Advanced Studies in Pure Mathematics 47-2, 2007
Asymptotic Analysis and Singularities
pp. 785–805

Interfacial analysis to a chemotaxis model equation with growth in three dimension

Tohru Tsujikawa

Abstract.

We consider the limiting system to a chemotaxis model equation with growth term by using singular limit analysis, which describes aggregation of biological individuals. The conditions to the linearized stability of symmetric localized stationary solutions of this system in $\mathbf{R}^3$ is shown.

§1. Introduction

We consider the following model equations which describes the movement of the biological individuals by the diffusion and chemotaxis effects in [4, 5];

$$\begin{cases} \dfrac{\partial u}{\partial \tau} = d_u \Delta u - \nabla(u \nabla \chi(v)) + f(u) \\ \\ \dfrac{\partial v}{\partial \tau} = d_v \Delta v + h(u,v) \end{cases} \qquad \tau > 0, \quad \mathbf{x} \in \mathbf{R}^N, \tag{1.1}$$

where $u(\tau, \mathbf{x})$ and $v(\tau, \mathbf{x})$ are respectively the population density and the concentration of chemotactic substance at time τ and position $\mathbf{x} \in \mathbf{R}^N$. d_u and d_v are diffusion rates of u and v.$\nabla \chi(v)$ is the velocity of the direct movement of u due to chemotaxis, which generally satisfies $\chi(v) > 0$ and $\chi'(v) \geq 0$ for $v > 0$. Here we specify the growth term $f(u)$ as $f(u) = (g(u) - \alpha)u$ where $g(u)$ is the growth rate with cooperation and competition effects and α is the degradation rate due to exterior forces such as predation or intoxication. Though the functional form of $f(u)$ is basically classified into several cases depending on $g(u)$ and α, we

---

Received October 30, 2005.
Revised January 23, 2006.
Partially supported by Grant-in-Aid for Scietific Research (No. 17540125) by Japan Society for the Promotion of Science.

consider the cubic-like form, which has three roots 0, $\underline{u}$ and $\overline{u}$ of $f(u) = 0$. The term $h(u, v)$ in (1.1) is simply specified as $h(u, v) = \beta u - \gamma v$ with the production rate $\beta > 0$ and the degradation rate $\gamma > 0$. For (1.1), we show the existence of the nonnegative global solution in 2–dimensional domain and the exponential atractor with finite dimension [8].

In [4, 5], we studied (1.1) assuming the situation that the movement of individuals is mainly due to chemotaxis and that the chemotactic substance diffuses so fast compared with the migration of individuals which move by diffusion and chemotaxis, so we introduce a small parameter $\varepsilon > 0$. By using the suitable transformations [1], the equations (1.1) can be rewritten as

$$\begin{cases} \dfrac{\partial u}{\partial \tau} = \varepsilon^2 \Delta u - \varepsilon k \nabla (u \nabla \chi(v)) + f(u) \\ \dfrac{\partial v}{\partial \tau} = \Delta v + u - \gamma v \end{cases} \qquad \tau > 0, \quad \mathbf{x} \in \mathbf{R}^N, \qquad (1.2)$$

where k is a positive constant such that $\chi(v)$ is sutably normalized. As was stated above, $f(u)$ satisfies $f(0) = f(a) = f(1) = 0$ for some $0 < a < 1$, $f(u) < 0$ for $0 < u < a$, $f(u) > 0$ for $a < u < 1$ and $f'(0) < 0$, $f'(1) < 0$. Here, we assume $\int_0^1 f(u)du > 0$. The boundary and initial conditions are taken to be

$$\lim_{|\mathbf{x}| \to +\infty} (u(\tau, \mathbf{x}), v(\tau, \mathbf{x})) = (0, 0) \qquad \tau > 0 \qquad (1.3)$$

and

$$(u(0, \mathbf{x}), v(0, \mathbf{x})) = (u_0(\mathbf{x}), v_0(\mathbf{x})) \qquad \mathbf{x} \in \mathbf{R}^N. \qquad (1.4)$$

In [4, 5], the existence and numerical stability of the radially symmetric stationary solutions of (1.2) – (1.4) in $\mathbf{R}^N$ ($N = 1, 2$) are studied for small $\varepsilon > 0$. Moreover, we have the limiting system of (1.2) – (1.4) as $\varepsilon \downarrow 0$ and show that by solving it the stability of the stationary solutions is suggested for small $\varepsilon > 0$.

In 3–dimensional domain, it is interesting to study the pattern formation of the chemotaxis model from the biological view point. But, it seems to be difficult to do the numerical simulation of (1.2) – (1.4). From this reason, we first consider the limiting system and by solving this problem we suggest the existence of the realistic stationary patterns in this paper.

In Section 2, we introduce the limiting system as $\varepsilon \downarrow 0$. In Section 3, the existence of radially symmetric stationary solutions of the limiting system in $\mathbf{R}^N$ ($N = 2, 3$) is shown. In Sections 4 and 5, we consider the stability of two different stationary solutions in $\mathbf{R}^3$, one is the solution constructed by extending the 2–dimensional radially symmetric solution uniformly toward another axis, which we call *cylindrical solution*, another one is a 3–dimensional radially symmetric solution, and show the dependency of the parameter k, the forms of $f(u)$ and $\chi(v)$ to the stability. Through this paper, we treat with the specified forms of $\chi(v)$ and $f(u)$, that is, $\chi(v) = sv^2/(s^2+v^2)$ and $f(u) = u(1-u)(u-0.1)$, $\varepsilon = 0.05, \gamma = 1.0$ for the numerical simulations. Finally, the derivation of the limiting system is written up in Appendix.

§2. Limiting System as $\varepsilon \downarrow 0$

In order to study the pattern-dynamics arising in solutions to (1.2)–(1.4) with small $\varepsilon > 0$, we derive the limiting system from (1.2) when $\varepsilon \downarrow 0$. To do it, we introduce the new time variable t with $\tau = t/\varepsilon$. Then (1.2) is rewritten as

$$\begin{cases} \varepsilon u_t = \varepsilon^2 \Delta u - \varepsilon k \nabla(u \nabla \chi(v)) + f(u) \\ \varepsilon v_t = \Delta v + u - \gamma v \end{cases} \quad t > 0, \quad \mathbf{x} \in \mathbf{R}^N. \tag{2.1}$$

Using the well known two-timing methods, one can intuitively understand that the time evolution of the solution of (2.1) consists of two stages. In the first stage, the solution is approximately described by the following system:

$$\begin{cases} u_t = \dfrac{1}{\varepsilon} f(u) \\ v_t = \dfrac{1}{\varepsilon}\{\Delta v + u - \gamma v\} \end{cases} \quad t > 0, \quad \mathbf{x} \in \mathbf{R}^N. \tag{2.2}$$

Since the system for u is bistable from the assumption of $f(u)$, the solution $u(t, \mathbf{x})$ tends, in short time, to 0 in one region, say $\Omega_{0\varepsilon}$ where $0 \le u_0(\mathbf{x}) < a$, while it tends to 1 in the other region, say $\Omega_{1\varepsilon}$ where $a < u_0(\mathbf{x})$. This implies the occurrence of layer regions, say R_ε, which is the boundary between two regions $\Omega_{0\varepsilon}$ and $\Omega_{1\varepsilon}$, that is, $\mathbf{R}^N$ is decomposed into $\mathbf{R}^N = \Omega_{0\varepsilon} \cup \Omega_{1\varepsilon} \cup R_\varepsilon$. In these two subregions, $\Omega_{0\varepsilon}$ and $\Omega_{1\varepsilon}$, the second variable v approximately satisfies the following stationary problems:

$$0 = \Delta v + g_i(v) \qquad \text{in } \Omega_{i\varepsilon} \ (i = 0, 1), \tag{2.3}$$

where $g_0(v) = -\gamma v$ and $g_1(v) = 1 - \gamma v$.

