Advanced Studies in Pure Mathematics 47-2, 2007
Asymptotic Analysis and Singularities
pp. 647–660

Speed estimate for a periodic rotating wave in an undulating zone on the sphere

Bendong Lou

Abstract.

On the unit sphere S^2 given a zone with periodically undulating boundaries, we consider periodic rotating waves (curves) in this zone which are driven by geodesic curvature. We state without proof the existence of periodic rotating waves. Then we study how the average rotating speed depends on the geometry of the boundaries, give the estimate of this average speed by using its homogenization limit.

§1. Introduction

Many kinds of curvature flows in manifolds have been studied recently. To name only a few, [3], [4], [5], [6], [7], etc. studied mean curvature flows on the plane; [1], [12], etc. studied mean curvature flows in manifolds; [2] etc. studied Gauss curvature flows. Besides these, there are also some studies about geodesic flows under Ricci curvature, geodesic curvature flows, etc..

Most of these works concern the existence and asymptotic behavior of the flows. As far as we know, very little is known about (periodic) traveling/rotating surfaces in manifolds, though traveling/rotating wave solutions of reaction diffusion equations in Euclidean spaces have been studied a lot (cf. [13] and references therein).

In this paper we study a geodisic curvature flow in a zone with undulating boundaries on the sphere. More precisely, define domain Ω_m as the following: Let $b(s)$ be 2π-periodic smooth functions satisfying

$$b(0) = b(2\pi) = 0, \quad b(s) \geq 0, \quad \max_s b'(s) = \tan\alpha, \quad \min_s b'(s) = -\tan\beta.$$

Received August 10, 2005.
Revised October 12, 2005.
Partially supported by the VBL program in Hokkaido University, NNSF of China and the Project-sponsored by SRF for ROCS, SEM..

for some $\alpha, \beta \in (0, \frac{\pi}{2})$. Given $\theta_0 \in (0, \frac{\pi}{2})$, for any $m \in \mathbb{N}$ define

$$b_m(s) := \frac{\cos\theta_0}{m} \cdot b(ms).$$

Let $S^2 := \left\{(\cos\theta\cos\varphi, \cos\theta\sin\varphi, \sin\theta) \in \mathbb{R}^3 \;\middle|\; \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right],\ \varphi \in \mathbb{R}\right\}$ be the unit sphere. For convenience, in the following we use spherical coordinates (θ, φ) to denote point $(\cos\theta\cos\varphi, \cos\theta\sin\varphi, \sin\theta) \in S^2$. Zone $\Omega_m \subset S^2$ is defined as

$$\Omega_m := \left\{(\theta, \varphi) \in S^2 \;\middle|\; -\theta_0 - b_m(\varphi) < \theta < \theta_0 + b_m(\varphi), \quad \varphi \in \mathbb{R}\right\}.$$

Denote the boundaries $\theta = -\theta_0 - b_m(\varphi)$ and $\theta = \theta_0 + b_m(\varphi)$ by $\partial^-\Omega_m$ and $\partial^+\Omega_m$, respectively (see Figure 1).

We consider the motion of curves immersed in Ω_m, which is driven by

$$V = \kappa_g, \tag{1.1}$$

where for a time-dependent simple curve Γ_t immersed in Ω_ε, V denotes the velocity of the curve at point $P \in \Gamma_t$ along the normal direction on the tangent plane $T_P S^2$, κ_g denotes the geodesic curvature of Γ_t at P. To avoid sign confusion, the normal vector ν to Γ_t on TS^2 will always be chosen to be the increasing direction of φ, the sign of the normal velocity V and the geodesic curvature κ_g will be understood in accordance with this choice (see details below).

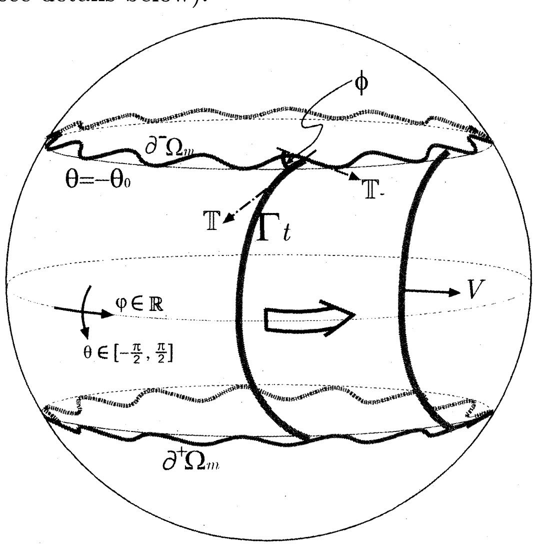

Fig. 1 Zone with undulating boundaries on unit sphere

By a solution of (1.1) we mean a time-dependent simple curve $\Gamma_t \subset \Omega_m$ which satisfies (1.1) and contacts the boundaries $\partial^{\pm}\Omega_m$ with angle $\phi \in (0, \frac{\pi}{2})$ (see details below).

