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Evolution of stable phases in forward-backward parabolic equations

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Abstract.

We review some recent work concerning an ill–posed *forward–backward* parabolic equation, which arises *e.g.* in the theory of phase transitions. Some new results are also presented, concerning local existence and uniqueness of solutions within a certain class of physical interest, and a hint of their proofs is given.

§1. Introduction

We study initial-boundary value problems for the *forward–backward* parabolic equation:

$$u_t = \Delta\phi(u)\,, \tag{1}$$

where $\phi \in C^2(\mathbb{R})$ is a *nonmonotone* cubic-type function satisfying the following assumptions:

$$(H_1) \qquad \begin{cases} (i) \;\; \phi'(u) > 0 \;\text{ if }\; u \in (-\infty, b) \cup (a, \infty), \\ \qquad \phi'(u) < 0 \;\text{ if }\; u \in (b, a); \\ (ii) \; \phi(u) \to \pm\infty \text{ as } u \to \pm\infty \end{cases}$$

(see Fig.1). The main feature of equation (1) is that it is *ill–posed forward in time* for u in the *spinoidal interval* (b, a).

The purpose of this paper is twofold. Firstly, we review some relevant recent work concerning equation (1); secondly, we present some new results about local existence and uniqueness for a special class of solutions, which we call *two-phase solutions* (see Section 5). While outlining proofs here, we refer the reader to [12] for an exhaustive presentation.

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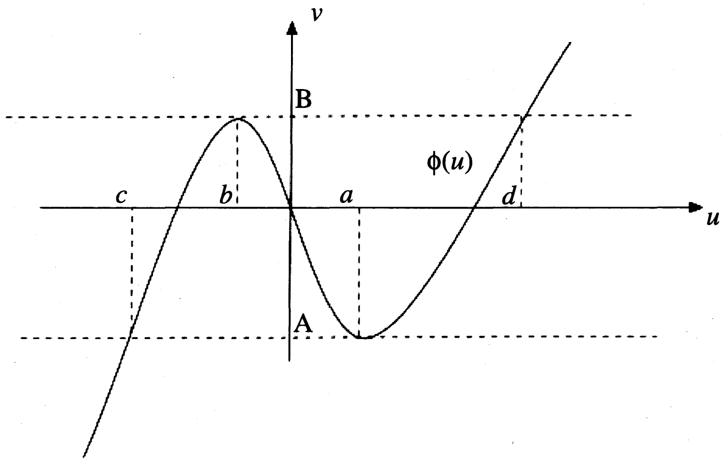

Fig. 1. Graph of $v = \phi(u)$.

1.1. Motivations

Equations of type (1) naturally arise in the theory of phase transitions; in fact, if the function u represents the *enthalpy* and $\phi(u)$ the *temperature* of the medium, equation (1) is a consequence of Fourier's law (*e.g.*, see [5]). Heuristically, the half-lines $(-\infty, b)$ and $(a, +\infty)$ correspond to *stable* phases and the interval (b, a) to an *unstable* phase, thus equation (1) describes the dynamics of transition between stable phases.

An independent motivation for studying equation (1) comes from population dynamics; in this case equation (1) arises as a quasi-continuous approximation to a discrete model for *aggregating* populations (see [16], [17] and references therein). In this framework $u \geq 0$ represents a *population density*; a cubic-type function ϕ was considered in [17], a non-monotone ϕ of the type $\phi(u) = u\exp(-u)$ in [16].

Let us also mention that a model for mass transfer in a stably stratified turbulent shear flow, investigated in [1] in one space dimension, gives for the *concentration* $v \geq 0$ the equation:

$$v_t = \big(\phi(v_x)\big)_x \,. \tag{2}$$

Since the *effective mass diffusivity* decreases very rapidly for large values of the concentration gradient, the function ϕ is again of the type $\phi(u) = u\exp(-u)$. The transformation $u := v_x$ points out the relationship between equations (2) and (1).

More generally, observe that equation (2) can be regarded as the formal L^2-gradient system associated with a *nonconvex* energy density in one space dimension (*e.g.*, see [2]). Hence the dynamics described by equation (1) is relevant in various settings where nonconvex functionals arise, including nonlinear elasticity and image processing among others (see [14], [18]).

1.2. Entropy conditions

A widely accepted idea is that ill–posedness derives from neglecting some relevant information, which underlies a correct mathematical modelling of the physical phenomenon. A general strategy is to restore this information by introducing *additional relations*, which define some restricted class of *admissible solutions*; in such class the problem is expectedly well–posed.

As is well-known, such ideas apply when studying the Cauchy problem for the *first order hyperbolic conservation law*:

$$(3)\qquad \begin{cases} u_t + \big(f(u)\big)_x = 0 & \text{in } \mathbb{R}\times(0,T] =: S_T \\ u = u_0 & \text{in } \mathbb{R}\times\{0\} \end{cases}$$

(we consider one space dimension for simplicity). Such problem is ill-posed in the class of bounded solutions, while the same problem for the corresponding *viscous conservation law*, namely:

$$(4)\qquad \begin{cases} u_t + \big(f(u)\big)_x = \epsilon u_{xx} & \text{in } S_T \\ u = u_0 & \text{in } \mathbb{R}\times\{0\} \end{cases} \qquad (\epsilon > 0)$$

is well-posed. For any $\epsilon > 0$ the weak solution of problem (4) satisfies an *entropy inequality*, which carries over to weak solutions of problem (3) in the *vanishing viscosity* limit $\epsilon \to 0$ (*e.g.*, see [23]). The latter inequality defines the class of *entropy solutions*, which is shown to be a well–posedness class for problem (3).

As pointed out in [7], there is a remarkable analogy between the above situation and the present case. In fact, it is physically meaningful to associate with equation (1) its *Sobolev regularization*:

$$(5)\qquad u_t = \Delta\phi(u) + \epsilon\Delta u_t \qquad (\epsilon > 0)$$

(see Subsection 1.3). The Neumann initial-boundary value problem for equation (5) was studied in [15], proving well-posedness in the class of bounded solutions; moreover, for any $\epsilon > 0$ its solution satisfies a suitable inequality (see (19)), which is natural to call *entropy inequality*. As $\epsilon \to 0$, solutions of the Neumann problem converge to those of an associated limiting problem in a sense made precise in [19]-[21]. By analogy, we call the latter *entropy solutions* of the problem, for they satisfy a suitable limiting *entropy inequality*; they also provide a *weak entropy solution in the sense of Young measures* of problem (21) (see Section 3; see also [6], [24]).

In this general framework, a specific class of solutions has been investigated in [9], which describe the transition between stable phases and exhibit interesting *hysteresis effects*. In view of their regularity, such solutions (which, as already mentioned, we call two-phase solutions) can be regarded as the counterpart of the *piecewise smooth solutions* in the theory of hyperbolic conservation laws. Like the latter, they exhibit an *interface* which evolves according to the *Rankine-Hugoniot condition*, obeying to *admissibility conditions* which follow from the entropy inequality (see [9] and Section 4 below).

Although very interesting, the above results do not give information about the actual *existence* or *uniqueness* of two-phase solutions. This point is addressed in Section 5, where we discuss uniqueness and local existence of solutions of this kind (see Theorems 22 and 23).

1.3. Related problems

Consider the *Cahn-Hilliard equation*:

$$u_t = \Delta\phi(u) - \kappa\Delta^2 u\,,$$

which describes isothermal phase separation of a binary mixture quenched into an unstable homogeneous state (*e.g.*, see [5]). Looking for a better agreement with experimental data, a *linear* model was derived in [3] assuming that the chemical potential also contains an integral relaxation term. This gives for the concentration the following equation:

$$u_{tt} + \gamma u_t = \Delta\Big(\gamma u - \gamma c_1 \Delta u + c_2 u_t - c_1 \Delta u_t\Big)\,. \tag{6}$$

If $c_1 = 0$ and $c_2/\gamma =: \epsilon > 0$ is fixed as $\gamma \to \infty$, from (6) we obtain the pseudoparabolic equation:

$$u_t = \Delta u + \epsilon\Delta u_t\,,$$

which can be regarded as a *viscous regularization* of equation (1) with $\phi(u) = u$. By analogy, we will associate with equation (1) its viscous regularization (5); the latter can be also regarded as a variant of the Cahn-Hilliard equation including viscous effects (see [15]). In this connection, let us mention that a fourth order viscous regularization of equation (2) has been considered in [24] and recently studied in [2].

