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Determination of the limit sets of trajectories of the Gierer-Meinhardt system without diffusion

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Abstract.

We consider a reaction-diffusion system consisting of an activator and an inhibitor which models biological pattern formation. A complete description of the entire dynamics of the kinetic system, i.e., the system without diffusion terms, is given. In particular, the α-limit sets and the ω-limit sets of all trajectories are determined.

§1. Introduction and Statement of Main Results

One of the central problems in developmental biology is to understand the mechanism of the formation of a spatial pattern in tissue structure, starting from almost homogeneous states. As a model of morphogenesis, A. Gierer and H. Meinhardt ([1]) proposed an activator-inhibitor system:

$$(1)\qquad \begin{cases} \dfrac{\partial u}{\partial t} = \varepsilon^2 \Delta u - u + \dfrac{u^p}{v^q} + \sigma(x) & \text{for } x \in \Omega,\ t > 0, \\ \tau \dfrac{\partial v}{\partial t} = d\Delta v - v + \dfrac{u^r}{v^s} & \text{for } x \in \Omega,\ t > 0, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 & \text{for } x \in \partial\Omega,\ t > 0, \end{cases}$$

where $\Delta = \Sigma_{j=1}^N \partial^2/\partial x_j^2$ is the Laplace operator in $\mathbb{R}^N$, Ω is a bounded smooth domain in $\mathbb{R}^N$, ν is the unit outer normal to $\partial\Omega$, σ is a non-negative function, τ, ε and d are positive constants. The exponents p,

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q, r and s satisfy

$$(2) \qquad p > 1,\ q > 0,\ r > 0,\ s \geq 0 \quad \text{and} \quad 0 < \frac{p-1}{r} < \frac{q}{s+1}.$$

The system (1) is derived based on the idea of "short-range activation, long-range inhibition", which means that the inhibitor, although activated by the activator, spreads by diffusion faster than the activator and blocks the production of the activator in distant places. As a results, striking patterns of the activator concentration are expected to emerge. For the past two decades there have been several papers on the existence of solutions of the initial-boundary value problem (1) ([2]–[5], [7], [8]). In particular, solutions to (1) are proved to exist and are bounded for all $t > 0$ provided $p - 1 < r$ when $\sigma > 0$ ([2, 8]). However, our understanding of the dynamics of (1) is far from complete.

It is our purpose to give a complete understanding of the dynamics of the following kinetic system which is obtained by removing the diffusion terms from the Gierer-Meinhardt system:

$$(\mathrm{K}) \qquad \begin{cases} \dfrac{du}{dt} = -u + \dfrac{u^p}{v^q} & \text{for } t > 0, \\ \tau \dfrac{dv}{dt} = -v + \dfrac{u^r}{v^s} & \text{for } t > 0, \\ u(0) = u_0,\ v(0) = v_0, \end{cases}$$

where $u_0 \geq 0$, $v_0 > 0$ and the exponents p, q, r, s satisfy

$$(\mathrm{C}) \qquad p > 1,\ q > 0,\ r > 0,\ s > -1 \quad \text{and} \quad 0 < \frac{p-1}{q} < \frac{r}{s+1},$$

which is slightly more general than (2) as we now assume that $s > -1$ in (C) instead of $s \geq 0$ in (2).

It turns out that the dynamics of (K) exhibits various interesting behaviors including convergence to the equilibria $(0,0)$ or $(1,1)$, periodic solutions, unbounded oscillating global solutions, and finite time blow-up solutions. At this stage it is too early to draw biologically meaningful conclusions from our results, but we expect that they stimulate further mathematical studies on the original Gierer-Meinhardt system towards a better understanding of the mechanism of pattern formation.

We begin by introducing some notation to state our main results.

1.1. Preliminaries

We shall assume throughout this paper that the initial value is in the first quadrant $\mathcal{Q} = \{(u,v) \in \mathbb{R}^2 \mid u > 0,\ v > 0\}$. By $\partial\mathcal{Q}$ we mean the set $\{(u,v) \mid uv = 0,\ u \geq 0,\ v \geq 0\}$. By ∞ we mean the point at infinity $(u,v) = (+\infty, +\infty)$ and by O the origin $(u,v) = (0,0)$. For $P \in \mathcal{Q}$, let $(T_-(P), T_+(P))$ denote the maximal existence interval of the solution of the initial value problem for (K) with initial value $(u(0), v(0)) = P$. Moreover, let $\gamma(P)$ denote the orbit

$$\gamma(P) = \{(u(t), v(t)) \mid T_-(P) < t < T_+(P)\}$$

of the solution $(u(t), v(t))$ with $(u(0), v(0)) = P$. We denote the ω-limit set and the α-limit set of $\gamma(P)$ by $L^+(\gamma(P))$ and $L^-(\gamma(P))$, respectively.

The two curves $\mathcal{C}_1$ and $\mathcal{C}_2$, defined by $v = u^{(p-1)/q}$(on which $u_t = 0$) and $v = u^{r/(s+1)}$(on which $v_t = 0$) respectively, intersect at exactly one point $E = (1,1)$. Hence it is an unique equilibrium point in $\mathcal{Q}$. These two curves divide the first quadrant into four regions $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$ and $\mathcal{R}_4$:

$$\begin{aligned}
\mathcal{R}_1 &= \{(u,v) \mid 0 < u < \infty,\ v < \min\{u^{(p-1)/q}, u^{r/(s+1)}\}\},\\
\mathcal{R}_2 &= \{(u,v) \mid 1 < u < \infty,\ u^{(p-1)/q} < v < u^{r/(s+1)}\},\\
\mathcal{R}_3 &= \{(u,v) \mid 0 < u < \infty,\ v > \max\{u^{(p-1)/q}, u^{r/(s+1)}\}\},\\
\mathcal{R}_4 &= \{(u,v) \mid 0 < u < 1,\ u^{r/(s+1)} < v < u^{(p-1)/q}\}.
\end{aligned}$$

First, we deal with the stability of the equilibrium point $E = (1,1)$ of (K). To do this, we define an important quantity by

$$\alpha = \frac{qr}{p-1} - (s+1). \tag{3}$$

Note that $\alpha > 0$ by (C).

Lemma 1. *The equilibrium $E = (1,1)$ of (K) is locally asymptotically stable if $\tau < \tau_E$, and, it is unstable if $\tau > \tau_E$. To be more precise, we have*

(i) E is a stable node if $\tau \leq \tau_1$,
(ii) E is a stable focus if $\tau_1 < \tau < \tau_E$,
(iii) E is an unstable focus if $\tau_E < \tau < \tau_2$,
(iv) E is an unstable node if $\tau_2 \leq \tau$,

where

$$\tau_E = \frac{s+1}{p-1}, \tag{4}$$

$$\tau_1 = \frac{s+1+2\alpha - 2\sqrt{\alpha(s+1+\alpha)}}{p-1}, \quad \tau_2 = \frac{s+1+2\alpha + 2\sqrt{\alpha(s+1+\alpha)}}{p-1}.$$

1.2. Main results

We now come to the precise statements of the results.

