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PDE and differential geometry in study of motion of elastic wires

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Abstract.

We study some PDE's of motion of elastic wires. These PDE's naturally require differential geometry. We mainly treat closed curves.

§1. Equation of Caflish and Maddocks

There are several PDE's of elastic wires. The most classic one was given by Courant and Hilbert [1, p. 246]:

$$\gamma_{xxxx} + \gamma_{tt} = \text{force}.$$

When we consider more honest (nonlinear) ones, the most classic one was given by Caflish and Maddocks. It is an equation of arc-length parametrized curve $\gamma(x,t)$ of length 1 in the euclidean plane $\mathbf{R}^2$. The constrained condition $|\gamma_x| \equiv 1$ means that the curve doesn't expand nor shrink.

We define the potential energy of the elastic wire $\gamma(x)$ by the elastic energy, i.e., the square integral $U = \int_0^1 |\gamma_{xx}|^2 \, dx$ of the curvature $|\gamma_{xx}|$. And, we define the kinetic energy of the motion of the wire by $E = \int_0^1 (|\gamma_t|^2 + |\gamma_{xt}|^2) \, dx$. Here, the term $|\gamma_t|^2$ means the usual velocity energy, and the term $|\gamma_{xt}|^2$ comes from the thickness of the wire.

Following to Hamilton's principle, the equation of motion of our curve is given as critical points of the variational problem defined by the functional

$$F = \int_0^T E - U \, dt = \int_0^T \int_0^1 (|\gamma_t|^2 + |\gamma_{xt}|^2 - |\gamma_{xx}|^2) \, dxdt.$$

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The Euler-Lagrange equation of this functional becomes:

$$\text{(CM)} \qquad \begin{cases} \gamma_{ttxx} - \gamma_{xxxx} - \gamma_{tt} = (u\gamma_x)_x, \\ |\gamma_x| \equiv 1, \end{cases}$$

where the function $u(x,t)$ is a Lagrange's unknown function, which is determined by the constrained condition.

They solved this equation as follows. Integrate equation (CM) in x,

$$\gamma_{ttx} - \gamma_{xxx} - \int_0^x \gamma_{tt}\, dx - \gamma_{tt}(0,t) = u\gamma_x$$

and take the tangential parts to S^1. It eliminates the unknown function u. Introduce a new unknown function $\theta(x,t)$ by

$$\gamma_x = (\cos\theta, \sin\theta)$$

and roughly put

$$\theta_{tt} - \theta_{xx} \sim: \phi.$$

Then we get a coupled equation:

$$\begin{cases} \theta_{tt} - \theta_{xx} = \phi + \text{nonlinear}(\theta_x, \theta_t), \\ \phi(x) + \displaystyle\int_0^1 \kappa(x,y)\phi(y)\, dy = \text{nonlinear}(\theta_x, \theta_t); \\ \qquad \kappa(x,y) := 1 - \max\{x,y\}. \end{cases}$$

Both of the wave equation and the integral equation have good properties, and we can solve equation (CM).

Theorem 1.1 (Caflish and Maddocks [2]). *The equation* (CM) *has a unique global solution, under suitable boundary conditions.*

Now, let's generalize this result to 3 dimensional case. The problem is that we don't have good parameters as θ, since $\gamma_x \in S^2$. Putting $\xi := \gamma_x$, calculate as above. We get:

$$\begin{cases} (\xi_{tt} - \xi_{xx})^T = \phi + \text{nonlinear}, \\ \phi(x) + \displaystyle\int_0^1 \kappa(x,y)\phi(y)\, dy = \text{nonlinear}, \end{cases}$$

where $*^T$ denotes the tangential part to the sphere S^2. The term $(\xi_{xx})^T$ is called covariant derivative on the sphere, and denoted by $\nabla_x \xi_x$. In local coordinates $\{y^1, y^2\}$ of S^2, it is expressed as

$$(\nabla_x \xi_x)^i = \xi^i_{xx} + \Gamma_j{}^i{}_k(\xi)\xi^j_x\xi^k_x,$$
$$(\xi_{tt} - \xi_{xx})^T = \nabla_t\xi_t - \nabla_x\xi_x = (\xi^i_{tt} - \xi^i_{xx})\frac{\partial}{\partial y^i} + \text{lower nonlinear},$$

where $\Gamma_j{}^i{}_k$ is Christoffel's symbol.

