorem for Landsberg surfaces, expressed earlier in this section, y-global examples on compact boundaryless 2-dimensional manifolds are particularly coveted.

§5. Ricci flow for Finsler metrics

5.1. Flag curvatures and Berwald's formula

We now turn to the Riemann curvature

$${}^bR_j{}^i{}_{kl} = \frac{\delta\, {}^b\Gamma^i{}_{jl}}{\delta x^k} + {}^b\Gamma^i{}_{hk}\, {}^b\Gamma^h{}_{jl} - (\text{terms with } k, l \text{ interchanged})$$

of the Berwald connection. According to the second variation of travel time $\int F(\sigma, \dot{\sigma})ds$, only a piece of that curvature is relevant, namely

$$R_{ik} := \ell^j \, {}^bR_{jikl} \, \ell^l \, .$$

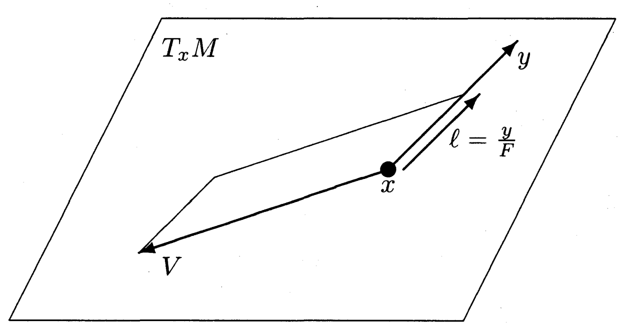
Most importantly, this piece is independent of all named-brand connections. For instance, the second variation is treated in detail in [Bao et al. 2000]; though that reference uses the Chern connection exclusively, it does give the relation between the Riemann curvatures of the Chern and Berwald connections, from which one can check that the resulting R_{ik} are identical. Because of this, we can borrow from [Bao et al. 2000] the symmetry property

$$R_{ki} = R_{ik} \, ,$$

whose derivation requires a certain Bianchi identity that we will not present here.

We have seen three roles played by the fundamental tensor g_{ij}. First, as a Riemannian metric $g := g_{ij} \, dx^i \otimes dx^j$ for Chern's pulled-back vector bundle π^*TM. Second, as a Riemannian metric $\hat{g} := g_{ij} \, dx^i \otimes dx^j + g_{ij} \, \delta y^i \otimes \delta y^j / F^2$ on the manifold of nonzero tangent vectors $TM \setminus 0$. Third, as a Riemannian metric $\tilde{g}_x := g_{ij} \, dy^i \otimes dy^j$ for the punctured tangent space $T_xM \setminus 0$. Remarkably, there is yet a fourth hat worn by this versatile object, one manifest in the definition of the flag curvature, a natural generalisation of the Riemannian sectional curvature.

A flag consists of the following data: a location $x \in M$ at which we are to plant the flagpole; a nonzero $y \in T_xM$ that serves as the flagpole; another nonzero $V \in T_xM$ that is transverse to y. This V, together with $\ell = y/F$, shall represent the actual "cloth" part of the flag.



The flagpole y singles out the inner product

$$g_y := g_{ij}(x,y)\, dx^i \otimes dx^j \,,$$

with respect to which the cloth part of the flag has area

$$g_y(\ell,\ell)\, g_y(V,V) - g_y^2(\ell,V) = \frac{1}{F^2(x,y)} \left\{ g_y(y,y)\, g_y(V,V) - g_y^2(y,V) \right\}.$$

This allows us to define the flag curvature $K(x,y,V)$ by a formula that becomes structurally the same as that for the sectional curvature:

$$\begin{aligned} K(x,y,V) \;&:=\; \frac{V^i\, R_{ik}\, V^k}{g_y(\ell,\ell)\, g_y(V,V) - g_y^2(\ell,V)} \\ &=\; \frac{V^i\, \ell^j\, {}^bR_{jikl}\, \ell^l\, V^k}{g_y(\ell,\ell)\, g_y(V,V) - g_y^2(\ell,V)} \\ &=\; \frac{V^i\, y^j\, {}^bR_{jikl}\, y^l\, V^k}{g_y(y,y)\, g_y(V,V) - g_y^2(y,V)} \,. \end{aligned}$$

(For a totally different approach to the flag curvature, see the exposition [Foulon 2002] and references therein.)

In practice, however, computations seldom survive to the ${}^bR_{jikl}$ stage, due to the enormous sizes of the intermediate expressions. This is where **Berwald's formula**

$$F^2\, R^i{}_k = 2\,(G^i)_{x^k} - (G^i)_{y^j}\,(G^j)_{y^k} - y^j\,(G^i)_{x^j y^k} + 2\, G^j\,(G^i)_{y^j y^k}$$

comes to the rescue. One leisurely derivation of this formula can be found in [Bao–Robles 2004].

Computing with Berwald's formula, the flag curvatures of the Numata type metric $F(x,y) := \sqrt{\delta_{ij}\, y^i y^j} + f_{x^i}\, y^i$ are shown to be *independent* of the transverse edges V:

$$K(x,y) = \frac{3}{4\,F^4}\left(f_{x^i x^j}\, y^i y^j\right)^2 - \frac{1}{2\,F^3}\left(f_{x^i x^j x^k}\, y^i y^j y^k\right) \,,$$

but explicitly dependent on the flagpoles y. An outline of the calculation can be found in [Bao et al. 2000].

For Finsler metrics, we have the Schur lemma due to [Berwald 1947], [del Riego 1973], and [Matsumoto 1986]. It says that in dimension at least 3, if the flag curvature function depends *neither* on the transverse edges *nor* on the flagpoles, then it must in fact be constant. The above example shows that the Schur lemma does *not* rule out cases where $K = K(x,y)$.

Finsler metrics with flag curvatures of the type K = constant or $K = K(x, y)$ are abundant. A computationally useful characterisation [Bao et al. 2000] of such metrics is

$$F^2\, R^i{}_k = K\, \{F^2\, \delta^i{}_k - y^i\, (g_{ks}\, y^s)\}\,.$$

- Randers metrics with constant flag curvature have been classified in [Bao et al. 2004], with a historical account of the problem in [Bao 2004]. Key background works in this regard are [Matsumoto–Shimada 2002] and [Bao–Robles 2003]. The geodesics of such spaces are classified in [Robles 2005].
- Randers metrics with $K = K(x, y)$ include the ones of Numata type mentioned above. Many more interesting examples in this category are given in [Chern–Shen 2005], together with an informative exposition. (Their V means something different.)
- Finsler metrics with constant flag curvature but which are not of Randers type have been extensively studied by Bryant. For interesting examples and a refreshing viewpoint, see the articles [Bryant 1996], [Bryant 2002].

When the flag curvature (is possibly non-constant and) falls within a certain precise range, we have the sphere theorem [Rademacher 2004]; see that article for a detailed exposition and references therein, especially to Dazord's work on the special case where $F(x, -y) = F(x, y)$.

We conclude this section with a discussion of the geometrical meaning of the flag curvature.

To that end, fix $x \in M$ and take any 2-plane Π which passes through the origin of T_xM. Let us limit our choices of the flagpole y to this Π and, for each selected y, choose our transverse edge V from Π as well. Each flagpole y generates a geodesic which emanates from x with initial velocity y. The totality of all such geodesics gives rise to a surface S in M, and the flag curvature function $K(x, y, V)$ is the Finslerian analogue of the Gaussian curvature of S. Since in this case span$\{y, V\}$ is always Π, we may think of this restricted K as depending only on x and y.

For Riemannian surfaces, the Gaussian curvature $K(x)$ dictates the behaviour of geodesic rays that emanate from x. This behaviour is quantified by the use of Jacobi fields. It says, in essence, that nearby rays tend to focus if K is positive, and tend to diverge if K is negative. One finds in [Bao et al. 2000] the details that re-establish these features for the Finslerian realm, again through the use of Jacobi fields.