All the possible choices of the exponents (p, q, r, s) satisfying (C) fall into the following three cases:

Case I: $r > p-1,\ s+1 > q,\quad (\text{example}:\ (p,q,r,s) = (2,4,2,4)),$
Case II: $r \geq p-1,\ s+1 \leq q,$
$(\text{example}:\ (p,q,r,s) = (2,2,2,0) \text{ or } (2,1,2,0)),$
Case III: $r < p-1,\ s+1 < q,\quad (\text{example}:\ (p,q,r,s) = (4,2,2,0)).$

Notice that (C) is never satisfied in the remaining case $r < p-1$ and $s+1 \geq q$. Moreover, in Case II, the possibility of $(r, s+1) = (p-1, q)$ is ruled out by (C). It turns out that these three cases differ in the possible α- and ω-limit sets of solution orbits. In addition, in each of the cases, solutions of (K) behave quite differently as the relaxation parameter τ varies.

In addition to (4) we define the following special values of τ:

$$\tau_\infty \equiv \frac{s+1}{r}, \quad \tau_O \equiv \frac{q}{p-1}. \tag{5}$$

Theorem 2 (Case I). *Let $r > p-1$ and $s+1 > q$. Then $\tau_\infty < \tau_O < \tau_E$. In the following statements (a)–(e), we assume $P \in \mathcal{Q}\backslash\{E\}$.*

(a) If $\tau < \tau_\infty$, then there are special orbits γ_c and γ_s for which the following (i)–(iii) hold. (i) $L^+(\gamma_c) = \{E\}$ and $L^-(\gamma_c) = \{\infty\}$, while $L^+(\gamma_s) = \{E\}$ and $L^-(\gamma_s) = \{O\}$. The combined curve $\gamma_c \cup \{E\} \cup \gamma_s$ separates $\mathcal{Q}$ into two subdomains $\mathcal{A}$ and $\mathcal{B}$, where boundary $\partial\mathcal{A}$ contains the positive v-axis and $\partial\mathcal{B}$ contains the positive u-axis. (ii) If $(u(0), v(0)) = P$ is in $\mathcal{A}$, then $L^+(\gamma(P)) = \{E\}$ and $L^-(\gamma(P)) = \{\infty\}$. (iii) If P is in $\mathcal{B}$, then there exists a positive number u_P depending only on P such that $L^+(\gamma(P)) = \{E\}$ and $L^-(\gamma(P)) = \{(u_P, 0)\}$. Conversely for any positive number ξ, there is a $P \in \mathcal{B}$ such that $L^-(\gamma(P)) = \{(\xi, 0)\}$.

(b) If $\tau_\infty \leq \tau \leq \tau_O$, then for each $P \in \mathcal{Q}$, there exists a non-negative number u_P depending only on P such that $L^+(\gamma(P)) = \{E\}$ and $L^-(\gamma(P)) = \{(u_P, 0)\}$. Conversely for any non-negative number ξ, there is a $P \in \mathcal{Q}$ such that $L^-(\gamma(P)) = \{(\xi, 0)\}$.

(c) If $\tau_O < \tau < \tau_E$, then there is a special orbit γ_s for which the following (i)–(iii) hold. (i) $L^+(\gamma_s) = \{O\}$ and $L^-(\gamma_s) = \{(u_s, 0)\}$, where u_s is a positive number. Let $\mathcal{D}$ be the domain enclosed by γ_s and the u-axis. (ii) If P is in $\mathcal{D}$, then there exists a u_P with $0 \leq u_P < u_s$ depending only on P such that $L^+(\gamma(P)) = \{E\}$ and $L^-(\gamma(P)) = \{(u_P, 0)\}$. Conversely for any non-negative number ξ with $0 \leq \xi < u_s$, there is a $P \in \mathcal{D}$ such that $L^-(\gamma(P)) = \{(\xi, 0)\}$.

(iii) If P is in $\mathcal{Q}\backslash\mathcal{D}$, then there exists a positive number u_P with $u_P > u_s$ depending only on P such that $L^+(\gamma(P)) = \{O\}$ and $L^-(\gamma(P)) = \{(u_P, 0)\}$. Conversely for any positive number ξ with $\xi > u_s$, there is a $P \in \mathcal{Q}\backslash\mathcal{D}$ such that $L^-(\gamma(P)) = \{(\xi, 0)\}$.

(d) *If $\tau = \tau_E$, then there is a special orbit γ_s for which the following (i)–(iii) hold. (i) $L^+(\gamma_s) = \{O\}$ and $L^-(\gamma_s) = \{O\}$. Let $\mathcal{D}$ be the domain enclosed by γ_s. (ii) If P is in $\mathcal{D}$, then $\gamma(P)$ is a closed orbit. (iii) If P is in $\mathcal{Q}\backslash\mathcal{D}$, then there exists a positive number u_P depending only on P such that $L^+(\gamma(P)) = \{O\}$ and $L^-(\gamma(P)) = \{(u_P, 0)\}$. Conversely for any positive number ξ, there is a $P \in \mathcal{Q}\backslash\mathcal{D}$ such that $L^-(\gamma(P)) = \{(\xi, 0)\}$.*

(e) *If $\tau > \tau_E$, then there is a special orbit γ_s for which the following (i)–(iii) hold. (i) $L^+(\gamma_s) = \{O\}$ and $L^-(\gamma_s) = \{O\}$. Let $\mathcal{D}$ be the domain enclosed by γ_s. (ii) If P is in $\mathcal{D}$, then $L^+(\gamma(P)) = \{O\}$ and $L^-(\gamma(P)) = \{E\}$. (iii) If P is in $\mathcal{Q}\backslash\mathcal{D}$, then there exists a positive number u_P depending only on P such that $L^+(\gamma(P)) = \{O\}$ and $L^-(\gamma(P)) = \{(u_P, 0)\}$. Conversely for any positive number ξ, there is a $P \in \mathcal{Q}\backslash\mathcal{D}$ such that $L^-(\gamma(P)) = \{(\xi, 0)\}$.*

Furthermore, if $L^-(\gamma(P)) = \{(u_P, 0)\}$, then $-\infty < T_-(P)$, while $T_-(P) = -\infty$ if $L^-(\gamma(P)) = \{\infty\}$ or $\{E\}$. On the other hand, $T_+(P) = +\infty$ for all $P \in \mathcal{Q}$.