Therefore, essentially we recover a wave equation. However, if we use concrete local coordinates such as polar coordinates, the nonlinear term would become very complicated. Moreover, the solution may go beyond the coordinate neighbourhood at finite time. Using covariant differentiation, the complication becomes milder.

We assume boundary point condition (B) and boundary direction condition (E):

(B) the boundary point $\gamma(0)$ belongs to an affine subspace P_0 of $\mathbf{R}^n$ and the vector $\gamma(1) - \gamma(0)$ belongs to an affine subspace P_1.

(E) the boundary direction $\gamma_x(0)$ and $\gamma_x(1)$ are free or fixed (4 types), or $\gamma_x(1) = \gamma_x(0)$.

The equation we get is

$$\nabla_t\xi_t - \nabla_x\xi_x = \phi,$$
$$\phi + L_{P_0}(\phi)^\top - v^\top$$
$$= -\xi(1)^\top + (1-x)\{\xi_x(0)^{P_0^\perp} + [\xi]_0^{1\,P_0^T}\}^\top + L_{P_0}((|\xi_t|^2 - |\xi_x|^2)\xi)^\top,$$
$$\{\int_0^1 \phi\, dx\}^{P_1^\perp} = \{\int_0^1 (|\xi_t|^2 - |\xi_x|^2)\xi\, dx\}^{P_1^\perp} - [\xi_x]_0^{1\,P_1^\perp},$$

where P_i^T is the sub vector space give by translating P_i to the origin, $P_i^\perp$ is the orthogonal complement vector space to P_i^T, and $*^{P_i^T}$ (resp. $*^{P_i^\perp}$) is the projection to P_i^T (resp. $P_i^\perp$), and operator L_{P_0} is defined by

$$L_{P_0}(f) := \int_0^1 \kappa(x,y)f(y)\, dy - \{\int_0^1 (1-x)(1-y)f(y)\, dy\}^{P_0^T}.$$

The function $v = v(t)$ is a $P_1^\perp$-valued unknown.

And, we can prove the existence of solutions.

Theorem 1.2 ([12]). *Equation* (CM) *in the n-dimensional euclidean space (more precisely, the above equation) has a unique global solution, under suitable boundary conditions.*

The "suitable boundary conditions" include the free boundary condition, the fixed boundary condition, and closed curve condition. More precisely, we consider the following boundary conditions.

(a) $P_0 = \mathbf{R}^N$ and ($P_1 = \mathbf{R}^N$ or ($\gamma_x(0)$ and $\gamma_x(1)$ are free)),
(b) $\gamma_x(0)$ is free and ($P_1 = \mathbf{R}^N$ or $\gamma_x(1)$ is free),
(c) $P_0 = \mathbf{R}^N$ and $P_1 = \{0\}$ and $\gamma_x(0) = \gamma_x(1)$.

We can prove the global existence under these boundary conditions.

§2. 1-dimensional plate equation

From now on, we treat only closed curves. In equation (CM), each term corresponds to an energy in Hamilton's principle.

γ_{tt}	usual kinetic (velocity) energy	$\lvert\gamma_t\rvert^2$
γ_{xxxx}	usual potential (elastic) energy	$\lvert\gamma_{xx}\rvert^2$
γ_{ttxx}	kinetic energy caused by the thickness of the wire	$\lvert\gamma_{tx}\rvert^2$

When the wire is thick, changing its tangent line moves mass away from the center of the wire. When we treat infinitely thin elastic wire, we have to drop the term γ_{ttxx}. And get a 1-dimensional plate equation:

$$\text{(PL)} \qquad -\gamma_{xxxx} - \gamma_{tt} = (u\gamma_x)_x.$$

The linear version of this equation is nothing but the equation proposed by Courant and Hilbert.

