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Method of the blowup envelope and applications

Takashi Suzuki

Abstract.

We describe the proof of mass quantization of collapse in the simplified system of chemotaxis [31], and study the blowup rate in connection with the free energy transmission (emergence).

§1. Introduction

The purpose of the present paper is to describe recent progress in the study of the simplified system of chemotaxis [16, 19], which has several backgrounds in statistical mechanics, non-equilibrium thermodynamics, and biology [31]. In the context of biology, it is associated with the chemotactic feature of cellular slime molds [17], typically,

$$
\begin{aligned}
& u_t = \nabla \cdot (\nabla u - u\nabla v) \\
& -\Delta v = u - \frac{1}{|\Omega|}\int_\Omega u \qquad \text{in } \Omega \times (0,T) \\
& \frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \qquad \text{on } \partial\Omega \times (0,T) \\
& \int_\Omega v = 0,
\end{aligned}
\tag{1}
$$

where $\Omega \subset \mathbf{R}^n$ $(n = 2, 3)$ is a bounded domain with smooth boundary $\partial\Omega$ and ν is the outer unit normal vector.

In this case, $u = u(x,t)$ stands for the density of the cellular slime molds. Thus, the first equation of (1) describes mass conservation,

$$u_t = -\nabla \cdot j,$$

with

$$j = -\nabla u + u\nabla v$$

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indicating the flux of u, and therefore, null flux condition,

$$\nu \cdot j = 0$$

is imposed on the boundary. On the other hand, $v = v(x,t)$ is the concentration of a chemical substance secreted by the slime molds, with the production law described by the second equation of (1). The above form of j indicates that v acts as a carrier of u, and the diffusion $-\nabla u$ is competing the chemotaxis $u\nabla v$ for u to vary.

System (1) is *well-posed* locally in time [39, 2, 29, 31]. Thus, if the initial value of u, denoted by u_0, is sufficiently regular, then there is a unique (classical) solution locally in time, and therefore, if $T_{\max}$ denotes the supremum of its existence time, we obtain $T_{\max} > 0$. In case $T = T_{\max} < +\infty$, this T is called the *blowup time*, and actually, it holds that

$$\lim_{t\uparrow T} \|u(\cdot,t)\|_\infty = +\infty.$$

Here and henceforth, $\|\cdot\|_p$ denotes the standard L^p norm:

$$\|f\|_p = \begin{cases} \left\{\int_\Omega |f|^p\right\}^{1/p} & (1 \le p < \infty) \\ ess.\sup_{x\in\Omega} |f(x)| & (p = \infty). \end{cases}$$

We obtain the following.

Theorem 1. *If $n = 2$ and $T = T_{\max} < +\infty$, then it holds that*

$$u(x,t)dx \rightharpoonup \sum_{x_0\in S} m(x_0)\delta_{x_0}(dx) + f(x)dx \tag{2}$$

as $t \uparrow T$, where

$$S = \{x_0 \in \overline{\Omega} \mid \text{there exists } (x_k,t_k) \to (x_0,T) \text{ such that } u(x_k,t_k) \to +\infty\}$$

denotes the blouwp set of u and

$$0 \le f = f(x) \in L^1(\Omega) \cap C(\overline{\Omega} \setminus S).$$

Theorem 2. *We have*

$$m(x_0) = m_*(x_0) \equiv \begin{cases} 8\pi & (x_0 \in \Omega) \\ 4\pi & (x_0 \in \partial\Omega) \end{cases}$$

and hence it holds that

$$2\sharp(S \cap \Omega) + \sharp(S \cap \partial\Omega) \leq \|u_0\|_1 /(4\pi).$$

The convergence in (2) is $*$-weakly in the set of measures on $\overline{\Omega}$, denoted by $M(\overline{\Omega})$. This means

$$\lim_{t \uparrow T} \int_\Omega u(\cdot, t)\varphi = \sum_{x_0 \in S} m(x_0)\varphi(x_0) + \int_\Omega f\varphi$$

for any $\varphi \in C(\overline{\Omega})$. This phenomenon, the appearence of the delta function singularity in $u(x,t)dx$, is called the formation of *collapse*, and the equality $m(x_0) = m_*(x_0)$ is referred to as the *mass quantization.*

In 1970, Keller-Segel [17] proposed a system of parabolic equations with a chemotaxis term competing diffusion, and system (1) is a simplified form introduced by Jäger-Luckhaus [16] in 1992. In 1973, Nanjundiah [23] conjectured the formation of collapse in such a system, while in 1981 Childress-Percus [5] tried semi-analysis and obtained the other conjectures that the formation of collapse holds only in the case of $n = 2$, and that 8π is the threshold of $\lambda = \|u_0\|_1$ for the existence of the solution global in time. In more detail, $\lambda < 8\pi$ will imply $T_{\max} = +\infty$, while there will exist $u_0 \geq 0$ such that $\|u_0\|_1 > 8\pi$ and $T_{\max} < +\infty$. In 1995, Nagai [19] showed that this 8π threshold conjecture is affirmative in the radially symmetric case, but later 4π is proven as the threshold value of the non-radially symmetric case [21, 2, 8, 20, 26]. The above mentioned Theorems 1 and 2 are obtained by [27, 31]. Detailed description of the related study is also given in the latter monograph.

The purpose of the present paper is to describe the background, story of the proof, and related topics concerning Theorems 1 and 2. Thus, describing the fundamental properties of this system in §2, we draw the outline of the proof of these theorems in sections 3 and 4. Then, we discuss free energy transmission and time relaxization in sections 5 and 6, respectively.

§2. Mean field hierarchy

In the context of physics, $u = u(x,t)$ is the distribution function of self- gravitating particles, and $v = v(x,t)$ is the field created by them. In more detail, the second equation of (1) is equivalent to

$$v(x,t) = \int_\Omega G(x,x')u(x',t)dx',$$

where $G = G(x,x')$ denotes the Green's function associated with the elliptic boundary value problem

$$-\Delta v = u - \frac{1}{|\Omega|}\int_\Omega u, \qquad \left.\frac{\partial v}{\partial \nu}\right|_{\partial\Omega} = 0, \qquad \int_\Omega v = 0,$$

and this Green's function behaves like the gravitational potential,

$$G(x,x') \approx \Gamma(x-x')$$

for

$$\Gamma(x) = \begin{cases} \frac{1}{4\pi}\cdot\frac{1}{|x|} & (n=3) \\ \frac{1}{2\pi}\log\frac{1}{|x|} & (n=2). \end{cases}$$

Actually, system (1) is contained in the hierarchy of the mean field of many self-gravitating particles subject to the second law of thermodynamics. More precisely, it is a macroscopic description of this mean field associated with the micoscopic *Langevin equation*, and the mezoscopic *Fokker-Planck-Poisson equation* [1, 35, 36].