Theorem 3 (Case II). *Let $r \geq p - 1$ and $s + 1 \leq q$. Then $\tau_\infty \leq \tau_E \leq \tau_O$. For all $P \in \mathcal{Q}$, $T_+(P) = +\infty$ and $T_-(P) = -\infty$. Moreover the following statements (a)–(e) hold for any $P \in \mathcal{Q}\backslash\{E\}$.*

(a) *If $\tau < \tau_\infty$, then $L^+(\gamma) = \{E\}$ and $L^-(\gamma) = \{\infty\}$.*
(b) *If $\tau_\infty \leq \tau < \tau_E$, then $L^+(\gamma) = \{E\}$ and $L^-(\gamma) = \overline{\partial\mathcal{Q}}$.*
(c) *If $\tau = \tau_E$, then all the orbits are closed.*
(d) *If $\tau_E < \tau \leq \tau_O$, then $L^+(\gamma) = \overline{\partial\mathcal{Q}}$ and $L^-(\gamma) = \{E\}$.*
(e) *If $\tau > \tau_O$, then $L^+(\gamma) = \{O\}$ and $L^-(\gamma) = \{E\}$.*

Here γ stands for $\gamma(P)$ and $\overline{\partial\mathcal{Q}}$ denotes the extended boundary of $\mathcal{Q}$:

$$\begin{aligned}\overline{\partial\mathcal{Q}} =& \{(u, 0) \mid u \geq 0\} \cup \{(0, v) \mid v \geq 0\} \\ & \cup \{(u, +\infty) \mid u \geq 0\} \cup \{(+\infty, v) \mid v \geq 0\}.\end{aligned}$$

Theorem 4 (Case III). *Let $r < p - 1$ and $s + 1 < q$. Then $\tau_E < \tau_\infty < \tau_O$. In the following statements (a)–(e), we assume $P \in \mathcal{Q}\backslash\{E\}$.*

(a) *If $\tau < \tau_E$, then there is a special orbit γ_s for which the following (i)–(iii) hold. (i) $L^+(\gamma_s) = \{\infty\}$ and $L^-(\gamma_s) = \{\infty\}$. The orbit γ_s separates $\mathcal{Q}$ into two subdomains $\mathcal{A}$ and $\mathcal{B}$, where $\mathcal{A}$ contains E. (ii) If P is in $\mathcal{A}$, then $L^+(\gamma(P)) = \{E\}$ and $L^-(\gamma(P)) = \{\infty\}$. (iii) If P is in $\mathcal{B}$, then there exists a v_P with $0 < v_P <$*

$+\infty$ depending only on P such that $L^+(\gamma(P)) = \{(+\infty, v_P)\}$ and $L^-(\gamma(P)) = \{\infty\}$. Conversely for any positive number η, there is a $P \in \mathcal{B}$ such that $L^+(\gamma(P)) = \{(+\infty, \eta)\}$.

(b) If $\tau = \tau_E$, then there is a special orbit γ_s for which the following (i)–(iii) hold. (i) $L^+(\gamma_s) = \{\infty\}$ and $L^-(\gamma_s) = \{\infty\}$. The orbit γ_s separates $\mathcal{Q}$ into two subdomains $\mathcal{A}$ and $\mathcal{B}$, where $\mathcal{A}$ contains E. (ii) If P is in $\mathcal{A}$, then $\gamma(P)$ is a closed orbit. (iii) If P is in $\mathcal{B}$, then there exists a v_P with $0 < v_P < +\infty$ depending only on P such that $L^+(\gamma(P)) = \{(+\infty, v_P)\}$ and $L^-(\gamma(P)) = \{\infty\}$. Conversely for any positive number η, there is a $P \in \mathcal{B}$ such that $L^+(\gamma(P)) = \{(+\infty, \eta)\}$.

(c) If $\tau_E < \tau < \tau_\infty$, then there is a special orbit γ_s for which the following (i)–(iii) hold. (i) $L^+(\gamma_s) = \{(+\infty, v_s)\}$ and $L^-(\gamma_s) = \{\infty\}$, where v_s is a number with $0 < v_s < +\infty$. The orbit γ_s separates $\mathcal{Q}$ into two subdomains $\mathcal{A}$ and $\mathcal{B}$, where $\mathcal{A}$ contains E. (ii) If P is in $\mathcal{A}$, then there exists a v_P with $v_s < v_P \le +\infty$ depending only on P such that $L^+(\gamma(P)) = \{(+\infty, v_P)\}$ and $L^-(\gamma(P)) = \{\infty\}$. Conversely for any positive number η with $v_s < \eta \le +\infty$, there is a $P \in \mathcal{A}$ such that $L^+(\gamma(P)) = \{(+\infty, \eta)\}$. (iii) If P is in $\mathcal{B}$, then there exists a v_P with $0 < v_P < v_s$ depending only on P such that $L^+(\gamma(P)) = \{(+\infty, v_P)\}$ and $L^-(\gamma(P)) = \{\infty\}$. Conversely for any positive number η with $0 < \eta < v_s$, there is a $P \in \mathcal{B}$ such that $L^+(\gamma(P)) = \{(+\infty, \eta)\}$.

(d) If $\tau_\infty \le \tau \le \tau_O$, then there exist a v_P with $0 < v_P \le +\infty$ depending only on P such that $L^+(\gamma(P)) = \{(+\infty, v_P)\}$ and $L^-(\gamma(P)) = \{E\}$ for all $P \in \mathcal{Q}\backslash\{E\}$. Conversely for any number η with $0 < \eta \le +\infty$, there is a $P \in \mathcal{Q}$ such that $L^+(\gamma(P)) = \{(+\infty, \eta)\}$.

(e) If $\tau > \tau_O$, then there are special orbits γ_c and γ_s for which the following (i)–(iii) hold. (i) $L^+(\gamma_c) = \{O\}$ and $L^-(\gamma_c) = \{E\}$, while $L^+(\gamma_s) = \{\infty\}$ and $L^-(\gamma_s) = \{E\}$. The combined curve $\gamma_c \cup \{E\} \cup \gamma_s$ separates $\mathcal{Q}$ into two subdomains $\mathcal{A}$ and $\mathcal{B}$, where boundary $\partial\mathcal{A}$ contains the positive v-axis and $\partial\mathcal{B}$ contains the positive u-axis. (ii) If P is in $\mathcal{A}$, then $L^+(\gamma(P)) = \{O\}$ and $L^-(\gamma(P)) = \{E\}$. (iii) If P is in $\mathcal{B}$, then there exists a v_P with $0 < v_P < +\infty$ depending only on P such that $L^+(\gamma(P)) = \{(+\infty, v_P)\}$ and $L^-(\gamma(P)) = \{E\}$. Conversely for any positive number η, there is a $P \in \mathcal{Q}$ such that $L^+(\gamma(P)) = \{(+\infty, \eta)\}$.

Furthermore, if $L^+(\gamma(P)) = \{(+\infty, v_P)\}$ with $0 < v_P \le +\infty$, then $T_+(P) < +\infty$, while $T_+(P) = +\infty$ if $L^+(\gamma(P)) = \{E\}$ or $\{O\}$. On the other hand, $T_-(P) = -\infty$ for all $P \in \mathcal{Q}$.

