Advanced Studies in Pure Mathematics 47-2, 2007
Asymptotic Analysis and Singularities
pp. 421–440

Hydrodynamic limit and nonlinear PDEs with singularities

Tadahisa Funaki

Abstract.

We shall discuss the derivation, via a scaling limit starting from large-scale interacting systems, of a certain class of nonlinear partial differential equations, especially with singular structures, such as the Stefan problem, the free boundary problem of elliptic type, an evolutionary variational inequality and a stochastic partial differential equation with reflection. A family of independent Brownian particles is first taken up as a simple example to explain the idea behind the scaling limit. Then, we survey several results concerning two kinds of model: the interacting particle systems on $\mathbb{Z}^d$ called lattice gases and the $\nabla\varphi$ interface model, which is a microscopic system for interfaces separating two distinct phases. The entropy method, which plays an essential role in the proof of the hydrodynamic limit, is explained in brief.

§1. Introduction

When we write down nonlinear partial differential equations (PDEs), the explanations are sometimes begun with a picture of underlying particle systems, such as the Laplacian Δ represents a diffusive motion of particles and a nonlinear term comes from interactions among them and so on. One of our goals, especially for those who are working in probability theory, is to lay mathematically rigorous foundations to such picture starting from various large-scale microscopic interacting systems. In the process to achieve the goal, the basic role is played by the so-called **local ergodicity** or, in other words, the **local equilibria**. In a large-scale interacting system, after a (microscopically) long time passes, detailed informations of dynamics are, in a sense, averaged out and one can eventually observe only a slow motion of conserved quantities, which is

Received November 18, 2005.
Revised December 26, 2005.
The author is partially supported by JSPS, Grant-in-Aids for Scientific Research (B) 14340029 and for Exploratory Research 17654020.

usually described by a nonlinear PDE at a macroscopic level of space and time.

This procedure is customarily called the **hydrodynamic limit**. An approach via the entropies and their time-derivatives called the entropy productions, which is rather universally applicable, was founded by Varadhan and his coauthors [25] in 1988, see also [28] which gives a good review of the theory of the hydrodynamic limit. Due to their method, several kinds of nonlinear PDE are derived from microscopic systems by now as we shall see below. The (relative) entropy measures the distance of the present states (at microscopic level) from the equilibrium ones (Gibbs measures) and acts as a Lyapunov function. Section 6 gives a flavor of this method.

Before starting our main discussions, Section 2 provides a simple example, i.e. a system of Brownian particles without interaction, and briefly explains how one can derive the macroscopic equation, i.e. heat (or diffusion) equation in this system, under a scaling limit which connects microscopic to macroscopic levels due to the local ergodicity or the local equilibria held at microscopic level. Then, as well-studied interacting systems, the present paper discusses the lattice gases and the $\nabla\varphi$ interface model.

Section 3 surveys the results on the **lattice gases** sometimes called **Kawasaki dynamics**, which are the discretization of interacting Brownian particles, i.e. a system of particles performing random walks on $\mathbb{Z}^d$ with exclusion rule. It is discussed under the diffusive scaling limit the derivation of reaction-diffusion equation from Glauber-Kawasaki dynamics (i.e. Kawasaki dynamics with creation and annihilation), nonlinear diffusion equation with a bulk diffusion coefficient characterized by a variational formula (Green-Kubo formula) and one or two phases' **Stefan problems** with absorption or reflection from Kawasaki dynamics with two types of particles. The derivation of an inviscid Burgers' equation under the hyperbolic scaling limit is also discussed.

Sections 4 and 5 deal with the **$\nabla\varphi$ interface model**, especially, considered over the wall or under the weak effects of additional self-potentials. The results are classified into two types:

(1) (Static theory, Section 4) For the corresponding equilibrium states (Gibbs measures), by establishing large deviation principles, one can deduce the asymptotic behavior of the measures and the limits are described by variational problems which characterize, for instance, Wulff shape, Winterbottom shape, **free boundary problems** (of Alt and Caffarelli's or Alt, Caffarelli and Friedman's type) and others.

(2) (Dynamic theory, Section 5) The corresponding dynamics are defined through the Langevin equation (of non-conservative or conservative type; a kind of stochastic differential equation). The **motion by mean curvature** (MMC) with spatial anisotropy is derived under the hydrodynamic space-time scaling limit and the corresponding dynamic large deviation principle is obtained. Then, the hydrodynamic limit and the equilibrium fluctuation are discussed under a presence of hard walls. The limit is characterized by an **evolutionary variational inequality**, i.e., a nonlinear PDE with solutions conditioned to stay over the wall (i.e. the MMC with anisotropy and obstacle). The dynamic fluctuation is described by a stochastic PDE (PDE with random term) of Nualart and Pardoux type (SPDE with reflection) in one dimension.

The results concerning the interacting particle systems (especially the so-called zero-range processes) and the $\nabla\varphi$ interface model are reviewed in the book [28] and the lecture notes [17], respectively.

§2. An example — independent Brownian particles

The historical reason that the scaling limit we are concerned is called the hydrodynamic limit may be understood by reminding the procedure of the derivation of hydrodynamic equations from the Boltzmann equation. It is a perturbative method known for a long time as Hilbert or Chapman-Enskog expansions [5] and its mathematical foundation was given by Ukai and others, [4], [42], [30]. In these expansions, the Maxwellian distributions play the role of local equilibria and the evolutional law for the components of conserved quantities (i.e. mass, momentum, energy) is governed by the hydrodynamic equations. However, the Boltzmann equation itself is derived already under certain scaling limit and the ergodicity was used in an essential way in the derivation. Our goal is to start with the particle systems at molecular level which underlie behind the Boltzmann equation. Such idea, at least in the sense to try to discuss in mathematically rigorous manner, goes back to Morrey [31].

In order to explain the mathematical idea behind the hydrodynamic limit, we introduce a simple model which is indeed almost trivial from the probabilistic point of view. Namely, as a microscopic particle system, we consider a system of **independent Brownian particles** $\{y_i(s) = y_i + B_i(s)\}_{i=1,2,\ldots}, s \geq 0$ in $\mathbb{R}^d$ with the position $y_i(s)$ of ith particle starting at y_i. Let us introduce the corresponding macroscopic system $\{x_i^\epsilon(t) := \epsilon y_i(t/\epsilon^2)\}_{i=1,2,\ldots}$ by means of the (diffusive) scaling in time and space: $t = \epsilon^2 s, x = \epsilon y$, where ϵ represents the ratio of macroscopic and microscopic spatial lengths, which goes to 0 eventually. The

macroscopic density field (random measure on $\mathbb{R}^d$) is associated with $\{x_i^\epsilon(t)\}$ by

$$\xi_t^\epsilon(dx) = \epsilon^d \sum_{i=1}^{\infty} \delta_{x_i^\epsilon(t)}(dx), \tag{2.1}$$

i.e. for $A \subset \mathbb{R}^d$, $\xi_t^\epsilon(A) = \epsilon^d \times \sharp\{i; x_i^\epsilon(t) \in A\}$ describes the total mass of particles inside the region A at macroscopic level; under the scaling, the mass of a single particle is also scaled as ϵ^d.