The plate equation is unstable under perturbation of lower derivatives. Therefore, we cannot see whether it has solutions, if we only look it. In fact, from the condition $|\gamma_x| = 1$, u satisfies an ODE:

$$-u_{xx} + |\gamma_{xx}|^2 u = 2(|\gamma_{xx}|^2)_{xx} - |\gamma_{xxx}|^2 + |\gamma_{tx}|^2,$$

hence u is at least bad as γ_{xx}, and $(u\gamma_x)_x$ is bad as γ_{xxx}.

As usual, we perturb (PL) to a parabolic type equation with principal terms:

$$\begin{aligned}&\gamma_{tt} - 2\varepsilon\gamma_{txx} + (1+\varepsilon^2)\gamma_{xxxx}\\ &= \{\frac{\partial}{\partial t} - (\varepsilon + \sqrt{-1})\frac{\partial^2}{\partial x^2}\}\{\frac{\partial}{\partial t} - (\varepsilon - \sqrt{-1})\frac{\partial^2}{\partial x^2}\}\gamma.\end{aligned}$$

and we can solve this equation for each $\varepsilon > 0$. But we cannot control the solution when $\varepsilon \to 0$. The existence of γ_{xxx} disturbs us.

We have to use covariant differentiation. We introduce a new unknown $\xi := \gamma_x \in S^2$. The the "bad term" becomes ξ_{xx}. We perturb

(PL) to a parabolic type principal terms

$$\nabla_t \xi_t - 2\varepsilon \nabla_x^2 \xi_t + (1+\varepsilon^2)\nabla_x^3 \xi_x.$$

Note that the term $\nabla_x^3 \xi_x$ still contains 3rd derivative $\Gamma_j{}^i{}_k(\xi)\xi_x^j \xi_{xxx}^k$ in local coordinates. However, it's a kind of magic, we can compute as though there were no such terms in integration by parts:

$$\int_0^1 (\nabla_x^3 \xi_x, f)\, dx = -\int_0^1 (\nabla_x^2 \xi_x, \nabla_x f)\, dx.$$

Due to this fact, we can get ε-uniform estimation of solutions of perturbed equation.

Theorem 2.1 ([8]). *Equation* (PL) *has a unique local solution.*

The author doesn't know whether the solution exists globally or diverges at finite time. Note that equation (CM) has global solutions.

This result is generalized to the riemannian version. The equation on riemannian manifold is given by

$$-\nabla_x^3 \gamma_x - \nabla_t \gamma_t = \nabla_x(u\gamma_x) - R(\gamma_x, \nabla_x \gamma_x)\gamma_x,$$

where R is the curvature tensor of the riemannian metric. For example, $R(X,Y)Z = (Y,Z)X - (X,Z)Y$ on the standard sphere S^n.

Theorem 2.2 ([10]). *The above equation has a unique local solution.*

§3. Singular perturbation

Now we add a resistance to (PL):

$$(\mathrm{PL}^\mu) \qquad -\gamma_{xxxx} - \gamma_{tt} = (u\gamma_x)_x + \mu\gamma_t.$$

When the resistance μ is very large, the curve γ moves very slowly. Therefore, to study the behavior of solutions when $\mu \to \infty$, we have to change time variable to $\tau := \mu^{-1}t$. The equation becomes

$$(\mathrm{PL}^\tau) \qquad -\gamma_{xxxx} - \mu^{-2}\gamma_{\tau\tau} = (u\gamma_x)_x + \gamma_\tau,$$

and it converges to a parabolic equation

$$(\mathrm{PB}) \qquad -\eta_{xxxx} = (u\eta_x)_x + \eta_\tau.$$

This equation is nothing but "method of steepest decent" for seeking elastica. The parabolic equation (PB) is not difficult to solve.