In fact, Helmholtz' free energy F is given by the inner energy minus entropy if the temperature is normalized as 1. This implies

$$F = F(u) = \int_\Omega u(\log u - 1) - \frac{1}{2}\int\int_{\Omega\times\Omega} G(x,x')u(x)u(x')dxdx',$$

because $\mu(dx,t) = u(x,t)dx$ stands for the particle distribution. The inner force is self-attractive in this case, and therefore, $-\frac{1}{2}$ is multiplied in the second term. This term is provided with the symmetric potential $G(x',x) = G(x,x')$ following Newton's third law.

The first variation $\delta F(u)$ of $F(u)$, on the other hand, is defined by

$$\langle w, \delta F(u)\rangle = \left.\frac{d}{ds}F(u+sw)\right|_{s=0}.$$

Identifying this paring $\langle\ ,\ \rangle$ with the L^2 inner product, we obtain

$$\delta F(u) = \log u - v$$

for

$$v = \int_\Omega G(\cdot,x')u(x')dx'.$$

Thus, (1) is nothing but

$$\begin{aligned} &u_t = \nabla\cdot(u\nabla\delta F(u)) && \text{in } \Omega\times(0,T) \\ &u\frac{\partial}{\partial\nu}\delta F(u) = 0 && \text{on } \partial\Omega\times(0,T). \end{aligned} \tag{3}$$

This is a *model B* equation [11, 15, 25, 7] derived from the free energy $F = F(u)$, and consequently, it follows that

$$\frac{d}{dt}\int_\Omega u = -\int_{\partial\Omega} u\frac{\partial}{\partial\nu}\delta F(u) = 0 \tag{4}$$

$$\frac{d}{dt}F(u) = -\int_\Omega u\,|\nabla\delta F(u)|^2 \leq 0. \tag{5}$$

Inequality (5) means the decrease of the free energy, while equality (4), combined with $u = u(x,t) \geq 0$, assures the total mass conservation,

$$\lambda \equiv \|u_0\|_1 = \|u(\cdot,t)\|_1 \qquad (t \in [0, T_{\max})). \tag{6}$$

Relation (6) leads to the selection $n = 2$ for the formation of collapse, using the dimension analysis [5]. In more detail, if u is concenrated on a region with the radius $\delta > 0$, then it is of order δ^{-n} because of this property. Then, we replace δ^{-1} by ∇ in (1), and take $\delta^{1-n/2}$ and $\delta^{1+n/2}$ for v and t, respectively, which results in

$$\delta^{-(1+3n/2)}\left(\delta^0, \delta^0, \delta^{-1+n/2}\right) = 0$$
$$\delta^{-n}\left(\delta^0, \cdot, \delta^{-1+n/2}\right) = 0$$

by

$$u_t - \nabla\cdot(u\nabla v) - \Delta v = 0$$
$$u - \frac{1}{|\Omega|}\int_\Omega u + \Delta v = 0.$$

Balance of these relations, thus, implies $n = 2$.

Theorem 2 of mass quantization, on the other hand, describes the "local" L^1 threshold for the post-blowup continuation. The reasons why 8π was conjectured first and why it was modified later to 4π are in the structure of the total set of the stationary states, which we do not describe here. See [31].

§3. Localization - Symmetrization

Henceforth, $n = 2$ is always assumed. Theorem 1 of the formation of collapse is proven by localizing the criterion concerning the existence of the solution globally in time [21, 2, 8]:

Theorem 3. *If* $\lambda = \|u_0\|_1 < 4\pi$, *then it holds that* $T_{\max} = +\infty$.

The proof of the above theorem proposed by [31], uses the dual Trudinger-Moser inequality

$$\inf \{F(u) \mid u \geq 0, \ \|u\|_1 = 4\pi\} > -\infty. \tag{7}$$

Then, it is easy to derive

$$\sup_{t\in[0,T_{\max})} \int_\Omega u(\log u - 1)(\cdot, t) \leq C$$

with a constant $C > 0$ in the case of $\lambda < 4\pi$, which guarantees $T_{\max} = +\infty$ with

$$\sup_{t\geq 0} \|u(\cdot, t)\|_\infty < +\infty$$

by the parabolic regularity.

The proof of Theorem 1 is based on the methods of *localization* and *symmetrization* [27], and we discuss in the following way.

(1) Using nice cut-off functions, we show the formation of collapse at each *isolated* blowup point.

(2) Gagliardo-Nirenberg inequality guarantees ε-regularity. In more detail, there is an ablsolute constant, denoted by $\varepsilon_0 > 0$, such that

$$\limsup_{t\uparrow T_{\max}} \|u(\cdot, t)\|_{L^1(\Omega\cap B(x_0,R))} < \varepsilon_0 \Rightarrow x_0 \notin S$$

holds for some $R > 0$. This means

$$x_0 \in S \Rightarrow \limsup_{t\uparrow T_{\max}} \|u(\cdot, t)\|_{L^1(\Omega\cap B(x_0,R))} \geq \varepsilon_0 \tag{8}$$

for any $R > 0$.

(3) If we can replace $\limsup_{t\uparrow T_{\max}}$ by $\liminf_{t\uparrow T_{\max}}$ in (8), then we obtain $\sharp S < +\infty$, because of the total mass conservation (6). In this case, any blowup point becomes isolated, and the formation of collapse, (2), will be proven with $m(x_0) \geq m_*(x_0)$.

(4) The above replacement is justified by the weak formulation of the problem,

$$\begin{aligned} &\frac{d}{dt}\int_\Omega u(\cdot, t)\varphi = \int_\Omega u(\cdot, t)\Delta\varphi \\ &+\frac{1}{2}\int\int_{\Omega\times\Omega} \rho_\varphi(x, x')u(x, t)u(x', t)dxdx, \end{aligned} \tag{9}$$

obtained by the method of symmetrization, where

$$\varphi \in C^2(\overline{\Omega}), \qquad \left.\frac{\partial \varphi}{\partial \nu}\right|_{\partial\Omega} = 0$$

and

$$\rho_\varphi(x,x') \equiv \nabla\varphi(x)\cdot\nabla_x G(x,x') + \nabla\varphi(x')\cdot\nabla_{x'}G(x,x') \in L^\infty(\Omega\times\Omega).$$

(5) In more detail, we obtain

$$\left|\frac{d}{dt}\int_\Omega u(\cdot,t)\varphi\right| \leq C_\varphi(\lambda+\lambda^2) \tag{10}$$

from (9), and therefore,

$$\lim_{t\uparrow T}\int_\Omega u(\cdot,t)\varphi$$

exists for such φ. Using this, we can replace (8) by

$$x_0 \in S \Rightarrow \liminf_{t\uparrow T_{\max}} \|u(\cdot,t)\|_{L^1(\Omega\cap B(x_0,R))} \geq \varepsilon_0$$

for any $R > 0$.

Mass quantization, on the other hand, is a "local" blowup criterion, while the global blowup criterion also follows from the weak formulation (9), using the second moment. A plot-type argument is given by Biler-Hilhorst-Nadzieja [3], and we obtain the following [26, 31].

