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L_p–L_q maximal regularity of the Neumann problem for the Stokes equations in a bounded domain

Yoshihiro Shibata [1] **and Senjo Shimizu** [2]

Abstract.

We consider the Neumann problem for the Stokes equations with non-homogeneous boundary and divergence conditions in a bounded domain. We obtain a global in time L_p-L_q maximal regularity theorem with exponential stability. To prove the L_p-L_q maximal regularity, we use the Weis operator valued Fourier multiplier theorem.

§1. Introduction and Results

This paper is concerned with the L_p-L_q maximal regularity of the Neumann problem for the Stokes equations in a bounded domain Ω in $\mathbb{R}^n$ $(n \geqq 2)$:

$$(1.1)\qquad \begin{aligned} &v_t - \operatorname{Div} S(v,\theta) = f && \text{in } \Omega \times (0,T),\\ &\operatorname{div} v = g = \operatorname{div} \tilde{g} && \text{in } \Omega \times (0,T),\\ &S(v,\theta)\nu = h && \text{on } \Gamma \times (0,T),\\ &v\big|_{t=0} = v_0 && \text{in } \Omega. \end{aligned}$$

Here, Γ is a $C^{2,1}$ boundary of Ω; ν is the unit outward normal to Γ; $v = (v_1, \ldots, v_n)^*$ and θ are unknown velocity and pressure, respectively, where M^* denotes the transpose of M. f, g, $\tilde{g}$, h and v_0 are given functions; $S(v,\theta)$ is the stress tensor defined by the formula:

$$S(v,\theta) = D(v) - \theta I,$$

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where $D(v)$ is the deformation tensor of the velocities with element $D_{ij}(v) = \partial_i v_j + \partial_j v_i$, $\partial_i = \partial/\partial x_i$, and I is the $n \times n$ identity matrix. This problem is obtained as a linearized problem of some time dependent problem with free surface for the Navier-stokes equations which describes the motion of an isolated finite volume of viscous incompressible fluid without taking surface tension into account. Such free boundary problem was first studied by Solonnikov [8]. In our forthcoming paper [7], we treat the problem by using the results obtained in the present paper.

First of all, in order to state our main results precisely we introduce function spaces and some symbols which will be used throughout the paper. For any domain D in $\mathbb{R}^n$, integer m and $1 \leqq q \leqq \infty$, $L_q(D)$ and $W_q^m(D)$ denote the usual Lebesgue space and Sobolev space of functions defined on D with norms: $\|\cdot\|_{L_q(D)}$ and $\|\cdot\|_{W_q^m(D)}$, respectively. And also, for any Banach space X, interval I, integer ℓ and $1 \leqq p \leqq \infty$, $L_p(I,X)$ and $W_p^\ell(I,X)$ denote the usual Lebesgue space and Sobolev space of the X-valued functions defined on I with norms: $\|\cdot\|_{L_p(I,X)}$ and $\|\cdot\|_{W_p^\ell(I,X)}$, respectively. Set

$$
\begin{aligned}
& W_{q,p}^{\ell,m}(D \times I) = L_p(I, W_q^\ell(D)) \cap W_p^m(I, L_q(D)), \\
& \qquad \|u\|_{W_{q,p}^{\ell,m}(D\times I)} = \|u\|_{L_p(I,W_q^\ell(D))} + \|u\|_{W_p^m(I,L_q(D))}, \\
& W_q^0(D) = L_q(D), \quad W_p^0(I,X) = L_p(I,X), \\
& W_{p,0}^\ell((0,T),X) = \{u \in W_p^\ell((-\infty,T),X) \mid u = 0 \text{ for } t < 0\}, \\
& W_{p,0}^0((0,T),X) = L_{p,0}((0,T),X).
\end{aligned}
$$

Given $\alpha \in \mathbb{R}$, we set

$$
\begin{aligned}
& < D_t >^\alpha u(t) = \mathcal{F}^{-1}[(1+s^2)^{\alpha/2}\mathcal{F}u(s)](t), \\
& H_p^\alpha(\mathbb{R},X) = \{u \in L_p(\mathbb{R},X) \mid < D_t >^\alpha u \in L_p(\mathbb{R},X)\}, \\
& \qquad \|u\|_{H_p^\alpha(\mathbb{R},X)} = \| < D_t >^\alpha u\|_{L_p(\mathbb{R},X)} + \|u\|_{L_p(\mathbb{R},X)}.
\end{aligned}
$$