Macroscopic view point. We call $u(x)$ an asymptotic density field (ADF) of a sequence of random measures $\{\xi^\epsilon(dx)\}$ if a weak law of large numbers: $\langle \xi^\epsilon, \varphi\rangle \to \langle u, \varphi\rangle$ holds as $\epsilon \downarrow 0$ in probability or in L^2-sense for all $\varphi \in C_0^\infty(\mathbb{R}^d)$.

Theorem 2.1 (Macroscopic formulation). *If ξ_0^ϵ admits an ADF u_0 (in L^1-sense), then $\{\xi_t^\epsilon\}$ defined by* (2.1) *also has an ADF $u(t)$ (in L^2-sense) which is a solution of the heat equation: $\partial u/\partial t = \Delta u/2$ with initial condition $u(0) = u_0$.*

Proof. The proof is easy. In fact, we have

$$\begin{aligned} E[\langle \xi_t^\epsilon, \varphi\rangle] &= \epsilon^d \sum_{i=1}^{\infty} E[\varphi(x_i^\epsilon(t))] = \epsilon^d \sum_{i=1}^{\infty} E[\varphi_t(x_i^\epsilon(0))] \\ &= E[\langle \xi_0^\epsilon, \varphi_t\rangle] \to \langle u_0, \varphi_t\rangle = \langle u(t), \varphi\rangle, \end{aligned} \tag{2.2}$$

as $\epsilon \downarrow 0$, where φ_t denotes the solution of the heat equation with initial value φ. Moreover,

$$\begin{aligned} &E\left[\left\{\langle \xi_t^\epsilon, \varphi\rangle - E[\langle \xi_t^\epsilon, \varphi\rangle]\right\}^2\right] \\ &\quad = E\left[\left\{\epsilon^d \sum_{i=1}^{\infty} \left(\varphi(x_i^\epsilon(t)) - E[\varphi(x_i^\epsilon(t))]\right)\right\}^2\right] \\ &\quad = \epsilon^{2d} \sum_{i=1}^{\infty} E\left[\left\{\varphi(x_i^\epsilon(t)) - E[\varphi(x_i^\epsilon(t))]\right\}^2\right] \\ &\quad \le \epsilon^{2d} \sum_{i=1}^{\infty} E\left[\varphi^2(x_i^\epsilon(t))\right] = \epsilon^d E[\langle \xi_t^\epsilon, \varphi^2\rangle] \longrightarrow 0, \end{aligned} \tag{2.3}$$

as $\epsilon \downarrow 0$. The second equality is from the independence of particles, while the last term tends to 0 because of (2.2). The proof is complete by combining (2.3) with (2.2). Q.E.D.

Microscopic view point. We have the following characterization of all equilibrium states of independent motions (see, e.g. [11]):

Proposition 2.2. *The equilibrium states of the independent Brownian particles are necessarily superpositions of the* **Poisson point field** *$\{\mu_\rho\}$ parametrized by the density $\rho \in [0, \infty)$.*

Here, μ_ρ is a probability measure on the space:

$$\mathcal{A} = \{\zeta = \{y_i\}_{i=1,2,\dots}; \text{infinite but locally finite sets of points in } \mathbb{R}^d\}$$

such that $\{\mu_\rho(\sharp(A \cap \zeta) = k)\}_{k=0,1,2,\dots}$ is Poisson distributed for $A \subset \mathbb{R}^d$ with mean $\rho|A|$, where $|A|$ stands for the Lebesgue measure of A, and $\{\sharp(A_j \cap \zeta)\}_{j=1,2,\dots,n}$ are mutually independent under μ_ρ if $\{A_j \subset \mathbb{R}^d\}_{j=1,2,\dots,n}$ are disjoint. This proposition is related to the conservation law of particles' number under the time evolution $\{y_i(s)\}_{i=1,2,\dots}$ of the independent Brownian particles.

Microscopic state of $\{y_i(s)\}_{i=1,2,\dots}$ at macroscopic time t and space x is defined by the distribution $\nu^\epsilon_{t,x}$ of $\{y_i(t/\epsilon^2) - x/\epsilon\}_{i=1,2,\dots}$ on $\mathcal{A}$. We say in general that a sequence of families of measures $\{\nu^\epsilon_x\}_{x \in \mathbb{R}^d}$ on $\mathcal{A}$ is a **local equilibrium state** with profile $\rho = \rho(x)$ if ν^ϵ_x weakly converges to the Poisson point field $\mu_{\rho(x)}$ as $\epsilon \downarrow 0$ for every $x \in \mathbb{R}^d$.

Theorem 2.3 (Microscopic formulation). *If $\{\nu^\epsilon_{0,x}\}$ is a local equilibrium state with profile u_0 (and fulfills a certain integrability condition), then $\{\nu^\epsilon_{t,x}\}$ is also a local equilibrium state and its profile $u(t)$ is the solution of the heat equation with initial data u_0.*

For the proof of Theorem 2.3, one may compute the limit of the Laplace transform of $\nu^\epsilon_{t,x}$. The computation is easy because of the independence of the particles. The details are omitted.

The model treated here is quite simple, but suggestive in:

(1) The macroscopic equation describes the change of the conserved component in space and time. In particular, after a course graining, macroscopic quantities are projected only to such component.

(2) The reason for this is due to the averaging effect based on the local ergodicity (from macroscopic point of view) or due to the establishment of the local equilibrium states (from microscopic point of view).

In general, for interacting systems, it is expected that the microscopic formulation (as in Theorem 2.3) combined with the ergodicity of the equilibrium states in space implies the macroscopic formulation (as in Theorem 2.1).

The hydrodynamic limit and the equilibrium fluctuation for **interacting Brownian particles** were studied by Varadhan [43] (in one dimension) and Spohn [37], respectively. In the interacting case, the Poisson point fields are replaced by the **canonical Gibbs measures**.

§3. Lattice gases

3.1. Independent random walks on a d-dimensional integer lattice $\mathbb{Z}^d$ (which is a special case of zero-range processes) are discussed in Chapter 1 (an introductory example) of the book [28] instead of the independent Brownian particles. A configuration of particles on $\mathbb{Z}^d$ in such system is denoted by $\eta = \{\eta_i\}_{i\in\mathbb{Z}^d} \in \mathbb{Z}_+^{\mathbb{Z}^d}$, where $\mathbb{Z}_+ = \{0,1,2,\ldots\}$, with η_i indicating the number of particles sitting on the site $i \in \mathbb{Z}^d$. The evolution $\eta(s) = \{\eta_i(s)\}_{i\in\mathbb{Z}^d}, s \geq 0$ of particles is specified by giving a constant rate of jumps of each particle to neighboring sites, for which the **space-time diffusive scaling** is defined by

$$\xi_t^\epsilon(dx) = \epsilon^d \sum_{i\in\mathbb{Z}^d} \eta_i(t/\epsilon^2)\delta_{\epsilon i}(dx).$$

Then, the macroscopic equation, obtained as the limit of ξ_t^ϵ as $\epsilon \downarrow 0$, is the (linear) diffusion equation. The equilibrium states for the microscopic dynamics $\eta(s)$ are the product measures, **Poisson measures** on $\mathbb{Z}_+^{\mathbb{Z}^d}$.