Theorem 4. *There is an absolute constant $\eta > 0$ such that if we have $x_0 \in \partial\Omega$ and $0 < R \ll 1$ satisfying*

$$\frac{1}{R^2}\int_{\Omega\cap B(x_0,2R)} |x-x_0|^2 u_0(x)dx < \eta$$

$$\int_{\Omega\cap B(x_0,R)} u_0(x)dx > 4\pi,$$

then $T_{\max} < +\infty$. More precisely, it holds that $T_{\max} = o(R^2)$ as $R \downarrow 0$. The same conclusion follows if $x_0 \in \Omega$ and

$$\frac{1}{R^2}\int_{\Omega\cap B(x_0,2R)} |x-x_0|^2 u_0(x)dx < \eta$$

$$\int_{\Omega\cap B(x_0,R)} u_0(x)dx > 8\pi.$$

Based on the above theorem, we can infer as follows.

(1) The argument employed in the proof of Theorem 4 is valid to the *weak solution.* In particular, formation of the over-quantized collapse in finite time,

$$m(x_0) > m_*(x_0)$$

in (2), means the complete blowup of the solution. In other words, weak post-blow up continuation of this $u = u(x,t)$ assures mass quantization,

$$m(x_0) = m_*(x_0).$$

(2) The Fokker-Planck-Poisson equation, on the other hand, admits a weak solution globally in time for appropriate initial values [34, 4, 24], and therefore, we use the rescaled vaiables to complete the proof of Theorem 1, regarding the hierarchy of the mean field of particles.

§4. Rescaling

If $T = T_{\max} < +\infty$ and $x_0 \in S$, then

$$y = (x - x_0)/(T-t)^{1/2}, \quad s = -\log(T-t), \quad t < T$$

is the standard backward self-similar variables. The formal blowup rate, on the other hand, is $(T-t)^{-1/(p-1)}$ if the nonlinearity is of degree p, and $p = 2$ in this system of chemotaxis (1). Thus, we define

$$z(y,s) = (T-t)u(x,t), \qquad w(y,s) = v(x,t)$$

and obtain

$$\begin{aligned} & z_s = \nabla \cdot (\nabla z - z\nabla w - yz/2) \\ & 0 = \Delta w + z - e^{-s}\lambda/|\Omega| \qquad \text{in } \textstyle\bigcup_{s>-\log T} e^{s/2} \times (\Omega - \{x_0\}) \times \{s\} \\ & \frac{\partial z}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0 \qquad \text{on } \textstyle\bigcup_{s>-\log T} e^{s/2} (\partial\Omega - \{x_0\}) \times \{s\}. \end{aligned} \tag{11}$$

Here, we use the following ingredients for the proof of Theorem 2.

(1) parabolic envelope.
(2) generation of the weak solution.
(3) (reverse) second moment.
(4) forward self-similar transformation.

Parabolic envelope

(1) Taking a nice cut-off function around $x_0 \in S$ with the support radius $2R > 0$, denoted by $\varphi_{x_0,R}$, we can refine (10) as

$$\left| \frac{d}{dt} \int_\Omega u(\cdot,t)\varphi_{x_0,R} \right| \leq C(\lambda + \lambda^2)R^{-2}$$

with a constant $C > 0$ independent of $0 < R < 1$. This implies

$$\begin{aligned} &|\langle \varphi_{x_0,R}, \mu(\cdot,T)\rangle - \langle \varphi_{x_0,R}, \mu(\cdot,t)\rangle| \\ &\quad \leq C(\lambda+\lambda^2)R^{-2}(T-t) \end{aligned} \tag{12}$$

because $u(x,t)dx = \mu(dx,t)$ is regarded as a $*$-weakly continuous function on $[0,T]$, i.e.,

$$\mu(dx,t) \in C_*([0,T], M(\overline{\Omega})).$$

(2) Since $0 < R < 1$ is arbitrary in (12), we can put

$$R = bR(t)$$

for given $b > 0$, provided that $0 < R(t) \equiv (T-t)^{1/2} < b^{-1}$, that is,

$$\left| \left\langle \varphi_{x_0,bR(t)}, \mu(\cdot,T) \right\rangle - \left\langle \varphi_{x_0,bR(t)}, \mu(\cdot,t) \right\rangle \right| \leq Cb^{-2}.$$

This implies

$$\limsup_{t\uparrow T} \left| m(x_0) - \left\langle \varphi_{x_0,bR(t)}, \mu(\cdot,t) \right\rangle \right| \leq Cb^{-2}$$

for $x_0 \in S$ by

$$\mu(dx,T) = \sum_{x_0 \in S} m(x_0)\delta_{x_0}(dx) + f(x)dx,$$

and therefore,

$$\lim_{b\uparrow+\infty} \limsup_{t\uparrow T} \left| \left\langle \varphi_{x_0,bR(t)}, \mu(\cdot,t) \right\rangle - m(x_0) \right| = 0. \tag{13}$$

Relation (13) indicates that infinitely wide parabolic region concerning the backward self-similar variables, called *parabolic enevelope*, contains the whole blowup mechanism.

Generation of the weak solution

(1) Given $s_k \uparrow +\infty$, we have $\{s'_k\} \subset \{s_k\}$ and $\zeta(dy,s)$ such that

$$z(y,s+s'_k)dy \rightharpoonup \zeta(dy,s)$$

in $C_*(-\infty,+\infty;M(\mathbf{R}^2))$, taking 0-extension to $z(y,s)$ where it is not defined. This $\zeta = \zeta(dy,s)$ satisfies supp $\zeta(\cdot,s) \subset \overline{L}$, and is a weak solution to

$$z_s = \nabla\cdot\left(\nabla z - z\nabla\left(w + |y|^2/4\right)\right) \qquad \text{in } L\times(-\infty,+\infty)$$
$$\left.\frac{\partial z}{\partial\nu}\right|_{\partial L} = 0$$
$$\nabla w(y,s) = \int_L \nabla\Gamma(y-y')z(y',s)dy',$$

where L is $\mathbf{R}^2$ and a half space with ∂L parallel to the tangent line of $\partial\Omega$ at x_0 if $x_0 \in \Omega$ and $x_0 \in \partial\Omega$, respectively, and

$$\Gamma(y) = \frac{1}{2\pi}\log\frac{1}{|y|}.$$

(2) Using the even extension to $\zeta(dy,s)$ if $x_0 \in \partial\Omega$, all the above cases are reduced to $L = \mathbf{R}^2$:

$$z_s = \nabla\cdot\left(\nabla z - z\nabla\left(w + |y|^2/4\right)\right)$$
$$\nabla w(y,s) = \int_{\mathbf{R}^2} \nabla\Gamma(y-y')z(y',s)dy'$$
$$\text{in } \mathbf{R}^2\times(-\infty,+\infty). \tag{14}$$

Thus, we obtain a full-orbit weak solution

$$\zeta = \zeta(dy,s) \in C_*(-\infty,+\infty;M(\mathbf{R}^2))$$

to (14). This $\zeta(dy,s)$ is a Radon measure on $\mathbf{R}^2$ for each $s \in (-\infty,+\infty)$, satisfying

$$\zeta(\mathbf{R}^2,s) = \begin{cases} m(x_0) & (x_0 \in \Omega) \\ 2m(x_0) & (x_0 \in \partial\Omega) \end{cases} \tag{15}$$

for each $s \in (-\infty,+\infty)$, from the parabolic envelope.