Here and hereafter, $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and its inverse, respectively. Set

$$
\begin{aligned}
& H_{q,p}^{1,1/2}(D \times \mathbb{R}) = H_p^{1/2}(\mathbb{R}, L_q(D)) \cap L_p(\mathbb{R}, W_q^1(D)), \\
& \qquad \|u\|_{H_{q,p}^{1,1/2}(D\times\mathbb{R})} = \|u\|_{H_p^{1/2}(\mathbb{R},L_q(D))} + \|u\|_{L_p(\mathbb{R},W_q^1(D))}, \\
& H_{q,p,0}^{1,1/2}(D \times (0,\infty)) = \{u \in H_{q,p}^{1,1/2}(D \times \mathbb{R}) \mid u = 0 \text{ for } t < 0\}.
\end{aligned}
$$

Finally, given $0 < T \leqq \infty$ we set

$$
\begin{aligned}
&H^{1,1/2}_{q,p,0}(D \times (0,T)) \\
&= \{u \mid {}^{\exists} v \in H^{1,1/2}_{q,p,0}(D \times (0,\infty)),\ u = v \text{ on } D \times (0,T)\}, \\
&\|u\|_{H^{1,1/2}_{q,p,0}(D\times(0,T))} = \inf \{\|v\|_{H^{1,1/2}_{q,p}(D\times\mathbb{R})} \mid \\
&\qquad v \in H^{1,1/2}_{q,p,0}(D \times (0,\infty)) \text{ with } v = u \text{ on } D \times (0,T)\}.
\end{aligned}
$$

Given Banach space X with norm $\|\cdot\|_X$, we set

$$
X^n = \{v = (v_1, \ldots, v_n)^* \mid v_j \in X\}, \quad \|v\|_X = \sum_{j=1}^n \|v_j\|_X.
$$

The dot $\cdot$ denotes the inner-product of $\mathbb{R}^n$. $F = (F_{ij})$ means the $n \times n$ matrix whose i-th row and j-th column component is F_{ij}. For the differentiation of the $n \times n$ matrix of functions $F = (F_{ij})$, the n-vector of functions $u = (u_1, \ldots, u_n)^*$ and the scalar function θ, we use the following symbols: $\theta_t = \partial_t\theta = \partial\theta/\partial t$, $\partial_j\theta = \partial\theta/\partial x_j$,

$$
\begin{gathered}
\nabla\theta = (\partial_1\theta, \ldots, \partial_n\theta)^*, \quad u_t = \partial_t u = (\partial_t u_1, \ldots, \partial_t u_n), \quad \nabla u = (\partial_i u_j), \\
\operatorname{div} u = \sum_{j=1}^n \partial_j u_j, \quad \operatorname{Div} F = (\sum_{j=1}^n \partial_j F_{1j}, \ldots, \sum_{j=1}^n \partial_j F_{nj})^*.
\end{gathered}
$$

The inner products $(\cdot,\cdot)_\Omega$ and $(\cdot,\cdot)_\Gamma$ are defined by

$$
(u, v)_\Omega = \int_\Omega u(x) \cdot v(x)\, dx, \quad (u, v)_\Gamma = \int_\Gamma u(x) \cdot v(x)\, d\sigma,
$$

where $d\sigma$ denotes the surface element of Γ. We denote by C a generic constant and $C_{a,b,\ldots}$ denotes a constant depending on the quantities a, b, ... The constants C and $C_{a,b,\ldots}$ may change from line to line.

To state our main results concerning the unique existence of solutions to (1.1), first of all we discuss an analytic semigroup approach to the initial-boundary value problem:

$$
\begin{aligned}
&v_t - \operatorname{Div} S(v,\theta) = 0, \quad \operatorname{div} v = 0 \quad \text{in } \Omega \times (0,\infty), \\
&S(v,\theta)\nu|_\Gamma = 0, \quad v|_{t=0} = v_0.
\end{aligned} \tag{1.2}
$$

Set

$$
\begin{aligned}
J_q(\Omega) &= \{w = (w_1, \ldots, w_n)^* \in L_q(\Omega)^n \mid \operatorname{div} w = 0 \quad \text{in } \Omega\}, \\
G_q(\Omega) &= \{\nabla\omega \mid \omega \in W^1_q(\Omega),\ \omega|_\Gamma = 0\}.
\end{aligned}
$$

Then, by Grubb and Solonnikov [3], we know the second Helmholtz decomposition corresponding to (1.2):

$$L_q(\Omega)^n = J_q(\Omega) \oplus G_q(\Omega)$$

for $1 < q < \infty$, where $\oplus$ denotes the direct sum. Let P_q be the solenoidal projection: $L_q(\Omega)^n \to J_q(\Omega)$ along $G_q(\Omega)$ and we consider the resolvent problem corresponding to (1.2):

$$\text{(1.3)} \qquad \lambda v - \operatorname{Div} S(v,\theta) = P_q f, \ \operatorname{div} v = 0 \quad \text{in } \Omega, \quad S(v,\theta)\nu|_\Gamma = 0.$$