It is worth pointing out an interesting relationship between equation (1) (in one space dimension) and the *p-system*:

$$(7) \qquad \begin{cases} u_t - v_x = 0 \\ v_t - \left(\phi(u)\right)_x = 0\,, \end{cases}$$

whence we obtain the hyperbolic equation:

$$u_{tt} = \left(\phi(u)\right)_{xx}\,,$$

formally analogous to (1). Observe that system (7) is *hyperbolic* if $\phi'(u) > 0$, *elliptic* if $\phi'(u) < 0$. Heuristically, it is expected that the elliptic phase is unstable and both hyperbolic phases are stable, so that a phase transition between the latter occurs (*e.g.*, see [13] and references therein).

Adding a damping term in (7) obtains the system:

$$\begin{cases} u_t - v_x = 0 \\ v_t - \left(\phi(u)\right)_x = -v\,, \end{cases}$$

which gives the following equation, closely related to (1):

$$u_{tt} + u_t = \left(\phi(u)\right)_{xx}\,.$$

In fact, if ϕ is *increasing*, solutions of the latter equation are known to behave like those of (1) as $t \to \infty$ (see [11]). It is an interesting open problem, whether the same is true when ϕ is nonmomotone.

§2. Viscous regularization

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$. Following [15], we investigate the Neumann initial-boundary value problem for equation (5). Introducing the *chemical potential*

$$(8) \qquad v := \phi(u) + \epsilon u_t \qquad (\epsilon > 0)\,,$$

the problem reads:

$$(9) \qquad \begin{cases} u_t = \Delta v & \text{in } \Omega \times (0,T] =: Q_T \\[2ex] \dfrac{\partial v}{\partial \nu} = 0 & \text{in } \partial\Omega \times (0,T] \\[2ex] u = u_0 & \text{in } \Omega \times \{0\}, \end{cases}$$

where $T > 0$. Let us make the following

Definition 1. *Let $u_0 \in C(\bar{\Omega})$. By a* solution *to problem (8)-(9) in Q_T we mean any couple $u \in C^1([0,T];C(\bar{\Omega}))$, $v \in C([0,T];C(\bar{\Omega}) \cap W^{2,p}_{loc}(\Omega))$ with $p > n$, $\Delta v \in C([0,T];C(\bar{\Omega}))$ which satisfies (8)-(9) in the classical sense. A solution is said to be* global *if it is a solution in Q_T for any $T > 0$.*

Concerning well-posedness of problem (8)-(9), the following holds (see also [15]).

Theorem 2. *Let $u_0 \in C(\bar{\Omega})$. Then for any $\epsilon > 0$ there exists a unique global solution (u^ϵ, v^ϵ) of problem (8)-(9). Moreover, there exists $C_1 > 0$ such that for any $\epsilon > 0$*

$$\sup_{t\geq 0} ||u^\epsilon(\cdot,t)||_{L^\infty(\Omega)} \leq C_1 \,. \tag{10}$$

2.1. Local existence and uniqueness

Let us first prove the following result.

Theorem 3. *Let $u_0 \in C(\bar{\Omega})$. Then for any $\epsilon > 0$ there exists $T_\epsilon > 0$ such that problem (8)-(9) has a unique solution (u^ϵ, v^ϵ) in Q_{T_ϵ}.*

Consider the problem:

$$\begin{cases} \Big(I - \epsilon\Delta\Big)w = h & \text{in } \Omega \\ \frac{\partial w}{\partial \nu} = 0 & \text{in } \partial\Omega. \end{cases} \tag{11}$$

Arguing as in [25], we can prove the following

Lemma 4. *For any $h \in C(\bar{\Omega})$ there exists a unique $w \in C(\bar{\Omega}) \cap W^{2,p}_{loc}(\Omega)$ with $p > n$ such that: (i) $\Delta w \in C(\bar{\Omega})$; (ii) w satisfies problem (11) in the classical sense.*

Proof. Since $h \in L^p(\Omega)$ for any $p \in [1,\infty)$, there exists a unique solution $w \in W^{2,p}(\Omega) \subseteq C^{1+\alpha}(\bar{\Omega})$ of problem (11) ($\alpha \in (0,1)$). Claim (i) follows from the first equation in (11); hence the conclusion. ♣

For any $p > n$ consider the following realization of the Laplacian with homogeneous Neumann boundary conditions:

$$\begin{cases} D(A_p) := \Big\{u \in C(\bar{\Omega}) \,\Big|\, u \in W^{2,p}_{loc}(\Omega), \Delta u \in C(\bar{\Omega}), \dfrac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega\Big\} \\ -A_p u := \Delta u \qquad\qquad (u \in D(A_p)) \,. \end{cases}$$

In view of Lemma 4, the inverse operator $(I + \epsilon A_p)^{-1} : C(\bar{\Omega}) \to C(\bar{\Omega})$ is well-defined and bounded.

Now we can prove Theorem 3.

Proof of Theorem 3. Consider the abstract Cauchy problem:

$$(12)\qquad \begin{cases} u_t = -\frac{1}{\epsilon}\Big(I - (I+\epsilon A_p)^{-1}\Big)\phi(u) & \text{for } t > 0 \\ \\ u = u_0 & \text{for } t = 0\,. \end{cases}$$

The right-hand side of the first equation in (12) is a locally Lipschitz continuous function from $C(\bar\Omega)$ into itself; hence there exists $T_\epsilon > 0$ such that problem (12) has a unique solution $u^\epsilon \in C^1([0,T_\epsilon]; C(\bar\Omega))$. ¿From the same equation we obtain:

$$v^\epsilon(t) := \phi(u^\epsilon(t)) + \epsilon u^\epsilon_t(t) = (I+\epsilon A_p)^{-1}\phi^\epsilon(u(t)) \in D(A_p)$$

for any $t \in [0, T_\epsilon]$; moreover, the map from $[0, T_\epsilon]$ to $C(\bar\Omega)$, $t \to \Delta v^\epsilon(t) = -A_p(I+\epsilon A_p)^{-1}\phi(u^\epsilon(t))$ is continuous. Then the conclusion follows. ♣

Remark 5. The bounded operator $\frac{1}{\epsilon}\Big(I-(I+\epsilon A_p)^{-1}\Big)$ is the *Yosida approximation* of the operator A_p .

Remark 6. The above results can be extended to other initial or initial-boundary value problems. In fact, the results in [25] do not require the set Ω to be bounded. If Ω is unbounded, the space $C(\bar\Omega)$ has to be replaced by

$$C^\natural(\bar\Omega) := \{u \in C(\bar\Omega) \mid \lim_{|x|\to\infty} u(x) = 0\}\,;$$

in particular, the Cauchy problem can be investigated in this framework. In the case of Dirichlet boundary conditions, the results in [26] can be used.

2.2. A priori estimates

Following [15], set for any $g \in C^1(\mathbb{R})$, $g' \geq 0$:

$$(13)\qquad G(u) := \int_0^u g(\phi(s))ds + K \qquad (K \in \mathbb{R})\,.$$

Let $\epsilon > 0$ be fixed; let (u^ϵ, v^ϵ) be the solution to problem (8)-(9) in Q_{T_ϵ}, whose existence is asserted in Theorem 3. In Q_{T_ϵ} we have:

$$\begin{aligned}\big(G(u^\epsilon)\big)_t &= g(\phi(u^\epsilon))u^\epsilon_t = g(v^\epsilon)\Delta v^\epsilon + \Big[g(\phi(u^\epsilon)) - g(v^\epsilon)\Big]\Delta v^\epsilon \\ (14)\qquad &= \operatorname{div}\Big[g(v^\epsilon)\nabla v^\epsilon\Big] - g'(v^\epsilon)|\nabla v^\epsilon|^2 + \underbrace{\Big[g(\phi(u^\epsilon)) - g(v^\epsilon)\Big]\frac{v^\epsilon - \phi(u^\epsilon)}{\epsilon}}_{\leq 0}\,.\end{aligned}$$

Then integrating in Ω we obtain:

$$\frac{d}{dt}\int_{\Omega} G(u^{\epsilon}(x,t))\,dx \leq 0 \quad \text{in } (0,T_{\epsilon})\,. \tag{15}$$

The above inequality is crucial to prove the existence of *positively invariant regions* for problem (8)-(9). This is the content of the following

Proposition 7. *Assume*

$$\phi(u_1) \leq \phi(u) \leq \phi(u_2) \qquad \text{for any } u \in [u_1, u_2]\,; \tag{16}$$

moreover, let $u_0(x) \in [u_1, u_2]$ *for any* $x \in \Omega$. *Then* $u^{\epsilon}(x,t) \in [u_1, u_2]$ *for any* $(x,t) \in Q_{T_{\epsilon}}$.