In this paper we are interested in those curves rotating along Ω_m periodically, also we estimate the average rotating speed for homogenization limit problem. For simplicity, we consider the case that each curve is the graph of a function $\varphi = u(\theta, t)$, that is, the curve is $\Gamma_t = \{(\theta, u(\theta, t))\} | -\theta_0 - b_m(u) \leq \theta \leq \theta_0 + b_m(u)\} \subset \Omega_m$. The unit tangent vector (pointing to the positive direction of θ) of Γ_t is

$$\mathbb{T} = \frac{1}{\sqrt{1 + u_\theta^2 \cos^2\theta}} \begin{pmatrix} -\sin\theta\cos u - \cos\theta\sin u \cdot u_\theta \\ -\sin\theta\sin u + \cos\theta\cos u \cdot u_\theta \\ \cos\theta \end{pmatrix}.$$

For a curve $\{(\theta(s), \varphi(s))\}_{s=s_1}^{s=s_2} \subset \Omega_m$ with parameter s, its geodesic curvature is

$$\kappa_g = \cos\theta \cdot \det \begin{pmatrix} \dfrac{d\theta}{ds} & \dfrac{d^2\theta}{ds^2} + \sin\theta\cos\theta \left(\dfrac{d\varphi}{ds}\right)^2 \\ \dfrac{d\varphi}{ds} & \dfrac{d^2\varphi}{ds^2} - 2\tan\theta \, \dfrac{d\theta}{ds}\dfrac{d\varphi}{ds} \end{pmatrix}.$$

So for curve $\varphi = u(\theta, t)$, its geodesic curvature is

$$\kappa_g = \cos\theta \cdot \frac{u_{\theta\theta} - 2\tan\theta \cdot u_\theta - \sin\theta\cos\theta \cdot {u_\theta}^3}{(1 + u_\theta^2 \cos^2\theta)^{3/2}}$$

because when we use arc length s as parameter we have

$$\frac{d\theta}{ds} = \frac{1}{\sqrt{1 + {u_\theta}^2 \cos^2\theta}}, \qquad \frac{d\varphi}{ds} = \frac{u_\theta}{\sqrt{1 + {u_\theta}^2 \cos^2\theta}},$$

$$\frac{d^2\theta}{ds^2} = \frac{u_\theta^2 \sin\theta\cos\theta - u_\theta u_{\theta\theta}\cos^2\theta}{(1 + u_\theta^2\cos^2\theta)^2}, \qquad \frac{d^2\varphi}{ds^2} = \frac{u_{\theta\theta} + u_\theta^3 \sin\theta\cos\theta}{(1 + u_\theta^2\cos^2\theta)^2}.$$

The unit normal vector of Γ_t on TS^2 is

$$\nu = \frac{1}{\sqrt{1 + u_\theta^2 \cos^2\theta}} \begin{pmatrix} \sin\theta\cos\theta\cos u \cdot u_\theta - \sin u \\ \sin\theta\cos\theta\sin u \cdot u_\theta + \cos u \\ -\cos^2\theta \cdot u_\theta \end{pmatrix},$$

and so

$$V = \begin{pmatrix} -\cos\theta\sin u \cdot u_t \\ \cos\theta\cos u \cdot u_t \\ 0 \end{pmatrix} \cdot \nu = \frac{\cos\theta \cdot u_t}{\sqrt{1 + u_\theta^2\cos^2\theta}}.$$

Thus (1.1) is equivalent to

(1.2)
$$u_t = \frac{u_{\theta\theta} - 2\tan\theta \cdot u_\theta - \sin\theta\cos\theta \cdot {u_\theta}^3}{1 + u_\theta^2 \cos^2\theta} \qquad \text{for } \eta_-(t) < \theta < \eta_+(t),\ t > 0,$$

where $\eta_\pm(t)$ (with $\eta_-(t) < 0 < \eta_+(t)$) denote the θ-coordinates of the end points of Γ_t lying on $\partial^\pm\Omega_m$, i.e. $\eta_-(t) = -\theta_0 - b_m(u(\eta_-(t), t))$, $\eta_+(t) = \theta_0 + b_m(u(\eta_+(t), t))$.

Denote the unit tangent vector along $\partial^\pm\Omega_m$ by $\mathbb{T}_\pm$, both toward the increasing direction of φ, then

$$\mathbb{T}_+ = \frac{1}{\sqrt{{b'_m}^2 + \cos^2\theta}} \begin{pmatrix} -b'_m \sin\theta\cos\varphi - \cos\theta\sin\varphi \\ -b'_m \sin\theta\sin\varphi + \cos\theta\cos\varphi \\ -b'_m \cos\theta \end{pmatrix} \quad (\theta = \theta_0 + b_m(\varphi)),$$

$\mathbb{T}_-$ is calculated similarly. Hereinafter we say that curve Γ_t contacts $\partial^\pm\Omega_m$ with angle $\phi \in (0, \frac{\pi}{2})$ in the sense that $\cos\phi = -\mathbb{T}\cdot\mathbb{T}_-$ on $\partial^-\Omega_m$, $\cos\phi = \mathbb{T}\cdot\mathbb{T}_+$ on $\partial^+\Omega_m$. These are nothing but our *boundary conditions*, which can be expressed as