In the second stage, the solution is no longer described by (2.2), (2.3) so that the layer regions must change. This means that $\Omega_{0\varepsilon}$, $\Omega_{1\varepsilon}$ and R_ε vary as time goes on. We now assume the situation in the limit $\varepsilon \downarrow 0$ such that there is an $(N-1)$–dimensional hypersurface $\Gamma(t)$, which means the interface of u, in $\mathbf{R}^N$ such that $R_\varepsilon(t) \to \Gamma(t)$ holds as $\varepsilon \downarrow 0$, that is, $\mathbf{R}^N = \Omega_0(t) \cup \Omega_1(t) \cup \Gamma(t)$ where $\Omega_{i\varepsilon} \to \Omega_i(t) = \{\mathbf{x} \in \mathbf{R}^N, u(t,\mathbf{x}) = i\}$ $(i = 0, 1)$. Letting V^* be the normal velocity of the intrface $\Gamma(t)$, we can derive the equation to describe the dynamics of $\Gamma(t)$ as follows (see Appendix):

$$\begin{cases} V^* = c^* + k\chi'(v)\dfrac{\partial v}{\partial n} - \varepsilon(N-1)\kappa + \varepsilon G & t > 0, \quad \mathbf{x} \in \Gamma(t), \\ 0 = \Delta v + g_i(v) & t > 0, \quad \mathbf{x} \in \Omega_i(t), \end{cases}$$

where n means the outerward unit normal vector from $\Omega_1(t)$ to $\Omega_0(t)$ on $\Gamma(t)$, κ is the mean curvature at the interface. Here, c^* is the velocity of the traveling front solution given by Lemma in Appendix. Although $G = O(1)$ for small ε in general, we neglect this term in order to study the effect of the curvature to the motion of the interface as the first step. Therefore, the equation is rewritten as

$$\begin{cases} V^* = c^* + k\chi'(v)\dfrac{\partial v}{\partial n} - \varepsilon(N-1)\kappa & t > 0, \quad \mathbf{x} \in \Gamma(t), \\ 0 = \Delta v + g_i(v) & t > 0, \quad \mathbf{x} \in \Omega_i(t). \end{cases} \tag{2.4}$$

The smoothness of v on the interface Γ is imposed to satisfy $v \in C^1$, that is,

$$v(t,\cdot) \in C^1(\mathbf{R}^N) \qquad t > 0. \tag{2.5}$$

(2.4), (2.5) is proposed in Appendix, which we call the *singular limit system* or simply the *interface equation* of (2.1). It clearly shows that the dynamics of the interface is determined by three effects; the velocity of the 1–dimensional traveling front solution, the chemotactic effect due to the gradient of $\chi(v)$ and the geometric effect of the interface. Moreover, from (1.3), we assume that

$$\lim_{|\mathbf{x}|\to\infty} v(t,\mathbf{x}) = 0, \quad t > 0. \tag{2.6}$$

In the previous paper [4], we show the existence of radially symmetric stationary solutions $(u(r), v(r))$ of the interface equation (2.4)–(2.6) in $\mathbf{R}^N$ ($N = 1, 2, 3$) with $|\mathbf{x}| = r$ where the center and the interface locate at the origin and $r = \eta$, respectively. Moreover, the stability of these solutions was discussed for $N = 1, 2$.

Bonami et al. [1] treated with the case where the equation for v is stationary and the potentials of two equilibria $(0,0)$ and $(1,1/\gamma)$ are almost all same, that is, c^*, effects of chemotaxis and curvature are same of order with respect to ε. In this situation, the solution of the interface equation is good approximation to one of the original reaction–diffusion equation.

§3. Existence of the radially symmetric stationary solutions in $\mathbf{R}^N (N=2,3)$

In this section, we consider the existence of radially symmetric stationary solutions of the interface equation (2.4)–(2.6). In order to show that, we first treat with the following problem:

$$\begin{cases} 0 = c^* + k\chi'(v)v_r - \dfrac{(N-1)\varepsilon}{r}, & r=\eta \\ 0 = v_{rr} + \dfrac{N-1}{r}v_r + g_i(v), & r\in\Omega_i, \quad (i=0,1) \\ v_r(0)=0, \quad \lim\limits_{r\to\infty} v(r)=0 & \text{and} \quad v\in C^1(\mathbf{R}_+), \end{cases} \tag{3.1}$$

where $|\mathbf{x}|=r$, $\Omega_1=(0,\eta)$ and $\Omega_0=(\eta,\infty)$.

For $N=2$, the solution $(\eta, v(r;\eta))$ from the second and third equations of (3.1) is explicitly described by

$$v(r;\eta) = \begin{cases} \frac{1}{\gamma} + \left(\alpha - \frac{1}{\gamma}\right)\frac{\mathbf{I}_0(\sqrt{\gamma}r)}{\mathbf{I}_0(\sqrt{\gamma}\eta)}, & r\in(0,\eta) \\ \alpha\frac{\mathbf{K}_0(\sqrt{\gamma}r)}{\mathbf{K}_0(\sqrt{\gamma}\eta)}, & r\in(\eta,\infty) \end{cases}$$

with $\alpha = v(\eta;\eta) = \eta\mathbf{I}_1(\sqrt{\gamma}\eta)\mathbf{K}_0(\sqrt{\gamma}\eta)/\sqrt{\gamma}$, where $\mathbf{I}_\nu(r)$ and $\mathbf{K}_\nu(r)$ are the modified Bessel function of the ν-th order. In order to obtain the solution of (3.1), we need to solve the equation

$$\begin{aligned} c^* \quad &- \quad k\chi'\left(\tfrac{\eta\mathbf{I}_1(\sqrt{\gamma}\eta)\mathbf{K}_0(\sqrt{\gamma}\eta)}{\sqrt{\gamma}}\right)\eta\mathbf{I}_1(\sqrt{\gamma}\eta)\mathbf{K}_1(\sqrt{\gamma}\eta) - \tfrac{\varepsilon}{\eta} \\ &\equiv \quad H_2(\eta,k,\varepsilon)=0. \end{aligned} \tag{3.2}$$

Next, the solutions $(\eta, v(r;\eta))$ except for the first equations of (3.1) for $N=3$ is described by

$$v(r;\eta) \equiv \begin{cases} \frac{1}{\gamma} + (\alpha - \frac{1}{\gamma})\frac{\eta\sinh\sqrt{\gamma}r}{r\sinh\sqrt{\gamma}\eta} & r\in(0,\eta) \\ \frac{\alpha\eta}{r}e^{-\sqrt{\gamma}(r-\eta)} & r\in(\eta,\infty) \end{cases} \tag{3.3}$$

with $\alpha = v(\eta;\eta) = \eta\mathbf{K}_{\frac{1}{2}}(\sqrt{\gamma}\eta)\mathbf{I}_{\frac{3}{2}}(\sqrt{\gamma}\eta)/\sqrt{\gamma}$. Substituting (3.3) into the first equation in (3.1), we obtain

$$\begin{aligned} c^* \quad - \quad & k\chi'\left(\frac{\eta\mathbf{I}_{\frac{3}{2}}(\sqrt{\gamma}\eta)\mathbf{K}_{\frac{1}{2}}(\sqrt{\gamma}\eta)}{\sqrt{\gamma}}\right)\eta\mathbf{I}_{\frac{3}{2}}(\sqrt{\gamma}\eta)\mathbf{K}_{\frac{3}{2}}(\sqrt{\gamma}\eta) - \frac{2\varepsilon}{\eta} \\ \equiv \quad & H_3(\eta,k,\varepsilon) = 0. \end{aligned} \tag{3.4}$$

By using the solution η of $H_N(\eta,k,\varepsilon) = 0$ ($N = 2,3$), one easily finds that the solution of (3.1) is represented by $(\eta, v(r;\eta))$.

Theorem 1. [4] *Let $k^* > 0$ be a constant to satisfy $c^* - \frac{k^*}{2\sqrt{\gamma}}\chi'(\frac{1}{2\gamma}) = 0$. For fixed small $\varepsilon > 0$, there exists a constant $\overline{k}(\varepsilon)$ ($> k^*$) such that for $k^* < k < \overline{k}(\varepsilon)$ there are at least two solutions $(\overline{\eta}, v(r;\overline{\eta}))$ and $(\underline{\eta}, v(r;\underline{\eta}))$ such that $\overline{\eta} = O(1)$ and $\underline{\eta} = O(\varepsilon)$, for $0 < k < k^*$, there is at least one solution $(\underline{\eta}, v(r,\underline{\eta}))$ with $\underline{\eta} = O(\varepsilon)$ for $N = 2,\ 3$, respectively.*

Letting $\eta = \eta(k)$ be a solution of (3.2) or (3.4), we define the pair of functions $(u^0(r), v^0(r))$ by

$$\begin{cases} u^0(r) = \begin{cases} 1 & r \in (0,\eta) \\ 0 & r \in (\eta,\infty), \end{cases} \\ \\ v^0(r) = v(r;\eta) \qquad r \in (0,\infty) \end{cases} \tag{3.5}$$

and call it a *radially symmetric stationary solution* of the interface equation (2.4)–(2.6) for $N = 2,\ 3$, respectively.