Note, however, that there is a crucial difference between the two settings. For Riemannian surfaces, the Gaussian curvature $K(x)$ depends only on the location x. Thus, if nearby geodesic rays behave in a certain way when heading out from x along some sector, then the *same* behaviour must be exhibited by those that are heading out from x along other sectors. For Finslerian surfaces, the y-dependence in $K(x, y)$ breaks this isotropy.

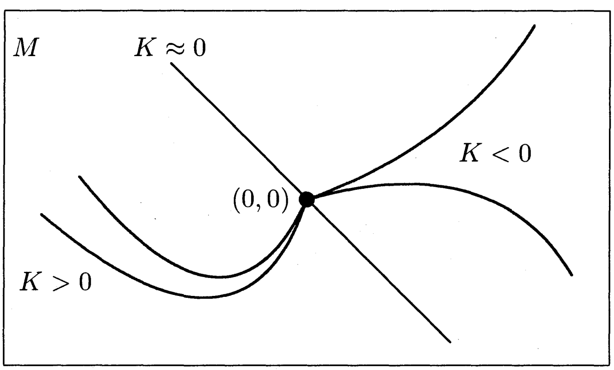
As a concrete example, take again our Numata type metric $F(x, y) = \sqrt{\delta_{ij}\, y^i y^j} + f_{x^i}\, y^i$, this time in dimension 2, with (x^1, x^2) abbreviated as (s, t), (y^1, y^2) abbreviated as (p, q), and $f(x) := s^3 + s^2 t + st^2 + t^3$. Then the stated formula for its flag curvature function becomes

$$\frac{3}{F^4}\left[(3s + t)p^2 + 2(s + t)pq + (s + 3t)q^2\right]^2 - \frac{3}{F^3}(p^3 + p^2 q + pq^2 + q^3),$$

where $F(x, y) = \sqrt{p^2 + q^2} + (3s^2 + 2st + t^2)\, p + (s^2 + 2st + 3t^2)\, q$. As a result, $K(x, y) = K(s, t; p, q)$ is approximately given by

$$\frac{-3}{(\sqrt{p^2 + q^2})^3}\,(p^3 + p^2 q + pq^2 + q^3)$$

at locations (s, t) close to the origin $(0, 0)$, and is exactly equal to this expression at the origin. The said expression is zero where $p + q = 0$, negative where $p + q > 0$, and positive where $p + q < 0$. Accordingly, the behaviour of geodesic rays emanating from $(0, 0)$ is depicted as follows.



In the above sketch, the underlying manifold M is a small neighbourhood of the origin in $\mathbb{R}^2$, on which the pointwise Euclidean norm of the differential *df* is everywhere less than 1; this is to ensure the strong convexity of the Finsler metric F.

Imagine that this sketch is a map prepared for mountain climbers. We can picture a mountain ridge above the locus of $K \approx 0$, with perhaps the summit directly above the point $(0, 0) \in M$. The slope of the mountain on the left of this ridge sits above the region in M with $K > 0$; similarly, the slope on the right of the ridge sits above the region in M with $K < 0$. The climate and the vegetation on the two slopes are typically quite different, so much so that the most efficient paths of descent from the summit project onto the geodesics shown. Any attempt to capture this anisotropy from the Riemannian perspective would necessarily introduce cusps. The Finslerian model can therefore be viewed as the unfolding or the blowing up of these singularities.

5.2. Ricci-constant metrics and Chern's question

For any fixed flagpole y in T_xM, we use the inner product g_y to find $n-1$ orthonormal transverse edges $\{e_\mu : \mu = 1, \ldots, n-1\}$ perpendicular to y. The flag curvatures $K(x, y, e_\mu)$ corresponding to these transverse edges simplify to $(e_\mu)^i \, R_{ik} \, (e_\mu)^k$. Next, introduce $e_n := y/F$ to complete $\{e_\mu\}$ into a g_y-orthonormal basis for T_xM, and note that $(e_n)^i \, R_{ik} \, (e_n)^k = \ell^i \, R_{ik} \, \ell^k = 0$ because ${}^bR_{jikl}$ is skew-symmetric in its last two indices. Defining the **Ricci scalar** $\mathcal{R}ic$ as the sum of those $n-1$ flag curvatures, we have

$$\begin{aligned} \mathcal{R}ic &:= \sum_{\mu=1}^{n-1} K(x, y, e_\mu) = \sum_{\mu=1}^{n-1} (e_\mu)^i \, R_{ik} \, (e_\mu)^k \\ &= \sum_{a=1}^{n} (e_a)^i \, R_{ik} \, (e_a)^k = \sum_{a=1}^{n} R_{aa} \\ &= R^a{}_a = R^i{}_i \,. \end{aligned}$$

Thus, while conceptually the Ricci scalar is a sum of $n-1$ flag curvatures, from a tensor analysis viewpoint it is simply the trace of R_{ik}. By virtue of this realisation, Berwald's formula immediately gives the following computational shortcut for the Ricci scalar:

$$F^2 \, \mathcal{R}ic = 2 \, (G^i)_{x^i} - (G^i)_{y^j} \, (G^j)_{y^i} - y^j \, (G^i)_{x^j y^i} + 2 \, G^j \, (G^i)_{y^j y^i} \,.$$

A companion of the Ricci scalar is the Ricci tensor

$$\mathrm{Ric}_{ij} := \left(\tfrac{1}{2} F^2 \, \mathcal{R}ic \right)_{y^i y^j}$$

proposed in [Akbar-Zadeh 1995]. If F happens to be Riemannian, this definition reproduces the standard formula for the Ricci tensor, namely $R_i{}^s{}_{sj}$. Also, thanks to Euler's theorem, we have

$$\mathcal{R}ic = \mathrm{Ric}_{ij} \, \ell^i \ell^j = \frac{1}{F^2} \left(\mathrm{Ric}_{ij} \, y^i y^j \right) .$$

So the Ricci scalar and the Ricci tensor are content-wise equivalent. Furthermore, this last expression for $\mathcal{R}ic$ shows that it generalises a familiar Riemannian object, namely, the so-called "Ricci curvature in the direction of y".

Generically, $\mathcal{R}ic$ depends on both x and y and is therefore a function on the manifold $TM \setminus 0$ of nonzero tangent vectors. Due to its invariance under positive rescaling $y \mapsto \lambda y$, $\lambda > 0$, the Ricci scalar in fact lives on the manifold of rays SM. Finsler metrics for which $\mathcal{R}ic$ is a constant function are therefore nothing short of being remarkable, and are said to be **Ricci-constant**. It is straightforward to check that

$$\mathcal{R}ic = \text{constant } C \qquad \Leftrightarrow \qquad \text{Ric}_{ij} = C\, g_{ij}\,,$$

the latter characterisation being one familiar to Riemannian geometers.

The conceptual definition of $\mathcal{R}ic$ makes transparent the statement that every Finsler metric of constant flag curvature K has constant Ricci scalar $(n-1)K$. Thus, metrics with constant flag curvature lie within the family of Ricci-constant metrics. As in the Riemannian setting, this inclusion is proper, and can be ascertained with the following explicit example.

* Set $M := S^m(\sqrt{m-1}) \times S^n(\sqrt{n-1})$. Since the factors have constant sectional curvatures $1/(m-1)$ and $1/(n-1)$, respectively, they are Ricci-constant with Ricci scalar equal to 1. A moment's thought about the sectional curvatures of product metrics tells us that the product Riemannian metric h on M has constant Ricci scalar 1 as well, and is *not* of constant sectional curvature.
* Regard points of each of the above spheres as row vectors in $\mathbb{R}^{m+1}$ and $\mathbb{R}^{n+1}$. Let $\Omega_1 \in so(m+1)$, $\Omega_2 \in so(n+1)$ be skew-symmetric real matrices of the indicated sizes. To each $(p,q) \in M$, we assign the element $(p\Omega_1, q\Omega_2) \in T_{(p,q)}M$. This assignment gives a Killing vector field W of h. Since M is compact, we can scale W by a constant, if necessary, to achieve $h(W,W) < 1$.
* As explained in [Bao–Robles 2004], the Finsler metric

$$\frac{\sqrt{[h(y,W)]^2 + h(y,y)\{1 - h(W,W)\}} - h(y,W)}{1 - h(W,W)}$$

of Randers type, with navigation data (h, W), is a strongly convex Ricci-constant metric with $\mathcal{R}ic = 1$, and is *not* of constant flag curvature.