In many of the cases above, the behavior of orbits may be concisely described in terms of "connecting orbits". For example, in (a) of Case I, (i) γ_s is the unique homoclinic orbit from O to O, (ii) each point P in $\mathcal{A}$ is on an orbit connecting ∞ to E, and (iii) $\mathcal{B} = \bigcup_{\xi>0} \gamma_\xi$, where γ_ξ is an orbit that connects the point $(\xi, 0)$ to E. To help the reader visualize the behavior of the solutions, we have included the graphics of the orbits in the above three theorems in Appendix.

We see that if $T_+(P) < +\infty$, then the solution $(u(t), v(t))$ with initial value P must blow up, i.e., at least $u(t)$ diverges to $+\infty$ as $t \uparrow T_+(P)$. However, if $T_-(P) > -\infty$, then the solution remains bounded as $t \downarrow T_-(P)$. Moreover, we see that there is no blow-up solution in Case I and Case II, while in Case III, there exists a solution which blows up in finite time for any $\tau > 0$. Moreover we observe that the behavior of solutions to (K) changes drastically as the parameter τ passes through τ_E, τ_O or τ_∞. In particular, at $\tau = \tau_E$ the stability of the equilibrium E changes, while there are solutions approaching the origin as $t \to +\infty$ if and only if $\tau > \tau_O$, and there appear solutions tending to ∞ as $t \to -\infty$ if and only if $\tau < \tau_\infty$.

From Theorem 2 and 4 we notice that Case I and Case III seem to share the same degree of complexity. As a matter of fact, these two cases are "dual" in the sense that will be made clear in the next section. This is the key to the proof of theorems.

§2. Transformation among the (K) family

For the complete classification of the solution orbits, the following transformation is crucial. Indeed, it reveals the underlying beautiful mathematical structure of (K). It is important to regard (K) as a family of differential equations labelled by (p, q, r, s, τ). Hence, for the moment we denote the problem by $(\mathrm{K}; p, q, r, s, \tau)$.

If we set

$$\tilde{u} = \frac{1}{v}, \quad \tilde{v} = \frac{1}{u} \quad \text{and} \quad \tilde{t} = -\frac{t}{\tau}, \tag{6}$$

then the system (K) is converted into the following system:

$$(\widetilde{\mathrm{K}}) \qquad \begin{cases} \dfrac{d\tilde{u}}{d\tilde{t}} = -\tilde{u} + \dfrac{\tilde{u}^{\tilde{p}}}{\tilde{v}^{\tilde{q}}}, \\[2ex] \tilde{\tau}\dfrac{d\tilde{v}}{d\tilde{t}} = -\tilde{v} + \dfrac{\tilde{u}^{\tilde{r}}}{\tilde{v}^{\tilde{s}}}, \end{cases}$$

where

$$\tilde{\tau} = \frac{1}{\tau}, \quad \tilde{p} = s + 2, \quad \tilde{q} = r, \quad \tilde{r} = q \quad \text{and} \quad \tilde{s} = p - 2. \tag{7}$$

It is easily verified that $\tilde{p},\ \tilde{q},\ \tilde{r} > 0,\ \tilde{s} > -1$ and

$$0 < \frac{\tilde{p} - 1}{\tilde{q}} < \frac{\tilde{r}}{\tilde{s} + 1}.$$

Hence, (K) becomes the same system $(\widetilde{\mathrm{K}})$ by transformation (6). More precisely, the transformation

$$(u, v, p, q, r, s, t, \tau) \mapsto (\tilde{u}, \tilde{v}, \tilde{p}, \tilde{q}, \tilde{r}, \tilde{s}, \tilde{t}, \tilde{\tau})$$

defined by (6) and (7) maps the problem $(\mathrm{K}; p, q, r, s, \tau)$ to $(\mathrm{K}; \tilde{p}, \tilde{q}, \tilde{r}, \tilde{s}, \tilde{\tau})$. We state here a few important properties of the transformation (6).

The two curves $\widetilde{\mathcal{C}}_1, \widetilde{\mathcal{C}}_2$, defined by $\tilde{v} = \tilde{u}^{(\tilde{p}-1)/\tilde{q}}$(where $\tilde{u}_{\tilde{t}} = 0$) and $\tilde{v} = \tilde{u}^{\tilde{r}/(\tilde{s}+1)}$(where $\tilde{v}_{\tilde{t}} = 0$) respectively, intersect at exactly one point $E = (1, 1)$. Then it is the unique equilibrium point of $(\widetilde{\mathrm{K}})$ in $\widetilde{\mathcal{Q}} \equiv \{(\tilde{u}, \tilde{v}) \mid \tilde{u} > 0,\ \tilde{v} > 0\}$. These two curves divide the first quadrant $\widetilde{\mathcal{Q}}$ into four regions $\widetilde{\mathcal{R}}_1, \widetilde{\mathcal{R}}_2, \widetilde{\mathcal{R}}_3$ and $\widetilde{\mathcal{R}}_4$:

$$\begin{aligned}
\widetilde{\mathcal{R}}_1 &= \{(\tilde{u}, \tilde{v}) \mid 0 < \tilde{u} < \infty,\ \tilde{v} < \min\{\tilde{u}^{(\tilde{p}-1)/\tilde{q}}, \tilde{u}^{\tilde{r}/(\tilde{s}+1)}\}\},\\
\widetilde{\mathcal{R}}_2 &= \{(\tilde{u}, \tilde{v}) \mid 1 < \tilde{u} < \infty,\ \tilde{u}^{(\tilde{p}-1)/\tilde{q}} < \tilde{v} < \tilde{u}^{\tilde{r}/(\tilde{s}+1)}\},\\
\widetilde{\mathcal{R}}_3 &= \{(\tilde{u}, \tilde{v}) \mid 0 < \tilde{u} < \infty,\ \tilde{v} > \max\{\tilde{u}^{(\tilde{p}-1)/\tilde{q}}, \tilde{u}^{\tilde{r}/(\tilde{s}+1)}\}\},\\
\widetilde{\mathcal{R}}_4 &= \{(\tilde{u}, \tilde{v}) \mid 0 < \tilde{u} < 1,\ \tilde{u}^{\tilde{r}/(\tilde{s}+1)} < \tilde{v} < \tilde{u}^{(\tilde{p}-1)/\tilde{q}}\}.
\end{aligned}$$

By simple computations, we see that (6) gives a bijection between (i) $\mathcal{R}_1$ and $\widetilde{\mathcal{R}}_1$, (ii) $\mathcal{R}_2$ and $\widetilde{\mathcal{R}}_4$, (iii) $\mathcal{R}_3$ and $\widetilde{\mathcal{R}}_3$, (iv) $\mathcal{R}_4$ and $\widetilde{\mathcal{R}}_2$. Moreover, it follows from the definition that $\tilde{r} - (\tilde{p} - 1) = q - (s + 1)$ and $\tilde{s} + 1 - \tilde{q} = (p - 1) - r$. Therefore, we see that (6) changes the conditions on the exponents in Case I, Case II and Case III into those in Case III, Case II and Case I, respectively. We obtain the following proposition:

Proposition 5. *The transformation* (6) *gives the following dualities:*

(i) *Solution trajectories of* (K) *in (a), (b), (c), (d) and (e) of Theorem 2 become those in (e), (d), (c), (b) and (a) of Theorem 4, respectively.*

(ii) *Solution trajectories of* (K) *in (a), (b), (c), (d) and (e) of Theorem 3 become those in (e), (d), (c), (b) and (a) of Theorem 3, respectively.*

(iii) *Solution trajectories of* (K) *in (a), (b), (c), (d) and (e) of Theorem 4 become those in (e), (d), (c), (b) and (a) of Theorem 2, respectively.*

This proposition means that Case I and Case III are symmetric, while Case II and itself are symmetric. Therefore, if we know the behavior of solutions for $\tau \le \tau_E$ in Case II completely, then we obtain at the same time the complete knowledge of the behavior of solutions for $\tau > \tau_E$, while if we have complete understanding of the behavior of solutions in Case I, then so do we also for Case III.

We notice that the direction of the time variable is reversed by the transformation (6). Therefore, in virtue of Proposition 5, for the proofs of Theorems 2–4, it suffices to verify only the assertions for L^+ in all cases (a)–(e) of the three theorems.

§3. Outline of the proof of theorems

In this section, we give an outline of the proof of Theorem 2–4. For the details, see [6]. We start by stating some general observations which hold true for all cases under (C).

Lemma 6. *Each solution in $\mathcal{R}_2$ must enter $\mathcal{R}_3$. Similarly, each solution in $\mathcal{R}_4$ must enter $\mathcal{R}_1$.*

Lemma 7. *If (K) has a solution $(u(t), v(t))$ converging to $(0,0)$ as $t \to +\infty$, then it is necessary that $\tau > \tau_O$.*

Lemma 8. *If $\tau > \tau_O$, then the solution (u, v) must tend to $(0,0)$ monotonically (i.e., $u \searrow 0$ and $v \searrow 0$) in $t > 0$ if $(u(0), v(0)) \in \overline{\mathcal{R}}_3$ and $v(0) = \frac{1}{\varepsilon} u^{(p-1)/q}(0)$ with $[\frac{q}{\tau} - (p-1)] + (p-1)\varepsilon^q < 0$.*

Lemma 9. *If $\tau \ne \tau_E$, then (K) does not possess any periodic solution.*

Lemma 10. *If a solution (u, v) of (K) stays inside $\mathcal{R}_1$ for $t \in (0, T_+)$, then it must either converge to $E = (1,1)$ or blow up in finite time.*

Lemma 11. *Let $(u(t), v(t))$ be a solution of* (K) *such that it neither converge to $(0,0)$ as $t \to +\infty$ nor blows up in finite time. If $\tau > \tau_E$, then its ω-limit set contains no interior point of $\mathcal{Q}$.*

3.1. Proof of Theorem 2

First, we prepare the following lemma.

Lemma 12. *Let $p-1<r$ and $q<s+1$. Then every solution trajectory cannot stay entirely in $\mathcal{R}_1$ as $t\to T_+$ unless it converges to E as $t\to T_+$ (and in that case $T_+=\infty$).*

Now, we are ready to sketch the proof of Theorem 2. We shall prove here only the assertion for $L^+(\gamma)$. The proof will be completed when we finish the proof of those for $L^+(\gamma)$ of Theorem 4 (see the remark immediately after Proposition 5).

Proof of Theorem 2 for $L^+(\gamma)$. We discuss the following four cases I–IV separately:

I. *If $\tau\le\tau_O$, then $L^+(\gamma)=\{E\}$.*

Lemma 7 guarantees that (u,v) cannot go to $(0,0)$ in $\mathcal{R}_3$. There is no periodic solution by Lemma 9. Denote two consecutive intersections of the trajectory with the line segment $L\equiv\{(u,1)\mid 0<u<1\}$ by $Q_1=(u(T_1),v(T_1))$ and $Q_2=(u(T_2),v(T_2))$. It follows from the stability of E that Q_2 lies in between E and Q_1, which implies that the solution (u,v) must converge to E as $t\to+\infty$. Therefore, all solutions must converge to E. The assertions for $L^+(\gamma)$ in (a) and (b) have been proved.

To treat the remaining three cases, we rely heavily on Lemma 8. We consider the solution with initial value $(u(0),v(0))\in L=\{(u,1)\mid 0<u<1\}$. Lemma 8 shows that if $u(0)$ is small enough, the solution must converge to $(0,0)$ monotonically in $t>0$ as $t\to+\infty$. Now, set

(8) $$U_*=\sup\{u(0)<1\mid (u,v)\to(0,0)\text{ monotonically in }t>0\}.$$

It follows from Lemma 1 that $U_*<1$ if $\tau<\tau_2$. Then we see easily that for any initial value $(u(0),v(0))$ with $u(0)<U_*$ and $v(0)=1$, the solution (u,v) tends to $(0,0)$ monotonically in $t>0$, and for any $(u(0),v(0))$ with $U_*<u(0)<1$ and $v(0)=1$, the solution (u,v) does not tend to $(0,0)$ monotonically (although it may still tend to $(0,0)$ eventually).

We denote the solution with initial value $Q_0=(U_*,1)$ by (u_*,v_*).

Lemma 13. *If $\tau_2>\tau>\tau_O$, then the solution (u_*,v_*) defined above must tend to $(0,0)$ monotonically in $t>0$. Moreover, if we extend (u_*,v_*) backwards in $t<0$, then it must enter $\mathcal{R}_2$ and then $\mathcal{R}_1$ as $t<0$ decreases further.*

We now turn to the proof of the following:

II. *If $\tau_O < \tau < \tau_E$, then there exists a special orbit γ_s for which the following* (i)–(ii) *hold.* (i) $L^+(\gamma_s) = \{O\}$, $L^-(\gamma_s) = \{(u_s, 0)\}$ *for some* $u_s \geq 0$. (ii) *Let $\mathcal{D}$ be the domain surrounded by γ_s and the positive u-axis. If $P \in \mathcal{D}$, then $L^+(\gamma(P)) = \{E\}$, while $L^+(\gamma(P)) = \{O\}$ if $P \in \mathcal{Q}\backslash\mathcal{D}$.*

We check that the solution (u_*, v_*) gives rise to the desired orbit γ_s. Indeed, from the examination of the vector field defined by (K), we can prove that (u_*, v_*) tends to $(u_s, 0)$ as $t \to T_-$ with $u_s \geq 0$.