(1.3)
$$u_\theta(\theta, t) = \mp \frac{\cos\phi\ \cos\theta - b'_m(u)\ \sin\phi}{\cos\theta\ (\sin\phi\ \cos\theta + b'_m(u)\ \cos\phi)} =: \mp\mathcal{F}(u) \quad \text{for} \quad \theta = \eta_\mp(t).$$

Let $\Omega_0 = \{(\theta, \varphi) \in S^2 \mid -\theta_0 < \theta < \theta_0\}$ be a trivial zone which is formally a limit of Ω_m as $m \to \infty$. For Ω_0 we consider problem (1.2) with boundary condition

$$u_\theta(\pm\theta_0, t) = \pm\frac{\cot\phi}{\cos\theta_0}. \tag{1.4}$$

As is shown in subsection 2.1, there exists a unique c_0 such that problem (1.2), (1.4) has a unique *rotating wave* $U_0(\theta) + c_0 t$, which has profile U_0 and rotating speed c_0.

On the other hand, in Ω_m, as Γ_t propagates, its shape and speed fluctuate along with the undulation of the domain Ω_m. In such a situation, we adopt a generalized definition of rotating waves. A solution $U_m(\theta, t)$ of (1.2)-(1.3) is called a *periodic rotating wave* if it satisfies

$$U_m(\theta, t + T_m) = U_m(\theta, t) + \frac{2\pi}{m} \qquad \text{for some} \quad T_m > 0.$$

The *average rotating speed* of a periodic rotating wave is defined by

$$c_m = \frac{2\pi}{mT_m}.$$

In what follows we concentrate on periodic rotating waves with average speed of order 1 as $m \to \infty$.

Before stating our results, we give two assumptions.

$$\textbf{(H1)} \qquad \phi+\alpha < \pi/2.$$

$$\textbf{(H2)} \qquad \alpha+\beta < \phi, \quad 2\beta < \phi.$$

Roughly speaking these conditions require that α and β are not large, that is, the undulation of the boundaries is gradual. (H1) guarantees the existence of lower solutions rotating in a positive speed (see Lemma 2.2 below). Conditions $\alpha + \beta < \frac{\pi}{2}$ in (H2) exclude the possible singularity that the curve touches the boundaries besides at two endpoints, otherwise the curve may split into multiple components; $\beta < \phi$ in (H2) ensure that $|u_\theta|$ is bounded on the boundaries.

About the existence of periodic rotating waves, using the standard theory of quasilinear parabolic equations (cf. [8], [9]) one can show that (refer also to [10]):

Proposition 1.1. *Assume* (H1) *and* (H2) *holds, then when m is large,* (1.2)-(1.3) *has a periodic rotating wave, which is unique up to time-shift.*

In fact, the unique periodic rotating wave is asymptotically stable. We refer reader to general theory in [11] or to [10].

Our main purpose in this paper is to study how the average speed of the periodic rotating wave depends on the shape of the boundaries. Speed estimate is an important problem in the study of traveling/rotating waves. So far, very little is known for periodic traveling/rotating waves of curvature flow equations.

THERUOREM **1.1.** *Assume* (H1) *and* (H2) *hold. Then when m is large,*

(i) *there exists $C > 0$ independent of m such that*

$$c^* - \frac{C}{m} < c_m < c^* + \frac{C}{\sqrt{m}} < c_0, \tag{1.5}$$

where $c^* = c^*(\alpha, \phi) > 0$ is given by the unique solution $(c^*, \Phi^*(\theta; c^*))$ of

$$\begin{cases} c = \dfrac{\Phi_{\theta\theta} - 2\tan\theta \cdot \Phi_\theta - \sin\theta\cos\theta \cdot \Phi_\theta^3}{1 + {\Phi_\theta}^2 \cos^2\theta}, & -\theta_0 < \theta < \theta_0, \\[2ex] \Phi_\theta(\pm\theta_0) = \pm\dfrac{\cot(\phi+\alpha)}{\cos\theta_0}, \quad \Phi(0) = 0, & \end{cases} \tag{1.6}$$

c_0 *is given by the unique rotating wave* $U_0(\theta) + c_0 t$ *of* (1.2), (1.4) *in* Ω_0.

(ii) *As* $m \to \infty$, *periodic rotating wave* $U_m(\theta, t) \to \Phi^*(\theta; c^*) + c^* t + C$ *in* $C^{2,1}([-\theta_0, \theta_0] \times [-T, T])$ *for any* $T > 0$, *where* C *is a constant independent of* T.

$c_m < c_0$ in (1.5) implies that boundary undulation always lowers the speed of the rotating wave, $c^* < c_0$ implies that the effect of spatial inhomogeneity of c_m is left to the homogenization limit. Moreover, the fact that c^* depends mainly on α (besides ϕ) is a notable result, the dependence on other information of the boundaries should appear in the error $\frac{C}{\sqrt{m}}$.

In section 2 we prove Theorem 1.1: estimate the average rotating speed by constructing a lower solution and a precise upper solution. We point out that our upper solution is only a *temporary* one (only on $t \in [0, 1]$), but it is good enough to give the upper bound of the average rotating speed. In section 3, we give some remarks.