Professor Chern had asked, on several occasions, whether every smooth manifold admits a Ricci-constant Finsler metric? Chern's question has already been settled in the affirmative for dimension 2 because every 2-manifold admits a complete Riemannian metric of constant Gaussian curvature. See the book [Besse 1987] for an explicit construction, and references therein.

Much less is known about dimension 3. As a concrete example, consider $M := S^2 \times S^1$.

We first show that for this M, the Ricci scalar $\mathcal{R}ic$ of any Finsler metric can not be a positive function. Indeed, suppose the conclusion was violated by some F. The compactness of M would imply that F is complete; the compactness of SM would imply that the Ricci scalar of F is in fact *uniformly* positive. Through local diffeomorphisms, the universal cover $\tilde{M} := S^2 \times \mathbb{R}$ of M would inherit a forward geodesically complete Finsler metric with uniformly positive Ricci scalar. Then the Bonnet–Myers theorem ([Auslander 1955], [Dazord 1969]; see [Bao et al. 2000] for an exposition), applied to $\tilde{M}$, would tell us that the latter is compact, which is a contradiction. This establishes the claim. So, $\mathcal{R}ic$ must be $\leqslant 0$ somewhere; thus, between the two flag curvatures that sum up to $\mathcal{R}ic$, one of them must be $\leqslant 0$ there.

Incidentally, the existence of Riemannian metrics with *non*-constant $\mathcal{R}ic \leqslant 0$ on $S^2 \times S^1$ has already been demonstrated in [Gao–Yau 1986] by smoothing the singularities of certain special metrics. On the other hand, in [Lohkamp 1994] the same result is proved via local deformations; most importantly, Lohkamp's method works in all dimensions $\geqslant 3$.

Next, we show that on our M, the flag curvatures of any Finsler metric can not be bounded above by zero. Again, suppose the conclusion was violated by some F, which would have to be complete because M is compact. The simply-connected universal cover $\tilde{M} := S^2 \times \mathbb{R}$ would inherit a forward geodesically complete Finsler metric with all non-positive flag curvatures. Then the Cartan–Hadamard theorem ([Auslander 1955], [Dazord 1969]; see also [Bao et al. 2000]), applied to $\tilde{M}$, would tell us that the latter is contractible, which is absurd. Thus the claim holds. In particular, between the two flag curvatures that sum up to the Ricci scalar $\mathcal{R}ic$, one of them must be positive somewhere.

If we were addressing Chern's question about $S^2 \times S^1$ within Riemannian geometry, our quest would be finished and the answer would have to be negative. The reason is that in dimension 3, the automatic vanishing of the Weyl conformal tensor forces the rigidity that having constant Ricci scalar $\mathcal{R}ic$ is the same as having constant sectional curvature $\mathcal{R}ic/2$; but, as we discovered above, the two flag (sectional) curvatures that sum up to $\mathcal{R}ic$ can not even have the same sign everywhere, let alone being identical! In the Finslerian realm, however, such rigidity is not yet known to be true, thereby leaving open the window that there may be Finsler metrics with constant non-positive Ricci scalar $\mathcal{R}ic$ on $S^2 \times S^1$. The next section proposes a deformation procedure for tackling this type of question.

5.3. Ricci flow

The geometric evolution equation [Hamilton 1982]

$$\partial_t(g_{ij}) \;=\; -2\,\mathrm{Ric}_{ij}\,, \qquad g(t=0) = g_o$$

is known as the un-normalised Ricci flow in Riemannian geometry. For an exposition of the rich geometric analysis behind this equation, see [Chow–Knopf 2004]. Here, we have put parentheses around g_{ij} for a pedantic reason: if the local coordinates should happen to depend on t, then $\partial_t(g_{ij})$ and $(\partial_t\, g)_{ij}$ represent different objects conceptually, and are in general numerically unequal!

In principle, the same equation can be used in the Finsler setting, because both g_{ij} and Ric_{ij} have been generalised to that broader framework, albeit gaining a y dependence in the process. However, there are two reasons why we shall refrain from doing so.

(1) Not every symmetric covariant 2-tensor $g_{ij}(x,y)$ arises from a Finsler metric $F(x,y)$. As explained in [Bao–Robles 2004], the essential integrability criterion is the total symmetry of $(g_{ij})_{y^k}$ on all three indices i, j, k. Having to incorporate this criterion into every step of the analysis is at best inconvenient.

(2) There is more than one geometrical context in which g_{ij} makes sense. Namely, as a fibre Riemannian metric on Chern's pulled-back vector bundle π^*TM, as a Riemannian metric on the manifold of non-zero tangent vectors and the manifold of rays, as a Riemannian metric on punctured tangent spaces, and as a direction dependent inner product on each tangent space.

Instead of this tensor evolution equation, we prefer a scalar one without any integrability criterion, and one for which the geometrical context is canonically clear. Note that contracting $\partial_t(g_{ij}) = -2\,\mathrm{Ric}_{ij}$ with $y^i y^j$ gives, via Euler's theorem, $\partial_t F^2 = -2\,F^2\,\mathcal{R}ic$. That is,

$$\partial_t \log F \;=\; -\,\mathcal{R}ic\,, \qquad F(t=0) = F_o\,.$$

This scalar equation directly addresses the evolution of the Finsler metric F, and makes geometrical sense on both the manifold of nonzero tangent vectors $TM \smallsetminus 0$ and the manifold of rays SM. It is therefore suitable as an **un-normalised Ricci flow** for Finsler geometry.

If M is compact, then so is SM, and we can normalise the above equation by requiring that the flow keeps the volume of SM constant. Recalling the Hilbert form $\omega := F_{y^i}\, dx^i$, that volume is

$$\mathrm{Vol}_{SM} \;:=\; \int_{SM} \frac{(-1)^{(n-1)(n-2)/2}}{(n-1)!}\, \omega \wedge (d\omega)^{n-1} \;=:\; \int_{SM} dV_{SM}\,.$$

During the evolution, F, ω, and consequently the volume form dV_{SM} and the volume Vol_{SM}, all depend on t. On the other hand, the domain of integration SM, being the quotient space of $TM \setminus 0$ under the equivalence relation $z \sim y \Leftrightarrow z = \lambda y$ for some $\lambda > 0$, is totally independent of any Finsler metric, and hence does *not* depend on t.

Thanks to the work and insights in [Akbar-Zadeh 1995], we have

$$\partial_t\, dV_{SM} = \{ g^{ij}\, \partial_t(g_{ij}) - n\, \partial_t \log F \}\; dV_{SM}\,,$$

together with the surprising realisation that the scalar $g^{ij}\, \partial_t(g_{ij})$ is merely $2n\, \partial_t \log F$ plus a covariant divergence on SM. If M is boundaryless, besides being compact, then so is SM. In that case, the said covariant divergence integrates to zero on SM and one obtains the remarkable formula

$$\partial_t\, \mathrm{Vol}_{SM} \;=\; \int_{SM} \{ n\, \partial_t \log F \}\; dV_{SM}\,.$$

An exposition of these derivations, at times adopting approaches different from those used by Akbar-Zadeh, is relegated to the Appendix.