Proof. It is not restrictive (possibly changing the definition of ϕ out of $[u_1, u_2]$) to assume $\phi(u) < \phi(u_1)$ if $u < u_1$, $\phi(u) > \phi(u_2)$ if $u > u_2$. Fix $g \in C^1(\mathbb{R})$, $g' \geq 0$ such that $g \equiv 0$ in $[\phi(u_1), \phi(u_2)]$, $g(z) < 0$ if $z < \phi(u_1)$, $g(z) > 0$ if $z > \phi(u_2)$; moreover, choose the constant K in (13) such that $G(u) := \int_{u_1}^{u} g(\phi(s))ds$. Plainly, this implies $G \equiv 0$ in $[u_1, u_2]$, $G > 0$ in $(-\infty, u_1) \cup (u_2, \infty)$. Then by inequality (15) the conclusion follows. ♣

Let us now prove Theorem 2.

Proof of Theorem 2. In view of assumption $(H_1) - (ii)$, there exists $C_1 > ||u_0||_{\infty}$ such that inequality (16) holds with $[u_1, u_2] = [-C_1, C_1]$. Then Proposition 7 implies the a priori estimate (10); hence the conclusion follows. ♣

Let us prove some other a priori estimates, which will be of use in the sequel.

Proposition 8. *Let* $(u^{\epsilon}, v^{\epsilon})$ *be the global solution of problem (8)-(9)* $(\epsilon > 0)$. *Then there exists* $C_2 > 0$ *such that for any* $\epsilon > 0$, $T > 0$:

$$||v^{\epsilon}||_{L^2((0,T),H^1(\Omega))} + ||\sqrt{\epsilon}u_t^{\epsilon}||_{L^2(Q_T)} \leq C_2\,. \tag{17}$$

Proof. Choosing $g(z) = z$ in (14) gives:

$$\left(G(u^{\epsilon})\right)_t = v^{\epsilon}\Delta v^{\epsilon} - \frac{|v^{\epsilon} - \phi(u^{\epsilon})|^2}{\epsilon} = v^{\epsilon}\Delta v^{\epsilon} - \epsilon|u_t^{\epsilon}|^2\,,$$

whence plainly by estimate (10) the result follows. ♣

Proposition 9. *Let (u^ϵ, v^ϵ) be the global solution of problem (8)-(9) $(\epsilon > 0)$. Then for any $T > 0$ there exists $C_3 = C_3(T) > 0$ such that for any $\epsilon > 0$:*

$$\|v^\epsilon\|_{L^\infty(Q_T)} \leq C_3 \,. \tag{18}$$

Proof. Since the function ϕ is locally Lipschitz continuous and (10) holds, the map $t \to \|\phi(u^\epsilon)(\cdot,t)\|_{L^\infty(\Omega)}$ is bounded in $[0,\infty)$. Moreover, $\|u^\epsilon_t\|_{L^\infty(Q_T)} < \infty$ since $u^\epsilon \in C^1([0,T]; C(\bar\Omega))$. Hence the result follows. ♣

2.3. Entropy inequality

In view of equality (14), for any $\epsilon > 0$ the global solution (u^ϵ, v^ϵ) of problem (8)-(9) satisfies the inequality:

$$\big(G(u^\epsilon)\big)_t \leq \operatorname{div}\Big[g(v^\epsilon)\nabla v^\epsilon\Big] - g'(v^\epsilon)|\nabla v^\epsilon|^2 \quad \text{in} \quad \Omega \times [0,\infty)\,.$$

It is convenient for further purposes to rewrite the above inequality in the weak form, namely:

$$\iint_{Q_T} \Big\{G(u^\epsilon)\psi_t - g(v^\epsilon)\nabla v^\epsilon \cdot \nabla\psi - g'(v^\epsilon)|\nabla v^\epsilon|^2\psi\Big\}\, dxdt \geq 0 \tag{19}$$

for any $T > 0$, $\psi \in C_0^\infty(Q_T)$, $\psi \geq 0$ and $g \in C^1(\mathbb{R})$, $g' \geq 0$.

Inequality (19) will be referred to as the *entropy inequality* for equation (5), in view of its analogy with the entropy inequality for the viscous conservation law in (4), namely:

$$\iint_{S_T} \Big\{E(u^\epsilon)\big(\psi_t + \epsilon\psi_{xx}\big) + F(u^\epsilon)\psi_x\Big\}\, dxdt \geq 0\,; \tag{20}$$

the latter holds for any couple *entropy-flux* (E, F) and any $\psi \in C_0^\infty(S_T)$, $\psi \geq 0$ (*e.g.*, see [23]).

§3. Vanishing viscosity

In this section we discuss the Neumann initial-boundary value problem for equation (1), namely:

$$(21)\qquad \begin{cases} u_t = \Delta\phi(u) & \text{in } Q_T \\ \dfrac{\partial\phi(u)}{\partial\nu} = 0 & \text{in } \partial\Omega \times (0,T] \\ u = u_0 & \text{in } \Omega \times \{0\}\,. \end{cases}$$

In view of Remark 6, similar considerations apply to other initial value problems for equation (1); we omit the details.

Following [20], we make use of the viscous regularization discussed in the previous section. It turns out that the limit of solutions of problem (8)-(9) as $\epsilon \to 0$ gives rise to a *weak solution in the sense of Young measures* of problem (21) (see (29)-(31) below). By analogy with the theory of hyperbolic conservation laws, this limit can be regarded as an *entropy solution* of problem (21), for it satisfies a limiting form of inequality (19) as $\epsilon \to 0$ (see (33)).

3.1. Young measures

Let us first make the following

Definition 10. *Let $u_0 \in L^\infty(\Omega)$. By a* weak solution *to problem (9) in Q_T we mean any couple $u \in L^\infty(Q_T)$, $v \in L^\infty(Q_T) \cap L^2((0,T); H^1(\Omega))$ such that:*

$$(22)\qquad \iint_{Q_T} \Big\{ u\psi_t - \nabla v \cdot \nabla\psi \Big\} dxdt + \int_\Omega u_0(x)\psi(x,0)dx = 0$$

for any $\psi \in C^1(\overline{Q}_T)$, $\psi(\cdot,T) = 0$ in Ω.

Then we have the following existence result.

Theorem 11. *Let $u_0 \in L^\infty(\Omega)$. Then there exists a global weak solution (u,v) of problem (9).*

Proof. For any $\epsilon > 0$ there is $u_0^\epsilon \in C(\bar\Omega)$ such that $||u_0^\epsilon||_\infty \le ||u_0||_\infty$, $||u_0^\epsilon - u_0||_1 < \epsilon$. Then the global solution (u^ϵ, v^ϵ) of problem (8)-(9) with $u_0 = u_0^\epsilon$ satisfies the equality:

$$(23)\qquad \iint_{Q_T} \Big\{ u^\epsilon\psi_t - \nabla v^\epsilon \cdot \nabla\psi \Big\} dxdt + \int_\Omega u_0^\epsilon(x)\psi(x,0)dx = 0$$

for any $\psi \in C^1(\overline{Q}_T)$, $\psi(\cdot, T) = 0$ in Ω and any $\epsilon > 0$. Clearly, the solution (u^ϵ, v^ϵ) satisfies the a priori estimates (10) and (17)-(18). Hence there exist sequences $\{u^{\epsilon_n}\}$, $\{v^{\epsilon_n}\}$ and a couple (u, v) with $u \in L^\infty(Q_T)$, $v \in L^\infty(Q_T) \cap L^2((0,T); H^1(\Omega))$ such that for any $T > 0$:

$$u^{\epsilon_n} \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty(Q_T)\,, \tag{24}$$

$$v^{\epsilon_n} \overset{*}{\rightharpoonup} v \quad \text{in } L^\infty(Q_T)\,, \tag{25}$$

$$v^{\epsilon_n} \rightharpoonup v \quad \text{in } L^2((0,T), H^1(\Omega))\,. \tag{26}$$

Taking the limit as $\epsilon \to 0$ in equality (23) gives the result. ♣

To obtain from Theorem 11 a weak solution of problem (21), we should like to have $v = \phi(u)$; however, no conclusion about the limit of the sequence $\phi(u^{\epsilon_n})$ can be drawn from (24)-(26). Nevertheless, since the sequence $\{u^{\epsilon_n}\}$ is uniformly bounded in $L^\infty(Q_T)$, there exist a subsequence (denoted again $\{u^{\epsilon_n}\}$) and a family of probability measure $\nu_{(x,t)}$ defined for almost every $(x,t) \in Q_T$, such that for any $f \in C(\mathbb{R})$:

$$f(u^{\epsilon_n}) \overset{*}{\rightharpoonup} \overline{f} \quad \text{in } L^\infty(Q_T)\,; \tag{27}$$

here

$$\overline{f}(x,t) := \int_{\mathbb{R}} f(\tau)\, d\nu_{(x,t)}(\tau) \qquad \text{for a.e. } (x,t) \in Q_T$$

(*e.g.*, see [8]).

