Advanced Studies in Pure Mathematics 47-2, 2007
Asymptotic Analysis and Singularities
pp. 613–622

Global solutions to a one-dimensional nonlinear parabolic system modeling colonial formation by chemotactic bacteria

Khin Phyu Phyu Htoo, Masayasu Mimura and Izumi Takagi

Abstract.

We consider a diffusion-chemotaxis-growth system which models pattern formation by a bacterial colony. In the case of spatial dimension one we prove that the initial-boundary value problem for the system has a unique solution on the entire time interval $(0, +\infty)$, and that the solution remains bounded.

§1. Statement of Results

Budrene and Berg ([1], [2]) observed in experiments that a chemotactic strain of *E. coli* generate surprisingly complex and ordered spatial patterns. In order to understand analytically why they generate such patterns, the second author and his group proposed a mesoscopic model governed by a diffusion-chemotaxis-growth system. They suggest by numerical simulation that the resulting patterns are possibly generated in a self-organized way. However, the rigorous study of this system has not been done yet. As a first step, we will study the fundamental question of existence of solutions of the initial and boundary value problem for the system in the case where the spatial dimension is one.

We assume that the bacteria have two states. The active bacteria move around randomly, and take nutrients in the environment. In addition, they release a certain chemical which causes a directed movement toward its higher concentration (chemotaxis). Some of the active bacteria become inactive at a rate depending on the population of the active bacteria and the nutrient concentration. The proposed model comprises the population density of the active bacteria $u(x,t)$, the population density of the inactive bacteria $w(x,t)$, the density of nutrient $n(x,t)$, and the concentration of the chemoattractant $c(x,t)$ at position $x \in (0,l)$

Received October 31, 2005.
Revised January 27, 2006.

and time $t \in [0, \infty)$. The diffusion-chemotaxis-growth system in the one dimensional case becomes:

$$
\text{(P)} \qquad \begin{cases}
u_t = d_u u_{xx} + \omega g(u)nu - (u(\chi(c))_x)_x - a(u)b(n)u, \\
n_t = d_n n_{xx} - g(u)nu, \\
c_t = d_c c_{xx} + \alpha u - \beta c, \\
w_t = a(u)b(n)u, \\
u_x(0,t) = u_x(l,t) = 0, \quad n_x(0,t) = n_x(l,t) = 0, \\
c_x(0,t) = c_x(l,t) = 0, \\
u(x,0) = u_0(x), \quad n(x,0) = n_0(x), \quad c(x,0) = c_0(x), \\
w(x,0) = w_0(x).
\end{cases}
$$

Here, $d_u > 0$, $d_n > 0$ and $d_c > 0$ are the diffusion rates of u, n and c, respectively. $\alpha > 0$ and $\beta > 0$ are the production by the bacteria and degradation rates of c, respectively. d_u, d_n, d_c, α, β are positive constants and ω is a nonnegative constant. $\chi(c)$ is the sensitivity function and is assumed to be of class $C^3[0, \infty)$ and satisfy $0 < \chi(c) \le M$, $0 < \chi'(c) \le M$ and $|\chi''(c)| + |\chi'''(c)| \le M$ for all $c \ge 0$ with some positive constant M. The active bacteria are assumed to become inactive at a rate $a(u)b(n)$ per capita, where $a(u)$ and $b(n)$ are monotone decreasing smooth positive functions defined for $u \ge 0$ and $n \ge 0$, respectively. Finally, we assume that the growth rate of bacteria is of the form $\omega g(u)n$, where $g(u)$ is a smooth bounded positive function such that $g'(u) \ge 0$. To be specific, in this paper we will assume that

$$
a(u) = \frac{a_0}{1 + \dfrac{u}{a_1}}, \qquad b(n) = \frac{b_0}{1 + \dfrac{n}{b_1}},
$$

$$
g(u) = \frac{g_0 u^m}{(u^*)^m + u^m}, \qquad \chi(c) = \frac{kc^2}{\theta^2 + c^2}
$$

for some positive constants a_0, a_1, b_0, b_1, g_0, m, u^*, k and θ.