Now, the trajectory of (u_*, v_*) separates $\mathcal{Q}$ into two regions: a bounded $\mathcal{D}$ and its complement. It is obvious that a solution with initial value in $\mathcal{D}$ must stay in $\mathcal{D}$ and therefore tends to E as $t \to +\infty$ by the definition of (u_*, v_*), while a solution with initial value in the complement of $\overline{\mathcal{D}}$ must converge to $(0, 0)$ as $t \to +\infty$. The assertion for $L^+(\gamma)$ in (c) has been proved.

III. *If $\tau_E < \tau$, then $L^+(\gamma) = \{O\}$.*

This follows from Lemma 1, Lemma 12, Lemma 6, Lemma 8 and Lemma 11, and the assertion for $L^+(\gamma)$ in (e) has been proved.

IV. *If $\tau = \tau_E$, then there exists a special orbit γ_s such that $L^+(\gamma_s) = \{O\}$ and $L^-(\gamma_s) = \{O\}$. Let $\mathcal{D}$ be the bounded domain enclosed by γ_s. If $P \in \mathcal{D}$, then $\gamma(P)$ is a closed orbit, while $L^+(\gamma(P)) = \{O\}$ if $P \in \mathcal{Q}\backslash\mathcal{D}$.*

This is verified by examining the level curves of the following function:

(9)

$$H(u, v) = \frac{v^{s+1}}{u^{p-1}} + \frac{p-1}{r-(p-1)}(u^{r-(p-1)} - 1) - \frac{s+1}{s+1-q}(v^{s+1-q} - 1).$$

The assertion for $L^+(\gamma)$ in (d) has been proved. Q.E.D.

3.2. Proof of Theorem 3

We need the following lemma.

Lemma 14. *Let $r \geq p-1$ and $q \geq s+1$. If a solution lies in $\mathcal{R}_1$ at some t, then it must either enter $\mathcal{R}_2$ or converge to E as $t \to +\infty$.*

As was pointed out in Section 2, it is sufficient to verify the assertions for $L^+(\gamma)$ of (a) – (e).

Proof of Theorem 3 for $L^+(\gamma)$. We treat the following four cases I–IV separately:

I. *If* $\tau < \tau_E$, *then* $L^+(\gamma) = \{E\}$.

By Lemma 14 and Lemma 7, the conclusion follows immediately. The assertions for $L^+(\gamma)$ of (a) and (b) have been proved.

II. *If* $\tau_E < \tau \leq \tau_O$, *then* $L^+(\gamma) = \{uv = 0\}$.

Observe that $\tau_2 > \tau_O$. Therefore, Lemma 1 implies that near E, every solution (u, v) must spiral outward. By virtue of Lemma 14 and Lemma 11, the assertion for $L^+(\gamma)$ of (d) has been proved.

III. *If* $\tau > \tau_O$, *then* $L^+(\gamma) = \{O\}$.

We notice that E is unstable. By Lemma 14, our conclusion follows immediately. The assertion for $L^+(\gamma)$ of (e) has been proved.

IV. *If* $\tau = \tau_E$, *then all the orbits are closed.*

For the case $r > p - 1$ and $s + 1 < q$, we use the same function H as (9). For the case $r = p - 1$ and $s + 1 < q$, we use a slightly different potential, namely,

$$(10) \quad H_1(u, v) \equiv \frac{v^{s+1}}{u^{p-1}} + (p-1)\log u + \frac{s+1}{q-(s+1)}\left(\frac{1}{v^{q-(s+1)}} - 1\right),$$

which is obtained by letting $r - (p - 1) \to 0$ in (9). Finally for the case $r > p - 1$ and $s + 1 = q$, if we transform (u, v, t) to $(\tilde{u}, \tilde{v}, \tilde{t})$, then $(\tilde{u}, \tilde{v})$ is in the previous case. We see that these functions are a constant along a solution trajectory and every level curve of these functions is compact. Therefore, all solution of (K) must be periodic. The proof of (c) of Theorem 3 is now complete. Q.E.D.

3.3. Proof of Theorem 4

Before proving Theorem 4, we study blow-up solutions. Assume that the exponents satisfy the condition $r < p - 1$, $s + 1 < q$.

From Lemma 10, we have already known that if there is a solution which stays in $\mathcal{R}_1$ for all $0 < t < T_+$, then it blows up in finite time unless it converge to $\{E\}$ as $t \to +\infty$.

We first prove that, for any $\tau > 0$, there is always a blow-up solution. To this end, we set $a = (p - 1) - r$, $b = q - (s + 1)$, $c = \frac{b}{a\tau} + 1$ and

$$S_c = \{(u, v) \in \overline{\mathcal{R}}_1 \mid u^a > cv^b\}$$

throughout this entire subsection. It is obvious that from (C) we have $a/b < (p-1)/q$. Our result reads as follows.

Proposition 15. *Let $(u(t), v(t))$ be a solution with initial value $(u(0), v(0)) \in S_c$. Then it must blow up in finite time.*

Next, we consider the behavior of blow-up solutions. We have the following two possibilities as $t \to T_+$:

(1) $(u(t), v(t)) \to (+\infty, v_0)$ with $0 < v_0 < +\infty$,
(2) $(u(t), v(t)) \to (+\infty, +\infty)$.

We have the following two theorems.

Theorem 16. *We assume that $v_0 < +\infty$. For each $0 < v_0 < +\infty$, there is a unique solution of* (K) *such that*

$$(u(t), v(t)) \to (+\infty, v_0) \quad \textit{as} \quad t \to T_+.$$

Such a solution is unique up to translations of the time variable t.

Theorem 17. *We assume that $v_0 = +\infty$. For given T_+, there is a unique solution (u, v) of* (K) *such that*

$$(u(t), v(t)) \to (+\infty, +\infty) \quad \textit{as} \quad t \to T_+.$$

Theorem 16 is proved by using the contraction mapping theorem. Indeed, put

$$U(t) = \frac{1}{u^{p-1}}, \quad V(t) = v^{s+1}, \quad \tilde{U}(\zeta) = U(T_+ - \zeta), \quad \tilde{V}(\zeta) = V(T_+ - \zeta).$$

We note that

$$\tilde{U}(0) = \lim_{\zeta \to 0} \tilde{U}(\zeta) = 0, \quad \tilde{V}(0) = \lim_{\zeta \to 0} \tilde{V}(\zeta) = \tilde{V}_0 < +\infty,$$
$$\lim_{\zeta \to 0} \tilde{U}'(\zeta) = \frac{p-1}{\tilde{V}_0^{q/(s+1)}}.$$

Therefore, we can regard $\tilde{U}(\zeta)$ as a continuously differentiable function at $\zeta = 0$. Hence, we have

$$\tilde{U}(\zeta) = \frac{p-1}{\tilde{V}_0^{q/(s+1)}}(1 + \Psi(\zeta))\zeta,$$

where $\Psi(\zeta) = o(1)$ as $\zeta \to 0$. Using the notation above, we change (K) into the following system:

(11)
$$\begin{cases} \Psi(\zeta) &= \dfrac{\tilde{V}_0^{q/(s+1)}}{\zeta} e^{-(p-1)\zeta} \displaystyle\int_0^\zeta \frac{e^{(p-1)\xi}}{\tilde{V}^{q/(s+1)}(\xi)}\, d\xi - 1, \\ \tilde{V}(\zeta) &= \tilde{V}_0\, e^{\frac{s+1}{\tau}\zeta} - \dfrac{s+1}{\tau}\left(\dfrac{\tilde{V}_0^{q/(s+1)}}{p-1}\right)^{r/(p-1)} \displaystyle\int_0^\zeta \frac{\xi^{-r/(p-1)} e^{\frac{s+1}{\tau}(\zeta-\xi)}}{(1+\Psi(\xi))^{r/(p-1)}}\, d\xi, \end{cases}$$

with initial data $\Psi(0) = 0$, $\tilde{V}(0) = \tilde{V}_0$.

Therefore, in order to prove the assertion in Theorem 16, it is sufficient to verify that (11) has a unique solution $(\Psi, \tilde{V})$ for each $\tilde{V}_0 > 0$.

Next, in order to prove Theorem 17, we change (K) into the following system:

$$\begin{cases} \dfrac{dW}{dt} &= \left[\dfrac{q}{\tau} - (p-1)\right] W + (p-1) - \dfrac{q}{\tau} z, \\ \dfrac{dz}{dt} &= z \left[\left(\dfrac{b}{\tau} - a\right) + \dfrac{1}{W}\left(a - \dfrac{b}{\tau} z\right)\right], \end{cases} \tag{12}$$

where we have set

$$W(t) = \frac{v^q(T_+ - t)}{u^{p-1}(T_+ - t)} \quad \text{and} \quad z(t) = \frac{v^b(T_+ - t)}{u^a(T_+ - t)}.$$

We have the following two lemmas.

Lemma 18. *If* (u, v) *blows up at* $t = T_+$, *then*

$$(W(t), z(t)) \to (0, z_0) \quad \text{as} \quad t \to 0$$

with $0 \le z_0 \le +\infty$.

Lemma 19. *If* $z(0) = 0$, *then there exists a constant* $C > 0$ *such that*

$$v(t) \le C \quad \text{for } t \text{ close to } T_+.$$

It follows from Lemma 18 and Lemma 19 that there is no solution which derived from a solution of (K) converging to ∞, among solutions of (12) with its initial value $(W(0), z(0)) = (0, 0)$. Therefore, to prove Theorem 17, it suffices to verify that there is a unique solution (W, z) of (12) with its initial value on the positive z-axis. Hence, the following proposition yields our conclusion:

Proposition 20.
(i) If there exists a solution of (12) *with its initial value* $(W(0), z(0)) = (0, z_0)$ *for some* $z_0 > 0$*, then* $z_0 = a\tau/b$.
(ii) There is a unique solution of (12) *with its initial value* $(W(0), z(0)) = (0, a\tau/b)$.

Finally we come to the proof of Theorem 4. We notice also that it suffices to verify the assertions for $L^+(\gamma)$ of (a)–(e). We shall prove assertions (a)–(e) separately in this order.

Proof of Theorem 4. (a) We have already showed that the existence and uniqueness of a solution converging to ∞. Letting this solution trajectory be γ_s, we easily see that $L^+(\gamma(P)) = \{(+\infty, v_P)\}$ with $0 < v_P < +\infty$ if $P \in \mathcal{B}$.

Assuming $P \in \mathcal{A}$, we see that the solution with initial value P does not converge to ∞. Moreover, it does not tend to O by Lemma 7. Therefore, by applying the same arguments as in the verification of Assertion I of the proof of Theorem 2, we conclude that $(u(t), v(t))$ tends to E as $t \to +\infty$.

(b) Set $\tilde{p} = s + 2$, $\tilde{q} = r$, $\tilde{r} = q$, $\tilde{s} = p - 2$, $\tilde{u} = 1/v$, $\tilde{v} = 1/u$ and $\mathcal{H}(\tilde{u}, \tilde{v}) = H(u, v)$, where H is defined by (9). We have that $\tilde{r} > \tilde{p} - 1$, $\tilde{s} + 1 > \tilde{q}$, and

$$\mathcal{H}(\tilde{u}, \tilde{v}) = \frac{\tilde{v}^{\tilde{s}+1}}{\tilde{u}^{\tilde{p}-1}} - \frac{\tilde{s}+1}{\tilde{s}+1-\tilde{q}}\left(\tilde{v}^{\tilde{s}+1-\tilde{q}} - 1\right) + \frac{\tilde{p}-1}{\tilde{r}-(\tilde{p}-1)}\left(\tilde{u}^{\tilde{r}-(\tilde{p}-1)} - 1\right).$$

Hence, the qualitative behavior of the level curves of $\mathcal{H}(\mu, \nu)$ is exactly the same as that of H studied in the proof of Theorem 2. Translating $(\tilde{u}, \tilde{v})$ back into (u, v), we get the conclusion.

(c) Recall that we have proved the existence of the solution (u_*, v_*) which tends to $(0, 0)$ as $t \to +\infty$ and $(u_0, 0)$ as $t \to T_-$ (see in the verification of Assertion II of the proof of Theorem 2). It follows from the transformation (6) that this solution trajectory becomes the one leaving ∞ in $\mathcal{R}_3$, passing through $\mathcal{R}_4$ and going to $(+\infty, v_s)$ with $v_s > 0$ in $\mathcal{R}_1$. We notice that $v_s < +\infty$ in virtue of the uniqueness of a solution tending to ∞ by Lemma 7 and Lemma 11. It is easily seen that $L^+(\gamma(P)) = \{(+\infty, v_P)\}$ for some $0 < v_P < v_s$ if $P \in \mathcal{B}$. Now, let us consider the case $P \in \mathcal{A}$. Since E is unstable and the solution trajectory cannot intersect itself, it follows from Lemma 7 and Lemma

11 that $L^+(\gamma(P)) = \{(+\infty, v_P)\}$ for some $v_s < v_P \leq +\infty$ when $P \in \mathcal{A}$.

(d) The assertion follows immediately from the arguments similar to those in the proof of (c) .