Next, as $\chi(v) = sv^2/(s^2+v^2)$, we draw numerically the global picture of radially symmetric stationary solutions of (3.1) for $N = 3$ when k is varied in Figure 1 (it is already shown for $N = 1,\ 2$ in [4]). In this case, there are two critical values $0 < s_* < s^*$ of a parameter s of $\chi(v)$ such that for (i) $s_* < s < s^*$, there are three branches corresponding to $\underline{\eta}, \overline{\eta}$ and $\hat{\eta}$, while for (ii) $s^* < s$, there are two ones corresponding to $\underline{\eta}$ and $\overline{\eta}$ when k is varied. In Figure 2, the existence region of the solution with $\overline{\eta} = O(1)$ for small $\varepsilon > 0$ is shown in the (k,s) - plane for $N = 3$.

On the other hands, by the numerical simulations it does not able to suggest which stationary solution in $\mathbf{R}^3$ is realistic till now. Therefore, from the theoretical view point, we consider the stability of the radially symmetric stationary solutions of the interface equation for $N = 2,3$ in the 3–dimensional domain in the next two sections. Moreover, it is shown that the stationary solutions $(\underline{\eta}, v(r;\underline{\eta}))$ are at least unstable with respect to the disturbances of radial direction due to the discussion.

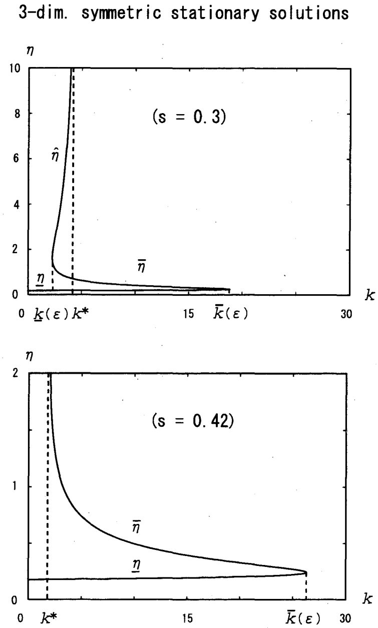

Figure 1

§4. Stability of the cylindrical stationary solutions in $\mathbf{R}^3$

In this section, according to the method in [7], we consider the stability of the cylindrical stationary solution in $\mathbf{R}^3$, that is, the solution is denoted by $(u(x,y,z),\ v(x,y,z)) = (u^0(r),\ v^0(r))$ where $(u^0(r),\ v^0(r))$ is given by (3.5) and $x = r\cos\theta,\ y = r\sin\theta\ (\ 0 \le \theta \le 2\pi\ ),\ z \in \mathbf{R}$. Hereafter, we only treat the solution with $\eta = O(1)$ for small $\varepsilon > 0$. To do it, the interface location is represented by $r = \eta + \zeta(\varphi, z, t)$ with a disturbance $\zeta(\varphi, z, t)$ where $\mathbf{r} = (r, \varphi, z)$ is the cylindrical coordinate. Then, $\Omega_i(t)$ $(i = 0, 1)$ are respectively denoted by

$$\Omega_1(t) = \{(r,\varphi,z)|\ 0 \le r < \eta + \zeta(\varphi,z,t),\ 0 \le \varphi < 2\pi,\ z \in \mathbf{R}\},$$

$$\Omega_0(t) = \{(r,\varphi,z)|\ \eta + \zeta(\varphi,z,t) < r,\ 0 \le \varphi < 2\pi,\ z \in \mathbf{R}\}.$$

It follows from (2.4) that

$$0 = \Delta v + H(\eta + \zeta(\varphi,z,t) - r) - \gamma v. \tag{4.1}$$

3-dim. symmetric stationary solutions

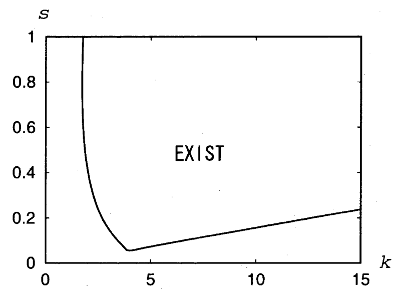

Figure 2

Putting $v_{\mathbf{q}}(t) = \int v(\mathbf{r}, t) e^{i\mathbf{q}\cdot\mathbf{r}} d\mathbf{r}$ where $\mathbf{q} = (q_x, q_y, q_z) \in \mathbf{R}^3$, we have

$$0 = -(\gamma + q^2) v_{\mathbf{q}} + H_{\mathbf{q}}(t)$$

where $H_{\mathbf{q}}(t) = \displaystyle\int H(\eta + \zeta(\varphi, z, t) - r) e^{i\mathbf{q}\cdot\mathbf{r}} d\mathbf{r}$ with $q = |\mathbf{q}|$, which implies $v_{\mathbf{q}}(t) = H_{\mathbf{q}}(t)/(\gamma + q^2)$. Therefore, $v(\mathbf{r}, t)$ is represented by

$$v(\mathbf{r}, t) = \frac{1}{(2\pi)^3} \int_{\mathbf{R}^3} \frac{H_{\mathbf{q}}(t)}{\gamma + q^2} e^{-i\mathbf{q}\cdot\mathbf{r}} d\mathbf{q}.$$

Here, we note the asymptotic expansion of $H_{\mathbf{q}}(t)$ with respect to ζ as follows:

$$H_{\mathbf{q}}(t) = H_{\mathbf{q}}^{(0)} + H_{\mathbf{q}}^{(1)} + O(\zeta^2) \quad \text{for small } \zeta$$

where

$$H_{\mathbf{q}}^{(0)} = \int_0^{2\pi} \int_0^{\eta} h e^{iqh\cos(\omega - \varphi)} dh d\varphi$$

and

$$H_{\mathbf{q}}^{(1)} = \eta \int_{\mathbf{R}} e^{izq_z} \int_0^{2\pi} e^{is\eta\cos(\omega - \varphi)} \zeta(\varphi, z, t) d\varphi dz$$

with $s = \sqrt{q_x^2 + q_y^2}$ and $q_x \cos\varphi + q_y \sin\varphi = s\cos(\omega - \varphi)$ for some angle ω. Then, $v(\mathbf{r}, t)$ is given by the representation as follows:

$$\begin{aligned} v(\mathbf{r}, t) &= \frac{1}{(2\pi)^3} \int_{\mathbf{R}^3} \frac{H_{\mathbf{q}}^{(0)} + H_{\mathbf{q}}^{(1)}}{\gamma + q^2} e^{-i\mathbf{q}\cdot\mathbf{r}} d\mathbf{q} + O(\zeta^2) \\ &= v_0 + v_1(\varphi, z, t) + v_2(\varphi, z, t) + O(\zeta^2) \quad \text{for small} \quad \zeta \end{aligned}$$

with $r = \eta + \zeta(\varphi, z, t)$ where

$$v_0 = \frac{\eta}{\sqrt{\gamma}}\mathbf{K}_0(\sqrt{\gamma}\eta)\mathbf{I}_1(\sqrt{\gamma}\eta),\ v_1(\varphi, z, t) = -\eta\mathbf{I}_1(\sqrt{\gamma}\eta)\mathbf{K}_1(\sqrt{\gamma}\eta)\zeta(\varphi, z, t),$$

$$v_2(\varphi, z, t) = \frac{1}{(2\pi)^3}\int_{\mathbf{R}^3}\frac{H_{\mathbf{q}}^{(1)}}{\gamma + q^2}e^{-i\{(q_x\cos\varphi + q_y\sin\varphi)\eta + q_z z\}}d\mathbf{q}.$$