3.2. Random walks with exclusion rule, i.e., lattice gases on $\mathbb{Z}^d$ or sometimes called Kawasaki dynamics are one of the simplest interacting systems. The **exclusion rule**, that at most one particle can occupy each site at once, gives an interaction among particles. The configuration space is now replaced by $\{0,1\}^{\mathbb{Z}^d}$. The macroscopic equation is the (linear) diffusion equation and the equilibrium states are the product measures, **Bernoulli measures** on $\{0,1\}^{\mathbb{Z}^d}$.

3.3. For the Kawasaki dynamics with creation and annihilation (Glauber-Kawasaki dynamics), the macroscopic equation is the **reaction-diffusion equation**:

$$\frac{\partial u}{\partial t} = \Delta u + f(u), \quad u \in [0,1],$$

where

$$f(u) = E^{\mu_u}[(1-2\eta_0)c_0(\eta)]$$

and the equilibrium states are the Bernoulli measures μ_u parametrized by the density $u \in [0,1]$, see [6].

The function $c_0(\eta) \geq 0$ represents a creation-annihilation rate at site $i = 0$ when the configuration is η. The scaling is taken in such a manner that the creation-annihilation rate (i.e. the time for Glauber part) is kept of $O(1)$.

3.4. We now consider the Kawasaki dynamics with general (finite-range and translation-invariant) interactions satisfying the so-called detailed balance condition (DBC, see Section 6) so that the system is symmetric and the equilibrium states are the **canonical Gibbs measures** $\{\mu_u\}_{u\in[0,1]}$ on $\{0,1\}^{\mathbb{Z}^d}$. The interaction comes into the jump rates $\{c_{i,j}(\eta)\}$ of particles from i to j. Each particle looks around and the jump rate is determined depending on the surrounded environment, e.g., if crowded, then it moves slowly or if sparse, it moves quickly, or other way around. The macroscopic equation is the **nonlinear diffusion equation**:

$$\begin{aligned}\frac{\partial u}{\partial t} &= \operatorname{div}\{D(u)\nabla u\} \\ &\equiv \sum_{k,k'=1}^{d} \frac{\partial}{\partial x_k}\left\{D_{kk'}(u)\frac{\partial u}{\partial x_{k'}}\right\}.\end{aligned} \tag{3.1}$$

The **bulk diffusion coefficient** $D = (D_{kk'}(u))_{1\le k,k'\le d}$ ($d\times d$ matrix) is determined by a variational formula (which is equivalent to Green-Kubo formula): For $\xi \in \mathbb{R}^d$,

$$\begin{aligned}(\xi, D(u)\xi) &= \frac{1}{4\chi(u)} \\ &\times \inf_{f=f(\eta)} E^{\mu_u}\left[\sum_{|i|=1} c_{0,i}(\eta)\left(\xi, i(\eta_i-\eta_0) - \pi^{0i}\sum_{j\in\mathbb{Z}^d}\tau_j f\right)^2\right],\end{aligned}$$

where $\chi(u) = \sum_{i\in\mathbb{Z}^d}(E^{\mu_u}[\eta_0\eta_i] - u^2)$ is the compressibility, π^{0i} is an exchange operator and τ_j is a shift operator, see Funaki, Uchiyama and Yau [23] (the case where μ_u are the Bernoulli measures and discussed on the lattice torus), Varadhan and Yau [45] (general case either on $\mathbb{Z}^d$ or on finite box under mixing conditions). It is known by now that the matrix $D(u)$ is positive definite (see [39]) and smooth enough as a function of u. Taking this model as an example, Section 6 outlines the entropy method which plays a basic role for establishing the so-called local equilibria.

3.5. From Kawasaki dynamics with two types (A/B) of particles, **one or two phases' Stefan problems** can be derived under the diffusive scaling limit, see Funaki [15] for (a), (b) and Komoriya [29] for (c).

(a) One phase Stefan problem with absorption (melting): The particle of type A is active, while the particle of type B is immobile. A dies when it meets B, while B disappears after receiving ℓth visit of A. The limit equation for the macroscopic density u of A particles is:

$$\begin{aligned}
&\frac{\partial u}{\partial t} = \Delta P_A(u) \quad \text{on} \quad \mathcal{L}(t) = \{u(t,x) > 0\}, \\
&u = 0 \qquad\qquad\quad \text{on} \quad \Sigma(t) = \partial \mathcal{L}(t), \\
&\ell V = \mathbf{n} \cdot \nabla P_A(u) \quad \text{on} \quad \Sigma(t),
\end{aligned}$$

where V is the speed of $\Sigma(t)$ to the direction $\mathbf{n}$ (outer normal of $\partial \mathcal{L}(t)$), $P_A(u)$ is determined from the jump rate $c_{i,j}(\eta)$ (satisfying the gradient condition additionally) and ℓ represents the latent heat.

(b) Two phase Stefan problem with absorption: Both A and B are active and disappear when they meet intermediate immobile states. The macroscopic densities u_A, u_B of A and B-particles fulfill:

$$\begin{aligned}
&\frac{\partial u_A}{\partial t} = \Delta P_A(u_A) \quad \text{on} \quad \mathcal{L}(t) = \{u_A > 0, u_B = 0\}, \\
&\frac{\partial u_B}{\partial t} = \Delta P_B(u_B) \quad \text{on} \quad \mathcal{S}(t) = \{u_A = 0, u_B > 0\}, \\
&u_A = u_B = 0 \qquad\quad \text{on} \quad \Sigma(t) = \partial \mathcal{L}(t) = \partial \mathcal{S}(t), \\
&(\ell - 1)V = \mathbf{n} \cdot (\nabla P_A(u_A) - \nabla P_B(u_B)) \quad \text{on} \quad \Sigma(t).
\end{aligned}$$