To normalise the proposed Ricci flow, we replace its equation by $\partial_t \log F = -\mathcal{R}ic + \mathcal{C}(t)$ and determine $\mathcal{C}(t)$ so that the volume of SM remains constant under the evolution of F; here, $\mathcal{C}(t)$ is a function of t only. Substituting this objective into the formula for $\partial_t\, \mathrm{Vol}_{SM}$, we get

$$0 \;=\; \int_{SM} n\, \{ -\mathcal{R}ic + \mathcal{C}(t) \}\; dV_{SM}\,,$$

from which we conclude that

$$\mathcal{C}(t) \;=\; \frac{1}{\mathrm{Vol}_{SM}} \int_{SM} \mathcal{R}ic\; dV_{SM} \;=:\; \mathrm{Avg}(\mathcal{R}ic)\,.$$

Hence we propose that a **normalised Ricci flow** for Finsler metrics is

$$\partial_t \log F \;=\; -\,\mathcal{R}ic + \mathrm{Avg}(\mathcal{R}ic)\,, \qquad F(t=0) = F_o$$

whenever the underlying manifold M is compact and boundaryless.

Ricci-constant metrics are exactly the fixed points of the above flow. Starting with any familiar metric on M as the initial data F_o (for instance, on $M = S^2 \times S^1$, F_o can be the Berwald metric of Randers type from Section 3.3), we may deform it using the proposed normalised Ricci flow, in the hope of arriving at a Ricci-constant metric. To that end, Berwald's formula for $\mathcal{R}ic$, presented earlier, will play an essential role in facilitating the computations, though the averaging of $\mathcal{R}ic$ over SM still remains a daunting challenge. It is hoped that this approach eventually proves to be viable for addressing Chern's question.

§6. Appendix: Calculus on the manifold of rays

6.1. Variation of volume forms

The manifold $TM \setminus 0$ of nonzero tangent vectors inherits a Sasaki type Riemannian metric

$$\hat{g} \;=\; g_{ij}\, dx^i \otimes dx^j + g_{ij}\, \frac{\delta y^i}{F} \otimes \frac{\delta y^j}{F}$$

from the Finsler structure F. Since every term on the righthand side is invariant under positive rescaling $y \mapsto \lambda y$, $\lambda > 0$, we may regard $\hat{g}$ as the description, via affine coordinates, of a Riemannian metric on the manifold SM of rays. Thus, both $TM \setminus 0$ and SM are endowed with Riemannian metrics that are canonically defined by F.

Calculations on SM can be done intrinsically, by writing out its Riemannian metric with the help of a special type of orthonormal coframes [Bao et al. 2000] on $TM \setminus 0$. For the purposes at hand, however, such intrinsic calculations are riddled with combinatorics and maneuvoers that are not instructive. Happily, there is a more illuminating, albeit *extrinsic*, way of achieving the same goals.

Each element of SM is a ray $[y] := \{\lambda y : \lambda > 0\}$, where $y \neq 0$; as such it corresponds uniquely to a direction, given by the unit vector y/F. The said orthonormal coframes do make clear that the map $[y] \mapsto y/F$ is a Riemannian isometry between SM and IM, the **submanifold** (in $TM \setminus 0$) **of unit tangent vectors** $\{(x, y) : F(x, y) = 1\}$, also known as the *indicatrix* bundle. (Incidentally, IM is given the notation Σ_F in [Bryant 2002].) The inter-relationships among these manifolds are summarised below:

$$\begin{array}{ccccc}
\frac{y}{F(x,y)} & \xleftarrow{\text{inclusion}} & \frac{y}{F(x,y)} & \xleftarrow{\text{diffeomorphism}} & [y] \\[2ex]
TM \setminus 0 & & IM & & SM \\[2ex]
y & \xrightarrow{\text{normalisation}} & \frac{y}{F(x,y)} & \xrightarrow{\text{quotient}} & [\frac{y}{F}] = [y]
\end{array}$$

Note that while $TM \setminus 0$ and SM are totally independent of F, IM is on the other hand equivalent to F in terms of mathematical content.

With respect to the natural (but non-holonomic) basis of 1-forms $\{dx^i; \delta y^i/F\}$, the components of the metric $\hat{g}$ on $TM \setminus 0$ are given by the block diagonal matrix

$\hat{g}_{IJ}$	$J = j$	$J = n + j$
$I = i$	g_{ij}	0
$I = n + i$	0	g_{ij}

whose inverse is

$\hat{g}^{IJ}$	$J = j$	$J = n + j$
$I = i$	g^{ij}	0
$I = n + i$	0	g^{ij}

The volume form on $TM \setminus 0$ is therefore

$$dV_{TM \setminus 0} \;=\; \sqrt{\det(\hat{g}_{IJ})}\; dx^1 \wedge \cdots \wedge dx^n \wedge \frac{\delta y^1}{F} \wedge \cdots \wedge \frac{\delta y^n}{F}\,,$$

where $\sqrt{\det(\hat{g}_{IJ})}$ numerically equals $\det(g_{ij})$.

Since IM is defined by one scalar condition $F(x, y) = 1$, it is a submanifold of codimension one, namely a hypersurface, of $TM \setminus 0$. For each fixed x, the chain rule and the identity $F_{y^i} = g_{is}\,\ell^s$ imply that the vector field $\ell^s(F\partial_{y^s})$ is $\hat{g}$-orthogonal to the $(n-1)$-dimensional indicatrix at x; the same vector field is also $\hat{g}$-orthogonal to $\delta/\delta x^1, \ldots, \delta/\delta x^n$. A little thought then convinces us that the unit outward-pointing normal field of the hypersurface IM is

$$\hat{n}_{\text{out}} \;=\; \ell^s \left(F\,\frac{\partial}{\partial y^s} \right).$$

Consequently, its volume form dV_{IM} can be obtained by contracting $\hat{n}_{\text{out}}$ into the first terms of the tensor products that comprise $dV_{TM \setminus 0}$, and the resulting formula reads

$$\begin{aligned} dV_{IM} = \sqrt{\det(\hat{g}_{IJ})}\, \tfrac{1}{F^n} \textstyle\sum_{j=1}^{n} \quad & (-1)^{n+j-1}\, y^j \\ & dx^1 \wedge \cdots \wedge dx^n \wedge \\ & \delta y^1 \wedge \cdots \wedge \delta y^{j-1} \wedge \delta y^{j+1} \wedge \cdots \wedge \delta y^n \,. \end{aligned}$$

Now consider a t-dependent family of Finsler metrics $F(t)$. Then IM and dV_{IM} both depend on t. Observe that:

$$\partial_t \sqrt{\det(\hat{g}_{IJ})} = \tfrac{1}{2}\, \hat{g}^{IJ}\, \partial_t(\hat{g}_{IJ}) \sqrt{\det(\hat{g}_{IJ})} = g^{ij}\, \partial_t(g_{ij}) \sqrt{\det(\hat{g}_{IJ})}\,,$$

$$\partial_t(\tfrac{1}{F^n}) = -n\, F^{-n-1} \partial_t F = -(n\, \partial_t \log F)\, \tfrac{1}{F^n}\,,$$

and

$$\partial_t\, \delta y^k = \partial_t(dy^k + N^k{}_p\, dx^p) = \partial_t(N^k{}_p)\; dx^p$$

which, when wedged with $dx^1 \wedge \cdots \wedge dx^n$, gives zero. Hence

$$\partial_t\, dV_{IM} = \{g^{ij}\, \partial_t(g_{ij}) - n\, \partial_t \log F\}\, dV_{IM}\,.$$

Compare with the derivation in [Akbar-Zadeh 1995].