(3) One point compactification of $\mathbf{R}^2$, denoted by $\overline{\mathbf{R}}^2$, is identified with the sphere $S^2 = \left\{(x_1,x_2,x_3) \in \mathbf{R}^3 \mid x_1^2+x_2^2+x_3^2 = 1\right\}$ by the stereographic projection $y_i = x_i/(1-x_3)$ $(i=1,2)$, and then the weak solution to (14) is defined as follows. This definition is slightly simpler than the original [31], but is sufficient for later arguments using second moment.

(a) There is $0 \leq \nu = \nu(s) \in L^\infty_*(-\infty, +\infty; E')$ satisfying

$$\nu(s)|_E = \zeta \otimes \zeta(dydy', s), \qquad \text{a.e. } s \in \mathbf{R},$$

where

$$E = \left[\{\rho^0_\varphi \mid \varphi \in C^2_0(\mathbf{R}^2) \oplus \mathbf{R}\}\right]^{L^\infty(\overline{\mathbf{R}}^2 \times \overline{\mathbf{R}}^2)} \subset C(\overline{\mathbf{R}}^2 \times \overline{\mathbf{R}}^2)$$
$$\rho^0_\varphi(y, y') = (\nabla\varphi(y) - \nabla\varphi(y')) \cdot \nabla\Gamma(y - y').$$

(b) For each $\varphi \in C^2_0(\mathbf{R}^2) \oplus \mathbf{R}$, the mapping

$$s \in \mathbf{R} \mapsto \langle \varphi, \zeta(dy, s) \rangle$$

is locally absolutely continuous, and it holds that

$$\frac{d}{ds} \langle \varphi, \zeta(\cdot, s) \rangle = \langle \Delta\varphi + y \cdot \nabla\varphi/2, \zeta(\cdot, s) \rangle + \frac{1}{2} \left\langle \rho^0_\varphi, \nu(s) \right\rangle_{E,E'},$$

a.e. $s \in \mathbf{R}$.

(4) From (15) and $m(x_0) \geq m_*(x_0)$, we have only to derive

$$\zeta(\mathbf{R}^2, 0) \leq 8\pi \tag{16}$$

to complete the proof of Theorem 2. For this purpose, we note that Kurokiba-Ogawa [18] obtained the non-existence of weighted L^2 solution globally in time for the pre-scaled system in the whole space, if the initial mass is greater than 8π. Using several technical tools, we follow their arguments in the following way.

Second moment - Forward self-similarity

(1) Since the (rescaled) weak solution $\zeta = \zeta(dy, s)$ is global in time, method of the second moment assures that a sufficient concentration at the origin implies (16). For instance, if we use smooth $c = c(s)$ satisfying

$$\begin{aligned} &0 \leq c'(s) \leq 1 \qquad (s \geq 0) \\ &-1 \leq c(s) \leq 0 \qquad (s \geq 0) \\ &c(s) = \begin{cases} s - 1 & (0 \leq s \leq 1/4) \\ 0 & (s \geq 4), \end{cases} \end{aligned}$$

then we can deduce

$$\frac{d}{ds} \left\langle c(|y|^2) + 1, \zeta(dy, s) \right\rangle \leq C \left\langle c(|y|^2) + 1, \zeta(dy, s) \right\rangle$$
$$+ C^{-1} \hat{m}(x_0) \left\{ 4 - \frac{\hat{m}(x_0)}{2\pi} \right\} \qquad \text{a.e. } s \in \mathbf{R}$$

with a constant $C \geq 1$, assuming $\hat{m}(x_0) \equiv \zeta(\mathbf{R}^2, 0) > 8\pi$. This implies that each $\varepsilon > 0$ admits $\delta > 0$ such that

$$
\begin{aligned}
& \hat{m}(x_0) = \zeta(\mathbf{R}^2, 0) \geq 8\pi + \varepsilon \\
& \left\langle c(|y|^2) + 1, \zeta(dy, 0) \right\rangle < \delta \\
\Rightarrow \quad & \left\langle c(|y|^2) + 1, \zeta(dy, s) \right\rangle < 0
\end{aligned}
$$

for $s \gg 1$, a contradiction, and therefore,

$$
\left\langle c(|y|^2) + 1, \zeta(dy, 0) \right\rangle < \delta \tag{17}
$$

$$
\Rightarrow \quad \zeta(\mathbf{R}^2, 0) < 8\pi + \varepsilon. \tag{18}
$$

(2) Problem (14) possesses the translation invariance in s and also a forward self-similarity. By these transformations, any initial data "looks like" concentrated at the origin. In other words, condition (17) is moved to obtain (18). This means (16), because $\varepsilon > 0$ is arbitraly.
(3) The above mentioned forward self-similarity of (14) is natural, because this equation is obtained through the following process:
  (a) Equation (1) with (potentially) forward self-similarity.
  (b) Backward self-similarly transformed equation (11).
  (c) Weak limit as $s'_k \uparrow +\infty$ defined on the whole space-time (14).
(4) Writing (14) as

$$
\begin{aligned}
& \tilde{z}_{\tilde{s}} - \tilde{\nabla} \cdot (\tilde{y}\tilde{z}/\tilde{s}) = \tilde{\nabla} \cdot \left(\tilde{\nabla}\tilde{z} - \tilde{z}\tilde{\nabla}\tilde{w} \right) \\
& \tilde{\nabla}\tilde{w}(\tilde{y}, \tilde{s}) = \int_{\mathbf{R}^2} \tilde{\nabla}\Gamma(\tilde{y} - \tilde{y}')\tilde{z}(\tilde{y}', \tilde{s}) d\tilde{y}'
\end{aligned} \tag{19}
$$

by

$$
\begin{aligned}
& \tilde{y} = e^{-s/2} y, \quad \tilde{s} = -e^{-s} \\
& \tilde{z}(\tilde{y}, \tilde{s}) = z(y, s), \quad \tilde{w}(\tilde{y}, \tilde{s}) = w(y, s)
\end{aligned}
$$

may be useful to detect actual self-similarity. In fact, (19) is invariant under

$$
\begin{aligned}
& \tilde{z}_\mu(\tilde{y}, \tilde{s}) = \mu^2 \tilde{z}(\mu\tilde{y}, \mu^2\tilde{s}) \\
& \tilde{w}_\mu(\tilde{y}, \tilde{s}) = \tilde{w}(\mu\tilde{y}, \mu^2\tilde{s}),
\end{aligned}
$$