The family $\nu_{(x,t)}$ is called the family of *Young measures* associated with the subsequence $\{u^{\epsilon_n}\}$. It was proved in [20] that *$\nu_{(x,t)}$ is a superposition of Dirac measures concentrated on the three monotone branches of the graph of $v = \phi(u)$*. In fact, define:

$$u := \beta_1(v),\ v \in (-\infty, B) \ \Leftrightarrow\ v = \phi(u)\,,\ u \in (-\infty, b)\,,$$

$$u := \beta_0(v),\ v \in (A, B) \ \Leftrightarrow\ v = \phi(u)\,,\ u \in (b, a)\,;$$

$$u := \beta_2(v),\ v \in (A, \infty) \ \Leftrightarrow\ v = \phi(u)\,,\ u \in (a, \infty)$$

(here $A := \phi(a)$, $B := \phi(b)$; see Fig.2). Then the following holds.

Theorem 12. *Let (u, v) be the global weak solution of problem (9) considered in Theorem 11. Then there exist $\lambda_i \in L^\infty(Q_T)$, $\lambda_i \geq 0$ $(i = 0, 1, 2)$ with the following properties:*

(i) $\sum_{i=0}^{2} \lambda_i(x,t) = 1$, *$\lambda_1(x,t) = 1$ if $v(x,t) < A$, $\lambda_2(x,t) = 1$ if $v(x,t) > B$ for almost every $(x,t) \in Q_T$;*

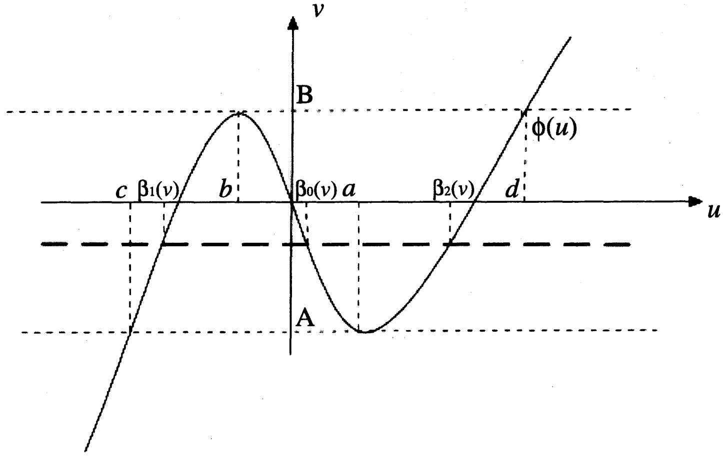

Fig. 2. The branches $\beta_i(v)$ $(i = 0, 1, 2)$

(ii) the family of Young measures $\nu_{(x,t)}$ associated with the (sub)sequence $\{u^{\epsilon_n}\}$ considered in the proof of Theorem 11 is of the form:

$$\nu_{(x,t)}(\tau) = \sum_{i=0}^{2} \lambda_i(x,t)\delta(\tau - \beta_i(v(x,t))) \tag{28}$$

for almost every $(x,t) \in Q_T$ and any $\tau \in \mathbb{R}$.

Let us point out the following consequence of Theorem 12. In view of equality (28), there holds:

$$\int_{\mathbb{R}} \tau d\nu_{(x,t)}(\tau) = \sum_{i=0}^{2} \lambda_i(x,t)\beta_i(v(x,t)) = u(x,t) \tag{29}$$

(this follows by (24) and (27) choosing $f(\tau) = \tau$); moreover,

$$\int_{\mathbb{R}} \phi(\tau) d\nu_{(x,t)}(\tau) = \sum_{i=0}^{2} \lambda_i(x,t)\phi(\beta_i(v(x,t))) = v(x,t) \tag{30}$$

for almost every $(x,t) \in Q_T$ (here use of Theorem 12-(i) has been made). Hence by (22) we have:

$$\iint_{Q_T}\Big\{\psi_t\int_{\mathbb{R}}\tau d\nu_{(x,t)}(\tau)-\nabla\psi\cdot\nabla\int_{\mathbb{R}}\phi(\tau)d\nu_{(x,t)}(\tau)\Big\}dxdt + \int_\Omega u_0(x)\psi(x,0)dx = 0\,. \tag{31}$$

This suggests to regard the weak solution (u,v) of problem (9) considered in Theorem 11 as a *weak solution in the sense of Young measures* to problem (21).

The above result can be heuristically interpreted as follows. The function u takes the *fraction* λ_i of its value at (x,t) on the branch $\beta_i(v)$ of the graph of $v=\phi(u)$. For instance, if $\lambda_1=1$, $\lambda_0=\lambda_2=0$ almost everywhere in Q_T, then $u=\beta_1(v)$ is a weak solution of problem (21) (see (30)-(31)). In general, in view of equality (29), u can be regarded as a *superposition of different phases*; for, $\lambda_0,\lambda_1,\lambda_2$ are referred to as *phase fractions.*

3.2. Entropy inequality

Consider inequality (19) along the sequence $\{(u^{\epsilon_n},v^{\epsilon_n})\}$ of global solutions to problem (8)-(9) used in the proof of Theorem 11. Using (24)-(27), as $n\to\infty$ we obtain the following result (see [20]).

Theorem 13. *Let (u,v) be the global weak solution of problem (9) considered in Theorem 11. Define*

$$G^*(x,t) := \sum_{i=0}^{2}\lambda_i G(\beta_i(v(x,t))) \qquad \text{for a.e. } (x,t)\in Q_T\,. \tag{32}$$

Then there holds:

$$\iint_{Q_T}\Big\{G^*\psi_t - g(v)\nabla v\cdot\nabla\psi - g'(v)|\nabla v|^2\psi\Big\}\,dxdt \ge 0 \tag{33}$$

for any $\psi\in C_0^\infty(Q_T)$, $\psi\ge 0$.

Let us first note the following

Lemma 14. *Let $\{(u^{\epsilon_n},v^{\epsilon_n})\}$ be the sequence of global solutions to problem (8)-(9) used in the proof of Theorem 11; then*

$$v^{\epsilon_n}\to v \qquad \text{in } L^2(Q_T)\,. \tag{34}$$

In particular, there exists a subsequence (denoted again $\{v^{\epsilon_n}\}$) such that:

$$v^{\epsilon_n}(x,t) \to v(x,t) \qquad \text{for a.e. } (x,t) \in Q_T\,. \tag{35}$$

Proof. Choosing $f = \phi$ in (27) and using (30) we obtain $\phi(u^{\epsilon_n}) \overset{*}{\rightharpoonup} v$ in $L^\infty(Q_T)$, thus $\phi(u^{\epsilon_n}) \rightharpoonup v$ in $L^2(Q_T)$. Similarly, choosing $f = \phi^2$ and using (27)-(28) gives $\left(\phi(u^{\epsilon_n})\right)^2 \overset{*}{\rightharpoonup} v^2$ in $L^\infty(Q_T)$, thus :

$$\|\phi(u^{\epsilon_n})\|^2_{L^2(Q_T)} = \iint_{Q_T} \left(\phi(u^{\epsilon_n})\right)^2 dxdt \to \iint_{Q_T} v^2\, dxdt = \|v\|^2_{L^2(Q_T)}\,.$$

This implies $\phi(u^{\epsilon_n}) \to v$ in $L^2(Q_T)$. In view of (8) and (17), we obtain:

$$\begin{aligned}\|v^{\epsilon_n} - v\|_{L^2(Q_T)} &\le \|\phi(u^{\epsilon_n}) - v\|_{L^2(Q_T)} + \epsilon_n \|u_t^{\epsilon_n}\|_{L^2(Q_T)}\\ &\le \|\phi(u^{\epsilon_n}) - v\|_{L^2(Q_T)} + \sqrt{\epsilon_n}\, C_2 \to 0\end{aligned}$$

as $n \to \infty$. This proves (34), whence (35) follows; hence the result. ♣

Proof of Theorem 13. We divide the proof into different steps, concerning the different terms in the left-hand side of inequality (19).
(i) Let $\{(u^{\epsilon_n}, v^{\epsilon_n})\}$ be the sequence of global solutions to problem (8)-(9) used in the proof of Theorem 11. In view of (27)-(28), there holds $G(u^{\epsilon_n}) \overset{*}{\rightharpoonup} G^*$ in $L^\infty(Q_T)$, with G^* defined in (32). In particular,

$$\lim_{n\to\infty} \iint_{Q_T} G(u^{\epsilon_n})\psi_t\, dxdt = \iint_{Q_T} G^*\psi_t\, dxdt \tag{36}$$

for any $\psi \in C_0^\infty(Q_T)$, $\psi \ge 0$.
(ii) Plainly,

$$\iint_{Q_T} \left\{ g(v^{\epsilon_n})\nabla v^{\epsilon_n} \cdot \nabla\psi - g(v)\nabla v \cdot \nabla\psi \right\} dxdt$$

$$= \iint_{Q_T} [g(v^{\epsilon_n}) - g(v)]\nabla v^{\epsilon_n} \cdot \nabla\psi\, dxdt - \iint_{Q_T} g(v)\nabla(v - v^{\epsilon_n}) \cdot \nabla\psi\, dxdt\,.$$