Since the interface stands for $\Gamma(t) = (r\cos\varphi, r\sin\varphi, z)$, it holds

$$V^* = c^* + k\chi'(v)\frac{\partial v}{\partial n} - 2\varepsilon\kappa =< \Gamma_t, n >= \zeta_t + O(\zeta^2). \tag{4.2}$$

Since

$$\frac{\partial v}{\partial n} = \frac{\partial v_0}{\partial r} + \frac{\partial}{\partial r}(v_1 + v_2) + O(\zeta^2),\ \kappa = \frac{1}{2}\left(\zeta_{zz} + \frac{\zeta_{\varphi\varphi}}{\eta^2} - \frac{1}{\eta} + \frac{\zeta}{\eta^2}\right) + O(\zeta^2),$$

it follows from $0 = c^* + k\chi'(v_0)\dfrac{\partial v_0}{\partial r} - \dfrac{\varepsilon}{\eta}$ that

$$\begin{aligned}\zeta_t = \;& k\left\{\chi'(v_0)\frac{\partial}{\partial r}(v_1 + v_2) + \chi''(v_0)(v_1 + v_2)\frac{\partial v_0}{\partial r}\right\} \\ & + \varepsilon\left(\zeta_{zz} + \frac{\zeta_{\varphi\varphi}}{\eta^2} - \frac{2\zeta}{\eta^2}\right) + O(\zeta^2).\end{aligned} \tag{4.3}$$

Here, we remark that

$$\begin{aligned}&\frac{1}{2\pi}\int_{\mathbf{R}}\int_0^{2\pi} v_2(\varphi, z, t)e^{imz - i\ell\varphi}d\varphi dz \\ &= \eta\mathbf{I}_\ell(\sqrt{\gamma + m^2}\eta)\mathbf{K}_\ell(\sqrt{\gamma + m^2}\eta)\zeta_{\ell,m}(t),\end{aligned}$$

where $\zeta_{\ell,m}(t) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}\int_{\mathbf{R}}\zeta(\varphi, z, t)e^{imz - i\ell\varphi}dzd\varphi$ for two disturbance modes ℓ and m.

Moreover,

$$\frac{\partial}{\partial r}v(\varphi, z, t) = \frac{\partial}{\partial r}(v_0 + v_1 + v_2) + O(\zeta^2) \quad \text{for small } \zeta.$$

From the easy computation, we obtain

$$\frac{\partial v_0}{\partial r} = -\eta\mathbf{I}_1(\sqrt{\gamma}\eta)\mathbf{K}_1(\sqrt{\gamma}\eta),$$

$$
\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi}\int_{\mathbf{R}}\frac{\partial v_1}{\partial r}e^{imz-i\ell\varphi}dzd\varphi \\
&= \left[\frac{\sqrt{\gamma}\eta}{2}(\mathbf{I}_1(\sqrt{\gamma}\eta)\mathbf{K}_0(\sqrt{\gamma}\eta)-\mathbf{I}_0(\sqrt{\gamma}\eta)\mathbf{K}_1(\sqrt{\gamma}\eta))\right. \\
&\qquad\qquad\qquad\qquad \left.+\mathbf{I}_1(\sqrt{\gamma}\eta)\mathbf{K}_1(\sqrt{\gamma}\eta)\right]\zeta_{\ell,m}(t)
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi}\int_{\mathbf{R}}\frac{\partial v_2}{\partial r}e^{imz-i\ell\varphi}dzd\varphi \\
&= \frac{\eta}{2}\sqrt{\gamma+m^2}\left[\mathbf{I}_{\ell+1}(\sqrt{\gamma+m^2}\eta)\mathbf{K}_\ell(\sqrt{\gamma+m^2}\eta)\right. \\
&\qquad \left.-\mathbf{I}_\ell(\sqrt{\gamma+m^2}\eta)\mathbf{K}_{\ell-1}(\sqrt{\gamma+m^2}\eta)\right]\zeta_{\ell,m}(t).
\end{aligned}
$$

Thus, it follows from (4.3) that for the balance of the lowest terms of ζ, it holds that

$$
\begin{aligned}
\frac{d}{dt}\zeta_{\ell,m} &= -\varepsilon\left(m^2+\frac{\ell^2}{\eta^2}-\frac{1}{\eta^2}\right)\zeta_{\ell,m} \\
&+ k\left[\chi'(v_0)\frac{1}{2\pi}\int_0^{2\pi}\int_{\mathbf{R}}\frac{\partial}{\partial r}(v_1+v_2)e^{imz-i\ell\varphi}dzd\varphi\right. \\
&+ \left.\chi''(v_0)\frac{\partial v_0}{\partial r}\frac{1}{2\pi}\int_0^{2\pi}\int_{\mathbf{R}}(v_1+v_2)e^{imz-i\ell\varphi}dzd\varphi\right].
\end{aligned}
$$

Therefore, the equation of $\zeta_{\ell,m}$ is obtained by

$$
\begin{aligned}
\frac{d}{dt}\zeta_{\ell,m} &= -\varepsilon\left(m^2+\frac{\ell^2}{\eta^2}-\frac{1}{\eta^2}\right)\zeta_{\ell,m} \\
&+ k\left[\chi'\left(\frac{\eta}{\sqrt{\gamma}}\mathbf{K}_0(\sqrt{\gamma}\eta)\mathbf{I}_1(\sqrt{\gamma}\eta)\right)\left\{\frac{\sqrt{\gamma}\eta}{2}(\mathbf{I}_1(\sqrt{\gamma}\eta)\mathbf{K}_0(\sqrt{\gamma}\eta)\right.\right. \\
&- \mathbf{I}_0(\sqrt{\gamma}\eta)\mathbf{K}_1(\sqrt{\gamma}\eta))+\mathbf{I}_1(\sqrt{\gamma}\eta)\mathbf{K}_1(\sqrt{\gamma}\eta) \\
&+ \frac{\eta}{2}\sqrt{\gamma+m^2}\left(\mathbf{I}_{\ell+1}(\sqrt{\gamma+m^2}\eta)\mathbf{K}_\ell(\sqrt{\gamma+m^2}\eta)\right.
\end{aligned}
$$

$$
\begin{aligned}
&- \quad \mathbf{I}_\ell(\sqrt{\gamma+m^2\eta})\mathbf{K}_{\ell-1}(\sqrt{\gamma+m^2\eta})\Big)\Big\} \\
&- \quad \chi''\left(\frac{\eta}{\sqrt{\gamma}}\mathbf{K}_0(\sqrt{\gamma}\eta)\mathbf{I}_1(\sqrt{\gamma}\eta)\right) \\
&\times \quad \eta^2\mathbf{I}_1(\sqrt{\gamma}\eta)\mathbf{K}_1(\sqrt{\gamma}\eta)\left\{\mathbf{I}_\ell(\sqrt{\gamma+m^2\eta})\mathbf{K}_\ell(\sqrt{\gamma+m^2\eta})\right. \\
&- \quad \mathbf{I}_1(\sqrt{\gamma}\eta)\mathbf{K}_1(\sqrt{\gamma}\eta)\}]\,\zeta_{\ell,m} \\
&\equiv \quad F_2(\ell,m,k,\varepsilon)\zeta_{\ell,m}.
\end{aligned}
$$

Definition (Linearlized stability of the stationary solution)

If $F_2(\ell,m,k,\varepsilon) < 0$ for all $\ell, m \in \mathbf{N}$ except for $(\ell,m) = (1,0)$, then the stationary solution $(\eta, v(r;\eta))$ is stable. If not, the solution is unstable.