As we remarked earlier, the manifold of rays SM is isometric with the hypersurface IM through the (t-dependent) diffeomorphism

$$(x,\, [y]) \quad \xrightarrow{\xi} \quad \left(x,\, \frac{y}{F(t,x,y)}\right).$$

Thus

$$dV_{SM} \;=\; \xi^* dV_{IM}$$

and

$$\partial_t\, dV_{SM} \;=\; \partial_t\,(\xi^* dV_{IM})\,.$$

Let $v_1, \ldots, v_{2n-1}$ be $2n-1$ arbitrary tangent vectors on SM. Since dV_{SM} is given by the pull-back $\xi^* dV_{IM}$, we have

$$\begin{aligned}
& (\partial_t\, dV_{SM})(v_1, \ldots, v_{2n-1}) \\
= \;& \partial_t\{(dV_{IM})(\xi_* v_1, \ldots, \xi_* v_{2n-1})\} \\
= \;& (\partial_t\, dV_{IM})(\xi_* v_1, \ldots, \xi_* v_{2n-1}) \\
& + \sum_{\kappa=1}^{2n-1} (dV_{IM})(\xi_* v_1, \ldots, \xi_* v_{\kappa-1},\, \partial_t\; \xi_* v_\kappa,\, \xi_* v_{\kappa+1}, \ldots, \xi_* v_{2n-1}) \\
= \;& \{\xi^*\,(\partial_t\, dV_{IM})\}(v_1, \ldots, v_{2n-1}) + \sum_{\kappa=1}^{2n-1} 0\,.
\end{aligned}$$

We digress to explain why each individual term in the sum vanishes. Take any tangent vector v of SM. It is the initial velocity of a curve of elements $(x_s,\, [y_s])$ of SM:

$$v \;=\; (\partial_s)_{|s=0}\; (x_s,\, [y_s])\,.$$

Thus

$$\xi_* v = (\partial_s)_{|s=0} \ \{\xi(x_s,\ [y_s])\} = (\partial_s)_{|s=0} \left(x_s,\ \frac{y_s}{F(t, x_s, y_s)} \right) .$$

Apply ∂_t to this equation, and note that it *does* commute with ∂_s. Then

$$\begin{aligned} & \partial_t \, \xi_* v \\ = \ & (\partial_s)_{|s=0} \ \partial_t \left(x_s,\ \frac{y_s}{F(t, x_s, y_s)} \right) \\ = \ & (\partial_s)_{|s=0} \left\{ 0 \, \partial_{x^i} - \frac{\partial_t F}{F^2} \, y^i \partial_{y^i} \right\} \\ = \ & (\partial_s)_{|s=0} \left\{ - \frac{\partial_t F}{F^2} \, \hat{n}_{\text{out}} \right\} , \end{aligned}$$

where $\hat{n}_{\text{out}}$ is the unit outward-pointing normal field of the hypersurface corresponding to $F(t)$. Consequently,

$$\begin{aligned} & (dV_{IM})(\xi_* v_1, \ldots, \xi_* v_{\kappa-1},\ \partial_t \, \xi_* v,\ \xi_* v_{\kappa+1}, \ldots, \xi_* v_{2n-1}) \\ = \ & (dV_{IM}) \left(\xi_* v_1, \ldots, (\partial_s)_{|s=0} \left\{ - \frac{\partial_t F}{F^2} \, \hat{n}_{\text{out}} \right\}, \ldots, \xi_* v_{2n-1} \right) \\ = \ & (\partial_s)_{|s=0} \left\{ - \frac{\partial_t F}{F^2} \, (dV_{IM})(\xi_* v_1, \ldots, \hat{n}_{\text{out}}, \ldots, \xi_* v_{2n-1}) \right\} \\ = \ & (\partial_s)_{|s=0} \left\{ - \frac{\partial_t F}{F^2} \, (dV_{TM \smallsetminus 0})(\hat{n}_{\text{out}}, \ldots, \hat{n}_{\text{out}}, \ldots) \right\} \\ = \ & (\partial_s)_{|s=0} \, \{0\} . \end{aligned}$$

The point here is that, even though ∂_t does *not* commute with ξ^*, the terms which correspond to the lack of commutation are zero nevertheless, due to the special nature of dV_{IM} and the diffeomorphism ξ. This ends our digression.

Resuming the main discussion, we now see that

$$\begin{aligned} & \partial_t \, dV_{SM} \\ = \ & \xi^* \, (\partial_t \, dV_{IM}) \\ = \ & \xi^* \, (\{g^{ij} \, \partial_t (g_{ij}) - n \, \partial_t \log F\} \, dV_{IM}) \\ = \ & \{g^{ij} \, \partial_t (g_{ij}) - n \, \partial_t \log F\} \, dV_{SM} . \end{aligned}$$

In conclusion:

$$\partial_t \, dV_{SM} = \{g^{ij} \, \partial_t (g_{ij}) - n \, \partial_t \log F\} \, dV_{SM} .$$

Note that we have deduced this statement by an extrinsic method, without directly differentiating dV_{SM}, which is a constant multiple of $\omega \wedge (d\omega)^{n-1}$, where $\omega := F_{y^i}\, dx^i = \ell_i\, dx^i$ is the Hilbert form. I would like to reassure the purists that an intrinsic derivation has been carried out as well.

6.2. Akbar-Zadeh's insight

Associated to the Riemannian metric $\hat{g}$ on $TM \setminus 0$ is its unique torsion-free metric-compatible connection $\hat{\omega}$, whose connection 1-forms can be expanded in terms of the non-holonomic basis $\{dx^k; \delta y^k / F\}$ as follows:

$$\begin{aligned} \hat{\omega}_J{}^I &= \hat{\omega}_J{}^I \left(\frac{\delta}{\delta x^k} \right) dx^k + \hat{\omega}_J{}^I \left(F\, \frac{\partial}{\partial y^k} \right) \frac{\delta y^k}{F} \\ &=: \hat{\Gamma}^I{}_{J\,k}\; dx^k + \hat{\Gamma}^I{}_{J\,n+k}\; \frac{\delta y^k}{F}\,. \end{aligned}$$

It is not too difficult to work out the forms $\hat{\omega}_J{}^I$ explicitly, thereby obtaining concrete formulae for the Christoffel symbols. The results read:

$\hat{\Gamma}^I{}_{J\,k}$	$J = j$	$J = n + j$
$I = i$	${}^b\Gamma^i{}_{jk} - \dot{A}^i{}_{jk}$	$-\{\frac{1}{2}\,\ell^s\, {}^bR_{sj}{}^i{}_k - A_j{}^i{}_k\}$
$I = n + i$	$\frac{1}{2}\,\ell^s\, {}^bR_s{}^i{}_{jk} - A^i{}_{jk}$	${}^b\Gamma^i{}_{jk} - \dot{A}^i{}_{jk}$

and

$\hat{\Gamma}^I{}_{J\,n+k}$	$J = j$	$J = n + j$
$I = i$	$A_{kj}{}^i + \frac{1}{2}\,\ell^s\, {}^bR_{skj}{}^i$	$\dot{A}^i{}_{jk}$
$I = n + i$	$-\dot{A}^i{}_{jk}$	$A_{kj}{}^i + (g_{kj}\, \ell^i - g_k{}^i\, \ell_j)$

These Christoffel symbols enable us to carry out explicit covariant differentiations $\hat{\nabla}$ on $TM \setminus 0$.