Using (17) we find for some $K_1 > 0$:

$$\left| \iint_{Q_T} [g(v^{\epsilon_n}) - g(v)]\nabla v^{\epsilon_n} \cdot \nabla\psi\, dxdt \right| \le K_1 \|g(v^{\epsilon_n}) - g(v)\|_{L^2(Q_T)} \longrightarrow 0$$

as $n \to \infty$; the above limit follows by (18), the local Lipschitz continuity of g and (35). Moreover, by (26) there holds:

$$\iint_{Q_T} g(v)\nabla(v - v^{\epsilon_n}) \cdot \nabla\psi\, dxdt \longrightarrow 0.$$

It follows that:

$$\lim_{n\to\infty}\iint_{Q_T} g(v^{\epsilon_n})\nabla v^{\epsilon_n}\cdot\nabla\psi\,dxdt = \iint_{Q_T} g(v)\nabla v\cdot\nabla\psi\,dxdt \tag{37}$$

for any ψ as above.
(*iii*) As in (*ii*), we write:

$$\iint_{Q_T} g'(v^{\epsilon_n})|\nabla v^{\epsilon_n}|^2\psi\,dxdt$$

$$= \iint_{Q_T}\big(g'(v^{\epsilon_n}) - g'(v)\big)|\nabla v^{\epsilon_n}|^2\psi\,dxdt + \iint_{Q_T} g'(v)|\nabla v^{\epsilon_n}|^2\psi\,dxdt\,.$$

In view of (17), (35) and the continuity of g', for some $K_2 > 0$ there holds:

$$\left|\iint_{Q_T}\big(g'(v^{\epsilon_n})-g'(v)\big)|\nabla v^{\epsilon_n}|^2\psi\,dxdt\right| \le K_2\|g'(v^{\epsilon_n})-g'(v)\|_{L^2(Q_T)} \longrightarrow 0$$

as $n\to\infty$. On the other hand,

$$\liminf_{n\to\infty}\iint_{Q_T} g'(v)|\nabla v^{\epsilon_n}|^2\psi\,dxdt \ge \iint_{Q_T} g'(v)|\nabla v|^2\psi\,dxdt\,;$$

the latter inequality follows observing that the limit (26) also holds in $L^2((0,T),H^1(\Omega))$ with measure $g'(v)\psi dxdt$. Hence we have:

$$\begin{aligned}&\liminf_{n\to\infty}\iint_{Q_T} g'(v^{\epsilon_n})|\nabla v^{\epsilon_n}|^2\psi\,dxdt\\ &= \lim_{n\to\infty}\iint_{Q_T}\big(g'(v^{\epsilon_n}) - g'(v)\big)|\nabla v^{\epsilon_n}|^2\psi\,dxdt\\ &\quad+\liminf_{n\to\infty}\iint_{Q_T} g'(v)|\nabla v^{\epsilon_n}|^2\psi\,dxdt\\ &\ge \iint_{Q_T} g'(v)|\nabla v|^2\,\psi\,dxdt\end{aligned}\tag{38}$$

for any ψ as above.

Finally, from (36)-(38) we obtain:

$$\iint_{Q_T}\Big\{G^*\psi_t - g(v)\nabla v\cdot\nabla\psi - g'(v)|\nabla v|^2\psi\Big\}\,dxdt$$

$$\ge \lim_{n\to\infty}\iint_{Q_T}\Big\{G(u^{\epsilon_n})\psi_t - g(v^{\epsilon_n})\nabla v^{\epsilon_n}\cdot\nabla\psi\Big\}\,dxdt$$

$$- \liminf_{n\to\infty} \iint_{Q_T} g'(v^{\epsilon_n})|\nabla v^{\epsilon_n}|^2 \psi \, dxdt$$

$$= \limsup_{n\to\infty} \iint_{Q_T} \Big\{ G(u^{\epsilon_n})\psi_t - g(v^{\epsilon_n})\nabla v^{\epsilon_n}\cdot\nabla\psi - g'(v^{\epsilon_n})|\nabla v^{\epsilon_n}|^2\psi \Big\} \, dxdt \geq 0 .$$

This completes the proof. ♣

Now recall that by definition *weak solutions* of the Cauchy problem (3) satisfy the equality:

$$\iint_{S_T} \Big\{ u\psi_t + f(u)\psi_x \Big\} dxdt + \int_{\mathbb{R}} u_0(x)\psi(x,0)dx = 0 , \tag{39}$$

analogue to equality (22). Moreover, *entropy solutions* to (3) satisfy by definition the inequality:

$$\iint_{S_T} \Big\{ E(u)\psi_t + F(u)\psi_x \Big\} dxdt \geq 0 \tag{40}$$

for any couple entropy-flux (E,F) and any $\psi \in C_0^\infty(S_T)$, $\psi \geq 0$. In view of the analogy between inequalities (19) and (20) in the viscous case, it is natural to regard inequality (33) (which is the vanishing viscosity limit of (19)) as the *entropy inequality* of problem (9) .

The above discussion suggests the following definition (see [20]).

Definition 15. *A* weak entropy solution in the sense of Young measures *to problem (21) in Q_T is given by $u, \lambda_0, \lambda_1, \lambda_2 \in L^\infty(Q_T)$, $v \in L^\infty(Q_T) \cap L^2((0,T), H^1(\Omega))$ such that:*

(a) $\sum_{i=0}^{2} \lambda_i = 1$, $\lambda_i \geq 0$ and there holds:

$$u = \sum_{i=0}^{2} \lambda_i \beta_i(v) \tag{41}$$

with $\lambda_1 = 1$ if $v < A$, $\lambda_2 = 1$ if $v > B$;

(b) the couple (u,v) is a weak solution to problem (9) in Q_T;

(c) inequality (33) is satisfied for any $\psi \in C_0^\infty(Q_T)$, $\psi \geq 0$ and $g \in C^1(\mathbb{R})$, $g' \geq 0$.

Then from Theorems 11-13 we obtain the following existence result.

Theorem 16. *There exists a weak entropy solution in the sense of Young measures to problem* (21).

Let us mention that no result concerning uniqueness of such solutions is available (in this connection, see [10]).

§4. Hysteresis and entropy conditions

4.1. Hysteresis loop

Let us first mention an important property of the phase fractions (see [20], [21]).

Theorem 17. *Let $(u, v, \lambda_0, \lambda_1, \lambda_2)$ be a weak entropy solution in the sense of Young measures to problem (21) in Q_T. Then $\lambda_i(x, \cdot) \in BV_{loc}(0, T)$ for almost every $x \in \Omega$ $(i = 0, 1, 2)$. Moreover, if*

$$\operatorname*{ess\,sup}_{t\in(t_1,t_2)} v(x,t) < B$$

for some interval $(t_1, t_2) \subseteq (0, T)$, then $\lambda_1(x, \cdot)$ is not decreasing in (t_1, t_2). Similarly, if

$$\operatorname*{ess\,inf}_{t\in(t_1,t_2)} v(x,t) > A$$

for some interval $(t_1, t_2) \subseteq (0, T)$, then $\lambda_2(x, \cdot)$ is not decreasing in (t_1, t_2).

It is informative to discuss the implications of the above theorem in a specific situation, already considered in [9]. Assume:

$$(H_2) \qquad \lambda_0 = 0 \text{ a.e. in } Q_T\,, \quad \lambda_i = 1 \text{ a.e. in } V_i \ \ (i = 1, 2)\,;$$

here $\overline{Q}_T = \overline{V}_1 \cup \overline{V}_2$, $V_1 \cap V_2 = \emptyset$ and the *interface* $\gamma := \overline{V}_1 \cap \overline{V}_2$ is a smooth n-dimensional surface. Since $\sum_{i=0}^2 \lambda_i = 1$ (see Definition 15-(a)), λ_1 and λ_2 only take the values $0, 1$.