§2. Estimate of Average Speed

2.1. Rotating waves in trivial zones

We first study rotating waves in trivial zones (zones with flat boundaries), select one of such rotating waves as lower solution. Denote

$$\bar{\theta} = \theta_0 + \max_s b_m(s).$$

Consider the following problem

$$(2.1) \quad \begin{cases} c = \dfrac{\Phi_{\theta\theta} - 2\tan\theta \cdot \Phi_\theta - \sin\theta\cos\theta \cdot \Phi_\theta{}^3}{1 + \Phi_\theta^2 \cos^2\theta}, & -\bar{\theta} < \theta < \bar{\theta}, \\ \Phi_\theta(\pm\bar{\theta}) = \pm B \in \mathbb{R}, \quad \Phi(0) = 0. \end{cases}$$

If there exist c and $\Phi(\theta)$ satisfy (2.1), then we call the pair $(c, \Phi(\theta))$ to be a solution of (2.1). This solution determines a rotating wave $\Phi(\theta) + ct$ of (1.2) in zone $\{(\theta, \varphi) | -\bar{\theta} < \theta < \bar{\theta}\}$. Assume the graph of $\Phi(\theta)$ contacts $\theta = \bar{\theta}$ with angle γ, then $B\cos\bar{\theta} = \cot\gamma$.

LEMMA **2.1.** *If* $B > 0$, *then* (2.1) *has a unique solution* $(c, \Phi(\theta))$. *Moreover,*

(i) $c = c(B) > 0$ *is increasing in* B;

(ii) $\Phi_\theta(\theta) \cdot \theta > 0$ *and* $\Phi_{\theta\theta}(\theta) > 0$ *for* $\theta \neq 0$.

Proof. (i). Set $\Psi(\theta) = \Phi_\theta(\theta)$, and consider the following initial value problem

(2.2)
$$\begin{cases} \Psi' = c(1+\Psi^2\cos^2\theta) + 2\Psi\tan\theta + \Psi^3\sin\theta\cos\theta, & \theta \geq -\bar{\theta}, \\ \Psi(-\bar{\theta}) = -B. \end{cases}$$

For each c, denote the solution of (2.2) by $\Psi(\theta; c)$. It is clear that $\Psi(\theta; c)$ is strictly increasing in c, and depends on c continuously.

When $c > 0$ is very large, we have $\Psi(\bar{\theta}; c) > B$. When $c = 0$, the solution of (2.2) is

$$\Psi(\theta; 0) = \frac{d}{\cos\theta\sqrt{\cos^2\theta - d^2}} \qquad (\theta \in [-\bar{\theta}, \bar{\theta}])$$

$$\text{with} \quad d = \frac{-B\cos^2\bar{\theta}}{\sqrt{1+B^2\cos^2\bar{\theta}}} < 0.$$

Hence $\Psi(\bar{\theta}; 0) < 0 < B$. Therefore there exists a unique $c > 0$ such that $\Psi(\bar{\theta}; c) = B$, which determines a solution of (2.1): $\Phi(\theta) = \int_0^\theta \Psi(\varsigma; c) d\varsigma$.

By the proof, one can see that $c = c(B)$ is increasing in B.

(ii). From above discussion it is easy to see that $\Psi(-\theta) = -\Psi(\theta)$. Moreover, $\Psi(\hat{\theta}) = 0$ implies that $\Psi'(\hat{\theta}) = c > 0$, this shows that $\Psi(\theta) = 0$ if and only if $\theta = 0$. Hence $\Psi(\theta) \cdot \theta > 0$ for $\theta \neq 0$, and so $\Psi_\theta(\theta) = \Phi_{\theta\theta}(\theta) > 0$.

■

2.2. Lower Solution

In this part, we show that for appropriate choice of B, the rotating wave $\Phi(\theta; c, B) + ct$, given by the unique solution of (2.1), is a lower solution of (1.2)-(1.3). In what follows, we shall use positive constants like C, ζ, etc., which may be different from line to line and may depend on some of b_m, θ_0, m. Denote

$$B^l = \frac{\cot(\phi+\alpha)}{\cos\theta_0}.$$

LEMMA **2.2.** *Assume* (H1) *holds and m is large, then* (1.2)-(1.3) *has a lower solution $\Phi^l(\theta) + c^l t$, and $c_0 > c^l > 0$.*

Proof. Consider (2.1) with $B = B^l - \zeta\frac{1}{m}$. (H1) implies that $B^l > 0$ and so $B > 0$ when m is large and $\zeta = O(1)$ as $m \to \infty$. Hence we have a unique solution of (2.1): $(c^l, \Phi^l(\theta))$, which determines a rotating wave $\Phi^l(\theta) + c^l t$ in the zone $[-\bar{\theta}, \bar{\theta}]$.