Theorem 3.1 ([4]). *Equation* (PB) *has a unique global solution, and the solution converges to one of elastica.*

Even if we replace the euclidean space to general riemannian manifold, we have the following.

Theorem 3.2 ([7]). *Let* (M, g) *be a compact real analytic riemannian manifold, and let* $\gamma_0(x)$ *be a closed arc-length parametrized curve of length* 1. *Suppose that there are no closed geodesics of length* 1 *on the manifold* M. *Then riemannian version of equation* (PB) *has a unique solution* $\gamma(x, t)$ *for all time, and the solution* $\gamma(*, t)$ *converges to an elastica when* $t \to \infty$.

Note that there are no closed geodesic in the euclidean space. The convergence when t tends to ∞ is rather delicate problem. There are counter examples if we only assume that the riemannian manifold is C^∞-class. However, even without real analyticity, the solution has a convergent subsequence that converges to an elastica. So the purpose of "method of steepest decent" is satisfied.

We ask main question here: Does the solution $\gamma^\mu(x, \tau)$ of (PL^τ) converge to the solution $\eta(x, \tau)$ of (PB)? Note that the initial velocity with respect to τ is $\gamma^\mu_\tau(x, 0) = \mu\gamma_1(x)$ and diverges when $\mu \to \infty$. The answer is:

Theorem 3.3 ([8]). *For any initial data* $\gamma_0(x)$, $\gamma_1(x)$ *for* (PL^μ), *and for any positive time* T, *if we take sufficiently large* μ, *then the solution* $\gamma^\mu(x, \tau)$ *of* (PL^τ) *with initial data* $\gamma^\mu(x, 0) = \gamma_0(x)$, $\gamma^\mu_\tau(x, 0) = \mu\gamma_1(x)$ *exists on the time interval* $[0, T]$. *Moreover,* $\gamma^\mu(x, \tau)$ *converges to the solution* $\eta(x, \tau)$ *of* (PB) *with initial data* $\eta(x, 0) = \gamma_0(x)$ *when* $\mu \to \infty$. *The convergence is uniform in t, and the convergence of* γ^μ_τ *is uniform in any compact set of* $(0, T]$.

To see what happens, we consider a simple ODE:

$$-\mu^{-2} v''(\tau) = v'(\tau); \; v(0) = 0, v'(0) = \mu.$$

The solution is $v(\tau) = \mu^{-1}(1 - e^{-\mu^2\tau})$. The initial value $v'(0)$ diverges as prescribed, but $v(\tau)$ converges to 0, and $v'(\tau) = \mu e^{-\mu^2\tau}$ converges to 0 where τ is away from 0.

The author believes that this phenomena of singular perturbation is very general. Let $(M; x^i)$ and $(N; y^p)$ be compact riemannian manifolds and consider a PDE of map $\varphi(x, t)$ from M to N:

$$\text{(WP)} \qquad \mu^{-2}\nabla_\tau\varphi_\tau + \varphi_\tau + \widetilde{\Delta}\varphi = 0,$$

where

$$\nabla_\tau \varphi_\tau := \{\varphi^p_{\tau\tau} + \Gamma_q{}^p{}_r(\varphi)\varphi^q_\tau \varphi^r_\tau\}\frac{\partial}{\partial y^p},$$

$$\widetilde{\Delta}\varphi := -g^{ij}(x)\{\frac{\partial^2 \varphi^p}{\partial x^i \partial x^j} - \Gamma_i{}^k{}_j(x)\frac{\partial \varphi^p}{\partial x^k} + \Gamma_q{}^p{}_r(\varphi)\frac{\partial \varphi^q}{\partial x^i}\frac{\partial \varphi^r}{\partial x^j}\}\frac{\partial}{\partial y^p}.$$

Its parabolic version ($\mu = \infty$):

$$\varphi_\tau + \widetilde{\Delta}\varphi = 0,$$

is called Eells-Sampson equation in differential geometry. (Note that sign convention of Laplacian is opposite of analysis.)