For $m = 0$, that is, this means the stability of the radially symmetric stationary solution in $\mathbf{R}^2$,

$$
\begin{aligned}
&F_2(\ell,0,k,\varepsilon) = \frac{\varepsilon}{\eta^2}(1-\ell^2) \\
&+ k\left[\chi'\left(\frac{\eta}{\sqrt{\gamma}}\mathbf{K}_0(\sqrt{\gamma}\eta)\mathbf{I}_1(\sqrt{\gamma}\eta)\right)\left\{\frac{\sqrt{\gamma}\eta}{2}\left(\mathbf{I}_1(\sqrt{\gamma}\eta)\mathbf{K}_0(\sqrt{\gamma}\eta)\right.\right.\right. \\
&- \mathbf{I}_0(\sqrt{\gamma}\eta)\mathbf{K}_1(\sqrt{\gamma}\eta)) + \mathbf{I}_1(\sqrt{\gamma}\eta)\mathbf{K}_1(\sqrt{\gamma}\eta) \\
&+ \left.\frac{\sqrt{\gamma}\eta}{2}(\mathbf{I}_{\ell+1}(\sqrt{\gamma}\eta)\mathbf{K}_\ell(\sqrt{\gamma}\eta) - \mathbf{I}_\ell(\sqrt{\gamma}\eta)\mathbf{K}_{\ell-1}(\sqrt{\gamma}\eta))\right\} \\
&- \chi''\left(\frac{\eta}{\sqrt{\gamma}}\mathbf{K}_0(\sqrt{\gamma}\eta)\mathbf{I}_1(\sqrt{\gamma}\eta)\right)\eta^2\mathbf{I}_1(\sqrt{\gamma}\eta)\mathbf{K}_1(\sqrt{\gamma}\eta)(\mathbf{I}_\ell(\sqrt{\gamma}\eta)\mathbf{K}_\ell(\sqrt{\gamma}\eta) \\
&- \left.\mathbf{I}_1(\sqrt{\gamma}\eta)\mathbf{K}_1(\sqrt{\gamma}\eta))\right].
\end{aligned}
$$

We already discussed this equation in [4].

On the other hand, for $m = 0$ and $\ell = 0$, we have $F_2(0,0,k,\varepsilon) = \frac{\partial}{\partial\eta}H_2(\eta,k,\varepsilon)$. Therefore, it follows that the stability with respect to disturbances for radial direction depends on the form of $H_2(\eta,k,\varepsilon)$.

Remark 1 *Let $(\hat{\eta}, v(r;\hat{\eta}))$ be the solutions corresponding to the upper branch for $s_* < s < s^*$ in Figure 1. Since $\frac{\partial}{\partial\eta}H_2(\hat{\eta},k,\varepsilon) > 0$, the stationary solution $(\hat{\eta}, v(r;\hat{\eta}))$ is unstable.*

For $m = 0$ and $\ell = 1$, $\zeta_{1,0}(t)$ satisfies

$$\frac{d}{dt}\zeta_{1,0} = k\chi'\left(\frac{\eta}{\sqrt{\gamma}}\mathbf{K}_0(\sqrt{\gamma}\eta)\mathbf{I}_1(\sqrt{\gamma}\eta)\right)\left\{\frac{\sqrt{\gamma}\eta}{2}\left(\mathbf{I}_2(\sqrt{\gamma}\eta)\mathbf{K}_1(\sqrt{\gamma}\eta)\right.\right.$$
$$\left.\left.-\mathbf{I}_0(\sqrt{\gamma}\eta)\mathbf{K}_1(\sqrt{\gamma}\eta)\right) + \mathbf{I}_1(\sqrt{\gamma}\eta)\mathbf{K}_1(\sqrt{\gamma}\eta)\right\}\zeta_{1,0} = 0,$$

which means that the solution has phase shift free in the whole plane.

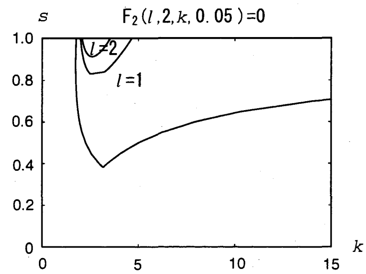

Figure 3

Next, we numerically treat with the functional form of $F_2(\ell, m, k, \varepsilon)$ for the solution $(\overline{\eta}, v(r; \overline{\eta}))$. In Figure 3, the curves of $F_2(\ell, m, k, \varepsilon) = 0$ for $\ell = 1, 2$ and $m = 2$ is shown in the (k, s)-plane. For small $s > 0$, the solution is stable and with any fixed $s > 0$, the solution is so for large $k > 0$. From these numerical results, we will consider the asymptotic behavior of $F_2(\ell, m, k, \varepsilon)$ as k tends to k^*, that is, $\overline{\eta}$ goes to infinity. Let

$$F(\ell, m, k^*, \varepsilon) = -\varepsilon\left(m^2 + \frac{\ell^2 - 1}{\eta^2}\right)$$

$$+k^*\left[\chi'\left(\frac{1}{2\gamma}\right) - \frac{1}{2\gamma}\chi''\left(\frac{1}{2\gamma}\right)\right]\frac{\sqrt{\gamma + m^2} - \sqrt{\gamma}}{4\sqrt{\gamma + m^2}\sqrt{\gamma}\eta}$$

$$+k^*\chi''\left(\frac{1}{2\gamma}\right)\left[\frac{\sqrt{\gamma} - \sqrt{\gamma + m^2}}{64\gamma^2\sqrt{\gamma + m^2}\eta^2} - \frac{3\sqrt{\gamma + m^2}(\gamma + m^2) - (4\ell^2 - 1)\sqrt{\gamma}\gamma}{32\gamma^2\sqrt{\gamma + m^2}(\gamma + m^2)\eta^2}\right]$$

with $k^* = 2\sqrt{\gamma}c^*/\chi'(\frac{1}{2\gamma})$.

Proposition 1 (Asymptotic behavior of $F_2(\ell, m, k, \varepsilon)$) *For $\eta = \overline{\eta}$, it holds that*

$$\lim_{k \to k^*} \frac{\{F_2(\ell, m, k, \varepsilon) + F(\ell, m, k^*, \varepsilon)\}}{\overline{\eta}^2} = 0.$$

Proof. Because of $\lim_{k \to k^*} \overline{\eta} = \infty$, we can prove this proposition by using the asymptotic behavior of the modified Bessel functions $\mathrm{I}_\ell(z)$ and $\mathrm{K}_\ell(z)$ as z tends to infinity.

Remark 2 *Let* $F^*(\varepsilon) = \varepsilon - k^* \chi''(\frac{1}{2\gamma})/(8\gamma^2)$. *For* $m = 0$, (*in the case of* 2*–dimensional domain*), *it holds that*

$$\lim_{k \to k^*} \left\{ F_2(\ell, 0, k, \varepsilon)\overline{\eta}^2 + (\ell+1)(\ell-1)F^*(\varepsilon) \right\} = 0.$$

If $F^*(\varepsilon) > 0$, *it follows from the proposition that for any integer* $\ell > 1$, *it holds* $F_2(\ell, 0, k, \varepsilon)\overline{\eta}^2 < 0$, *that is, the stationary solution becomes stable as* k *tends to* k^*.

As k *tends to* k^*, *it holds that if* $F^*(\varepsilon) > 0$, *then*

$$0 > F_2(2,0,k,\varepsilon) > F_2(3,0,k,\varepsilon) > \cdots$$
$$> F_2(\ell,0,k,\varepsilon) > F_2(\ell+1,0,k,\varepsilon) > \cdots ,$$

if $F^*(\varepsilon) < 0$, *then*

$$0 < F_2(2,0,k,\varepsilon) < F_2(3,0,k,\varepsilon) < \cdots$$
$$< F_2(\ell,0,k,\varepsilon) < F_2(\ell+1,0,k,\varepsilon) < \cdots .$$

For the numerical simulation, it holds that $F^*(\varepsilon) < 0$ for $0.98\cdots < s < 5.45\cdots$.

§5. Stability of the spherical stationary solutions in $\mathbf{R}^3$

In this section, we consider the stability of the radially symmetric stationary solution of (2.4)–(2.6) for $N = 3$, which satisfies $\eta = O(1)$ for small $\varepsilon > 0$.