Denote by $\eta^l_\pm(t)$ the θ-coordinate of the point where the graph of $\Phi^l(\theta)+c^lt$ meets $\partial^\pm\Omega_m$. Then $\eta^l_-(t)+\bar\theta=O(\frac{1}{m})$ and we have

$$\Phi^l_\theta(\eta^l_-(t))\geq\Phi^l_\theta(-\bar\theta)-\frac{C}{m}=-B^l+\zeta\frac{1}{m}-\frac{C}{m}\qquad\text{for some }\ C>0.$$

On the other hand, since $\frac{\partial\mathcal{F}(u)}{\partial(b'_m)}<0$ and $b'_m(u)\leq\cos\theta_0\tan\alpha$ we have

$$\begin{aligned}-\mathcal{F}(\Phi^l(\eta^l_-)) &= \frac{b'_m(u)\ \sin\phi-\cos\phi\ \cos\eta^l_-}{\cos\eta^l_-\ (\sin\phi\ \cos\eta^l_-+b'_m(u)\ \cos\phi)}\\ &\leq \frac{\cos\theta_0\tan\alpha\ \sin\phi-\cos\phi\ \cos\eta^l_-}{\cos\eta^l_-\ (\sin\phi\ \cos\eta^l_-+\cos\theta_0\tan\alpha\ \cos\phi)}\\ &\leq \frac{\cos\theta_0\tan\alpha\ \sin\phi-\cos\phi\ \cos\theta_0}{\cos\theta_0\ (\sin\phi\ \cos\theta_0+\cos\theta_0\tan\alpha\ \cos\phi)}+\frac{C}{m}\\ &= -B^l+\frac{C}{m}\leq\Phi^l_\theta(\eta^l_-(t))\end{aligned}$$

provided ζ is large. Similarly, $\Phi^l_\theta(\eta^l_+)\leq\mathcal{F}(\Phi^l(\eta^l_+))$ provided ζ is large.

Therefore $\Phi^l(\theta)+c^lt$ (for θ with $(\theta,\Phi^l(\theta)+c^lt)\in\Omega_m$) is a lower solution of (1.2)-(1.3). Moreover, when m is large, it is easy to see from Lemma 2.1 that $c_0>c^l>0$. ∎

Remark 2.1. Suppose $P_1=(-\theta^l,s^l)\in\partial^-\Omega_m$ such that $b'_m(s^l)=\cos\theta_0\tan\alpha$, then when the graph of $\Phi^l(\theta)+c^lt$ meets $\partial^-\Omega_m$ at P_1, we have

$$\Phi^l_\theta(-\theta^l)=-B^l+O\left(\frac{1}{m}\right)=-\mathcal{F}(\Phi^l(-\theta^l))+O\left(\frac{1}{m}\right).$$

Similar discussion is true on $\partial^+\Omega_m$. Hence $\Phi^l(\theta)+c^lt$ is a *good* lower solution of (1.2)-(1.3), which means that the graph of $\Phi^l(\theta)+c^lt$ contacts $\partial^\pm\Omega_m$ with angles not smaller than ϕ, and equals to $\phi+O\left(\frac{1}{m}\right)$ at some points.

Proof of the first inequality of (1.5). The fact that $\Phi^l(\theta)+c^lt$ is a lower solution implies that $c_m\geq c^l$.

Denote by $(c^*,\Phi^*(\theta))$ the solution of (1.6). Then from the proofs of Lemmas 2.1 and 2.2, it is easily seen that

$$c^*=c^l+O\left(\frac{1}{m}\right),\quad \Phi^*=\Phi^l+C+O\left(\frac{1}{m}\right),\quad \Phi^*_\theta=\Phi^l_\theta+O\left(\frac{1}{m}\right).$$

Therefore,

$$c_m>c^*-\frac{C}{m}\qquad\text{for some }\ C>0.\tag{2.3}$$

■

2.3. Upper solution

Now we use $\Phi^l(\theta) + c^l t$ to construct an upper solution. Let $U(\theta,t)$ be the periodic rotating wave of (1.2)-(1.3). We note that $U(\theta,t)\big|_{[-\theta_0,\theta_0]}$ is nothing but the solution of

(2.4)
$$\begin{cases} \tilde{u}_t = \dfrac{\tilde{u}_{\theta\theta} - 2\tan\theta\cdot\tilde{u}_\theta - \sin\theta\cos\theta\cdot\tilde{u}_\theta^3}{1+\tilde{u}_\theta^2\cos^2\theta}, & -\theta_0 < \theta < \theta_0,\ t>0, \\ \tilde{u}(-\theta_0,t) = U(-\theta_0,t),\quad \tilde{u}(\theta_0,t) = U(\theta_0,t), & t>0, \\ \tilde{u}(\theta,0) = U(\theta,0), & -\theta_0 < \theta < \theta_0. \end{cases}$$

Without loss of generality we assume $U(\theta,0) \preceq \Phi^l(\theta)$, "$\preceq$" means that $U(\theta,0) \le \Phi^l(\theta)$ for $\theta \in [-\theta_0,\theta_0]$ and $U(\hat{\theta},0) = \Phi^l(\hat{\theta})$ for some $\hat{\theta} \in [-\theta_0,\theta_0]$.

Define

$$(2.5)\quad w(\theta,t) = E\sqrt{\frac{1}{m}}\left(t+\frac{\theta^2}{2}\right) + tEF\sqrt{\frac{1}{m}} \qquad \text{for } |\theta|\le\theta_0,\ t\ge 0,$$

where $E = O(1)$ is determined later, and $F = \frac{2\pi\cot(\phi+\alpha)}{\cos\theta_0}$.