Take any (contravariant) vector field

$$\hat{Z} \;:=\; \hat{Z}^i\, \frac{\delta}{\delta x^i} + \hat{Z}^{n+i}\, F \frac{\partial}{\partial y^i}$$

on $TM \setminus 0$. Its equivalent covariant description is the 1-form

$$\hat{Z}^\flat \;=\; \hat{Z}_j\; dx^j + \hat{Z}_{n+j}\; \frac{\delta y^j}{F}\,,$$

where

$$\begin{aligned} \hat{Z}_j &:= \hat{g}_{jI}\, \hat{Z}^I &= g_{ji}\, \hat{Z}^i\,, \\ \hat{Z}_{n+j} &:= \hat{g}_{n+j\,I}\, \hat{Z}^I &= g_{ji}\, \hat{Z}^{n+i}\,. \end{aligned}$$

The divergence of Z can be computed either directly as $(\hat{\nabla}_I\, \hat{Z})^I$ (denoted $\hat{\nabla}_I\, \hat{Z}^I$ in tensor analysis), or as $\hat{g}^{IJ}\, (\hat{\nabla}_I\, \hat{Z}^\flat)_J$ (namely $\hat{g}^{IJ}\, \hat{\nabla}_I\, \hat{Z}_J$ in tensor analysis). Using the tabulated Christoffel symbols, we find that

$$\begin{aligned} (\hat{\nabla}_I\, \hat{Z})^I &= \left\{ \frac{\delta}{\delta x^i} \hat{Z}^i + \hat{Z}^s\, {}^b\Gamma^i{}_{si} \right\} + F \partial_{y^i}\, \hat{Z}^{n+i} - (n-1) \hat{Z}^{n+s}\, \ell_s \\ &\quad - 2\, \hat{Z}^s\, \dot{A}_s + 2\, \hat{Z}^{n+s}\, A_s\,, \end{aligned}$$

where $\dot{A}_s := \dot{A}^i{}_{is}$ and $A_s := A^i{}_{is}$. On the other hand,

$$\begin{aligned} & \hat{g}^{IJ}\, (\hat{\nabla}_I\, \hat{Z}^\flat)_J \\ = & \left\{ g^{ij} \frac{\delta}{\delta x^i} \hat{Z}_j - \hat{Z}_s\, {}^b\Gamma^{si}{}_i \right\} + g^{ij}\, F \partial_{y^i}\, \hat{Z}_{n+j} - (n-1) \hat{Z}_{n+s}\, \ell^s, \end{aligned}$$

where ${}^b\Gamma^{si}{}_i := g^{ij}\, {}^b\Gamma^s{}_{ji}$. It is apparent that the second formula is computationally simpler, even though the answers in the two cases are numerically equal.

In [Akbar-Zadeh 1995], we find the following insightful choice of $\hat{Z}$:

$$\hat{Z}^i = 0\,, \quad \hat{Z}^{n+i} = g^{ip}\, \ell^q\, \partial_t(g_{pq}) - 2\, \ell^i\, \partial_t \log F\,.$$

The components of $\hat{Z}^\flat$ are:

$$\hat{Z}_j = 0\,, \quad \hat{Z}_{n+j} = \ell^q\, \partial_t(g_{jq}) - 2\, \ell_j\, \partial_t \log F\,.$$

Note that

$$\hat{Z}_{n+s}\, \ell^s = (F\, \partial_t F)_{y^s y^q}\, \frac{y^q y^s}{F^2} - 2\, \partial_t \log F = 0\,,$$

by two applications of Euler's theorem. We will return to the geometrical meaning of this condition later. For the moment, it simplifies the formula for $\hat{g}^{IJ}(\hat{\nabla}_I \hat{Z}^\flat)_J$, which we then compute as follows:

$$\begin{aligned}
& \hat{g}^{IJ}(\hat{\nabla}_I \hat{Z}^\flat)_J \\
= \; & g^{ij} F \partial_{y^i} \hat{Z}_{n+j} \\
= \; & g^{ij} F \partial_{y^i} \{\ell^q \partial_t(g_{jq}) - 2\,\ell_j\, \partial_t \log F\} \\
= \; & g^{ij} \{(\delta^q{}_i - \ell^q \ell_i)\, \partial_t(g_{jq}) + \partial_t([g_{jq}]_{y^i}\, y^q) \\
& \quad -2\,(g_{ji} - \ell_j \ell_i)\, \partial_t \log F - 2\, F \ell_j\, (\partial_t \log F)_{y^i}\} \\
= \; & g^{qj}\, \partial_t(g_{jq}) - (F\, \partial_t F)_{y^j y^q}\, \ell^q \ell^j + g^{ij}\, \partial_t\{(\tfrac{1}{2}F^2)_{y^i y^j y^q}\, y^q\} \\
& - 2\,(n-1)\, \partial_t \log F - 2\,(\partial_t \log F)_{y^i}\, y^i \\
= \; & g^{ij}\, \partial_t(g_{ij}) - 2\, \partial_t \log F + 0 - 2\,(n-1)\, \partial_t \log F - 0 \\
= \; & g^{ij}\, \partial_t(g_{ij}) - 2n\, \partial_t \log F\,.
\end{aligned}$$

Here, Euler's theorem has been invoked three times. The conclusion reads

$$g^{ij}\, \partial_t(g_{ij}) \;=\; 2n\, \partial_t \log F + \operatorname{div}_{TM \setminus 0} \hat{Z}\,.$$

As promised, we now return to address the geometrical significance of the fact $\hat{Z}_{n+s}\, \ell^s = 0$. To that end, recall that the unit outward-pointing normal field of the hypersurface IM is $\hat{n}_{\text{out}} = \ell^s\,(F\, \partial_{y^s})$. Note that this vector field, like Akbar-Zadeh's $\hat{Z}$, is manifestly well-defined on all of $TM \setminus 0$, and

$$\hat{g}(\hat{Z}, \hat{n}_{\text{out}}) = g_{ij}\, \hat{Z}^{n+i}\, \ell^j = \hat{Z}_{n+j}\, \ell^j = 0\,.$$

Thus, $\hat{Z}_{n+s}\, \ell^s = 0$ tells us that $\hat{Z}$ and $\hat{n}_{\text{out}}$ are everywhere $\hat{g}$-orthogonal on $TM \setminus 0$. In particular, at points on the hypersurface IM, $\hat{Z}$ is tangent to IM. Since IM is a Riemannian manifold in its own right, the divergence $\operatorname{div}_{IM} \hat{Z}$ makes sense. It is then natural to wonder how this divergence is related to the ambient divergence $\operatorname{div}_{TM \setminus 0} \hat{Z}$.

In order to facilitate the comparison, let us take a $\hat{g}$-orthonormal frame $\{\hat{e}_1, \ldots, \hat{e}_n; \hat{e}_{n+1}, \ldots, \hat{e}_{n+n}\}$ on $TM \setminus 0$ with the following properties:

- $\hat{e}_{n+n} = \hat{n}_{\text{out}}$.
- The first $2n-1$ vector fields, when restricted to IM, are all tangent to the latter.

Refer to those first $2n-1$ vector fields as $\hat{e}_\alpha$, with $\alpha = 1, \ldots, 2n-1$. Also, the induced Riemannian metric on IM has its own Christoffel symbols, which in turn define a covariant derivative operator $\breve{\nabla}$. Then

$$\operatorname{div}_{IM} \hat{Z} = (\breve{\nabla}_{\hat{e}_\alpha} \hat{Z})^\alpha = (\hat{\nabla}_{\hat{e}_\alpha} \hat{Z})^\alpha$$

because the two covariant derivatives in question differ by a term proportional to $\hat{n}_{\text{out}}$, which has no components tangent to IM. On the other hand,

$$\operatorname{div}_{TM \setminus 0} \hat{Z} = (\hat{\nabla}_{\hat{e}_a} \hat{Z})^a .$$

Therefore the difference between the two divergences is the term

$$\begin{aligned}
(\hat{\nabla}_{\hat{e}_{n+n}} \hat{Z})^{n+n} &= (\hat{\nabla}_{\hat{n}_{\text{out}}} \hat{Z})^{n+n} \\
&= \hat{g}(\hat{\nabla}_{\hat{n}_{\text{out}}} \hat{Z}, \hat{n}_{\text{out}}) \\
&= \hat{\nabla}_{\hat{n}_{\text{out}}} \{\hat{g}(\hat{Z}, \hat{n}_{\text{out}})\} - \hat{g}(\hat{Z}, \hat{\nabla}_{\hat{n}_{\text{out}}} \hat{n}_{\text{out}}) \\
&= 0 - \hat{g}(\hat{Z}, \hat{\nabla}_{\hat{n}_{\text{out}}} \hat{n}_{\text{out}}) .
\end{aligned}$$