and hence the forward self-similar transformation to (14) is defined by

$$\begin{aligned}
&z(y,s) = e^{-s}A(y',s'), \quad w(y,s) = B(y',s') \\
&y' = e^{-s/2}y, \quad s' = -e^{-s} \\
&z^{\mu}(y,s) = e^{-s}A_{\mu}(y',s'), \quad w^{\mu}(y,s) = B_{\mu}(y',s') \\
&A_{\mu}(y',s') = \mu^2 A(\mu y', \mu^2 s'), \quad B_{\mu}(y',s') = B(\mu y', \mu^2 s'),
\end{aligned}$$

where $\mu > 0$ is a constant. Then, from the above result, we obtain $\eta > 0$ in case of $m(x_0) > m_*(x_0)$ such that

$$\left\langle c(|y|^2) + 1, z^{\mu}(dy,s) \right\rangle \geq \eta$$

for any $\mu > 0$ and $s \in \mathbf{R}$. This implies

$$\begin{aligned}
&\left\langle c\left((-s')^{-1}|y'|^2\right) + 1, A_{\mu}(dy',s') \right\rangle \geq \eta \\
&\left\langle c\left((-s')^{-1}\mu^{-2}|y'|^2\right) + 1, A(dy',s') \right\rangle \geq \eta \\
&\left\langle c\left(\mu^{-2}|y|^2\right) + 1, \zeta(dy,s) \right\rangle \geq \eta
\end{aligned}$$

in turn, a contradiction by $\mu \uparrow +\infty$.

(5) The above argument [31] can be replaced by using

$$\begin{aligned}
&z(y,s) = e^{-s}A(y',s'), \quad w(y,s) = B(y',s') \\
&y' = e^{-s/2}y, \quad s' = -e^{-s}
\end{aligned}$$

directly, which transforms (14) to the system studied by [18],

$$\begin{aligned}
&A_{s'} = \nabla' \cdot (\nabla' A - A \nabla' B) \\
&\nabla' B(\cdot, s') = \int_{\mathbf{R}^2} \nabla \Gamma(\cdot - y') A(y',s') dy' \qquad \text{in } \mathbf{R}^2 \times (-\infty, 0).
\end{aligned}$$

Then, $\zeta(\mathbf{R}^2, 0) \leq 8\pi$ is obtained similarly, using the second moment and the forward self-similar transformation.

§5. Free energy transmission

Fundamental concepts for the proof of Theorems 1 and 2 are thus, localization, symmetrization, and scaling. Classification of the *blowup rate*, on the other hand, is another issue.

Type (I) blowup point

(1) We say that $x_0 \in S$ is of type (I) if

$$\limsup_{t\uparrow T} \sup_{x\in\Omega\cap B(x_0,bR(t))} R(t)^2 u(x,t) < +\infty$$

for any $b > 0$, where $R(t) = (T-t)^{1/2}$. In this case, we obtain a classical solution $z = z(y,s) \geq 0$ to (14), satisfying $\|z(\cdot,s)\|_1 = 8\pi$ for all $s \in \mathbf{R}$.

This type of blowup point has an interesting feature, maybe called *emergence*. In more detail, the local free energy diverses to $+\infty$ around it, i.e.,

$$\lim_{t\uparrow T} F_{x_0,bR(t)}(u(\cdot,t)) = +\infty$$

for any $b > 0$, where

$$F_{x_0,R}(u) = \int_\Omega u(\log u - 1)\varphi_{x_0,R}$$
$$-\frac{1}{2}\int\int_{\Omega\times\Omega} G(x,x')(u\cdot\varphi_{x_0,R})(x)(u\cdot\varphi_{x_0,R})(x')dxdx'.$$

It also holds that

$$\limsup_{t\uparrow T} \int_\Omega \varphi_{x_0,bR(t)} u(\cdot,t) < m_*(x_0)$$

for any $b > 0$.

(2) Such an orbit, however, may not exist, and actually, type (I) blowup point has not been known so far. In this context, the recent work [10, 9] concerning the Navier-Stokes equation defined on the whole space $\mathbf{R}^2$ is worth mentioning. It guarantees the (forward) self-similaity gain of the vorticity as $t \uparrow +\infty$, using entropy decreasing, total vorticity conservation, compactness of the (forward) scaled semi-orbit, and the uniqueness of its stationary state (i.e., self-similar solution) with the prescribed total vorticity.

(3) Problem (14) has a similar structure to the above mentioned vorticity equation. First, it is formally provided with the Lyapunov function,

$$H(z) = \int_{\mathbf{R}^2} z\log(z/G)$$
$$-\frac{1}{2}\int\int_{\mathbf{R}^2\times\mathbf{R}^2} \Gamma(y-y')z(y,s)z(y',s)dydy', \tag{20}$$

where $G(y) = \exp(|y|^2/4)$. Second, the classical w stationary state of (14) is defined by

$$-\Delta w = 8\pi \frac{e^{w+|y|^2/4}}{\int_{\mathbf{R}^2} e^{w+|y|^2/4}},$$

or equivalently,

$$(21) \qquad -\Delta\varphi = e^{\varphi} - 1 \quad \text{in } \mathbf{R}^2, \qquad \int_{\mathbf{R}^2} e^{\varphi} = 8\pi,$$

in terms of

$$\varphi = w + |y|^2/4 + \log 8\pi - \int_{\mathbf{R}^2} e^{w+|y|^2/4}.$$

Here, it is easy to see that (21) admits no bounded radially symmetric solution [12, 13].

Finally, if $x(t)$ denotes the local maximizer of $u(\cdot,t)$ converging to $x_0 \in S$, then the condition

$$|x(t) - x_0| = O(R(t))$$

implies

$$\sup_{x \in \Omega \cap B(x_0, bR(t))} R(t)^2 u(x,t) \le C, \qquad (0 \le t < T)$$

with a constant $C > 0$ independent of $b \gg 1$, and therefore, it holds that

$$\sup_{s \in \mathbf{R}} \|z(\cdot,s)\|_{\infty} < +\infty.$$

This conclusion may assure the compactness of the orbit.

(4) Still, we have several obstructions to confirm the non-existence of the tpye (I) blowup point, using above mentioned properties of (14). Actually, justification of (20) itself is not obvious.

In fact, convergence of the first term is equivalent to

$$\lim_{b \uparrow +\infty} \limsup_{t \uparrow T} \int_{\Omega \cap B(x_0, bR(t))} u(x,t) \left| \log \frac{T-t}{e^{|x-x_0|^2/4(T-t)}} \right| dx < +\infty,$$

which follows from

$$\lim_{b \uparrow +\infty} \lim_{t \uparrow T} \log \frac{1}{T-t} \cdot \int_{\Omega \cap B(x_0, bR(t))} u(x,t) dx < +\infty.$$

Unfortunately, this is impossible by $\|z(\cdot,0)\|_{L^1(B(0,b))} > 0$.