For an integer $j \ge 2$, let $H_N^j(0, l)$ denote the Hilbert space of all L^2-functions whose derivatives up to order j are in L^2 and satisfy the homogeneous Neumann boundary conditions at $x = 0$ and $x = l$. Our results are stated as follows:

Theorem 1. *Let $u_0 \in H_N^2(0, l)$, $n_0 \in H_N^2(0, l)$, $c_0 \in H_N^3(0, l)$ be nonnegative functions. Moreover, let w_0 be a non-negative continuous function on $[0, l]$. Then, there exists a unique solution to* (P) *for all $t > 0$. Moreover, we have*

$$
\|u(t)\|_{H^1} + \|n(t)\|_{H^1} + \|c(t)\|_{H^2} \le M,
$$

and

$$\int_t^{t+1}\int_0^l (u_{xx}^2+n_{xx}^2+c_{xx}^2+u_t^2+n_t^2+c_t^2)\,dx\,ds \le M$$

for any $t \ge 0$*, where* M *is a positive constant independent of* t*.*

Remarks.

(i) The theorem implies that any solution of (P) remains bounded uniformly in $t > 0$.

(ii) The conclusion holds true also for more general $a(u)$, $b(n)$, $g(u)$ and $\chi(c)$.

(iii) In a forthcoming paper, as an application of the estimates derived in Theorem 1 we will study the asymptotic behavior of the solution as $t \to \infty$, and show that $u \to 0, c \to 0$ as $t \to \infty$ while n converges to a nonnegative constant.

§2. Proof of Theorem

First, we note that if the initial data u_0, n_0 and c_0 are nonnegative, then solution of (P) remains nonnegative by the Maximum Principle. Since the local-in-time existence of a solution and its uniqueness can be proved along the same line as in [3], we only verify the estimates. The proof is carried out in several steps.

Step 1. Integrating the equation (P) in x yields that

$$\frac{d}{dt}\int_0^l u\,dx = \omega\int_0^l g(u)nu\,dx - \int_0^l a(u)b(n)u\,dx, \tag{1}$$

$$\frac{d}{dt}\int_0^l n\,dx = -\int_0^l g(u)nu\,dx, \tag{2}$$

$$\frac{d}{dt}\int_0^l c\,dx = \alpha\int_0^l u\,dx - \beta\int_0^l c\,dx, \tag{3}$$

$$\frac{d}{dt}\int_0^l w\,dx = \int_0^l a(u)b(n)u\,dx. \tag{4}$$

Therefore,

$$\frac{d}{dt}\int_0^l u\,dx + \omega\frac{d}{dt}\int_0^l n\,dx + \frac{d}{dt}\int_0^l w\,dx = 0$$

whence follows that

$$(5)\quad \int_0^l u\,dx + \omega\int_0^l n\,dx + \int_0^l w\,dx = \int_0^l u_0 dx + \omega\int_0^l n_0\,dx + \int_0^l w_0\,dx.$$

Therefore,

$$(6)\qquad \int_0^l u\,dx \le M.$$

Here and hereafter M denotes various positive constants independent of t.

Step 2. Multiply both sides of (3) by $e^{\beta t}$ and integrate the resulting equation with respect to t. Then,

$$\int_0^l c\,dx = e^{-\beta t}\int_0^l c_0\,dx + \alpha\int_0^t e^{-\beta(t-s)}\int_0^l u\,dx\,ds.$$

This together with (6) implies

$$(7)\qquad \int_0^l c\,dx \le \int_0^l c_0\,dx + M.$$

Similarly, integrating the third equation of (P) multiplied by c with respect to x yields

$$(8)\qquad \frac{1}{2}\frac{d}{dt}\int_0^l c^2\,dx + d_c\int_0^l {c_x}^2\,dx + \beta\int_0^l c^2\,dx = \alpha\int_0^l uc\,dx.$$