(e) We know that there is a unique trajectory γ_s which goes to ∞. On the other hand, it follows from Lemma 8 that there are solutions tending to $(0,0)$ monotonically as $t \to +\infty$. Fix v_1 with $v_1 < 1$ and set $L_1 = \{(u, v_1) \mid 0 < u < (v_1)^{q/(p-1)}\}$. We consider a solution with initial value $(u(0), v(0)) \in L_1$. Now, we define U_* by

$$U_* = \sup\{u(0) < (v_1)^{q/(p-1)} \mid (u,v) \to (0,0) \text{ monotonically as } t \to +\infty\}.$$

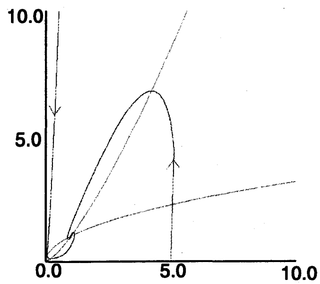
Denote the solution with initial value $\bar{Q} = (U_*, v_1)$ by (u_*, v_*). We see that (u_*, v_*) tends to $(0,0)$ monotonically as $t \to +\infty$ and denote the trajectory of (u_*, v_*) by γ_c.

If $P \in \mathcal{A}$, then a solution (u,v) of (K) with initial value P cannot converge to ∞ or E. In virtue of Lemma 11, it must tend to O as $t \to +\infty$. If $P \in \mathcal{B}$, then the solution of (K) with initial value P must converge to $\{(+\infty, v_P)\}$ for some v_P, $0 < v_P < +\infty$, since it cannot intersect γ_s. Q.E.D.

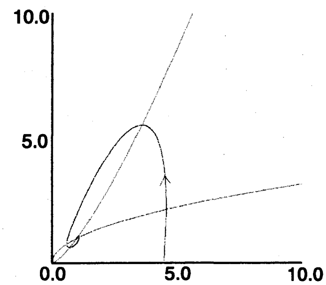
§ Appendix

Theorem 2

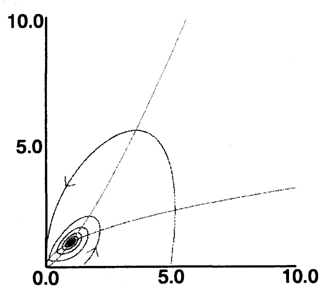
(a) $\tau < \tau_\infty$



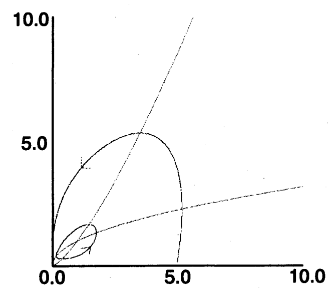
(b) $\tau_\infty \leq \tau \leq \tau_O$



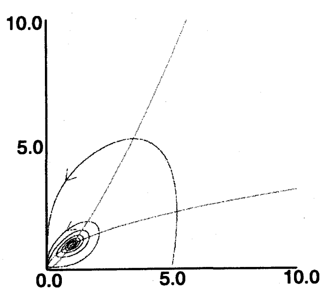
(c) $\tau_O < \tau < \tau_E$



(d) $\tau = \tau_E$

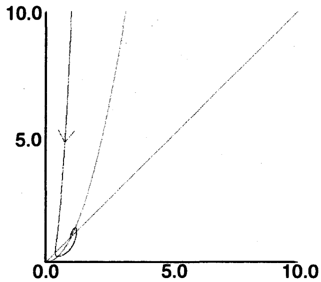


(e) $\tau > \tau_E$

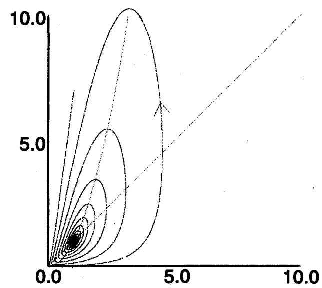


Theorem 3

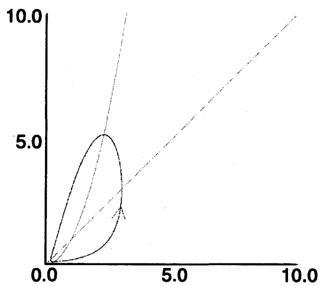
(a) $\tau < \tau_\infty$



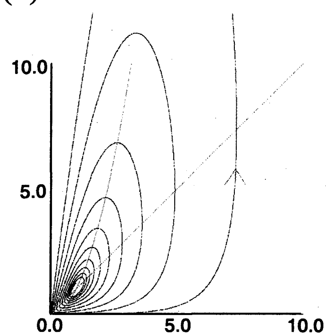
(b) $\tau_\infty \le \tau < \tau_E$



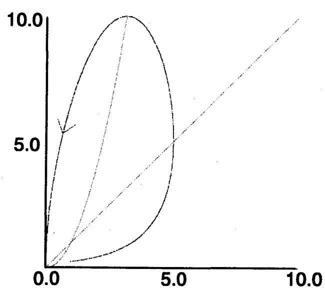
(c) $\tau = \tau$



(d) $\tau_E < \tau \le \tau_O$

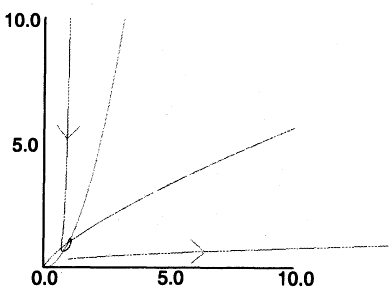


(e) $\tau > \tau_O$

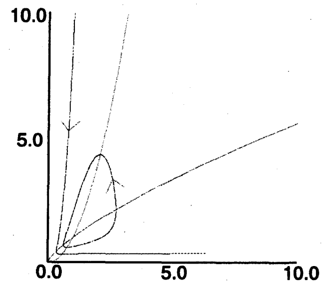


Theorem 4

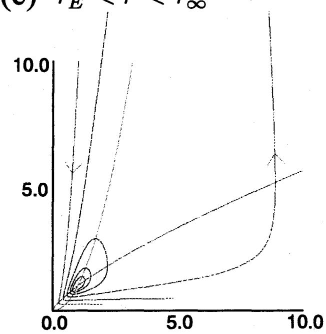
(a) $\tau < \tau_E$



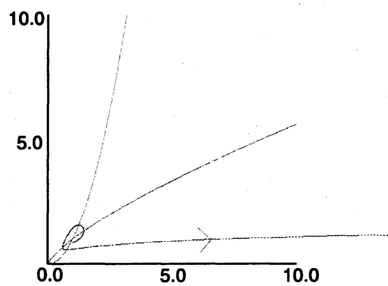
(b) $\tau = \tau_E$



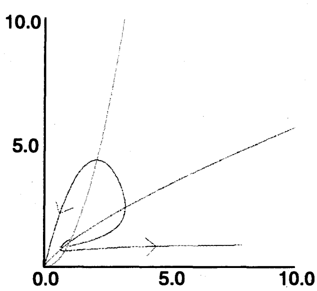
(c) $\tau_E < \tau < \tau_\infty$



(d) $\tau_\infty \leq \tau \leq \tau_O$



(e) $\tau > \tau_O$



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