(c) Two phase Stefan problem with reflection: This is studied in one dimension for constant jump rates c_A, c_B for A/B particles, respectively. Assume that, initially at time $s = 0$, the right/left hand sides of 0 are occupied only by A/B particles, respectively. Then, the limit is described by the equations:

$$\begin{aligned}
&\frac{\partial u}{\partial t} = c_A \Delta u, \quad x > a(t), \\
&\frac{\partial u}{\partial t} = c_B \Delta u, \quad x < a(t), \\
&\dot{a}(t) = -c_A \frac{1}{u} \frac{\partial u}{\partial x}\Big|_{x=a(t)+} = -c_B \frac{1}{u} \frac{\partial u}{\partial x}\Big|_{x=a(t)-}, \\
&c_A \frac{\partial u}{\partial x}\Big|_{x=a(t)+} = c_B \frac{\partial u}{\partial x}\Big|_{x=a(t)-}, \\
&u(t, a(t)+) = u(t, a(t)-),
\end{aligned}$$

where the free boundary $a(t)$, which separates A/B phases, moves in t.
3.6. Starting from the Glauber dynamics and others, Spohn [38] studied a pattern formed after proper scaling and derived the **motion by mean curvature**. The argument there is rather heuristic, but contains several suggestive conjectures. Presutti and others [8], [26] derived (isotropic) motion by mean curvature for the interfaces from the Glauber-Kawasaki

dynamics. Since one can derive the reaction-diffusion equation from the Glauber-Kawasaki dynamics as we saw in Section *3.3* and the motion by mean curvature can be obtained from the reaction-diffusion equation under singular limit, these results are thought of as the two scalings are accomplished at once. In [7], [27], the motion by mean curvature was derived from the Glauber dynamics corresponding to the Kac's type potential with long range interaction. See a review paper by Giacomin et al. [24].

3.7. So far, we have discussed symmetric dynamics under the diffusive scaling limit. From asymmetric exclusion process (Kawasaki dynamics) on $\mathbb{Z}$ (one dimension), one can derive a solution of the **inviscid Burgers' equation** satisfying the entropy condition under the **hyperbolic scaling limit**:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\big\{u(1-u)\big\} = 0.$$

See Varadhan [44] and Remark 5.1 in connection with the growing interface model. These models have a close relation to the theory of random matrices, see, e.g. Sections 16.4 and 16.5 of [17]. Systems of conservation laws are obtained from lattice gases with two conserved quantities by Fritz and Tóth [14], see also [13], [40], [41].

3.8. (non-Kawasaki type models) From Hamilton system with small noise, and with (modified) Hamiltonian growing at most linearly in momentum, Euler equation or that with Navier-Stokes correction are derived, see [33].

§4. Static theory for the $\nabla\varphi$ interface model

4.1. Let us introduce the **$\nabla\varphi$ interface model** briefly. We are concerned with a surface (interface) in $\mathbb{R}^{d+1}$, which separates two distinct pure phases, described by the height variables $\phi = \{\phi_i \in \mathbb{R}\}_{i\in\Gamma}$ measured from a reference hyperplane Γ located in $\mathbb{R}^d$. To avoid complications, we assume that the interface has no overhangs nor bubbles. The variables ϕ are microscopic objects, and the space Γ is discretized and taken as $\Gamma = D_N (\equiv ND \cap \mathbb{Z}^d)$, lattice torus $\mathbb{T}_N^d \big(\equiv (\mathbb{Z}/N\mathbb{Z})^d = \{1, 2, \dots, N\}^d \big)$ or $\mathbb{Z}^d$. Here D is a (macroscopic) bounded domain in $\mathbb{R}^d$ and N represents the size of the microscopic system.

An **energy** is associated with each height variable $\phi : \Gamma \to \mathbb{R}$ as the sum over all bonds $\langle i, j\rangle$ in Γ (or in $\Gamma \cup \partial^+\Gamma$)

$$H(\phi) \equiv H_\Gamma^\psi(\phi) = \sum_{\langle i,j\rangle \subset \Gamma (\text{or } \Gamma\cup\partial^+\Gamma)} V(\phi_i - \phi_j),$$

and the **equilibrium state (Gibbs measure)** is defined by

$$d\mu \equiv d\mu_\Gamma^\psi = Z^{-1} e^{-H(\phi)} \prod_{i\in\Gamma} d\phi_i, \tag{4.1}$$

where Z is a normalization constant and $\partial^+\Gamma = \{i \in \mathbb{Z}^d \setminus \Gamma; |i-j| = 1$ for some $j \in \Gamma\}$ denotes the outer boundary of Γ for $\Gamma \subset \mathbb{Z}^d$. The **potential** V is symmetric, smooth and strictly convex ($0 <^\exists c_- \le V'' \le^\exists c_+ < \infty$). Note that the boundary conditions $\psi = \{\psi_i\}_{i\in\partial^+\Gamma}$ are required to define $H(\phi)$ and μ when $\Gamma = D_N$. An infinite volume limit (thermodynamic limit) as $\Gamma \nearrow \mathbb{Z}^d$ exists when $d \ge 3$ and the limit measure μ has a long correlation. More information on the $\nabla\varphi$ interface model can be found in [17].

4.2. Our main interest is in studying the **scaling limit**, which passes from microscopic to macroscopic levels, defined by

$$h^N(x) = \frac{1}{N}\phi_{[Nx]}, \quad x \in D \text{ (or } \in \mathbb{T}^d \equiv [0,1)^d,\ \mathbb{R}^d),$$

where $[Nx]$ stands for the integral part of $Nx (\in \mathbb{R}^d)$ taken componentwise. The function h^N is the macroscopic height variable associated with the microscopic $\phi : \Gamma \to \mathbb{R}$. The **surface tension** $\sigma = \sigma(u)$ is the macroscopic energy for a surface with tilt $u \in \mathbb{R}^d$ determined by

$$\mu\left(\text{tilt of } h^N \sim u\right) \underset{N\to\infty}{\sim} \exp\{-N^d \sigma(u)\}.$$

Theorem 4.1. (**Large Deviation**, *Deuschel, Giacomin and Ioffe* [9]) *Consider the Gibbs measure $\mu^0_{D_N}$ on $\Gamma = D_N$ with 0-boundary conditions $\psi_i = 0, i \in \partial^+ D_N$. Then, the probability that h^N is close to a given macroscopic surface $h \in H^1_0(D)$ behaves as*

$$\mu^0_{D_N}\left(h^N \sim h\right) \underset{N\to\infty}{\sim} \exp\{-N^d \Sigma_D(h)\},$$

where $\Sigma_D(h)$ is the total surface tension of h defined by

$$\Sigma_D(h) = \int_D \sigma(\nabla h(x))\, dx. \tag{4.2}$$

This result is an analogue of Dobrushin, Kotecký and Shlosman [10] for the Ising model.