To study the stability, we represent deformations of the interface $r = \eta$ by the polar coordinate $(r, \theta, \varphi) = (\eta + \zeta(t, \theta, \varphi), \theta, \varphi)$ with the azimuthal angle (θ, φ), where u takes 1 for $(r, \theta, \varphi) \in (0, \eta + \zeta(t, \theta, \varphi)) \times (0, \pi) \times (0, 2\pi)$, while u takes 0 for $(r, \theta, \varphi) \in (\eta + \zeta(t, \theta, \varphi), \infty) \times (0, \pi) \times (0, 2\pi)$. For $\Gamma = (r \sin\theta \cos\varphi, r \sin\theta \sin\varphi, r\cos\theta)$, it follows from (4.2) that

$$\frac{r\sin\theta}{\sqrt{r^2\sin^2\theta + \zeta_\varphi^2 + \zeta_\theta^2 \sin^2\theta}}\zeta_t$$

$$= c + k\chi'(v)\frac{v_r r^2 \sin^2\theta - v_\theta \sin^2\theta\zeta_\theta - v_\varphi \sin^3\theta\zeta_\varphi}{r\sin\theta\sqrt{r^2\sin^2\theta + \zeta_\varphi^2 + \zeta_\theta^2\sin^2\theta}} - 2\varepsilon\kappa + O(\zeta^2).$$

By using the balance of the above equation with respect to lower parts of ζ and their derivatives, it holds that

$$\begin{aligned}\zeta_t = \; & k\left\{\chi'(v_0)\left(v_r^{(1)}+v_r^{(2)}\right)+\chi''(v_0)v_{0r}\left(v^{(1)}+v^{(2)}\right)\right\} \\ & +2\varepsilon\left(\frac{\zeta}{\eta^2}+\frac{\zeta_{\theta\theta}}{2\eta^2}+\frac{\zeta_{\varphi\varphi}}{2\eta^2\sin^2\theta}+\frac{\zeta_\theta\cos\theta}{2\eta^2\sin\theta}\right)+O(\zeta^2).\end{aligned} \tag{5.1}$$

Defining the completely orthonormal system $\{Y_{\ell,m}(\theta,\varphi)\}$ on the sphere by

$$Y_{\ell,m}(\theta,\varphi)=\sqrt{\frac{(\ell-|m|)!(2\ell+1)}{(\ell+|m|)!}}P_\ell^m(\cos\theta)\exp(-im\varphi)$$

where $P_\ell^m(x)=(1-x^2)^{\frac{|m|}{2}}\dfrac{d^{|m|}}{dx^{|m|}}P_\ell(x)$ and $P_\ell(x)=\dfrac{1}{2^\ell\ell!}\dfrac{d^\ell}{dx^\ell}(x^2-1)^\ell$, we have

$$\begin{aligned}&\frac{1}{4\pi}\int_0^{2\pi}\int_0^\pi \varepsilon\left\{\frac{2\zeta}{\eta_0^2}+\frac{\zeta_{\theta\theta}}{\eta^2}+\frac{\zeta_{\varphi\varphi}}{\eta_0^2\sin^2\theta}+\frac{\zeta_\theta\cos\theta}{\eta_0^2\sin\theta}\right\}Y_{\ell,m}(\theta,\varphi)\sin\theta d\theta d\varphi \\ &\qquad = -\frac{\varepsilon}{\eta^2}(\ell+2)(\ell-1)\zeta_{\ell,m}(t),\end{aligned}$$

$$\frac{1}{4\pi}\int_0^{2\pi}\int_0^\pi v^{(1)}Y_{\ell,m}(\theta,\varphi)\sin\theta d\theta d\varphi=\frac{d}{dr}v_0(\eta)\zeta_{\ell,m}(t),$$

$$\frac{1}{4\pi}\int_0^{2\pi}\int_0^\pi v^{(2)}Y_{\ell,m}(\theta,\varphi)\sin\theta d\theta d\varphi=\eta\mathbf{I}_{\ell+\frac12}(\sqrt{\gamma}\eta)\mathbf{K}_{\ell+\frac12}(\sqrt{\gamma}\eta)\zeta_{\ell,m}(t),$$

$$\begin{aligned}&\frac{1}{4\pi}\int_0^{2\pi}\int_0^\pi \frac{\partial}{\partial r}v^{(1)}Y_{\ell,m}(\theta,\varphi)\sin\theta d\theta d\varphi \\ &= -\frac{1}{3}\left[\frac12+\sqrt{\gamma}\eta\left(\mathbf{I}_{\frac12}(\sqrt{\gamma}\eta)\mathbf{K}_{\frac32}(\sqrt{\gamma}\eta)-2\mathbf{I}_{\frac32}(\sqrt{\gamma}\eta)\mathbf{K}_{\frac52}(\sqrt{\gamma}\eta)\right)\right]\zeta_{\ell,m}(t),\end{aligned}$$

$$\begin{aligned}&\frac{1}{4\pi}\int_0^{2\pi}\int_0^\pi \frac{\partial}{\partial r}v^{(2)}Y_{\ell,m}(\theta,\varphi)\sin\theta d\theta d\varphi \\ &= \frac{1}{2\ell+1}\left[\frac12+\sqrt{\gamma}\eta\left(\ell\mathbf{I}_{\ell-\frac12}(\sqrt{\gamma}\eta)\mathbf{K}_{\ell+\frac12}(\sqrt{\gamma}\eta)\right.\right., \\ &-(\ell+1)\mathbf{I}_{\ell+\frac12}(\sqrt{\gamma}\eta)\mathbf{K}_{\ell+\frac32}(\sqrt{\gamma}\eta)\Big]\zeta_{\ell,m}(t)\end{aligned}$$

where $\zeta_{\ell,m}(t)=\dfrac{1}{4\pi}\displaystyle\int_0^{2\pi}\int_0^\pi \zeta(\theta,\varphi,t)Y_{\ell,m}(\theta,\varphi)\sin\theta d\theta d\varphi.$

It follows from (5.1) that

$$\frac{d}{dt}\zeta_{\ell,m} = \Big\{k\chi''(v_0)v_{0r}\Big[v_{0r} + \eta \mathbf{I}_{\ell+\frac{1}{2}}(\sqrt{\gamma}\eta)\mathbf{K}_{\ell+\frac{1}{2}}(\sqrt{\gamma}\eta)\Big]$$

$$-k\chi'(v_0)\left[\frac{1}{3}\left[\frac{1}{2} + \sqrt{\gamma}\eta\left(\mathbf{I}_{\frac{1}{2}}(\sqrt{\gamma}\eta)\mathbf{K}_{\frac{3}{2}}(\sqrt{\gamma}\eta) - 2\mathbf{I}_{\frac{3}{2}}(\sqrt{\gamma}\eta)\mathbf{K}_{\frac{5}{2}}(\sqrt{\gamma}\eta)\right)\right]\right.$$

$$-\frac{1}{2\ell+1}\left[\frac{1}{2} + \sqrt{\gamma}\eta\left(\ell\mathbf{I}_{\ell-\frac{1}{2}}(\sqrt{\gamma}\eta)\mathbf{K}_{\ell+\frac{1}{2}}(\sqrt{\gamma}\eta)\right.\right.$$

$$\left.\left.\left. -\ (\ell+1)\mathbf{I}_{\ell+\frac{1}{2}}(\sqrt{\gamma}\eta)\mathbf{K}_{\ell+\frac{3}{2}}(\sqrt{\gamma}\eta)\right)\right]\right] - \frac{\varepsilon}{\eta^2}(\ell+2)(\ell-1)\Big\}\,\zeta_{\ell,m}(t)$$

$$\equiv F_3(\ell,k,\varepsilon)\zeta_{\ell,m}(t), \tag{5.2}$$

where $v_{0r} = -\eta \mathbf{I}_{\frac{3}{2}}(\sqrt{\gamma}\eta)\mathbf{K}_{\frac{3}{2}}(\sqrt{\gamma}\eta)$.

By a simple computation, we note that (i) $F_3(0,k,\varepsilon) = \frac{\partial}{\partial\eta}H_3(\eta;k,\varepsilon)$, that is, the stability of the spherical stationary solution under the radially symmetric disturbances is determined by the sign of $\frac{\partial}{\partial\eta}H_3(\eta;k,\varepsilon)$; (ii) $F_3(1,k,\varepsilon) = 0$, that is, the stationary solution has phase shift free in (2.4).