Theorem 3.4 ([5]). *For equation* (WP), *we have same type theorem with the above* (PL^τ)-(PB) *singular perturbation.*

Note that, however, the nonlinearity with respect to φ_τ is expressed as $\mu^{-1}\varphi^q_\tau$. When we artificially replace it by $\mu^{-1+a}\varphi^q_\tau$ ($a > 0$), we have a counter example of convergence. The expression $\mu^{-1}\varphi^q_\tau$ comes from the fact that the nonlinearity of lower derivatives in 2nd covariant derivative is a polynomial of degree 2. The author does not know the meaning of this exponent 2 from view of analysis. At least we can say: "Differential geometry is very well constructed."

The author gave very few literature concerning dynamics of elastica. Please consult reviews in WWW. For example, search "dynamic*" and/or "elastic*" in MathSciNet.

§4. Generalization to Riemannian manifold

When we consider equations of motion of elastic wire $\gamma(x,t)$, we are forced to consider a "curved space" S^n as a target manifold.

There is another situation when we consider riemannian manifold: Given a simple PDE of $\mathbf{R}^n$-valued function. We want to generalize the PDE to a riemannian version.

If the original PDE is stable under perturbation of lower derivatives, the local existence for riemannian version should be easy. And the main work should be done on long time existence, singular perturbation, and so on.

If the original PDE is unstable, then the local solution may not exist. So the main work should be done on it, at least firstly.

In both cases, the author believes, if the original PDE is good, we will get good results on riemannian version. If we are interested in semilinear PDE, let's try riemannian version.

As the last equation, consider the vortex filament equation:

$$\gamma_t = \gamma_x \times \gamma_{xx} \quad (|\gamma_x| = 1),$$

where $\times$ means the cross product in 3-dimensional euclidean space $\mathbf{R}^3$. Using $\xi := \gamma_x \in S^2$, this equation is written to

$$\xi_t = \xi_x \times \xi_x + \xi \times \xi_{xx} = J\big((\xi_{xx})^T\big),$$

where J means $\pi/2$ rotation on the each tangent space of S^2. It is called almost complex structure of S^2. In fact, it is nothing but the multiplication of $\sqrt{-1}$ via the stereographic projection: $S^2 \to \mathbf{R}^2 = \mathbf{C}$.

Therefore, this equation is converted to

$$\xi_t = J\nabla_x \xi_x$$

on S^2. Hence it can be viewed as a riemannian version of linear Schrödinger equation:

$$u_t = \sqrt{-1}u_{xx}.$$

The author noticed this fact far after Hasimoto "solved" the vortex filament equation. But still it proves the "well construction" of riemannian geometry. In fact, we can solve the equation $\xi_t = J\nabla_x \xi_x$, if these two structures: the riemannian structure g that defines the covariant differentiation ∇, and the complex structure J, are well compatible (Kähler manifold), then we have local existence theorem of solutions. Moreover, if these two structures have good symmetry, then we have long time existence theorem.

Theorem 4.1 ([6]). *Semilinear Schrödinger equation: $\xi_t = J\nabla_x \xi_x$ on any Kähler manifold has a local solution. On hermitian symmetric spaces (e.g. complex projective spaces), it has global solutions.*

To prove this theorem, we again use perturbation to parabolic equations:

$$\xi_t = \varepsilon \nabla_x \xi_x + J\nabla_x \xi_x.$$

The local solution of this equation can be uniformly estimated with respect to ε, hence has a convergent subsequence, whose limit is the solution we expected.