Corollary 4.2. (*Wall and constant volume conditions*) *For every $v \ge 0$, under the conditional probability $\mu^0_{D_N}\left(\cdot \,\middle|\, h^N \ge 0, \int_D h^N(x)dx \ge v\right)$,* *the law of large numbers* $h^N \longrightarrow \bar{h}_v$ (*as* $N \to \infty$) *holds, where* $\bar{h}_v$ *is the unique minimizer* (*called* **Wulff shape**) *of the variational problem*

$$\min\left\{\Sigma_D(h); h \in H_0^1(D), h \geq 0, \int_D h(x)dx = v\right\}.$$

We add a **weak self-potential** term to the energy $H_{D_N}^{\psi}(\phi)$:

$$H_{D_N}^{\psi,Q,W}(\phi) = H_{D_N}^{\psi}(\phi) + \sum_{i\in D_N} Q\left(\frac{i}{N}\right) W(\phi_i),$$

having the boundary conditions $\psi_i = Nf(i/N), i \in \partial^+ D_N$ determined from macroscopic function $f : \partial D \to \mathbb{R}$ (more precisely saying, defined on a neighborhood of ∂D), where $Q : D \to [0, \infty)$ and $W : \mathbb{R} \to \mathbb{R}$ satisfies $\alpha_\pm =^{\exists} \lim_{r\to\pm\infty} W(r)$ such that $\alpha_+ \wedge \alpha_- \leq W \leq \alpha_+ \vee \alpha_-$. The Gibbs measure is associated and defined by

$$d\mu_{D_N}^{\psi,Q,W} = Z_{\psi,Q,W}^{-1} e^{-H_{D_N}^{\psi,Q,W}(\phi)} \prod_{i\in D_N} d\phi_i.$$

Theorem 4.3. (**Large Deviation**, *Funaki and Sakagawa* [21]) *Assume* $A := \alpha_+ - \alpha_- \geq 0$. *Then,*

$$\mu_{D_N}^{\psi,Q,W}\left(h^N \sim h\right) \underset{N\to\infty}{\sim} \exp\{-N^d I_D^A(h)\},$$

where

$$I_D^A(h) = \Sigma_D^A(h) - \inf_{h'\in H_f^1(D)} \Sigma_D^A(h'),$$

$$\Sigma_D^A(h) = \Sigma_D(h) - A\int_D Q(x) 1_{\{h(x)\leq 0\}}\, dx,$$

and $H_f^1(D)$ *is the space of all* $h \in H^1(D)$ *having boundary conditions* f.

Corollary 4.4. *The law of large numbers* $h^N \longrightarrow \bar{h}_A$ (*as* $N \to \infty$) *holds under* $\mu_{D_N}^{\psi,Q,W}$, *if the minimizer* $\bar{h}_A$ *of the variational problem*

$$\min\left\{\Sigma_D^A(h); h \in H_f^1(D)\right\}$$

is unique.

Remark 4.1. (1) *The variational problem obtained in Corollary 4.4 was studied by Alt and Caffarelli* [1], *Alt, Caffarelli and Friedman* [2], *Weiss* [46] *and others. The minimizer fulfills the* **free boundary problem** *of elliptic type and the corresponding free boundary condition*

is called the **Young's relation**.
(2) *The case with two minimizers is studied by* [18] *in one dimension. The large deviation for the Gibbs measure with* δ*-pinning* (*defined by* (4.1) *with* $d\phi_i$ *replaced by* $e^J\delta_0(d\phi_i) + d\phi_i, J \in \mathbb{R}$) *instead of weak self-potentials is discussed by* [21] *in one dimension.*
(3) *Bolthausen and Ioffe* [3] *proved the law of large numbers for the Gibbs measure on the wall with* δ*-pinning and quadratic potential under the constant volume condition in two dimension. The limit called* **Winterbottom shape** *is uniquely* (*except translation*) *characterized by a certain variational problem.*

§5. Dynamic theory for the $\nabla\varphi$ interface model

5.1. One can introduce **microscopic dynamics** (stationary and reversible under the Gibbs measure μ) for the interfaces by the SDEs (Langevin equation)

$$d\phi_i(s) = -\frac{\partial H}{\partial \phi_i}(\phi(s))\,ds + \sqrt{2}dw_i(s), \ \ i \in \Gamma,$$

where $\{w_i(s)\}_{i\in\Gamma}$ is a family of independent Brownian motions and

$$\frac{\partial H}{\partial \phi_i}(\phi) = \sum_{j:|i-j|=1} V'(\phi_i - \phi_j).$$

The goal is to discuss the **space-time diffusive scaling limit** for $\phi(s) = \{\phi_i(s)\}_{i\in\Gamma}$ defined by

$$h^N(t,x) = \frac{1}{N}\phi_{[Nx]}(N^2t).$$

Theorem 5.1. (**Hydrodynamic Limit**, *Funaki and Spohn* [22] *on the torus* $\mathbb{T}^d$, *Nishikawa* [32] *on* D *with boundary conditions*) *As* $N \to \infty$, $h^N(t,x) \longrightarrow h(t,x)$. *The limit* $h(t,x)$ *is a unique weak solution of the nonlinear PDE* (*MMC with spatial anisotropy*)*:*

$$\text{(5.1)} \qquad \frac{\partial h}{\partial t}(t) = \text{div}\ \{\nabla\sigma(\nabla h(t))\} \equiv \sum_{k=1}^{d} \frac{\partial}{\partial x_k}\left\{\frac{\partial \sigma}{\partial u_k}(\nabla h(t))\right\}.$$

The surface tension has the following properties: $\sigma \in C^1(\mathbb{R}^d)$, $\nabla\sigma$ is Lipschitz continuous and σ is strictly convex. The PDE (5.1) can be regarded as the gradient flow for $\Sigma = \Sigma_{\mathbb{T}^d}$ or Σ_D, the total surface tension (4.2) on $\mathbb{T}^d$ or D:

$$\frac{\partial h}{\partial t}(t) = -\frac{\delta \Sigma}{\delta h(x)}(h(t)).$$

Theorem 5.2. (**Dynamic Large Deviation**, *Funaki and Nishikawa* [19] *on* $\mathbb{T}^d$) *The probability that $h^N(t)$ is close to $h(t)$ behaves as*

$$P\left(h^N(t) \sim h(t),\ t \le T\right) \underset{N\to\infty}{\sim} \exp\{-N^d I_T(h)\},$$

where $h(t) = h(t,x)$ is a given motion of surface and

$$I_T(h) = \frac{1}{4}\int_0^T dt \int_{\mathbb{T}^d} \left[\frac{\partial h}{\partial t} - \operatorname{div}\{\nabla\sigma(\nabla h(t))\}\right]^2 dx.$$

The relation to the static large deviation (Theorem 4.1) is given by

$$\lim_{T\to\infty} S_T(\bar h) = \Sigma_{\mathbb{T}^d}(\bar h), \quad \bar h = \bar h(x),$$

where

$$S_T(\bar h) = \inf\left\{I_T(h); h(T,x) = \bar h(x)\right\}.$$

5.2. **Dynamics on the wall** are introduced by SDEs of Skorohod type:

$$d\phi_i(s) = -\frac{\partial H}{\partial \phi_i}(\phi(s))\,ds + \sqrt{2}dw_i(s) + \frac{1}{N} f\left(\frac{s}{N^2}, \frac{i}{N}, \frac{1}{N}\phi_i(s)\right) ds + d\ell_i(s), \tag{5.2}$$

subject to the conditions

$$\phi_i(s) \ge 0, \quad \ell_i(s) \nearrow \quad \text{and} \quad \int_0^\infty \phi_i(s) d\ell_i(s) = 0,$$

where $f = f(t,x,h)$ is a given macroscopic external field. Note that $\ell_i(s)$ increases only when $\phi_i(s) = 0$. The unique invariant (stationary) measure (when $f = 0, \Gamma = D_N$ with 0-boundary conditions) is given by $\mu^0_{D_N}(\,\cdot\,|\phi \ge 0)$, which is reversible under the dynamics.