Next, we numerically treat with the functional form of $F_3(\ell,k,\varepsilon)$ for the solution $(\overline{\eta}, v(r;\overline{\eta}))$. In Figure 4, the curves of $F_3(\ell,k,\varepsilon)=0$ for $\ell = 2,3,4$ is shown in the (k,s)-plane. For small $s>0$, the solution is stable and with any fixed $s>0$, the solution is so for large $k>0$. Figure 5 shows the form of $F_3(\ell,k,\varepsilon)$ for $s = 0.6, 1.0$ and $\ell = 2,3,4$. It is known that these results are similar to that of the case for $N=2$ in [4].

Proposition 2. (Asymptotic behavior of $F_3(\ell,k,\varepsilon)$) *It holds that*

$$\lim_{k\to k^*}\left\{F_3(\ell,k,\varepsilon)\overline{\eta}^2 + (\ell+2)(\ell-1)F^*(\varepsilon)\right\} = 0,$$

where $F^*(\varepsilon)$ stands in Remark 2.

Proof. Because of $\lim_{k\to k^*}\overline{\eta} = \infty$, we can prove this proposition from (5.2) by using the asymptotic behavior of the modified Bessel functions $\mathrm{I}_{\ell+\frac{1}{2}}(z)$ and $\mathrm{K}_{\ell+\frac{1}{2}}(z)$ as z tends to infinity.

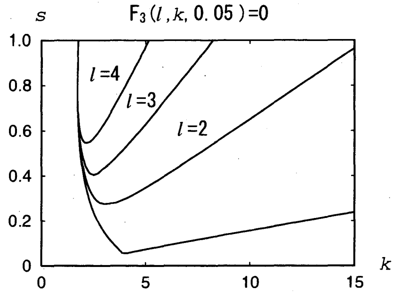

Figure 4.1

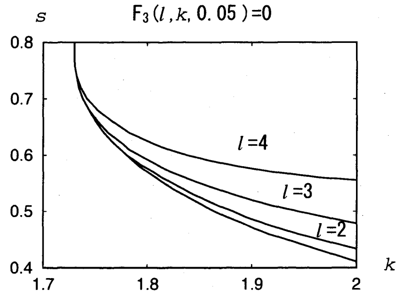

Figure 4.2

This propostion imples that the similar results to the last part in Remark 2 holds for $F_3(\ell, k, \varepsilon)$ depending on the sign of $F^*(\varepsilon)$. We note that $F^*(\varepsilon) < 0$ at $s = 1.0$ and $F^*(\varepsilon) > 0$ at $s = 0.6$ for $\chi(v) = sv^2/(s^2 + v^2)$. On the other hands, it is suggested that $F_3(\ell, k, \varepsilon) < 0$ for $\ell > 1$ as k tends to $\overline{k}(\varepsilon)$ in Figure 5. Since $\overline{k}(\varepsilon)$ is the turning point of the global branch of the stationary solution, we may assume that $\overline{\eta}(\varepsilon)$ becomes of order ε for small ε as k tends to $\overline{k}(\varepsilon)$ from Theorem 1. Then, we have

$$\frac{F_3(\ell, k, \varepsilon)}{\overline{\eta}(\varepsilon)} = -\frac{(\ell + 2)(\ell - 1)\mu}{\varepsilon} + O(1) \tag{5.3}$$

for some positive constant μ. Therefore, as k tends to $\overline{k}(\varepsilon)$, it follows from (5.3) that

$$0 > F_3(2, k, \varepsilon) > F_3(3, k, \varepsilon) > \cdots > F_3(\ell, k, \varepsilon) > F_3(\ell + 1, k, \varepsilon) > \cdots .$$

In this paper, we do not discuss the relation of the solutions between the interface equation (2.4) and the original reaction–diffusion equation (1.2). That is, the solution of (2.4) becomes the good approximation of the solution of (1.2). Moreover, there is the problem such that the asymptotic behavior of the critical eigenvalues of the linearized eigenvalue problem of (1.2) is represented by using $F_3(\ell, k, \varepsilon)$ (see [9]).

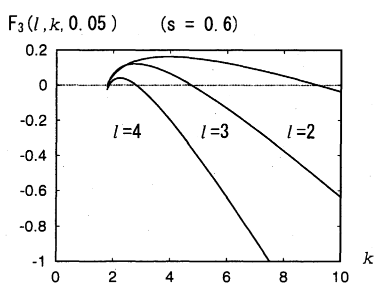

Figure 5.1

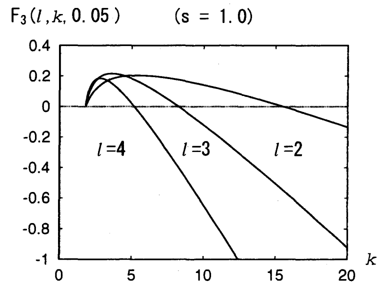

Figure 5.3

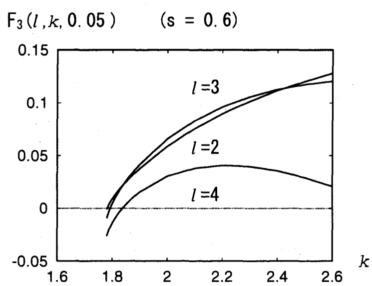

Figure 5.2

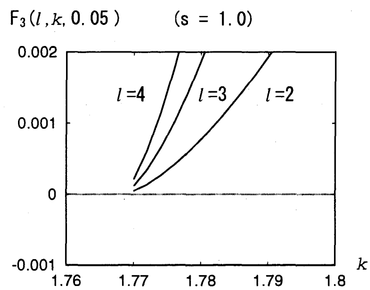

Figure 5.4

§6. Appendix

In this section, we obtain the equations (2.4), (2.5) which describes the motion of the interface according to [2, 6] by using the matched asymptotic expansion. To do so, we set the outer expansion as follows:

$$\begin{cases} u &= u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots \\ v &= v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + \cdots . \end{cases}$$

Substituting this expansion into (2.1), we shall have the outer solutions u_0, $u_1, \cdots$ and v_0, $v_1, \cdots$ as follows:

$O(1)$**- term**

Since u_0 satisfies $0 = f(u_0)$ and the original system is bistable, we may set

$$u_0(t, x) = \begin{cases} 0 & \text{in} \quad \Omega_0(t) \\ 1 & \text{in} \quad \Omega_1(t). \end{cases}$$

Here, we assume that $\Omega_0(t)$, $\Omega_1(t)$ and $\Gamma(t)$ satisfy $\Gamma(t) = \mathbf{R}^N \backslash \{\Omega_0(t) \cup \Omega_1(t)\}$ and $\Omega_0(t) \cap \Omega_1(t) = \phi$ where $\Omega_1(t)$ is a bounded domain and $\Gamma(t)$ is $(N-1)$-dimensional hypersurface.

On the other hand, v_0 satisfies the second equation of (2.4).

$O(\varepsilon)$**- term**

Since $0 = f'(u_0)u_1$ and $f'(u_0) < 0$ for $u_0 = 0,\ 1$, we have $u_1(x) \equiv 0$. Therefore, v_1 satisfies $v_{0t} = \Delta v_1 - \gamma v_1$ in $\mathbf{R}^N$ with $v_1 \in C^1(\mathbf{R}^N)$.

$O(\varepsilon^2)$**- term**

From $0 = f'(u_0)u_2$, we have $u_2(x) \equiv 0$. Therefore, v_2 satisfies $v_{1t} = \Delta v_2 - \gamma v_2$ in $\mathbf{R}^N$.

Because the constructed solution is discontinuous on the interface Γ, we need to consider the another approximation near Γ.