The second term has also an obstruction, caused by

$$\int_{\mathbf{R}^2} z(y,s)dy = 8\pi.$$

In this connection, we recall that if this value is zero, then it is regarded as a Hardy-BMO paring [6], or simply

$$\int_{\mathbf{R}^2} \frac{|\hat{z}(\xi,s)|^2}{|\xi|^2} d\xi,$$

using the Fourier transform $\hat{z}$ of z.

Type (II) blowup point

(1) We say that $x_0 \in S$ is of type (II) if it is not of type (I), i.e, if there is $t_k \uparrow T$ and $b > 0$ such that

$$\lim_{k\to\infty} \sup_{x\in\Omega\cap B(x_0,bR(t_k))} R(t_k)^2 u(x,t_k) = +\infty.$$

In this case, the associated limiting rescaled measure $\zeta(dy,0)$, formed from $\{s'_k\} \subset \{s_k\}$ for $s_k = -\log(T-t_k)$ has non-vanishing singular part, and then its regular part vanishes by $\zeta(\mathbf{R}^2, 0) = 8\pi$.

More precisely, since mass quantization of the collapse formed in infinite time holds to the rescaled equation [28], the singular part of $\zeta(dy,0)$, denoted by $\zeta_s(dy,0)$ is composed of delta functions with the quantized mass. This implies

$$\zeta(dy,0) = 8\pi\delta_{y(0)}(dy)$$

with some $y(0) \in \mathbf{R}^2$.

Then, the parabolic unique continuation theorem guarantees the vanishing of the regular part of $\zeta(dy,s)$ for any s, and thus, we obtain

$$\zeta(dy,s) = 8\pi\delta_{y(s)}(dy), \qquad -\infty < s < +\infty$$

with

$$s \in (-\infty,+\infty) \mapsto y(s) \in \mathbf{R}^2$$

locally absolutely continuous.

Next, this $y(s)$ is shown to be identically 0. In fact, otherwise the gradient factor $|y|^2/4$ of (14) attracts the particle far away, and then we obtain a contradiction to the parabolic envelope. This implies

$$z(y, s+s_k)dy \rightharpoonup m_*(x_0)\delta_0(dy) \tag{22}$$

in $C_*((-\infty, +\infty); M(\mathbf{R}^2))$ with the right-hand side called a *sub-collapse.*

In other words, around this type of blowup point, the whole blowup mechanism is contained in infinitely small parabolic region, called *hyper-parabola*, and it holds that

$$\lim_{k\to\infty} \int_\Omega \varphi_{x_0, bR(t_k)} u(\cdot, t_k) = m_*(x_0)$$

for any $b > 0$.

(2) The above mentioned formation of sub-collapse suggests the other scaling $r(t) \ll R(t)$ to describe the real blowup envelope for type (II) blowup point.

Herrero-Velázquez' solution [14] is such an example. It is a radially symmetric solution to (1) satisfying

$$u(x,t) = \frac{1}{r(t)^2}\overline{u}(x/r(t)) \cdot \left(1 + o(1) + O\left(\frac{e^{-\sqrt{2}|\log(T-t)|^{1/2}}}{|x|^2} \cdot 1_{x \geq r(t)}\right)\right)$$

as $t \uparrow T = T_{\max} < +\infty$ uniformly in $x \in B(0, bR(t))$ for any $b > 0$, where

$$r(t) = (T-t)^{1/2} e^{-\sqrt{2}|\log(T-t)|^{1/2}} \cdot |\log(T-t)|^{\frac{1}{4}\left(|\log(T-t)|^{-1/2} - 1\right)} (1 + o(1)) \ll R(t)$$

and

$$\overline{u}(y) = \frac{8}{\left(1 + |y|^2\right)^2}$$

are the blowup rate and the *entire stationary state*, respectively.

Still we have the profile of emergence concerning this blowup envelope,

$$\lim_{t \uparrow T} F_{0, br(t)}(u(\cdot, t)) = +\infty$$

for any $b > 0$. Thus, we can summarize as follows [31]:

"Mass and entropy are exchanged at the wedge of the blowup envelope, creating a clean, quantized self."

(3) Formation of the quantized collapse is observed also in the harmonic heat flow. If we take the flat torus $\Omega = \mathbf{R}^2/a\mathbf{Z} \times b\mathbf{Z}$ $(a, b > 0)$ and the $(n-1)$-dimensional sphere $S^{n-1} = \{x \in \mathbf{R}^n \mid |x| = 1\}$ as the domain and the target, respectively, for simplicity, then this flow is described by

$$u = u(x,t) : \Omega \times [0,T) \to S^{n-1} \subset \mathbf{R}^n$$

satisfying

$$u_t - \Delta u = u \left|\nabla u\right|^2, \qquad |u| = 1.$$

In this case, it holds that

$$\frac{1}{2}\frac{d}{dt} \|\nabla u\|_2^2 = - \|u_t\|_2^2 \leq 0$$

and if $T = T_{\max} < +\infty$, then we obtain the formation of collapse [30]

$$\begin{aligned} &\left|\nabla u(x,t)\right|^2 dx \rightharpoonup \mu(dx) \qquad \text{in } M(\Omega) \\ &\mu(dx) = \sum_{x_0 \in S} m(x_0)\delta_{x_0}(dx) + f(x)dx \end{aligned}$$

as $t \uparrow T$, where $0 \leq f = f(x) \in L^1(\Omega)$ and

$$\begin{aligned} S = \{x_0 \in \Omega \mid &\text{ there exist } t_k \uparrow T \text{ and } x_k \to x_0 \\ &\text{such that } \left|\nabla u(x_k, t_k)\right|^2 \to +\infty\Big\}. \end{aligned}$$

Mass quantization is also true [33], and this $m(x_0) > 0$ is the total energy of a non-constant stationary harmonic map defined on the sphere, i.e.,

$$\begin{aligned} &m(x_0) = \|\nabla \omega\|_2^2 \\ &\omega = \omega_{x_0}(x) : S^2 \to S^{n-1} \subset \mathbf{R}^{n-1} \\ &-\Delta \omega = \omega \left|\nabla \omega\right|^2, \qquad |\omega| = 1. \end{aligned}$$

In this harmonic heat flow, however, any blowup point is type (II). More precisely, we have $t_k \uparrow T$ satisfying $(T - t_k) \|u_t(t_k)\|_2^2 \to 0$ by

$$\int_0^T \|u_t(s)\|_2^2 \, ds \leq \frac{1}{2} \|\nabla u_0\|_2^2,$$

and then we obtain the formation of sub-collapse,

$$\left|\nabla_y z(y, s_k)\right|^2 dy \rightharpoonup m(x_0)\delta_0(dy) \qquad \text{in } M(\mathbf{R}^2)$$

for $s_k = -\log(T - t_k) \uparrow +\infty$, where

$$z(y, s) = u(x, t), \quad y = (x - x_0)/(T - t)^{1/2}, \quad s = -\log(T - t).$$

(4) Above mentioned property of the harmonic heat flow is derived from the non-negativity of the Lyapunov function,

$$E = -\frac{1}{2}\left\|\nabla u\right\|_2^2.$$

Thus, a similarl fact holds for (1), if the free energy is bounded.