Theorem 5.3. (**Hydrodynamic Limit**, *Funaki* [16] *on* $\mathbb{T}^d$) *As $N \to \infty$, $h^N(t,x) \longrightarrow h(t,x)$. The limit $h(t,x)$ is a unique solution of* *the* **evolutionary variational inequality** (*MMC with reflection (obstacle)):*

$$
\begin{aligned}
&\text{(a)} \quad h \in L^2(0,T;V),\ \frac{\partial h}{\partial t} \in L^2(0,T;V'), \quad {}^\forall T>0,\\
&\text{(b)} \quad \left(\frac{\partial h}{\partial t}(t), h(t)-v\right) + (\nabla\sigma(\nabla h(t)), \nabla h(t) - \nabla v)\\
&\qquad\qquad \le (f(t,h(t)), h(t)-v)\,, \quad a.e.\,t, \quad {}^\forall v \in V : v \ge 0,\\
&\text{(c)} \quad h(t,x) \ge 0, \qquad a.e.,\\
&\text{(d)} \quad h(0,x) = h_0(x),
\end{aligned}
$$

where $V = H^1(\mathbb{T}^d), H = L^2(\mathbb{T}^d), V' = H^{-1}(\mathbb{T}^d)$ *and* $(\cdot,\cdot)$ *denotes the inner product of* H *(or* H^d*) or the duality between* V' *and* V.

Remark 5.1. *Rezakhanlou* [34], [35] *derived a Hamilton-Jacobi equation under hyperbolic scaling from growing SOS dynamics (i.e.* $\phi_i \in \mathbb{Z}$*) with constraints on the gradients (e.g.* $(\nabla\phi)_i \le v$*). Related results were obtained by Evans and Rezakhanlou* [12] *and Seppäläinen* [36].

Let us consider the **equilibrium dynamics** $\phi(s)$ on the wall in one dimension, i.e., $\phi(s)$ is a solution of the SDE (5.2) with $d=1, \Gamma = \{1,2,\dots,N-1\}, f=0$ and with 0-boundary conditions $\phi_0(s) = \phi_N(s) = 0$, and an initial distribution $\mu_\Gamma^0(\,\cdot\,|\phi \ge 0)$. **Macroscopic fluctuation field** (around the hydrodynamic limit $h(t,x)=0$) is defined by

$$\Phi^N(t,x) = \sqrt{N} h^N(t,x)\ (\ge 0), \quad x \in [0,1].$$

Theorem 5.4. (**Equilibrium Fluctuation**, *Funaki and Olla* [20]) *As* $N \to \infty$, $\Phi^N(t,x) \Longrightarrow \Phi(t,x)$. *The limit* $\Phi(t,x)$ *is a unique weak stationary solution of the stochastic PDE with reflection of Nualart and Pardoux type:*

$$
\begin{aligned}
&\frac{\partial \Phi}{\partial t}(t,x) = q\frac{\partial^2 \Phi}{\partial x^2}(t,x) + \sqrt{2}\dot{B}(t,x) + \xi(t,x), \quad x \in [0,1],\\
&\Phi(t,x) \ge 0, \quad \int_0^\infty \int_0^1 \Phi(t,x)\,\xi(dtdx) = 0,\\
&\Phi(t,0) = \Phi(t,1) = 0, \quad \xi\text{: random measure},
\end{aligned}
$$

where $\dot{B}(t,x)$ *is a space-time white noise and*

$$1/q = \int_{\mathbb{R}} \eta^2 e^{-V(\eta)}\,d\eta \Big/ \int_{\mathbb{R}} e^{-V(\eta)}\,d\eta.$$

§6. Entropy method

This section presents the heart of the entropy method, which was exploited by [25], for the macroscopic equation and the microscopic system in parallel, to observe the analogy between them.

Entropy for macroscopic equation. Let us consider the nonlinear diffusion equation (3.1) with positive initial data $u(0,x) = u_0(x) > 0$ under the periodic boundary condition, i.e., on the torus $\mathbb{T}^d$. Note that the solution enjoys $u(t,x) > 0$ by the maximum principle and it holds the conservation law

$$\int_{\mathbb{T}^d} u(t,x)\,dx = \text{ const in } t.$$

For $u = \{u(x) > 0\}$, define the entropy

$$H(u) = \int_{\mathbb{T}^d} u(x) \log u(x)\,dx$$

and the entropy production (the Dirichlet integral of $\sqrt{u}$)

$$I(u) = \sum_{k,k'=1}^{d} \int_{\mathbb{T}^d} D_{k,k'}(u) \frac{\partial \sqrt{u}}{\partial x_k} \frac{\partial \sqrt{u}}{\partial x_{k'}}\,dx.$$

Jensen's inequality shows that $H(u) \geq \bar{u} \log \bar{u}$, where $\bar{u} = \int_{\mathbb{T}^d} u(x)dx$. Simple computations, using the integration by parts formula on $\mathbb{T}^d$ and the conservation law, lead us to the following lemma.

Lemma 6.1. *The solution $u(t)$ of the PDE* (3.1) *satisfies*

$$\frac{d}{dt} H(u(t)) = -4I(u(t)) \leq 0.$$

This lemma implies the H-theorem (the principle of increase of entropy): $-H(u(t))$ is non-decreasing in t and $\lim_{t\to\infty} I(u(t)) = 0$, since $H(u(t))$ is bounded below. Note that $I(u) = 0$ is equivalent to $\partial\sqrt{u}/\partial x_k \equiv 0$ (by the positive definiteness of D) so that $u = \text{const}$, which are the equilibrium solutions of (3.1). A similar argument works for the PDE (3.1) on D with smooth boundary ∂D under the boundary condition $(D(u)\nabla u, \mathbf{n}) = 0$ at ∂D for the outward unit normal vector $\mathbf{n}$. One may apply Gauss-Green theorem on D.