To do so, we introduce the new stretched variable $\xi = d(t,x)/\varepsilon$ where $d(t,x)$ is the signed distance function to Γ,

$$d(t,x) = \begin{cases} \mathrm{dist}(x,\Gamma) & \text{in } \Omega_0(t) \\ -\mathrm{dist}(x,\Gamma) & \text{in } \Omega_1(t). \end{cases}$$

We note that $d(t,x) = 0$ on $\Gamma(t)$ and $|\nabla d| = 1$. Then, we set the inner expansion as follows:

$$\begin{cases} d &= d_0 + \varepsilon d_1 + \varepsilon^2 d_2 + \cdot \\ U &= U_0 + \varepsilon U^1 + \varepsilon^2 U^2 + \cdots \\ V &= V_0 + \varepsilon V^1 + \varepsilon^2 V^2 + \cdots \end{cases}$$

To make the outer and inner expansions consistent, we require the matching condition on $\Gamma(t)$:

$$\lim_{\substack{y \to x \\ \xi \to \infty}} \{(U(t,\xi,x), V(t,\xi,x)) - (u(t,y), v(t,y))\} = (0,0),\ y \in \Omega_1,\ x \in \Gamma(t),$$

$$\lim_{\substack{y \to x \\ \xi \to -\infty}} \{(U(t,\xi,x), V(t,\xi,x)) - (u(t,y), v(t,y))\} = (0,0),\ y \in \Omega_0,\ x \in \Gamma(t).$$

Substituting these inner expansions into (2.1) and using the above matching condition, we shall have the inner solutions U_0, U_1, $\cdots$ and V_1, $V_2, \cdots$ as follows:

$O(\varepsilon^{-2})$**- term**

V_0 satisfies

$$0 = V_{0\xi\xi} \quad \text{and} \quad \lim_{\xi \to \pm\infty} V_0(t,\xi,x) = v_0(t,x).$$

From the condition $v_0 \in C^1(\mathbf{R}^N)$, we have $V_0(t,\xi,x) \equiv v_0(t,x)$.

$O(\varepsilon^{-1})$**- term**

V_1 satisfies

$$0 = V_{1\xi\xi} \quad \text{and} \quad \lim_{\xi \to \pm\infty} V_1(t,\xi,x) = v_1(t,x).$$

From the condition $v_1 \in C^1(\mathbf{R}^N)$, we have $V_1(t,\xi,x) \equiv v_1(t,x)$.

$O(1)$**- term**

U_0 satisfies

$$\begin{cases} d_{0t}U_{0\xi} = U_{0\xi\xi} - k\chi'(v_0)\nabla d_0 \nabla v_0 U_{0\xi} + f(U_0), & \xi \in \mathbf{R} \\ \lim_{\xi\to-\infty} U_0(\xi) = 1 \quad \text{and} \quad \lim_{\xi\to\infty} U_0(\xi) = 0 \end{cases}$$

where ∇v_0 is the gradient of v_0 on the interface Γ.

Lemma (Fife and McLeod [3]) *For some constant c^*, there is a monotone solution $W(\xi)$ of*

$$\begin{cases} 0 = W_{\xi\xi} + c^* W_\xi + f(W), & \xi \in \mathbf{R} \\ \lim_{\xi\to-\infty} W(\xi) = 1 \quad \text{and} \quad \lim_{\xi\to\infty} W(\xi) = 0. \end{cases} \tag{a.1}$$

Therefore, we set $U_0 = W$ and $-c^* = k\chi'(v_0)\nabla d_0 \nabla v_0 + d_{0t}$ on Γ, that is,

$$-c^* = k\chi'(v_0)\frac{\partial v_0}{\partial n} + d_{0t} \quad \text{on } \Gamma.$$

$O(\varepsilon)$**- term**

Let L be the linearized operator of (a.1) around the monotone solution $W(\xi)$ as follows:

$$L = \frac{d^2}{d\xi^2} + c^* \frac{d}{d\xi} + f'(U_0).$$

Then, $U_1(\xi)$ satisfies

$$\begin{aligned} LU_1 &= (d_{1t} - \Delta d_0)U_{0\xi} + k\{\chi'(v_0)[(\nabla d_1 \nabla v_0 + \nabla d_0 \nabla v_1 + V_{2\xi})U_{0\xi} \\ &\quad +(\Delta v_0 + V_{2\xi\xi})U_0] + \chi''(v_0)\left(\frac{\partial v_0}{\partial n} V_1 U_{0\xi} + (\nabla v_0)^2 U_0\right)\right\} \\ &\equiv \hat{G} \end{aligned}$$

where V_2 satisfies

$$V_{2\xi\xi} + U_0 = -\Delta v_0 + \gamma v_0 = u_0 = \begin{cases} 1 & (\xi < 0) \\ 0 & (\xi > 0). \end{cases}$$

Let L^* be the adjoint operator of L defined by

$$L^* = \frac{d^2}{d\xi^2} - c^* \frac{d}{d\xi} + f'(U_0).$$

Then, $\phi = U_{0\xi}$ and $\psi = e^{c^*\xi}\phi$ are eigenfunctions of 0 eigenvalues of L and L^*, respectively. From the solvability condition, $\hat{G}$ satisfies

$$0 = <\hat{G}, \psi>_{L^2(\mathbf{R})}$$

where $< \cdot, \cdot >_{L^2(\mathbf{R})}$ is the inner product of $L^2(\mathbf{R})$. Let V^* be the normal velocity of the interface Γ and κ be the mean curvature. In our case, we define that Γ is positive if Ω_1 is convex. From $\Delta d_0 = (N-1)\kappa$ and $V^* = -d_t$ on Γ, that is, $(V^* + d_{0\xi})/\varepsilon = -d_{1t} + O(\varepsilon)$. Then, we have

$$V^* = c^* + k\chi'(v_0)\frac{\partial v_0}{\partial n} - \varepsilon(N-1)\kappa + \varepsilon G \tag{a.2}$$

where

$$\begin{aligned} G = k\{&\chi'(v_0)\nabla d_1 \nabla v_0 + \nabla d_0 \nabla(\chi'(v_0)v_1) \\ &+[\chi'(v_0)\int_{-\infty}^{\infty} \psi(U_{0\xi}V_{2\xi} + U_0 V_{2\xi\xi})\, d\xi \\ &+\nabla(\chi'(v_0)\nabla v_0)\int_{-\infty}^{\infty} \psi U_0\, d\xi]/\int_{-\infty}^{\infty} \phi\psi\, d\xi \Big\} \end{aligned}$$

Therefore, we have the interface equation as follows:

$$\begin{cases} V^* = c^* + k\chi'(v_0)\dfrac{\partial v_0}{\partial n} - \varepsilon(N-1)\kappa + \varepsilon G \quad \text{on} \quad \Gamma(t),\ t>0 \\ 0 = \Delta v_0 + g_i(v_0) \quad \text{in} \quad \Omega_i(t)\ (i=0,\ 1),\ t>0 \\ \qquad v_0(t,\cdot) \in C^1(\mathbf{R}^N),\ t>0, \end{cases} \tag{a.3}$$

where $g_i(v) = i - \gamma v$.

References

[1] A. Bonami, D. Hilholst, E. Logak and M. Mimura, Singular limit of a chomotaxis growth model, Adv. Differential Equations, **6** (2001), 1173–1218.

[2] P. C. Fife, Dynamics of internal layers and diffusive interfaces, CBM-NSF Regional Conf. Ser. in Appl. Math., SIAM, Philadelphia, 1988.

[3] P. C. Fife and J. B. McLeod, The approach of solutions of nonlinear diffusion equations to travelling front solutions, Arch. Ration. Mech. Anal., **65** (1977), 335–361.

[4] M. Mimura and T. Tsujikawa, Aggregating pattern dynamics in a chemotaxis model including growth, Physica A, **230** (1996), 499–543.

[5] M. Mimura, T. Tsujikawa, R. Kobayashi and D. Ueyama, Dynamics of aggregating patterns in a chemotaxis–diffusion–growth model equation, Forma, **8** (1993), 179–195.

[6] K.-I. Nakamura, H. Matano, D. Hilholst and R. Schätzle, Singular limit of a reaction-diffusion equation with a spatially inhomogeneous reaction term, J. Statist. Phys., **95** (1999), 1165–1185.

[7] T. Ohta, M. Mimura and R. Kobayashi, Higher-dimensional localized patterns in excitable media, Phys. D, **34** (1989), 115–144.

[8] K. Osaki, T. Tsujikawa, A. Yagi and M. Mimura, Exponential attractors for a chemotaxis-growth system of equations, Nonlinear Anal., **51** (2002), 119–144.

[9] T. Tsujikawa, Singular limit analysis of planar equilibrium solutions to chemotaxis model equation with growth, Methods Appl. Anal., **3** (1996), 401–431.

Department of Applied Physics
University of Miyazaki
Miyazaki 889-2192
Japan