LEMMA **2.3.** $\bar{u}(\theta,t) := w(\theta,t) + \Phi^l(\theta) + c^l t$ *is an upper solution of* (2.4) *on time-interval* $t\in[0,1]$, *and hence*

$$(2.6)\qquad \bar{u}(\theta,t) \ge U(\theta,t) \qquad \text{for } \theta\in[-\theta_0,\theta_0],\ t\in[0,1].$$

PROOF: To prove the Lemma, it suffices to show that

(2.7)
$$\bar{u}_t \ge \frac{\bar{u}_{\theta\theta} - 2\tan\theta\cdot\bar{u}_\theta - \sin\theta\cos\theta\cdot\bar{u}_\theta^3}{1+\bar{u}_\theta^2\cos^2\theta} \qquad \text{for } -\theta_0<\theta<\theta_0,\ t>0,$$

and

$$(2.8)\quad U(-\theta_0,t) \le \bar{u}(-\theta_0,t), \qquad U(\theta_0,t)\le\bar{u}(\theta_0,t) \qquad \text{for } t\in[0,1].$$

We first prove (2.7). Since $\Phi^l_{\theta\theta} - 2\tan\alpha\Phi^l_\theta - \sin\theta\cos\theta(\Phi^l_\theta)^3 > 0$ and $\Phi^l_\theta(\theta)\cdot\theta \geq 0$ for $\theta \in [-\theta_0, \theta_0]$, a direct calculation shows that

$$\begin{aligned}
&\bar{u}_t - \frac{\bar{u}_{\theta\theta} - 2\tan\theta\cdot\bar{u}_\theta - \sin\theta\cos\theta\cdot\bar{u}_\theta^3}{1+\bar{u}_\theta^2\cos^2\theta}\\
\geq\ & (E+EF)\sqrt{\frac{1}{m}} + E\sqrt{\frac{1}{m}}\cdot\frac{2\theta\cdot\tan\theta + 3\theta\Phi^l_\theta\sin\theta\cos\theta}{1+\left(\Phi^l_\theta + E\sqrt{\frac{1}{m}}\theta\right)^2\cos^2\theta} + O\left(\frac{1}{m^2}\right)\\
\geq\ & (E+EF)\sqrt{\frac{1}{m}} - E\sqrt{\frac{1}{m}}\cdot 3|\theta\Phi^l_\theta\sin\theta\cos\theta|\\
\geq\ & E\left(1+F-\frac{2\pi\cot(\phi+\alpha)}{\cos\theta_0}\right)\sqrt{\frac{1}{m}} > 0.
\end{aligned}$$

Next we prove (2.8). Suppose that they hold on $t \in [0,\tau]$ for some $\tau < 1$, then $\bar{u}$ is upper solution on $t \in [0,\tau]$ and so

$$U(\theta,t) \leq \bar{u}(\theta,t) \qquad \text{for } \theta \in [-\theta_0,\theta_0],\ t\in[0,\tau]. \tag{2.9}$$

We show that (2.8) holds in fact on $t \in [0,1]$ (see Figure 2).

Construct a *great circle* $\varphi = \lambda(\theta)$ on S^2 as the following. Assume $b'_m(s_1) = \cos\theta_0\tan\alpha$ at $s_1 \in [0, \frac{2\pi}{m})$. Denote $\theta_1^* = -\theta_0 - b_m(s_1)$ and $P_1 = (\theta_1^*, s_1) \in \partial^-\Omega_m$. Choose $\lambda(\theta)$ to be the great circle (geodesic curvature is 0) contacting $\partial^-\Omega_m$ at P_1 with angle ϕ. This $\lambda(\theta)$ corresponds to a solution of (2.1) with $c = 0$, and so from the proof of Lemma 2.1 we have

$$\lambda(\theta) = \int_0^\theta \frac{d}{\cos\varsigma\ \sqrt{\cos^2\varsigma - d^2}}d\varsigma + C$$

for suitable d and C. Just as that in the boundary conditions (1.3), at P_1 we have

$$\begin{aligned}
\lambda_\theta(\theta_1^*) &= -\frac{\cos\phi\cos\theta_1^* - \cos\theta_0\tan\alpha\sin\phi}{\cos\theta_1^*\ (\sin\phi\cos\theta_1^* + \cos\theta_0\tan\alpha\cos\phi)}\\
&= -\frac{\cot(\phi+\alpha)}{\cos\theta_0} + O\left(\frac{1}{m}\right) = \Phi^l_\theta(-\bar{\theta}) + O\left(\frac{1}{m}\right).
\end{aligned}$$

Hence, there exists $R > 1$ such that

$$|\Phi^l_\theta(\theta) - \lambda_\theta(\theta)| \leq (R-1)\sqrt{\frac{1}{m}} \qquad \text{for } \theta \in \left[-\theta_0, -\theta_0 + \sqrt{\frac{1}{m}}\right], \tag{2.10}$$

and

$$|\lambda(-\theta_0)| = |\lambda(\theta_1^*) + \lambda_\theta(\theta_1^*)\cdot b_m(s_1) + O(m^{-2})| \leq \frac{R_1}{m} \qquad \text{for some } R_1 > 0.$$