Entropy for microscopic system. Let us consider the Kawasaki dynamics on the lattice torus $\mathbb{T}_N^d$ of side length N with configuration space $\mathcal{X}_N = \{0,1\}^{\mathbb{T}_N^d}$. Its generator is given by

$$L_N = \sum_b c_b(\eta)\pi^b,$$

where π^b is an exchange operator defined for each bond $b = \langle i, j \rangle$ by

$$\pi^b f(\eta) = f(\eta^b) - f(\eta)$$

with a configuration $\eta^b \in \mathcal{X}_N$ obtained from $\eta \in \mathcal{X}_N$ by exchanging the values of η_i and η_j. The jump rates $c_b(\eta) = c_{i,j}(\eta) > 0$ of particles between two neighboring sites i and j satisfy the conditions listed in Section *3.4* (on $\mathbb{T}_N^d$) (i.e. of finite-range independently of N, translation-invariant, DBC); the DBC (detailed balance condition) means that $c_b(\eta)\mu_N(\eta) = c_b(\eta^b)\mu_N(\eta^b)$ for every b and $\eta \in \mathcal{X}_N$ with some $\mu_N \in \mathcal{P}(\mathcal{X}_N) \equiv \{$probability measures on $\mathcal{X}_N\}$. This implies the symmetry of L_N under μ_N.

Define the (relative) entropy of $\nu \in \mathcal{P}(\mathcal{X}_N)$ by

$$H_N(\nu) = \int_{\mathcal{X}_N} \varphi \log \varphi \, d\mu_N, \quad \varphi = \frac{d\nu}{d\mu_N},$$

and its entropy production by

$$\begin{aligned} I_N(\nu) &= -\int_{\mathcal{X}_N} \sqrt{\varphi} L_N \sqrt{\varphi} \, d\mu_N \\ &= \frac{1}{2} \sum_b \int_{\mathcal{X}_N} c_b (\pi^b \sqrt{\varphi})^2 \, d\mu_N. \end{aligned}$$

Let $\nu_N(t) = \varphi_t d\mu_N$ be the distribution of the process (Kawasaki dynamics) η_t^N with the generator $N^2 L_N$ on $\mathcal{X}_N$, which is speeded up by the factor N^2.

Proposition 6.2. *We have*

$$\frac{d}{dt} H_N(\nu_N(t)) \le -4N^2 I_N(\nu_N(t)).$$

Proof. This inequality follows from

$$\begin{aligned} \frac{d}{dt} H_N(\nu_N(t)) &= \frac{d}{dt} \int_{\mathcal{X}_N} \varphi_t \log \varphi_t \, d\mu_N \\ &= \int_{\mathcal{X}_N} \frac{d\varphi_t}{dt} \log \varphi_t \, d\mu_N + \int_{\mathcal{X}_N} \frac{d\varphi_t}{dt} \, d\mu_N \\ &= N^2 \int_{\mathcal{X}_N} L_N \varphi_t \cdot \log \varphi_t \, d\mu_N \\ &= -\frac{N^2}{2} \sum_b \int_{\mathcal{X}_N} c_b (\pi_b \varphi_t)(\pi_b \log \varphi_t) \, d\mu_N \end{aligned}$$

$$\leq -4N^2 \int_{\mathcal{X}_N} \sqrt{\varphi_t} L_N \sqrt{\varphi_t}\, d\mu_N$$
$$= -4N^2 I_N(\nu_N(t)).$$

For the third line, we have used the conservation law:

$$\int_{\mathcal{X}_N} \frac{d\varphi_t}{dt}\, d\mu_N = \frac{d}{dt}\int_{\mathcal{X}_N} \varphi_t\, d\mu_N = 0$$

and the forward equation

$$\frac{d\varphi_t}{dt} = N^2 L_N \varphi_t,$$

note that $L_N^* = L_N$, which is symmetric with respect to μ_N. For the fifth line, the elementary inequality

$$(a-b)(\log a - \log b) \geq 4(\sqrt{a}-\sqrt{b})^2, \quad a, b > 0$$

has applied. Q.E.D.

Since $0 \leq H_N(\nu) \leq CN^d$, by convexity of I_N,

$$I_N(\bar{\nu}_N) \leq \frac{CN^d}{N^2} \tag{6.1}$$

for the space-time average of $\{\nu_N(t)\}_{0\leq t\leq T}$:

$$\bar{\nu}_N = \frac{1}{N^d}\sum_{i\in\mathbb{T}_N^d} \frac{1}{T}\int_0^T \nu_N(t)\circ\tau_i^{-1}\, dt,$$

where τ_i is the spatial shift operator. In particular, the estimate (6.1) proves that all limits as $N \to \infty$ of $\bar{\nu}_N$ restricted on a finite region (independently of N) are (finite-volume) canonical Gibbs measures. This guarantees the realization of the local equilibria in a weak sense. See [23], [28], [45] for details.

The entropy method relevantly works also in the $\nabla\varphi$ interface model, see [22] for the proof of Theorem 5.1.

References

[1] H. W. Alt and L. A. Caffarelli, Existence and regularity for a minimum problem with free boundary, J. Reine Angew. Math., **325** (1981), 105–144.

[2] H. W. Alt, L. A. Caffarelli and A. Friedman, Variational problems with two phases and their free boundaries, Trans. Amer. Math. Soc., **282** (1984), 431–461.

[3] E. Bolthausen and D. Ioffe, Harmonic crystal on the wall: a microscopic approach, Commun. Math. Phys., **187** (1997), 523–566.

[4] R. E. Caflisch, Fluid dynamics and the Boltzmann equation, In: Nonequilibrium phenomena. I, (eds. Lebowitz and Montroll), North-Holland, Amsterdam, 1983, pp. 193–223.

[5] C. Cercignani, Theory and application of the Boltzmann equation, Elsevier, New York, 1975, xii+415 pp.

[6] A. De Masi, P. A. Ferrari and J. L. Lebowitz, Reaction-diffusion equations for interacting particle systems, J. Statist. Phys., **44** (1986), 589–644.

[7] A. De Masi, E. Orlandi, E. Presutti and L. Triolo, Glauber evolution with Kac potentials, I. Mesoscopic and macroscopic limits, interface dynamics, Nonlinearity, **7** (1994), 633–696.

[8] A. De Masi, A. Pellegrinotti, E. Presutti and M. E. Vares, Spatial patterns when phases separate in an interacting particle system, Ann. Probab., **22** (1994), 334–371.

[9] J.-D. Deuschel, G. Giacomin and D. Ioffe, Large deviations and concentration properties for $\nabla\varphi$ interface models, Probab. Theory Relat. Fields, **117** (2000), 49–111.

[10] R. L. Dobrushin, R. Kotecký and S. Shlosman, Wulff Construction: a Global Shape from Local Interaction, Amer. Math. Soc. Transl. Ser., **104** (1992).

[11] R. L. Dobrushin and Ra. Siegmund-Schultze, The hydrodynamic limit for systems of particles with independent evolution, Math. Nachr., **105** (1982), 199–224.

[12] L. C. Evans and F. Rezakhanlou, A stochastic model for growing sandpiles and its continuum limit, Commun. Math. Phys., **197** (1998), 325–345.

[13] J. Fritz, Entropy pairs and compensated compactness for weakly asymmetric systems, In: Stochastic Analysis on Large Scale Interacting Systems, Proceedings of Shonan/Kyoto meetings, 2002, (eds. Funaki and Osada), Adv. Stud. Pure Math., **39**, Math. Soc. Japan, 2004, pp. 143–172.