Assume for simplicity $n = 1$; thus $Q_T = (-L, L) \times (0, T]$ for some $L > 0$, $V_1 := \{(x,t) \,|\, -L < x < \xi(t),\, t \in [0, T]\}$, $V_2 := Q_T \setminus \overline{V}_1$ and $\gamma = \{(\xi(t), t) \,|\, t \in [0, T]\}$ with $\xi \in \mathrm{Lip}\,([0, T])$, say. In view of Definition 15, we have:

$$u = \beta_i(v) \ \text{ a.e. in } V_i \quad (i = 1, 2); \tag{42}$$

$$\iint_{Q_T} \Big\{ u\psi_t - v_x\psi_x \Big\} dxdt = 0 \tag{43}$$

for any $\psi \in C_0^\infty(Q_T)$;

$$\iint_{Q_T} \left\{ G(u)\psi_t - g(v)v_x\psi_x - g'(v)|v_x|^2\psi \right\} dxdt \geq 0 \tag{44}$$

for any $\psi \in C_0^\infty(Q_T)$, $\psi \geq 0$ and $g \in C^1(\mathbb{R})$, $g' \geq 0$.

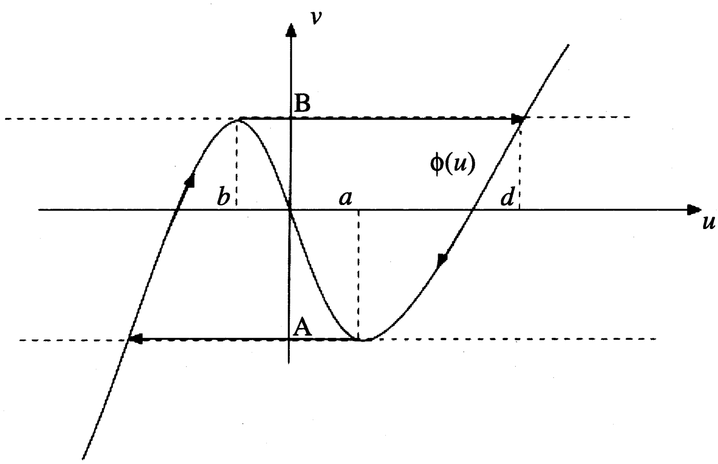

Fig. 3. Hysteresis loop

In the above situation, for almost every $(x,t) \in V_1$ we have $\lambda_1(x,t) = 1$ and $u(x,t) = \beta_1(v(x,t))$ (see (H_2) and (42)). As long as $v(x,t) < B$ there holds $\lambda_1(x,t) = 1$, since by Theorem 17 $\lambda_1(x,\cdot)$ cannot decrease; hence the point (x,t) stays in V_1 and no phase transition at (x,t) takes place. On the other hand, in view of the same theorem, if $v(x,t) = B > A$ the solution may suddenly "jump" from the branch $\beta_1(v)$ to $\beta_2(v)$ and a phase transition may occur at (x,t). In fact, in this case $\lambda_1(x,\cdot)$ can decrease, while $\lambda_2(x,\cdot)$ cannot; since both λ_1 and λ_2 can only take the values $0, 1$, we obtain $\lambda_1(x,t) = 0$ and $\lambda_2(x,t) = 1$. Hence the point (x,t) switches to the set V_2, thus $u(x,t) = \beta_2(v(x,t))$ (see (41)). The reverse situation occurs in V_2, depending as to whether $v(x,t) > A$ or $v(x,t) = A < B$. This gives rise to a *hysteresis effect*; observe that the *hysteresis loop* is clockwise oriented (see Fig.3).

Remark 18. Hysteresis effects could already be expected from problem (8)-(9) as $\epsilon \to 0$. In fact, by the very definition (8) of chemical potential we obtain:

$$u_t^\epsilon = \frac{v^\epsilon - \phi(u^\epsilon)}{\epsilon} \begin{cases} > 0 & \text{in the set } \{v^\epsilon > \phi(u^\epsilon)\} \\ < 0 & \text{in the set } \{v^\epsilon < \phi(u^\epsilon)\}. \end{cases}$$

As $\epsilon \to 0$, if v^ϵ is slowly varying, the system is driven onto the stable branches of the graph of $v = \phi(u)$, thus giving rise to the hysteresis loop.

4.2. Two-phase solutions

Let $(u, v, \lambda_0, \lambda_1, \lambda_2)$ be a weak entropy solution in the sense of Young measures to problem (21) in Q_T, which satisfies assumption (H_2). Let $n = 1$ and V_1, V_2, γ and ξ be defined as above. In view of (42)-(43) and standard regularity results, u is a classical solution of

$$u_t = \big(\phi(u)\big)_{xx} \quad \text{in } V_i \qquad (i = 1, 2). \tag{45}$$

In this situation it is natural to speak of *piecewise smooth entropy solutions* to problem (21); this suggests the following definition.

Definition 19. *By a* two-phase solution *to the Neumann initial-boundary value problem:*

$$\begin{cases} u_t = \big(\phi(u)\big)_{xx} & \text{in } (-L, L) \times (0, T] \\ \phi(u)_x = 0 & \text{in } \{-L, L\} \times (0, T) \\ u = u_0 & \text{in } (-L, L) \times \{0\} \end{cases} \tag{46}$$

we mean any triple (ξ, u, v) such that:
(a) $\xi : [0, T] \to (-L, L)$ is Lipschitz continuous, $\xi(0) = 0$;
(b) $u, v : Q_T \to \mathbb{R}$ are smooth in $Q_T \setminus \gamma$. Moreover, $v(\cdot, t)$ is continuous in $(-L, L)$ for almost every $t \in (0, T)$;
(c) u, v satisfy (42)-(44);
(d) initial and boundary conditions are satisfied in classical sense.

Any entropy solution in the sense of Young measures to problem (21), which satisfies (H_2) and is smooth in V_i $(i = 1, 2)$, is a two-phase solution of problem (46).

Once again, there is a remarkable analogy with the situation encountered when dealing with *piecewise C^1 entropy solutions* of the hyperbolic problem (3). In fact, assume $\overline{S_T} = \overline{V_1} \cup \overline{V_2}$, $V_1 \cap V_2 = \emptyset$ and $\gamma := \overline{V_1} \cap \overline{V_2} = \{(\xi(t), t) \,|\, t \in [0, T]\}$ as above. Set $h^\pm(t) := \lim_{\eta \to 0^+} h(\xi(t) \pm \eta, t)$ $(t \in [0, T])$; denote by $[h] \equiv [h](t) := h^+(t) - h^-(t)$ the jump across the curve γ of any piecewise continuous function h defined in S_T. In view of the very definition of weak solution (see (39)), we obtain by standard arguments the *Rankine-Hugoniot condition*:

$$\xi' = \frac{[f(u)]}{[u]} \quad \text{a.e. on } \gamma\,. \tag{47}$$

On the other hand, the entropy condition (40) becomes:

$$\xi'\,[E(u)] \geq [F(u)] \quad \text{a.e. on } \gamma\,. \tag{48}$$

Since $g' \geq 0$, comparing inequality (40) with (44) suggests the formal correspondence

$$E(u) \Leftrightarrow G(u)\,, \quad F(u) \Leftrightarrow -g(v)v_x\,, \tag{49}$$

which transforms (47) into (50) below, respectively (48) into (51) (observe that the couple $(u, f(u))$ corresponds to $(u, -v_x)$). Therefore the following result concerning problem (46) (see [9]) does not come as a surprise.

Theorem 20. *Let (ξ, u, v) be a two-phase solution of problem (46). Then:*
(i) there holds the Rankine-Hugoniot condition*:*

$$\xi' = -\frac{[v_x]}{[u]} \quad \text{a.e. on } \gamma\,. \tag{50}$$

(ii) there holds the entropy condition*:*

$$\xi'\,[G(u)] \geq -g(v)[v_x] \quad \text{a.e. on } \gamma\,. \tag{51}$$

It is first proved that $v_x(\cdot, \bar{t}) \in BV_{loc}(V_i \cap \{t = \bar{t}\})$ for almost every $\bar{t} \in (0, T)$, thus the traces $v_x^{\pm}$ at γ are well defined; then the proof of Theorem 20 is very similar to that of (47)-(48) (*e.g.*, see [23]). We refer the reader to [12] for details.