However, for the geometry at hand,

$$\hat{\nabla}_{\hat{n}_{\text{out}}} \hat{n}_{\text{out}} = 0 .$$

Indeed, keep in mind that $F\partial_{y^s}$ is the $(n+s)$th vector in the basis $\{\delta/\delta y^s ; F\partial_{y^s}\}$, we have

$$\begin{aligned}
&\hat{\nabla}_{\ell^k (F\partial_{y^k})} \{\ell^j (F\partial_{y^j})\} \\
&= \ell^k \hat{\nabla}_{F\partial_{y^k}} \{\ell^j (F\partial_{y^j})\} \\
&= \ell^k (F\partial_{y^k} \ell^j) F\partial_{y^j} + \ell^k \ell^j \hat{\nabla}_{F\partial_{y^k}} (F\partial_{y^j}) \\
&= \ell^k (\delta^j{}_k - \ell^j \ell_k) F \frac{\partial}{\partial y^j} + \ell^k \ell^j \hat{\nabla}_{F\partial_{y^k}} (F\partial_{y^j}) \\
&= 0 + \ell^j \ell^k \left(\hat{\Gamma}^i{}_{n+j\ n+k} \frac{\delta}{\delta x^i} + \hat{\Gamma}^{n+i}{}_{n+j\ n+k} F \frac{\partial}{\partial y^i} \right) \\
&= \ell^j \ell^k \left(\dot{A}^i{}_{jk} \frac{\delta}{\delta x^i} + \{A_{kj}{}^i + (g_{kj} \ell^i - g_k{}^i \ell_j)\} F \frac{\partial}{\partial y^i} \right) \\
&= 0 .
\end{aligned}$$

Thus,

$$(\hat{\nabla}_{\hat{e}_{n+n}} \hat{Z})^{n+n} = (\hat{\nabla}_{\hat{n}_{\text{out}}} \hat{Z})^{n+n} = 0 ,$$

from which we conclude that

$$\operatorname{div}_{IM} \hat{Z} = \operatorname{div}_{TM \setminus 0} \hat{Z} \text{ (at points of } IM\text{)}.$$

Consequently, a previous formula for $g^{ij} \partial_t (g_{ij})$, when restricted from $TM \setminus 0$ to the submanifold IM, gives a result of Akbar-Zadeh's:

$$g^{ij} \partial_t (g_{ij}) = 2n \, \partial_t \log F + \operatorname{div}_{IM} \hat{Z} .$$

Compare the treatment here with that in [Akbar-Zadeh 1995] and the secondary reference [Akbar-Zadeh 1979].

Since $\xi : SM \to IM$, mapping $[y]$ to y/F, is a Riemannian isometry, applying ξ^* to the above statement on IM yields

$$g^{ij}\,\partial_t(g_{ij}) \;=\; 2n\,\partial_t \log F + \mathrm{div}_{SM}\,\{(\xi^{-1})_*\,\hat{Z}\}\,.$$

6.3. Varying the volume of SM

If M is compact, then so is SM and its volume is given by

$$\mathrm{Vol}_{SM} \;:=\; \int_{SM} dV_{SM}\,.$$

Keep in mind that the Riemannian metric on SM is simply ξ^* applied to that on IM, which in turn is induced by the $\hat{g}$ of $TM \smallsetminus 0$. Thus, the geometry of SM certainly depends on the Finsler metric F, even though SM itself does not.

Let us be given a deformation $F(t)$ of some F_o. The domain of integration SM does *not* depend on the Finsler metrics, and hence it is independent of t. This fact allows our work from the previous two sections to effect the following:

$$\begin{aligned}
&\partial_t\,\mathrm{Vol}_{SM} \\
&= \int_{SM} \partial_t\, dV_{SM} \\
&= \int_{SM} \{g^{ij}\,\partial_t(g_{ij}) - n\,\partial_t \log F\}\; dV_{SM} \\
&= \int_{SM} \{2n\,\partial_t \log F + \mathrm{div}_{SM}\,[(\xi^{-1})_*\,\hat{Z}] - n\,\partial_t \log F\}\; dV_{SM} \\
&= \int_{SM} \{n\,\partial_t \log F + \mathrm{div}_{SM}\,[(\xi^{-1})_*\,\hat{Z}]\}\; dV_{SM}\,.
\end{aligned}$$

If, in adddition to being compact, M is also boundaryless, then so is SM. In that case the standard divergence lemma for Riemannian manifolds (see, for instance, [Bao et al. 2000] for an exposition) tells us that

$$\int_{SM} \{\mathrm{div}_{SM}\,[(\xi^{-1})_*\,\hat{Z}]\}\; dV_{SM} \;=\; 0\,.$$

This simplifies the above formula for $\partial_t\,\mathrm{Vol}_{SM}$ to

$$\partial_t\,\mathrm{Vol}_{SM} \;=\; \int_{SM} \{n\,\partial_t \log F\}\; dV_{SM}\,.$$

Compare this with a formally identical statement in [Akbar-Zadeh 1995], one with the domain of integration stipulated as the t-dependent IM instead of our t-independent SM.

References

[Akbar-Zadeh 1979] H. Akbar-Zadeh, Transformations infinitésimales conformes des variétés finslériennes compactes, Ann. Polon. Math., **36** (1979), 213–229.

[Akbar-Zadeh 1995] H. Akbar-Zadeh, Generalized Einstein manifolds, J. Geom. Phys., **17** (1995), 342–380.

[Aikou 2001] T. Aikou, A note on some special Finsler manifolds, unpublished manuscript, 2001.

[Antonelli et al. 1993] P. L. Antonelli, R. S. Ingarden and M. Matsumoto, The Theory of Sprays and Finsler Spaces with Applications in Physics and Biology, Fund. Theories Phys., **58**, Kluwer Academic Publishers, 1993.

[Asanov 1985] G. S. Asanov, Finsler Geometry, Relativity and Gauge Theories, Fund. Theories Phys., **12**, D. Reidel Publishing Co., 1985.

[Asanov 1995] G. S. Asanov, Finsler case of GF-space. Aequationes Math., **49** (1995), 234–251.

[Asanov 1998] G. S. Asanov, Finsler metric functions over the product $\mathbb{R} \times M$ and their potential applications, Rep. Math. Phys., **41** (1998), 117–132.

[Asanov 2006a] G. S. Asanov, Finsleroid-Finsler space with Berwald and Landsberg conditions, manuscript, 2006.

[Asanov 2006b] G. S. Asanov, Finsleroid-Finsler space and spray coefficients, manuscript, 2006.

[Asanov 2006c] G. S. Asanov, Finsleroid-Finsler spaces of positive-definite and relativistic types, Rep. Math. Phys., **58** (2006), 275–300.

[Auslander 1955] L. Auslander, On curvature in Finsler geometry, Trans. Amer. Math. Soc., **79** (1955), 378–388.

[Bao 2004] D. Bao, Randers space forms, Period. Math. Hungar., **48** (2004), 3–15.

[Bao 2006] D. Bao, Unicorns in Finsler geometry, In: Proceedings of the 40th Symposium on Finsler geometry, Sapporo (Hokkaido Tokai University), Japan, 2005, pp. 19–27.

[Bao–Chern 1996] D. Bao and S. S. Chern, A note on the Gauss–Bonnet theorem for Finsler spaces, Ann. of Math. (2), **143** (1996), 233–252.

[Bao et al. 2000] D. Bao, S. S. Chern and Z. Shen, An Introduction to Riemann–Finsler Geometry, Grad. Texts in Math., **200**, Springer, 2000. (Errata at `http://www.math.uh.edu/~bao`)

[Bao–Lackey 1996] D. Bao and B. Lackey, A Hodge decomposition theorem for Finsler manifolds, C. R. Acad. Sci. Paris Sér. I Math., **323** (1996), 51–56.

[Bao–Robles 2003] D. Bao and C. Robles, On Randers spaces of constant flag curvature, Rep. Math. Phys., **51** (2003), 9–42.

[Bao–Robles 2004] D. Bao and C. Robles, Ricci and flag curvatures in Finsler geometry, In: A Sampler of Riemann–Finsler Geometry, Math. Sci. Res. Inst. Publ., **50**, Cambridge Univ. Press, 2004, pp. 197–259.