Suppose at τ, $\lambda(\theta) + D(\tau)$ intersects $\bar{u}(\theta, \tau)$ at $\theta = -\theta_0 + \sqrt{\frac{1}{m}}$, i.e. $\bar{u}\left(-\theta_0 + \sqrt{\frac{1}{m}}, \tau\right) = \lambda\left(-\theta_0 + \sqrt{\frac{1}{m}}\right) + D(\tau)$. Then by (2.10) there exists $\tilde{\theta} \in \left[-\theta_0, -\theta_0 + \sqrt{\frac{1}{m}}\right]$ such that

$$\begin{aligned}
D(\tau) &= \bar{u}\left(-\theta_0 + \sqrt{\frac{1}{m}}, \tau\right) - \lambda\left(-\theta_0 + \sqrt{\frac{1}{m}}\right) \\
&= w\left(-\theta_0 + \sqrt{\frac{1}{m}}, \tau\right) \\
&\quad + c^l\tau + \Phi^l\left(-\theta_0 + \sqrt{\frac{1}{m}}\right) - \lambda\left(-\theta_0 + \sqrt{\frac{1}{m}}\right) \\
&= w(-\theta_0, \tau) - \frac{\theta_0 E}{m} + c^l\tau + \Phi^l(-\theta_0) \\
&\quad - \lambda(-\theta_0) + (\Phi^l_\theta(\tilde{\theta}) - \lambda_\theta(\tilde{\theta}))\sqrt{\frac{1}{m}} + o\left(\frac{1}{m}\right) \\
&< \bar{u}(-\theta_0, \tau) + \frac{R_1 + R - 1 - \theta_0 E}{m} + o\left(\frac{1}{m}\right) \\
&< \bar{u}(-\theta_0, \tau) - \frac{R_1 + 5\pi}{m}
\end{aligned}$$

provided we choose E large such that $\theta_0 E > R + 2R_1 + 5\pi$.

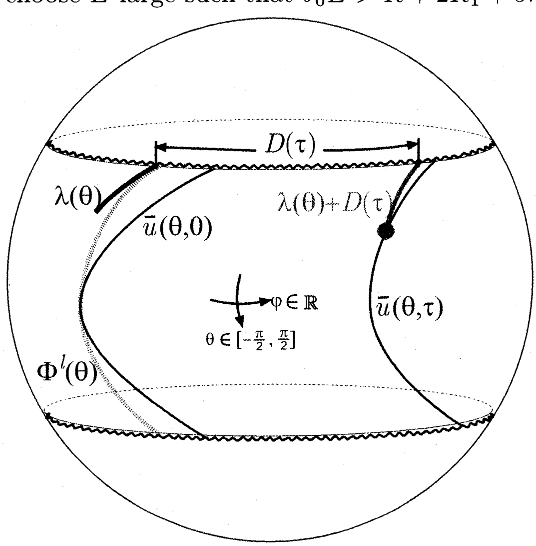

Fig. 2 Upper solution

Since $\lambda(\theta)$ contacts $\partial^-\Omega_m$ at P_1 with angle ϕ, there exists $\delta \in [0, \frac{2\pi}{m})$ such that $\lambda(\theta) + D(\tau) + \delta$ also contacts $\partial^-\Omega_m$ at some point with angle ϕ, so $\lambda(\theta) + D(\tau) + \delta$ is stationary. Therefore by

$$U\left(-\theta_0 + \sqrt{\frac{1}{m}}, \tau\right) \le \bar{u}\left(-\theta_0 + \sqrt{\frac{1}{m}}, \tau\right) \le \lambda\left(-\theta_0 + \sqrt{\frac{1}{m}}\right) + D(\tau) + \delta,$$

we have $U(\theta, \tau) \le \lambda(\theta) + D(\tau) + \delta$ for $-\theta_0 \le \theta \le -\theta_0 + \sqrt{\frac{1}{m}}$. Especially,

$$U(-\theta_0, \tau) \le \lambda(-\theta_0) + D(\tau) + \delta \le D(\tau) + \frac{R_1 + 2\pi}{m} \le \bar{u}(-\theta_0, \tau) - \frac{3\pi}{m}.$$

Therefore,

$$\bar{u}(-\theta_0, \tau + t) \ge \bar{u}(-\theta_0, \tau) \ge U(-\theta_0, \tau) + \frac{3\pi}{m} \ge U(-\theta_0, \tau + t) \qquad \text{for } t \in [0, T_m],$$

In other words, the first inequality of (2.8) holds at least on $[0, \tau + T_m]$. Similarly, the second inequality of (2.8) at $\theta = \theta_0$ holds on $[0, \tau + T_m]$. Consequently, (2.8) hold on $t \in [0, \tau + T_m]$ provided $\tau < 1$.