4.3. Admissibility conditions for the interface

Clearly, the mechanism of phase transitions described in Subsection 4.1 is related to the evolution of the interface γ. It is an interesting consequence of the entropy inequality (51) that this evolution must satisfy suitable *admissibility conditions.*

As is well known, a specific choice of the couple (E, F) in (48) is that of the *Kruzkov entropy* with related entropy flux. In view of the arbitrariness of (E, F), this gives the *Oleinik entropy condition*, which selects *admissibility conditions* for a piecewise C^1 solution of problem (3) to be entropic (*e.g.*, see [23]). In a similar vein, suitable choices of g

in inequality (51) select *admissible directions of propagation* of γ, thus *admissible phase changes*, which exhibit the hysteresis effects described in Subsection 4.1.

This is the content of the following theorem (see [9] for a slightly different proof).

Theorem 21. *Let (ξ, u, v) be a two-phase solution of problem (46). Then:*

$$(52)\qquad \begin{cases} (a) & \xi' \geq 0 \text{ if } v = A\,; \\ (b) & \xi' \leq 0 \text{ if } v = B\,; \\ (c) & \xi' = 0 \text{ if } v \neq A,\, v \neq B\,. \end{cases}$$

Proof. By a regularization procedure, we can choose

$$(53)\qquad g(t) = g_k(t) := \operatorname{sgn}(t-k)\,,$$

whence

$$G(u) = G_k(u) := \int_0^u \operatorname{sgn}(\phi(t)-k)dt \quad (k, t \in \mathbb{R})\,.$$

On the other hand, from (50)-(51) we obtain:

$$\xi' \Big\{ [G(u)] - g(v)[u] \Big\} \geq 0 \quad \text{a.e. on } \gamma$$

for any nondecreasing g. Hence the choice (53) gives:

$$\xi' \int_{u^-}^{u^+} \Big\{ \operatorname{sgn}(\phi(t)-k) - \operatorname{sgn}(\phi(u)-k) \Big\} dt \geq 0$$

for any $k \in \mathbb{R}$.

When $v = \phi(u) = A$ choose $k \in (A, B)$. Then

$$\phi(u) - k = A - k < 0 \Rightarrow \operatorname{sgn}(\phi(u)-k) = -1$$

$$\Rightarrow \operatorname{sgn}(\phi(t)-k) - \operatorname{sgn}(\phi(u)-k) \geq 0\,,$$

whence $\xi' \geq 0$ by the above inequality. This proves claim (a). To prove claim (b) we also choose $k \in (A, B)$; instead, to prove claim (c) we choose first $k \in (v, B)$, then $k \in (A, v)$. Then the conclusion follows. ♣

In view of Theorem 21, three cases are possible (see (52) and Fig.3):

$$(54)\qquad \begin{cases} (a) & v = A \Rightarrow u^- = c,\, u^+ = a,\, \xi' \geq 0\,; \\ (b) & v = B \Rightarrow u^- = b,\, u^+ = d,\, \xi' \leq 0\,; \\ (c) & v \neq A,\, v \neq B \Rightarrow \xi' = [v_x] = 0\,. \end{cases}$$

In cases $(a)-(b)$ we speak of *moving interface*, in case (c) of *steady interface.*

§5. Existence and uniqueness of two-phase entropy solutions

The present section is devoted to investigate existence and uniqueness of two-phase solutions of problem (46). Concerning the initial data, we assume the following:

$$(H_3)\begin{cases} (i)\ u_0 \le b \text{ in } (-L,0),\ u_0 \ge a \text{ in } (0,L); \\ (ii)\ \phi(u_0) \in C([-L,L]); \\ (iii)\ \text{there exists a finite number of solutions of the} \\ \qquad \text{equations } \phi(u_0(x)) = A,\ \phi(u_0(x)) = B\ \ (x \in (-L,L)). \end{cases}$$

As for uniqueness, we can prove the following result.

Theorem 22. *Let assumption (H_3) be satisfied. Then there exists at most one two-phase solution to problem (46).*

Proof. Let (ξ_1,u_1,v_1), (ξ_2,u_2,v_2) be two two-phase solutions of problem (46). Define $F : Q_T \setminus \{\gamma_1 \cup \gamma_2\} \to \mathbb{R}^2$ as follows:

$$F := \Big(|u_1 - u_2|, \operatorname{sgn}(u_1-u_2)(-v_{1x}+v_{2x})\Big)\,.$$

Formally we obtain:

$$\begin{aligned} \operatorname{div} F &:= |u_1-u_2|_t + \big[\operatorname{sgn}(u_1-u_2)(-v_{1x}+v_{2x})\big]_x \\ &= \delta_{\{u_1=u_2\}}(u_1-u_2)_x(-v_{1x}+v_{2x})\,. \end{aligned}$$

In view of the Saks lemma (*e.g.*, see [4], [22]), for any $\tau \in (0,T]$ there holds:

$$\iint_{Q_\tau} \Big\{|u_1-u_2|_t + \operatorname{sgn}(u_1-u_2)(-v_{1x}+v_{2x})\Big\}dxdt = 0\,, \tag{55}$$

where $Q_\tau = (-L,L)\times(0,\tau]$.

It is not restrictive to assume $\xi_1(t) \le \xi_2(t)$ for any $t \in (0,T]$ (otherwise we work with $\underline{\xi} := \min\{\xi_1,\xi_2\}$ and $\overline{\xi} := \max\{\xi_1,\xi_2\}$). We have for any $\tau \in (0,T]$:

$$Q_\tau = \Sigma^l_\tau \cup \Sigma^c_\tau \cup \Sigma^r_\tau\,,$$

where (see Fig.4):

$$\Sigma^l_\tau := \{(x,t) \in Q_\tau \mid -L < x \le \xi_1(t),\ t \in (0,\tau]\}\,,$$

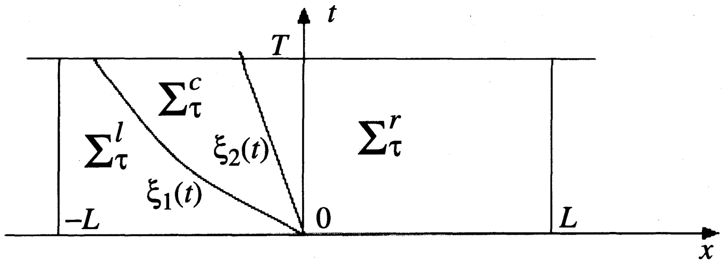

Fig. 4. The subsets Σ_τ^l, Σ_τ^c and Σ_τ^r

$$\Sigma_\tau^c := \{(x,t) \in Q_\tau \mid \xi_1(t) < x \le \xi_2(t),\, t \in (0,\tau]\},$$
$$\Sigma_\tau^r := \{(x,t) \in Q_\tau \mid \xi_2(t) < x < L,\, t \in (0,\tau]\}.$$

It is expedient to introduce the notation $h^{i,\pm}(t) := \lim_{\eta\to 0^+} h(\xi_i(t) \pm \eta, t)$ $(i = 1,2;\ t \in [0,T])$; moreover, denote by $[h]_i \equiv [h]_i(t) := h^{i,+}(t) - h^{i,-}(t)$ the jump across the curve γ_i of any piecewise continuous function h defined in Q_T. Plainly, there holds:

$$\iint_{\Sigma_\tau^l} \Big\{ |u_1 - u_2|_t + \operatorname{sgn}(u_1 - u_2)(-v_{1x} + v_{2x}) \Big\} dxdt$$
$$= \int_{-L}^{\xi_1(\tau)} |u_1(x,\tau) - u_2(x,\tau)|\, dx$$
$$+ \int_0^\tau \Big\{ -|u_1^{1,-} - u_2^{1,-}|\, \xi_1' + \operatorname{sgn}(u_1^{1,-} - u_2^{1,-})(-v_{1x}^{1,-} + v_{2x}^{1,-}) \Big\}\, dt\,;$$
$$\iint_{\Sigma_\tau^c} \Big\{ |u_1 - u_2|_t + \operatorname{sgn}(u_1 - u_2)(-v_{1x} + v_{2x}) \Big\} dxdt$$
$$= \int_{\xi_1(\tau)}^{\xi_2(\tau)} |u_1(x,\tau) - u_2(x,\tau)|\, dx$$
$$+ \int_0^\tau \Big\{ |u_1^{1,+} - u_2^{1,+}|\, \xi_1' - \operatorname{sgn}(u_1^{1,+} - u_2^{1,+})(-v_{1x}^{1,+} + v_{2x}^{1,+}) \Big\}\, dt$$
$$+ \int_0^\tau \Big\{ -|u_1^{2,-} - u_2^{2,-}|\, \xi_2' + \operatorname{sgn}(u_1^{2,-} - u_2^{2,-})(-v_{1x}^{2,-} + v_{2x}^{2,-}) \Big\}\, dt\,;$$
$$\iint_{\Sigma_\tau^r} \Big\{ |u_1 - u_2|_t + \operatorname{sgn}(u_1 - u_2)(-v_{1x} + v_{2x}) \Big\} dxdt$$

$$= \int_{\xi_2(\tau)}^{L} |u_1(x,\tau) - u_2(x,\tau)|\, dx$$

$$+ \int_0^\tau \Big\{ |u_1^{2,+} - u_2^{2,+}|\, \xi_2' - \mathrm{sgn}\,(u_1^{2,+} - u_2^{2,+})(-v_{1x}^{2,+} + v_{2x}^{2,+}) \Big\}\, dt\,.$$