[14] J. Fritz and B. Tóth, Derivation of the Leroux system as the hydrodynamic limit of a two-component lattice gas, Commun. Math. Phys., **249** (2004), 1–27.

[15] T. Funaki, Free boundary problem from stochastic lattice gas model, Ann. Inst. H. Poincaré, **35** (1999), 573–603.

[16] T. Funaki, Hydrodynamic limit for $\nabla\phi$ interface model on a wall, Probab. Theory Relat. Fields, **126** (2003), 155–183.

[17] T. Funaki, Stochastic Interface Models, In: Lectures on Probability Theory and Statistics, Ecole d'Eté de Probabilités de Saint-Flour XXXIII - 2003 (ed. J. Picard), Lect. Notes Math., **1869**, Springer, 2005, pp. 103–274.

[18] T. Funaki, Dichotomy in a scaling limit under Wiener measure with density, Electron. Comm. Probab., **12** (2007), 173–183.

[19] T. Funaki and T. Nishikawa, Large deviations for the Ginzburg-Landau $\nabla\phi$ interface model, Probab. Theory Relat. Fields, **120** (2001), 535–568.

[20] T. Funaki and S. Olla, Fluctuations for $\nabla\phi$ interface model on a wall, Stoch. Proc. Appl., **94** (2001), 1–27.

[21] T. Funaki and H. Sakagawa, Large deviations for $\nabla\varphi$ interface model and derivation of free boundary problems, In: Stochastic Analysis on Large Scale Interacting Systems, Proceedings of Shonan/Kyoto meetings, 2002, (eds. Funaki and Osada), Adv. Stud. Pure Math., **39**, Math. Soc. Japan, 2004, pp. 173–211.

[22] T. Funaki and H. Spohn, Motion by mean curvature from the Ginzburg-Landau $\nabla\phi$ interface models, Commun. Math. Phys., **185** (1997), 1–36.

[23] T. Funaki, K. Uchiyama and H. T. Yau, Hydrodynamic limit for lattice gas reversible under Bernoulli measures, In: Nonlinear Stochastic PDE's: Hydrodynamic Limit and Burgers' Turbulence, Univ. Minnesota, (eds. Funaki and Woyczynski), IMA Vol. Math. Appl., **77**, Springer, 1995, pp. 1–40.

[24] G. Giacomin, J. L. Lebowitz and E. Presutti, Deterministic and stochastic hydrodynamic equations arising from simple microscopic model systems, In: Stochastic Partial Differential Equations: Six Perspectives, (eds R. Carmona and B. Rozovskii), Math. Surveys Monogr., **64**, Amer. Math. Soc., 1999, pp. 107–152.

[25] M. Z. Guo, G. C. Papanicolaou and S. R. S. Varadhan, Nonlinear diffusion limit for a system with nearest neighbor interactions, Commun. Math. Phys., **118** (1988), 31–59.

[26] M. A. Katsoulakis and P. E. Souganidis, Interacting particle systems and generalized evolution of fronts, Arch. Rat. Mech. Anal., **127** (1994), 133–157.

[27] M. A. Katsoulakis and P. E. Souganidis, Generalized motion by mean curvature as a macroscopic limit of stochastic Ising models with long range interactions and Glauber dynamics, Commun. Math. Phys., **169** (1995), 61–97.

[28] C. Kipnis and C. Landim, Scaling Limits of Interacting Particle Systems, Springer, 1999.

[29] K. Komoriya, Free boundary problem from two component system on $\mathbb{Z}$, J. Math. Sci. Univ. Tokyo., **12** (2005), 141–163.

[30] M. Lachowicz, On the initial layer and the existence theorem for the nonlinear Boltzmann equation; differentiability of the solution of the corresponding system of linear equations, Arch. Mech., **38** (1986), 127–141.

[31] C. B. Morrey, On the derivation of the equations of hydrodynamics from statistical mechanics, Commun. Pure Appl. Math., **8** (1955), 279–326.

[32] T. Nishikawa, Hydrodynamic limit for the Ginzburg-Landau $\nabla\phi$ interface model with boundary conditions, Probab. Theory Relat. Fields, **127** (2003), 205–227.

[33] S. Olla, S. R. S. Varadhan and H. T. Yau, Hydrodynamical limit for a Hamiltonian system with weak noise, Commun. Math. Phys., **155** (1993), 523–560.

[34] F. Rezakhanlou, Continuum limit for some growth models, Stoch. Proc. Appl., **101** (2002), 1–41.

[35] F. Rezakhanlou, Continuum limit for some growth models II, Ann. Probab., **29** (2001), 1329–1372.

[36] T. Seppäläinen, Existence of hydrodynamics for the totally asymmetric simple K-exclusion process, Ann. Probab., **27** (1999), 361–415.

[37] H. Spohn, Equilibrium fluctuations for interacting Brownian particles, Commun. Math. Phys., **103** (1986), 1–33.

[38] H. Spohn, Interface motion in models with stochastic dynamics, J. Statis. Phys., **71** (1993), 1081–1132.

[39] H. Spohn and H. T. Yau, Bulk diffusivity of lattice gases close to criticality, J. Statis. Phys., **79** (1995), 231–241.

[40] B. Tóth and B. Valkó, Onsager relations and Eulerian hydrodynamic limit for systems with several conservation laws, J. Statist. Phys., **112** (2003), 497–521.

[41] B. Tóth and B. Valkó, Perturbation of singular equilibria of hyperbolic two-component systems: a universal hydrodynamic limit, Commun. Math. Phys., **256** (2005), 111–157.

[42] S. Ukai and K. Asano, The Euler limit and initial layer of the nonlinear Boltzmann equation, Hokkaido Math. J., **12** (1983), 311–332.

[43] S. R. S. Varadhan, Scaling limits for interacting diffusions, Commun. Math. Phys., **135** (1991), 313–353.

[44] S. R. S. Varadhan, Large deviations for the asymmetric simple exclusion process, In: Stochastic Analysis on Large Scale Interacting Systems, Proceedings of Shonan/Kyoto meetings, 2002, (eds. Funaki and Osada), Adv. Stud. Pure Math., **39**, Math. Soc. Japan, 2004, pp. 1–27.

[45] S. R. S. Varadhan and H. T. Yau, Diffusive limit of lattice gas with mixing conditions, Asian J. Math., **1** (1997), 623–678.

[46] G. S. Weiss, A free boundary problem for non-radial-symmetric quasi-linear elliptic equations, Adv. Math. Sci. Appl., **5** (1995), 497–555.

Tadahisa Funaki
Graduate School of Mathematical Sciences,
The University of Tokyo,
3-8-1 Komaba, Meguro-ku, Tokyo 153-8914, JAPAN
E-mail address: funaki@ms.u-tokyo.ac.jp