Theorem 5. *If $T = T_{\max} < +\infty$ and the total free energy is bounded in (1), then any blowup point is of type (II). More precisely, it holds that*

$$z(y, s + s')dy \rightharpoonup m_*(x_0)\delta_0(dy) \qquad \text{in } C_*(-\infty, +\infty; M(\mathbf{R}^2)) \tag{23}$$

as $s' \uparrow +\infty$.

In fact, we have

$$\frac{d}{dt}\int_\Omega \varphi_{x_0,R} u = -\int_\Omega u\nabla\delta F(u) \cdot \nabla\varphi_{x_0,R}$$

and therefore,

$$\begin{aligned}\left|\frac{d}{dt}\int_\Omega \varphi_{x_0,R} u\right| &\leq \left\|u^{1/2}\nabla\delta F(u)\right\|_2 \cdot \left\|\nabla\varphi_{x_0,R}\right\|_\infty \cdot \left\|u\right\|_1 \\ &\leq C\lambda R^{-1}\left\|u^{1/2}\nabla\delta F(u)\right\|_2.\end{aligned}$$

If

$$\lim_{t\uparrow T} F(u(\cdot, t)) > -\infty, \tag{24}$$

on the other hand, we obtain

$$\int_0^T \left\|u^{1/2}\nabla\delta F(u)(\cdot, t)\right\|_2^2 dt < +\infty$$

by (5), and therefore, we can replace (12) by

$$\begin{aligned}
&|\langle \varphi_{x_0,R}, \mu(dx,t)\rangle - \langle \varphi_{x_0,R}, \mu(dx,T)\rangle| \\
&\leq C\lambda R^{-1} \int_t^T \left\| u^{1/2}\nabla\delta F(u)(\cdot,s)\right\|_2 ds \\
&\leq C\lambda R^{-1}(T-t)^{1/2} \left\{ \int_t^T \left\| u^{1/2}\nabla\delta F(u)(\cdot,s)\right\|_2^2 ds \right\}^{1/2},
\end{aligned}$$

which guarantees

$$\left\langle \varphi_{x_0,bR(t)}, \mu(dx,t)\right\rangle = m_*(x_0) + o(1)$$

as $t \uparrow T$, where $b > 0$ is arbitrary. Then, any $s_k \uparrow +\infty$ admits $\{s'_k\} \subset \{s_k\}$ such that

$$z(y, s + s'_k)dy \rightharpoonup m_*(x_0)\delta_0(dy)$$

in $C_*(-\infty, +\infty; M(\mathbf{R}^2))$ for each $x_0 \in S$ from the previous arguments. This implies (23), and in particular, any $x_0 \in S$ is type (II).

We can confirm that Herrero-Velázquez' solution has the bounded total free enery.

(5) From the proof of the above theorem, on the other hand, $x_0 \in S$ satisfies (23), if, more weakly,

$$\int_0^T dt \cdot \int_\Omega \left[\varphi_{x_0,2bR(t)} - \varphi_{x_0,bR(t)}\right] \cdot u\left|\nabla(\log u - v)\right|^2(\cdot,t) < +\infty \tag{25}$$

for any $b > 0$. Note that (25) does not mean the boundedness of the local free energy,

$$\liminf_{t\uparrow T} F_{x_0,bR(t)}(u(\cdot,t)) > -\infty.$$

(6) If $x_0 \in S$ is type (I), then it holds that

$$\int_\Omega \left[\varphi_{x_0,2bR(t)} - \varphi_{x_0,bR(t)}\right] \cdot u\left|\nabla(\log u - v)\right|^2(\cdot,t) \sim (T-t)^{-1}$$

as $t \uparrow T$, and therefore, (25) is impossible. In other words, we can infer that $x_0 \in S$ is type (II) by (25), only from the classification of the blowup rate.

We obtain, however, the stronger conclusion (23), using the above argument. Determing the blowup rate and the control of the blowup mechanism using the stationary solution on the whole space may be possible under (25).

§6. Time relaxization

In this section, we review briefly the study on the full system of chemotaxis.

Its typical form is given by

$$
\begin{aligned}
& u_t = \nabla \cdot (\nabla u - u \nabla v) \\
& \tau v_t = \Delta v + u - \frac{1}{|\Omega|}\int_\Omega u \qquad \text{in } \Omega \times (0,T) \\
& \frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \qquad \text{on } \partial\Omega \times (0,T) \\
& \int_\Omega v = 0 \qquad (0 < t < T),
\end{aligned} \tag{26}
$$

associated with the relaxization time $\tau > 0$ in the left-hand side of the second equation. Thus, it describes a chemical process in the formation of the field.

(1) An important mathematical structure is the existence of the Lagrangian,

$$
L(u,v) = \int_\Omega u(\log u - 1) + \frac{1}{2}\|\nabla v\|_2^2 - \int_\Omega vu
$$

defined for

$$
\begin{aligned}
& u \geq 0, \quad \|u\|_1 = \lambda \\
& \int_\Omega v = 0,
\end{aligned}
$$

and (26) is equivalent to

$$
\begin{aligned}
& u_t = \nabla \cdot (u \nabla L_u(u,v)) \\
& \tau v_t = -L_v(u,v) \qquad \text{in } \Omega \times (0,T) \\
& u\frac{\partial}{\partial \nu} L_u(u,v) = 0 \qquad \text{on } \partial\Omega \times (0,T).
\end{aligned}
$$

This form guarantees the *dual variation*, and the linearly stable statioinary solution is dynamically stable [32].

(2) Well-posedness local in time and the blowup criterion are valid to this system. If $T = T_{\max} < +\infty$, then the blowup set of u, denoted by S, is not empty. Then, we have the global blowup criterion that $\lambda = \|u_0\|_1 < 4\pi$ implies $T_{\max} = +\infty$, and $\lambda < 8\pi$ is sufficient for radially symmetric case [21, 2, 8].

Local blowup criterion also holds, and we obtain

$$\limsup_{t\uparrow T} \|u(\cdot,t)\|_{L^1(\Omega\cap B(x_0,R))} \geq m_*(x_0)$$

for each $x_0 \in S$, where $R > 0$ is arbitrary [22].

However, any blowup criterion is not known.

(3) If we replace u_t by εu_t in (26) and then put $\varepsilon = 0$, then we obtain another simplified system of chemotaxis,

$$\tau v_t = \Delta v + \lambda \left(\frac{e^v}{\int_\Omega e^v} - \frac{1}{|\Omega|} \right), \quad \left.\frac{\partial v}{\partial \nu}\right|_{\partial\Omega} = 0, \quad \int_\Omega v = 0 \tag{27}$$

introduced by Wolansky [37, 38]. This system is provided with the Lyapunov function

$$J_\lambda(v) = \frac{1}{2}\|\nabla v\|_2^2 - \lambda \log \int_\Omega e^v, \qquad \int_\Omega v = 0,$$

and the formation of collapse is proven with $m(x_0) \geq m_*(x_0)$ if this function is bounded in time [32].