Adding the above equalities and taking equality (55) into account gives:

(56)
$$\int_{-L}^{L} |u_1(x,\tau) - u_2(x,\tau)|\, dx$$

$$= \int_0^\tau \{ -[|u_1 - u_2|]_1\, \xi_1' + [\mathrm{sgn}\,(u_1 - u_2)(-v_{1x} + v_{2x})]_1 \}\, dt$$

$$+ \int_0^\tau \{ -[|u_1 - u_2|]_2\, \xi_2' + [\mathrm{sgn}\,(u_1 - u_2)(-v_{1x} + v_{2x})]_2 \}\, dt\,.$$

Now assume $\xi_1(t) < \xi_2(t) < 0$ for any $t \in (0,\tau]$ (see Fig 4). In this case u_2 is regular along γ_1, thus

$$u_2^{1,-} = u_2^{1,+} = u_2(\xi_1(t), t) \le b < d \quad \text{in}\,(0,\tau]$$

(see (42) and Fig.1); moreover,

$$u_1^{1,-} = b\,, \quad u_1^{1,+} = d \quad \text{in}\,(0,\tau]$$

(see (54)). Then we have:

$$[|u_1 - u_2|]_1 = |u_1^{1,+} - u_2| - |u_1^{1,-} - u_2| = [u_1]_1\,;$$

$$[\mathrm{sgn}\,(u_1 - u_2)(-v_{1x} + v_{2x})]_1$$

$$= \mathrm{sgn}\,(u_1^{1,+} - u_2)(-v_{1x}^{1,+} + v_{2x}) - \mathrm{sgn}\,(u_1^{1,-} - u_2)(-v_{1x}^{1,-} + v_{2x}) = -[v_{1x}]_1\,.$$

In view of the Rankine-Hugoniot condition (50) we obtain:

$$-[|u_1 - u_2|]_1\, \xi_1' + [\mathrm{sgn}\,(u_1 - u_2)(-v_{1x} + v_{2x})]_1 = -[u_1]_1\, \xi_1' - [v_{1x}]_1 = 0\,;$$

hence the first integral in the right-hand side of equality (56) vanishes.

Concerning the second one, observe that the assumption $\xi_2(t) < 0$ for any $t \in (0,\tau]$ implies the existence of a sequence $t_n \to 0^+$ such that $\xi_2'(t_n) < 0$ strictly for any $n \in \mathbb{N}$. In view of Theorem 21, this implies $v_2(\xi_2(t_n), t_n) = B$, thus:

$$\lim_{t_n \to 0^+} v_2(\xi_2(t_n), t_n) = v_2(0,0) = \phi(u_0(0)) = B\,.$$

Then by assumption $(H_3) - (iii)$ there exists a right neighbourhood $(0,\bar{x})$ such that either $\phi(u_0(x)) < B$ or $\phi(u_0(x)) > B$ for any $x \in (0,\bar{x})$.

However, the former possibility is ruled out by Theorem 21, for it would imply $\xi_2'(t) \geq 0$ for any $t \in (0,\tau]$. Hence there holds $u_0(x) > d$ for any $x \in (0,\bar{x})$. By continuity (possibly choosing a smaller τ) this implies

$$u_1(\xi_2(t),t) > d > b \quad \text{in } (0,\tau]$$

(recall that u_1 is regular along $(t,\xi_2(t))$); moreover,

$$u_2^{2,-} = b\,, \quad u_2^{2,+} = d \quad \text{in } (0,\tau]\,.$$

Then we have:

$$[|u_1 - u_2|]_2 = |u_1 - u_2^{2,+}| - |u_1 - u_2^{2,-}| = -[u_2]_2\,;$$

$$[\operatorname{sgn}(u_1 - u_2)(-v_{1x} + v_{2x})]_2$$

$$= \operatorname{sgn}(u_1 - u_2^{2,+})(-v_{1x} + v_{2x}^{2,+}) - \operatorname{sgn}(u_1 - u_2^{2,-})(-v_{1x} + v_{2x}^{2,-}) = [v_{2x}]_2\,.$$

By the Rankine-Hugoniot condition (50) we have:

$$-[|u_1 - u_2|]_2\,\xi_2' + [\operatorname{sgn}(u_1 - u_2)(-v_{1x} + v_{2x})]_2 = [u_2]_2\,\xi_2' + [v_{2x}]_2 = 0\,.$$

Hence the second integral in the right-hand side of equality (56) vanishes, too. This implies $u_1(\cdot,\tau) = u_2(\cdot,\tau)$ a.e. in $(-L,L)$, thus $\xi_1(\tau) = \xi_2(\tau)$, a contradiction.

This completes the proof in the case $\xi_1(t) < \xi_2(t) < 0$ for any $t \in (0,\tau]$. The remaining cases can be dealt with similarly (see [12]); hence the result follows. ♣

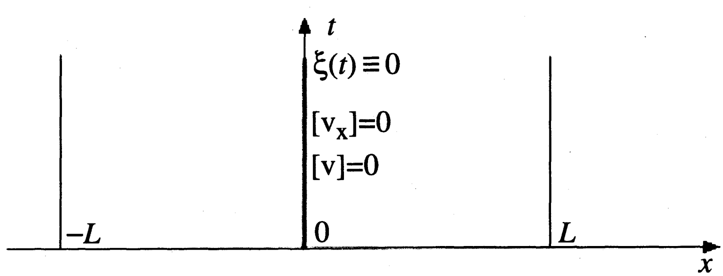

Fig. 5. Solution of the steady boundary problem

Local existence is the content of the following

Theorem 23. *Let (H_3) be satisfied; assume $[\phi(u_0)_x](0) \neq 0$. Then there exists $\tau > 0$ such that problem (46) has a two-phase solution in Q_τ.*

Sketch of the proof. (*i*) If $\phi(u_0)(0) \in (A, B)$, the solution of the steady boundary problem

$$\text{(57)} \qquad \begin{cases} u_t = v_{xx} & \text{in } V_i \quad (i = 1, 2) \\ [v] = [v_x] = 0 & \text{in } \{0\} \times (0, T] \\ v = \phi(u_0) & \text{in } (-L, L) \times \{0\} \\ v_x \equiv 0 & \text{in } \{-L, L\} \times (0, T) \end{cases}$$

(see Fig. 5) gives a local solution of problem (46). Hence the conclusion in this case.

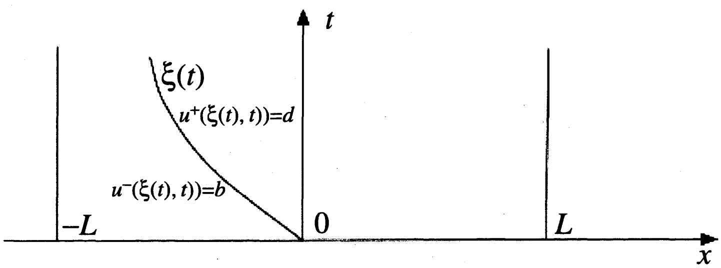

Fig. 6. Solution of the moving boundary problem

(*ii*) Let $\phi(u_0)(0) = B$ (the case $\phi(u_0)(0) = A$ is similar). If $[\phi(u_0)_x](0) > 0$, we find by iteration a solution (ξ, u, v) of the moving boundary problem:

$$\begin{cases} u_t = v_{xx} & \text{in } V_i \ (i=1,2) \\ [v](\xi(t), t) = 0 & \text{in } (0, T] \\ u^-(\xi(t), t) = b, \ u^+(v)(\xi(t), t) = d & \text{in } (0, T] \\ v = \phi(u_0) & \text{in } (-L, L) \times \{0\} \\ v_x \equiv 0 & \text{in } \{-L, L\} \times (0, T) \end{cases}$$

(see Fig. 6). As before, this gives a local solution of (46). If $[\phi(u_0)_x](0) < 0$, we solve as in (i) the steady boundary problem (57). This gives the result.

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