Finally, repeating the discussion stated above finite times we obtain (2.8) on $t \in [0, 1]$, and so (2.6) holds on $t \in [0, 1]$. ■

Proof of the second inequality of (1.5). From Lemma 2.3 we have

$$U(\theta, 1) \le \bar{u}(\theta, 1) \le \Phi^l(\theta) + E(1 + \frac{\theta_0^2}{2} + F)\sqrt{\frac{1}{m}} + c^l \le \Phi^l(\theta) + N\frac{2\pi}{m} \qquad \text{for } \theta \in [-\theta_0, \theta_0],$$

where $N := \left[\left(E(1 + \frac{\theta_0^2}{2} + F)\sqrt{\frac{1}{m}} + c^l\right) \cdot \frac{m}{2\pi} + 1\right]$, $[\cdot]$ denotes the Gauss function. On the other hand, $U(\theta, 0) \preceq \Phi^l(\theta)$ implies that

$$U(\theta, NT_m) \preceq \Phi^l(\theta) + N \cdot \frac{2\pi}{m} \qquad \text{for } \theta \in [-\theta_0, \theta_0],$$

Since $U(\theta, t)$ is strictly increasing in t (we omit the proof and refer to [10]), we have $NT_m \ge 1$ and so

$$\begin{aligned} c_m = \frac{2\pi}{mT_m} &\le \frac{2\pi}{m} \cdot \left\{\left(E(1 + \frac{\theta_0^2}{2} + F)\sqrt{\frac{1}{m}} + c^l\right) \cdot \frac{m}{2\pi} + 1\right\} \\ &\le c^l + E(1 + \theta_0^2 + F)\sqrt{\frac{1}{m}}. \end{aligned}$$

This proves the second inequality of (1.5).

Finally, it is not difficult to see that (ii) of Theorem 1.1 can be proved by (i) of Theorem 1.1 and regularity of U. ■

§3. Some Remarks

1. In [10], we studied periodic traveling waves of a mean curvature flow equation in an undulating band domain, obtained similar results as above. Problem in that paper is different from the present one in several points. First, since the boundaries of a zone on S^2 have period 2π anyway, here the existence result is true in fact even if $m = 1$, or the two boundaries of the zone are given by two different functions with different periods $\frac{2\pi}{m}$ and $\frac{2\pi}{n}$. In such cases, the period of periodic rotating wave is $\frac{2\pi}{(m,n)}$ ((m, n) is the greast common divisor), not necessarily to be small as that in [10]. Second, the problems and backgrounds are different. Mean curvature flows in an *unbounded* band domain is reduced from a traveling front or a traveling pulse, but problem in this paper is about geodesic curvature flows in *bounded* zone on sphere, which is more interesting in geometry.

2. (H1) is a necessary condition for the existence of periodic rotating waves. However, we do not think (H2) is essential in the speed estimate. In fact, we believe that Theorem 1.1 remains true even if the curve develops singularities near the boundaries.

3. So far, very little is know about periodic rotating/traveling wave surfaces in manifolds, we believe that other curvature flows in some other manifolds can be studied in a similar way as above.

Acknowledgements. This research was carried out when the author was in Hokkaido University as a VBL researcher. He would like to thank Professors Y. Nishiura and Y. Tonegawa for helpful discussions.

References

[1] N. D. Alikakos and A. Freire, The normalized mean curvature flow for a small bubble in a Riemannian manifold, J. Differential Geometry, **64** (2003), 247–303.

[2] B. Andrews, Motion of hypersurfaces by Gauss curvature, Pacific J. Math., **195** (2000), 1–34.

[3] S. Angenent, Parabolic equations for curves on surfaces. I. Curves with p-integrable curvature, Ann. of Math. (2), **132** (1990), 451–483.

[4] S. Angenent, Parabolic equations for curves on surfaces. II. Intersections, blow-up and generalized solutions, Ann. of Math. (2), **133** (1991), 171–215.

[5] K. S. Chou and X. P. Zhu, The curve shortening problem. Chapman & Hall/CRC, Boca Raton, FL, 2001.

[6] H. Gage and R. S. Hamilton, The heat equation shrinking convex plane curves, J. Differential Geom., **23** (1986), 69–96.

[7] M. A. Grayson, The heat equation shrinks embedded plane curves to round points, J. Differential Geom., **26** (1987), 285–314.

[8] O. A. Ladyzhenskia, V. A. Solonnikov and N. N. Uraltseva, Linear and Quasilinear Equations of Parabolic Type, Amer. Math. Soc., Providence, RI, 1968.

[9] G. M. Lieberman, Second Order Parabolic Differential Equations, World Scientific, 1996.

[10] H. Matano, K. I. Nakamura and B. Lou, Periodic traveling waves in a two-dimensional cylinder with saw-toothed boundary and their homogenization limit, Networks and Heterogeneous Media, **1** (2006), 537–568.

[11] T. Ogiwara and H. Matano, Monotonicity and convergence results in order-preserving systems in the presence of symmetry, Discrete and Continuous Dyn. Sys., **5** (1999), 1–34.

[12] M. T. Wang, Mean curvature flow of surfaces in Einstein four-manifolds, J. Differential Geometry, **57** (2001), 301–338.

[13] J. Xin, Front propagation in heterogeneous media, SIAM Rev., **42** (2000), 161–230.

Department of Mathematics
Tongji University
Siping Road 1239, Shanghai, P.R. China