[Bao et al. 2004] D. Bao, C. Robles and Z. Shen, Zermelo navigation on Riemannian manifolds, J. Differential Geom., **66** (2004), 377–435.

[Berwald 1929] L. Berwald, Über die n-dimensionalen Geometrien konstanter Krümmung, in denen die Geraden die kürzesten sind, Math. Z., **30** (1929), 449–469.

[Berwald 1947] L. Berwald, Ueber Finslersche und Cartansche Geometrie IV. Projektivkrümmung allgemeiner affiner Räume und Finslersche Räume skalarer Krümmung, Ann. of Math. (2), **48** (1947), 755–781.

[Besse 1987] A. L. Besse, Einstein manifolds, Springer-Verlag, Heidelberg, 1987.

[Bryant 1995] R. L. Bryant, Finsler surfaces with prescribed curvature conditions, preprint (part of his Aisenstadt lectures), 1995.

[Bryant 1996] R. L. Bryant, Finsler structures on the 2-sphere satisfying $K = 1$, Contemp. Math., **196** (1996), 27–41.

[Bryant 2002] R. L. Bryant, Some remarks on Finsler manifolds with constant flag curvature, Houston J. Math., **28** (2002), 221–262.

[Caratheodory 1999] C. Carathéodory, Calculus of Variations and Partial Differential Equations of the First Order, AMS Chelsea Publishing, Providence, RI, 1999.

[Chern 1990] S. S. Chern, Historical remarks on Gauss–Bonnet, In: Analysis, et cetera, volume dedicated to Jürgen Moser, Academic Press, 1990, pp. 209–217.

[Chern–Shen 2005] S. S. Chern and Z. Shen, Riemann–Finsler geometry. Nankai Tracts Math., **6**, World Scientific, 2005.

[Chow–Knopf 2004] B. Chow and D. Knopf, The Ricci Flow: An Introduction, Math. Surveys Monogr., **110**, Amer. Math. Soc., Providence, RI, 2004.

[Dazord 1969] P. Dazord, Propriétés globales des géodésiques des Espaces de Finsler, Theses, Université de Lyon, 1969.

[del Riego 1973] L. del Riego, Tenseurs de Weyl d'un spray de directions, Theses, Université Scientifique et Medicale de Grenoble, 1973.

[Fischer–Marsden 1975a] A. E. Fischer and J. E. Marsden, Linearization stability of nonlinear partial differential equations, Proc. Sympos. Pure Math., **27** (1975), 219–263.

[Fischer–Marsden 1975b] A. E. Fischer and J. E. Marsden, Deformations of the scalar curvature, Duke Math. J., **42** (1975), 519–547.

[Foulon 2002] P. Foulon, Curvature and global rigidity in Finsler manifolds, Houston J. Math., **28** (2002), 263–292.

[Gao–Yau 1986] L. Z. Gao and S. T. Yau, The existence of negatively Ricci curved metrics on three-manifolds, Invent. Math., **85** (1986), 637–652.

[Hamilton 1982] R. S. Hamilton, Three-manifolds with positive Ricci curvature, J. Differential Geom., **17** (1982), 255–306.

[Hashiguchi–Ichijyo 1975] M. Hashiguchi and Y. Ichijyo, On some special (α, β) metrics, Rep. Fac. Sci. Kagoshima Univ., **8** (1975), 39–46.

[Ichijyo 1976] Y. Ichijyo, Finsler manifolds modeled on a Minkowski space. J. Math. Kyoto Univ., **16** (1976), 639–652.

[Ichijyo 1978] Y. Ichijyo, On special Finsler connections with the vanishing hv-curvature tensor, Tensor (N. S.), **32** (1978), 149–155.

[Kikuchi 1979] S. Kikuchi, On the condition that a space with (α, β) metric be locally Minkowskian, Tensor (N. S.), **33** (1979), 242–246.

[Lackey 2002] B. Lackey, On the Gauss–Bonnet formula in Riemann–Finsler geometry, Bull. London Math. Soc., **34** (2002), 329–340.

[Landsberg 1907a] G. Landsberg, Über die Totalkrümmung, Jahresber. Deutsch. Math.-Verein., **16** (1907), 36–46.

[Landsberg 1907b] G. Landsberg, Krümmungstheorie und Variationsrechnung, Jahresber. Deutsch. Math.-Verein., **16** (1907), 547–557.

[Landsberg 1908] G. Landsberg, Über die Krümmung in der Variationsrechnung, Math. Ann., **65** (1908), 313–349.

[Lohkamp 1994] J. Lohkamp, Metrics of negative Ricci curvature, Ann. of Math. (2), **140** (1994), 655–683.

[Lovas 2005] R. L. Lovas, A note on Finsler–Minkowski norms, Houston J. Math., **33** (2007), 701–707.

[Matsumoto 1974] M. Matsumoto, On Finsler spaces with Randers' metric and special forms of important tensors, J. Math. Kyoto Univ., **14** (1974), 477–498.

[Matsumoto 1986] M. Matsumoto, Foundations of Finsler Geometry and Special Finsler Spaces, Kaiseisha Press, Japan, 1986.

[Matsumoto 1989] M. Matsumoto, A slope of a mountain is a Finsler surface with respect to a time measure. J. Math. Kyoto Univ., **29** (1989), 17–25.

[Matsumoto 1996] M. Matsumoto, Remarks on Berwald and Landsberg spaces. Contemp. Math., **196** (1996), 79–82.

[Matsumoto–Shimada 2002] M. Matsumoto and H. Shimada, The corrected fundamental theorem on Randers spaces of constant curvature, Tensor (N. S.), **63** (2002), 43–47.

[Rademacher 2004] H.-B. Rademacher, Nonreversible Finsler metrics of positive flag curvature, In: A sampler of Riemann–Finsler geometry, Math. Sci. Res. Inst. Publ., **50**, Cambridge Univ. Press, 2004, pp. 261–302.

[Randers 1941] G. Randers, On an asymmetrical metric in the four-space of general relativity, Phys. Rev. (2), **59** (1941), 195–199.

[Robles 2005] C. Robles, Geodesics in Randers spaces of constant curvature, Trans. Amer. Math. Soc., **359** (2007), 1633–1651.

[Rund 1959] H. Rund, The Differential Geometry of Finsler Spaces, Springer-Verlag, 1959.

[Rutz 1996] S. F. Rutz, Symmetry in Finsler spaces, Contemp. Math., **196** (1996), 289–300.

[Shen 2001] Z. Shen, Differential Geometry of Sprays and Finsler Spaces, Kluwer Academic Publishers, 2001.

[Shen 2003] Z. Shen, Finsler metrics with $K = 0$ and $S = 0$, Canad. J. Math., **55** (2003), 112–132.

[Shen 2004] Z. Shen, Landsberg curvature, S-curvature, and Riemann curvature. In: A sampler of Riemann–Finsler geometry, Math. Sci. Res. Inst. Publ., **50**, Cambridge Univ. Press, 2004, pp. 303–355.

[Shen 2006] Z. Shen, On a class of Landsberg metrics in Finsler geometry, Canad. J. Math., to appear.

[Shibata et al. 1977] C. Shibata, H. Shimada, M. Azuma and H. Yasuda, On Finsler spaces with Randers' metric, Tensor (N. S.), **31** (1977), 219–226.

[Szabó 1981] Z. I. Szabó, Positive definite Berwald spaces (structure theorems on Berwald spaces), Tensor (N. S.), **35** (1981), 25–39.

[Szabó 2006] Z. I. Szabó, Berwald metrics constructed by Chevalley's polynomials, preprint arXiv:math.DG/0601522, 2006.

[Zermelo 1931] E. Zermelo, Über das Navigationsproblem bei ruhender oder veränderlicher Windverteilung, Z. Angew. Math. Mech., **11** (1931), 114–124.

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