On the other hand, we can prove the existence of solutions of the original vortex filament equation, in a sense via Hasimoto's transformation, as follows. For a space arc-length parametrized curve $\gamma(x)$, we choose a unit vector field $E(x)$ along $\gamma(x)$ so that $E(x)$ is perpendicular to $T(x) := \gamma'(x)$ and $E'(x)$ is parallel to $T(x)$. This ODE of $E(x)$ has a unique solution when the initial value $E(0)$ is given. Let $F(x)$ be the unit vector field along $\gamma(x)$ defined by $F(x) := T(x) \times E(x)$. Then, it holds that $F'(x)$ is parallel to $T(x)$, and the frame field $\{T(x), E(x), F(x)\}$ becomes a orthogonal frame field along $\gamma(x)$.

Define a real valued function $p(x)$, $q(x)$ by $E'(x) = p(x)T(x)$, $F'(x) = q(x)T(x)$, and a complex valued function $z(x) := p(x)+q(x)\sqrt{-1}$. Then, we can easily check that $|z(x)| = \kappa(x)$, the curvature, and $(\arg z(x))' = \tau(x)$, the torsion of the curve. Therefore, we see that

$$z(x) = \kappa(x) \exp(\sqrt{-1} \int \tau(x)\, dx).$$

This transformation is nothing but Hasimoto's transformation. Note that the right hand side of the equality is not defined beyond the point where $\kappa(x) = 0$, but the left hand side $z(x)$ is well defined.

Using the frame $\{T, E, F\}$ and functions p, q, the vortex filament equation is converted to the nonlinear Schrödinger equation:

$$z_t = \sqrt{-1}\{z_{xx} + \frac{1}{2}|z^2|z\},$$

hence we can solve it.

Historically, Hasimoto [3] proved the existence using the function z defined by the curvature and the torsion. Hence his proof had a gap mathematically. Later, Nishiyama and Tani [13] proved the existence using a perturbation to parabolic equation.

Using our method, we can generalize the existence theorem to 3-dimensional space forms, i.e., on the standard sphere S^3 and on the standard hyperbolic space H^3. Moreover, we can generalize the result to any 3-dimensional riemannian manifolds.

Theorem 4.2 ([9, 11]). *The vortex filament equation* $\gamma_t = \gamma_x \times \nabla_x \gamma_x$ *in any* 3*-dimensional riemannian manifold* (M, g) *has local solutions. If the sectional curvature of* (M, g) *is bounded, then the equation has global solutions.*

References

[1] R. Courant and D. Hilbert, Methods of mathematical physics Vol. I, Interscience publishers INC, New York, 1966.

[2] R. E. Caflish and J. H. Maddocks, Nonlinear dynamical theory of the elastica, Proc. of the Royal Society of Edinburgh, **99A** (1984), 1–23.

[3] H. Hasimoto, A soliton on a vortex filament, J. Fluid Mech., **51** (1972), 477–485.

[4] N. Koiso, On the motion of a curve towards elastica, Actes de la table ronde de géométrie différentielle en l'honneur de Marcel Berger (Collection SMF Séminaires & Congrès no 1, ed. A. L. Besse). (1996), 403–436.
[5] N. Koiso, On singular perturbation of a semilinear hyperbolic equation, Calc. Var., **4** (1996), 89–101.
[6] N. Koiso, The vortex filament equation and a semilinear Schrödinger equation in a hermitian symmetric space, Osaka J. Math., **34** (1997), 199–214.
[7] N. Koiso, Convergence towards an elastica in a riemannian manifold, Osaka J. Math., **37** (2000), 467–487.
[8] N. Koiso, On motion of an elastic wire and singular perturbation of a 1-dimensional plate equation, Osaka J. Math., **37** (2000), 905–924.
[9] N. Koiso, Vortex filament equation in a riemannian manifold, Tohoku Math. J., **55** (2003), 311–320.
[10] N. Koiso, On motion of an elastic wire in a riemannian manifold, to appear.
[11] N. Koiso, Long time existence for vortex filament equation in a riemannian manifold, to appear.
[12] N. Koiso and M. Sugimoto, Motion of elastic wire with thickness, to appear.
[13] T. Nishiyama and R. Tani, Solvability of the localized induction equation for vortex motion, Commun. Math. Phys., **162** (1994), 433–445.

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