References

[1] F. Bavaud, Equilibrium properties of the Vlasov functional: the generalized Poisson-Boltzmann-Emden equation, Rev. Mod. Phys., **63** (1991), 129–149.

[2] P. Biler, Local and global solvability of some parabolic systems modelling chemotaxis, Adv. Math. Sci. Appl., **8** (1998), 715–743.

[3] P. Biler, D. Hilhorst and T. Nadzieja, Existence and nonexistence of solutions for a model of graviational interaction of particles, II, Colloq. Math., **67** (1994), 297–308.

[4] F. Bouchut, Existence and uniqueness of a global smooth solution for the Vlasov-Poisson-Fokker-Planck system in three dimensions, J. Funct. Anal., **111** (1993), 239–258.

[5] S. Childress and J. K. Percus, Nonlinear aspects of chemotaxis, Math. Biosci., **56** (1981), 217–237.

[6] R. Coifman, P. L. Lions, Y. Meyer and S. Semmes, Compensated compactness and Hardy sapces, J. Math. Pure Appl., **72** (1993), 247–286.

[7] P. C. Fife, Models for phase separation and their mathematics, Elec. J. Differential Equaitons, **2000** (2000), 1–26.

[8] H. Gajewski and K. Zacharias, Global behaviour of a reaction-diffusion system modelling chemotaxis, Math. Nachr., **195** (1998), 77–114.

[9] I. Gallagher and T. Gallay, Uniqueness for the two-dimensional Navier-Stokes equation with a measure as initial vortiticy, Math. Ann., **332** (2005), 287–327.

[10] T. Gallay and C. E. Wayne, Global stability of vortex solutions of the two-deimensional Navier-Stokes equation, Comm. Math. Phys., **255** (2005), 97–129.

[11] B. I. Halperin, P. C. Hohenberg and S.-k. Ma, Renormalization-group methods for critical dynamics: I, Recursion relations and effects of energy conservation, Phys. Rev. B, **10** (1974), 139–153.

[12] M. A. Herrero, E. Medina and J. J. L. Velázquez, Self-similar blow-up for a reaction-diffusion system, J. Comp. Anal. Math., **97** (1998), 99–119.

[13] M. A. Herrero, E. Medina and J. J. L. Velázquez, Finite-time aggregation into a single point in a reaction-diffusion system, Nonlinearity, **10** (1997), 1739–1754.

[14] M. A. Herrero and J. J. L. Velázquez, Singularity patterns in a chemotaxis model, Math. Ann., **306** (1996), 583–623.

[15] P. C. Hohenberg and B. I. Halperin, Theory of dynamic critical phenomena, Rev. Mod. Phys., **49** (1977), 435–479.

[16] W. Jäger and S. Luckhaus, On explosions of solutions to a system of partial differential equations modelling chemotaxis, Trans. Amer. Math. Soc., **329** (1992), 819–824.

[17] E. F. Keller and L. A. Segel, Initiation of slime mold aggregation viewed as an instability, J. Theor. Biol., **26** (1970), 399–415.

[18] M. Kurokiba and T. Ogawa, Finite time blow-up of the solution for a nonlinear parabolic equation of drift-diffusion type, Differential and Integral Equations, **16** (2003), 427–452.

[19] T. Nagai, Blow-up of radially symmetric solutions to a chemotaxis system, Adv. Math. Sci. Appl., **5** (1995), 581–601.

[20] T. Nagai, Blowup of nonradial solutions to parabolic-elliptic systems modelling chemotaxis in two-dimensional domains, J. Inequality and Applications, **6** (2001), 37–55.

[21] T. Nagai, T. Senba and K. Yoshida, Application of the Trudinger-Moser inequality to a parabolic system of chemotaxis, Funkcial. Ekvac., **40** (1997), 411–433.

[22] T. Nagai, T. Senba and T. Suzuki, Chemotactic collapse in a parabolic system of mathematical biology, Hiroshima Math. J., **30** (2000), 463–497.

[23] V. Nanjundiah, Chemotaxis, signal relaying, and aggregation morphology, J. Theor. Biol., **42** (1973), 63–105.

[24] K. Ono and W. Strauss, Regular solutions of the Vlasov-Poisson-Fokker-Planck system, Discrete and Continuous Dynamical Systems, **6** (2000), 751–772.

[25] O. Penrose and P. C. Fife, Thermodynamically consistent models of phase-field type for the kinetics of phase transition, Phys. D, **43** (1990), 44–62.

[26] T. Senba and T. Suzuki, Parabolic system of chemotaxis; blowup in a finite and in the infinite time, Methods Appl. Anal., **8** (2001), 349–368.

[27] T. Senba and T. Suzuki, Chemotactic collapse in a parabolic-elliptic system of mathematical biology, Adv. Differential Equations, **6** (2001), 21–50.

[28] T. Senba and T. Suzuki, Blowup behavior of solutions to re-scaled Läger-Luckhaus system, Adv. Differential Equations, **8** (2003), 787–820.

[29] T. Senba and T. Suzuki, Applied Analysis: Mathematical Methods in Natural Science, Imperial College Press, London, 2004.

[30] M. Struwe, On the evolution of harmonic mapping of Riemannian surfaces, Comment. Math. Helv., **60** (1985), 558–581.

[31] T. Suzuki, Free Energy and Self-Interacting Particles, Birkhüser, Boston, 2005.

[32] T. Suzuki, Mean Field Theories and Dual Variation, in preparation.

[33] P. Topping, Winding behaviour of finite-time sigularities of the harmonic map heat flow, Math. Z., 247 (2004), 279–302.

[34] H. D. Victory, Jr., On the existence of global weak solutions to Vlasov-Poisson-Fokker-Planck systems, J. Math. Anal. Appl., **160** (1991), 525–555.

[35] G. Wolansky, On the evolution of self-attracting clusters and applications to semilinear equations with exponential nonlinearity, J. Anal. Math., **59** (1992), 251–272.

[36] G. Wolansky, On steady distributions of self-attracting clusters under friction and fluctuations, Arch. Rational Meth. Anal., **119** (1992), 355–391.

[37] G. Wolansky, A critical parabolic estimate and application to nonlocal equations arising in chemotaxis, Applicable Analysis, **66** (1997), 291–321.

[38] G. Wolansky, A concentration theorem for the heat equation, Monatsh. Math., **132** (2001), 255–261.

[39] A. Yagi, Norm behavior of solutions to the parabolic system of chemotaxis, Math. Japonica, **45** (1997), 241–265.

Division of Mathematical Science
Department of System Innovation
Graduate School of Engineering Science
Osaka University

Added in Proof: Recently, Type II blowup at each blowup point is proren. See

T. Senba, Type II blowup of solutions to a sinplified Keller–Segel system in two dimensions, Nonlinear Analysis (to appear),

Y. Naito and T. Suzuki, Self-similarity in chemotaxis systems, (preprint).