Observe that

$$\begin{aligned} c(x,t) &= \frac{1}{l}\int_0^l c(x,t)\,dx + \int_{x_0}^x c_x(\xi,t)\,d\xi \\ &\le M + \sqrt{l\int_0^l c_x^2\,dx} \end{aligned}$$

where

$$c(x_0,t) = \frac{1}{l}\int_0^l c(x,t)\,dx.$$

Hence

$$(9)\quad \int_0^l uc\,dx \le \left(M + \sqrt{l\int_0^l c_x^2\,dx}\right)\int_0^l u\,dx \le M^2 + M\sqrt{l\int_0^l c_x^2\,dx}.$$

Combining (8) and (9), we obtain

$$\int_0^l c^2\,dx \le \int_0^l {c_0}^2\,dx + M. \tag{10}$$

Let $G(x,y,t)$ be the fundamental solution of the initial-boundary value problem

$$\begin{cases} c_t = d_c c_{xx} - \beta c & (0 \le x \le l,\ t > 0), \\ c_x(0,t) = c_x(l,t) = 0 & (t > 0), \\ c(x,0) = c_0(x) & (0 \le x \le l). \end{cases}$$

Moreover, let $H(x,y,t)$ denote the fundamental solution of the following problem:

$$\begin{cases} v_t = d_c v_{xx} - \beta v & (0 \le x \le l,\ t > 0), \\ v(0,t) = v(l,t) = 0 & (t > 0), \\ v(x,0) = v_0(x) & (0 \le x \le l). \end{cases}$$

Then

$$c(x,t) = \int_0^l G(x,y,t)c_0(y)\,dy + \alpha\int_0^t ds\int_0^l G(x,y,t-s)u(y,s)\,dy, \tag{11}$$

and

(12)

$$c_x(x,t) = \int_0^l H(x,y,t){c_0}'(y)\,dy + \alpha\int_0^t ds\int_0^l \frac{\partial G}{\partial x}(x,y,t-s)u(y,s)\,dy.$$

It is well-known that

$$0 < G(x,y,t) \le \frac{M_1}{\sqrt{t}}\ \exp\left(-\frac{|x-y|^2}{4d_c t} - \beta t\right), \tag{13}$$

$$0 \le H(x,y,t) \le \frac{M_2}{\sqrt{t}}\ \exp\left(-\frac{|x-y|^2}{4d_c t} - \beta t\right), \tag{14}$$

$$\left|G_x(x,y,t)\right| \le \frac{M}{t}\ \exp\left(-\kappa\frac{|x-y|^2}{4d_c t} - \beta t\right). \tag{15}$$

Here $0 < \kappa < 1$ and M depends on κ.

From (12), (13), (14), (15) and after a series of estimates we obtain

$$\int_0^l c_x^2\,dx \le M. \tag{16}$$

Therefore, by (10) and (15), we conclude that $c(x,t) \le M$ for any $0 \le x \le l$ and $t \ge 0$.

Step 3. Multiply the first equation of (P) by u and then integrate with respect to x. Then,

$$\frac{1}{2}\frac{d}{dt}\int_0^l u^2\,dx + d_u\int_0^l u_x^2\,dx + \int_0^l a(u)b(n)u^2\,dx = \omega\int_0^l g(u)nu^2\,dx + \int_0^l u(\chi(c))_x u_x\,dx,$$

so that

$$\frac{1}{2}\frac{d}{dt}\int_0^l u^2\,dx + \frac{d_u}{2}\int_0^l u_x^2\,dx + \int_0^l a(u)b(n)u^2\,dx \le M\int_0^l u^2\,dx + \frac{1}{2d_u}\int_0^l u^2(\chi'(c))^2 c_x^2\,dx.$$

Note that $n(x,t) \le \|n_0\|_{L^\infty}$ by the Maximum Principle.

By the Gagliardo-Nirenberg inequality,

$$\begin{aligned}\int_0^l u^2(\chi'(c))^2 c_x^2\,dx &\le M\|u\|_{L^4}^2\|c_x\|_{L^4}^2 \\ &\le M\|u\|_{H^1}\|u\|_{L^1}\|c_x\|_{H^1}^{\frac{1}{2}}\|c_x\|_{L^2}^{\frac{3}{2}} \\ &\le \epsilon\left(\|u\|_{H^1}^2 + \|c\|_{H^2}^2\right) + K_\epsilon\end{aligned}$$

with an arbitrary $\epsilon > 0$ and a positive constant K_ϵ depending on ϵ. Therefore,

$$\frac{1}{2}\frac{d}{dt}\int_0^l u^2\,dx + \frac{d_u}{2}\int_0^l u_x^2\,dx + \int_0^l a(u)b(n)u^2\,dx \le M\int_0^l u^2\,dx + \epsilon\left(\|u\|_{H^1}{}^2 + \|c\|_{H^2}{}^2\right) + K_\epsilon.$$

Similarly, multiplying the third equation of (P) by c_{xx} and integrating the result with respect to x, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_0^l c_x^2\,dx + d_c\int_0^l c_{xx}^2\,dx + \beta\int_0^l c_x^2\,dx = -\alpha\int_0^l u c_{xx}\,dx,$$

$$\frac{1}{2}\frac{d}{dt}\int_0^l c_x^2\,dx + \frac{d_c}{2}\int_0^l c_{xx}^2\,dx + \beta\int_0^l c_x^2\,dx \le \frac{\alpha^2}{2d_c}\int_0^l u^2\,dx.$$

Again by the Gagliardo-Nirenberg inequality, we have

$$\begin{aligned}\int_0^l u^2\,dx &\leq M\|u\|_{H^1}^{\frac{2}{3}}\|u\|_{L^1}^{\frac{4}{3}}\\ &\leq \epsilon\|u\|_{H^1}^2 + K_\epsilon\|u\|_{L^1}^2\end{aligned}$$

for an arbitrary $\epsilon > 0$ and a positive constant K_ϵ depending on ϵ. Therefore,

(17)

$$\frac{1}{2}\frac{d}{dt}\int_0^l c_x^2\,dx + \frac{d_c}{2}\int_0^l c_{xx}^2\,dx + \beta\int_0^l c_x^2\,dx \leq M\left(\epsilon\|u\|_{H^1}^2 + K_\epsilon\|u\|_{L^1}^2\right).$$

Hence,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_0^l \left(u^2 + c_x^2\right)\,dx + \int_0^l\left(\frac{d_u}{2}u_x^2 + \frac{d_c}{2}c_{xx}^2\right)\,dx\\ &\qquad\qquad + \int_0^l a(u)b(n)u^2\,dx + \beta\int_0^l c_x^2\,dx\\ &\leq M\int_0^l u^2\,dx + \epsilon\left(\|u\|_{H^1}^2 + \|c\|_{H^2}^2\right) + K_\epsilon + M\left(\epsilon\|u\|_{H^1}^2 + K_\epsilon\|u\|_{L^1}^2\right).\end{aligned}$$

Consequently, we obtain that

$$\|u\|_{L^2}^2 + \|c\|_{H^1}^2 + \int_t^{t+1}(\|u\|_{H^1}^2 + \|c\|_{H^2}^2)\,ds \leq M. \tag{18}$$

Step 4. We multiply the first equation of (P) by u_{xx} and then integrate with respect to x to get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_0^l u_x^2\,dx + d_u\int_0^l u_{xx}^2\,dx\\ &= \int_0^l a(u)b(n)uu_{xx}\,dx - \omega\int_0^l g(u)nuu_{xx}\,dx + \int_0^l (u(\chi(c))_x)_x u_{xx}\,dx.\end{aligned}$$

Therefore, we have

$$\frac{1}{2}\frac{d}{dt}\int_0^l u_x^2\,dx + \frac{d_u}{2}\int_0^l u_{xx}^2\,dx \leq K_\epsilon + 3\epsilon\left(\|u\|_{H^2}^2 + \|c\|_{H^3}^2\right) \tag{19}$$

by a calculation similar to that in Step 3.

On the other hand, we obtain from the third equation of (P)

$$(20)\quad \frac{1}{2}\frac{d}{dt}\int_0^l c_{xx}^2\,dx + \frac{d_c}{2}\int_0^l c_{xxx}^2\,dx + \beta\int_0^l c_{xx}^2\,dx \le \epsilon\|u\|_{H^2}^2 + K_\epsilon\|u\|_{L^2}^2.$$

Hence, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_0^l \left(u_x^2 + c_{xx}^2\right)\,dx + \int_0^l \left(\frac{d_u}{2}u_{xx}^2 + \frac{d_c}{2}c_{xxx}^2\right)\,dx + \beta\int_0^l c_{xx}^2\,dx \\ &\le\ K_\epsilon + 3\epsilon\Big(\|u\|_{H^2}^2 + \|c\|_{H^3}^2\Big) + \epsilon\|u\|_{H^2}^2 + K_\epsilon\|u\|_{L^2}^2.\end{aligned}$$

Therefore,

$$(21)\qquad \|u\|_{H^1}^2 + \|c\|_{H^2}^2 + \int_t^{t+1}(\|u\|_{H^2}^2 + \|c\|_{H^3}^2)\,ds \le M.$$

Step 5. Multiply the second equation of (P) by n and integrate the resulting equation with respect to x. We then have

$$\frac{1}{2}\frac{d}{dt}\int_0^l n^2\,dx + d_n\int_0^l n_x^2\,dx + \int_0^l g(u)n^2u\,dx = 0,$$

therefore

$$\int_0^l n^2\,dx \le \int_0^l n_0^2\,dx.$$

Next we multiply the second equation of (P) by n_{xx} and then integrate with respect to x, obtaining

$$\frac{1}{2}\frac{d}{dt}\int_0^l n_x^2\,dx + d_n\int_0^l n_{xx}^2\,dx = \int_0^l g(u)unn_{xx}\,dx.$$

Hence

$$\frac{1}{2}\frac{d}{dt}\int_0^l n_x^2\,dx + \frac{d_n}{2}\int_0^l n_{xx}^2\,dx \le M\int_0^l n^2\,dx.$$

From the Poincaré inequality, the above estimate implies

$$(22)\qquad \int_0^l n_x^2\,dx \le M.$$

Multiply the second equation of (P) by n_t and then integrate with respect to x. Then,

$$\int_0^l n_t^2\,dx = -d_n\int_0^l n_{xt}n_x\,dx - \int_0^l g(u)unn_t\,dx,$$

$$\int_0^l n_t^2\,dx + \frac{d_n}{2}\frac{d}{dt}\int_0^l n_x^2\,dx \le \frac{1}{2}\int_0^l g(u)^2u^2n^2\,dx + \frac{1}{2}\int_0^l n_t^2\,dx.$$

Integrating the above inequality in t, we have

$$\begin{aligned}\int_t^{t+1}\int_0^l n_t^2\,dx\,ds + d_n\int_0^l n_x(x,t+1)^2\,dx \\ \le\ d_n\int_0^l n_x(x,t)^2\,dx + \int_t^{t+1}\int_0^l g(u)^2u^2n^2\,dx\,ds.\end{aligned}$$

Therefore

$$\int_t^{t+1}\int_0^l n_t^2\,dx\,ds \le M. \tag{23}$$

Similarly we can prove that

$$\int_t^{t+1}\int_0^l c_t^2\,dx\,ds \le M,$$

and

$$\int_t^{t+1}\int_0^l u_t^2\,dx\,ds \le M.$$

Thus we have proved Theorem 1.

References

[1] E. O. Budrene and H. Berg, Complex patterns formed by motile cells of Escherichia coli, Nature, **349** (1991), 630–633.
[2] E. O. Budrene and H. Berg, Dynamics of formation of symmetrical patterns by chemotactic bacteria, Nature, **376** (1991), 49–53.
[3] K. Osaki and A. Yagi, Finite dimensional attractor for one-dimensional Keller-Segel equations, Funkcialaj Ekvacioj, **44** (2001), 441–469.

Khin Phyu Phyu Htoo
Mathematics Department
Monywa University
Monywa
Myanmar

Masayasu Mimura
Department of Mathematics
Meiji Institute for Mathematical Sciences
School of Science and Technology
Meiji University
1-1-1 Higashimita, Tamaku, Kawasaki 214-8571
Japan

Izumi Takagi
Mathematical Institute
Tohoku University
Sendai 980-8578
Japan