If we take the divergence of the first equation of (1.3) and take the inner product between the boundary condition and ν, we have

$$\text{(1.4)} \qquad \Delta\theta = 0 \quad \text{in } \Omega, \quad \theta|_\Gamma = \nu \cdot [S(v)\nu] - \operatorname{div} v|_\Gamma,$$

because $\nu \cdot \nu = 1$ on Γ. We know that for any $v \in W_q^2(\Omega)^n$ there exists a unique $\theta \in W_q^1(\Omega)$ such that θ solves (1.4) and enjoys the estimate:

$$\|\theta\|_{W_q^1(\Omega)} \leqq C\|v\|_{W_q^2(\Omega)}.$$

Let us define the map $K : W_q^2(\Omega) \to W_q^1(\Omega)$ by $\theta = K(v)$ for $v \in W_q^2(\Omega)$. We know that (1.2) is equivalent to the reduced Stokes equation:

$$\lambda v - \operatorname{Div} S(v, K(v)) = P_q f \quad \text{in } \Omega \quad S(v, K(v))\nu|_\Gamma = 0.$$

Set

$$A_q v = -\operatorname{Div} S(v, K(v)) \quad \text{for } v \in \mathcal{D}(A_q),$$
$$\mathcal{D}(A_q) = \{v \in J_q(\Omega) \cap W_q^2(\Omega)^n \mid S(v, K(v))\nu|_\Gamma = 0\}.$$

By Grubb and Solonnikov [3] and Shibata and Shimizu [5], we know the following theorem.

Theorem 1.1. *Let* $1 < q < \infty$. A_q *generates the analytic semigroup* $\{e^{-A_q t}(t)\}_{t \geqq 0}$ *on* $J_q(\Omega)$.

In order to state a global in time unique existence result, we introduce the rigid space $\mathcal{R}$ defined by

$$\mathcal{R} = \{Ax + b \mid A : n \times n \text{ anti-symmetric matrix}, \ b \in \mathbb{R}^n\}.$$

We know that u satisfies the condition: $S(u) = 0$ if and only if $u \in \mathcal{R}$. If $u \in \mathcal{R}$, then $\operatorname{div} u = 0$. Therefore, if $u \in \mathcal{R}$, then u satisfies (1.1) with $f = g = \tilde{g} = h = 0$ and $v_0 = u$. To obtain solutions of (1.1) in $W_{q,p}^{2,1}(\Omega \times (0,\infty))^n \times L_p((0,\infty), W_p^1(\Omega))$ decaying as $t \to \infty$, initial data and right members should be orthogonal to $\mathcal{R}$. To represent the orthogonality, we introduce a basis $\{p_\ell\}_{\ell=1}^M$ of $\mathcal{R}$ normalized as

$$(p_\ell, p_m)_\Omega = \delta_{\ell m}, \quad \ell, m = 1, \ldots, M,$$

where $\delta_{\ell m}$ is the Kronecker symbol such as $\delta_{\ell\ell} = 1$ and $\delta_{\ell m} = 0$ with $\ell \neq m$.

We can now state our main result which shows the L_p-L_q maximal regularity with exponential stability of solutions of (1.1) global in time.

Theorem 1.2. *Let $1 < p, q < \infty$. Set*

$$\mathcal{D}_{q,p}(\Omega) = [J_q(\Omega), \mathcal{D}(A_q)]_{1-1/p,p},$$

where $[\cdot,\cdot]_{\theta,p}$ denotes the real interpolation functor. Then, there exists a positive constant γ_0 such that if initial data v_0 and right members f, g, $\tilde{g}$ and h for (1.1) *satisfy the conditions:*

$$v_0 \in \mathcal{D}_{q,p}(\Omega), \ e^{\gamma t} f \in L_p((0,\infty), L_q(\Omega))^n, \ e^{\gamma t} g \in L_{p,0}((0,\infty), W_q^1(\Omega)),$$
$$e^{\gamma t}\tilde{g} \in W_{p,0}^1((0,\infty), L_q(\Omega))^n, \ e^{\gamma t} h \in H_{q,p,0}^{1,1/2}(\Omega \times (0,\infty))^n$$

for some $\gamma \in [0, \gamma_0]$ and

$$(v_0, p_\ell)_\Omega = 0, \quad (f(\cdot,t), p_\ell)_\Omega + (h(\cdot,t), p_\ell)_\Gamma = 0$$

for a.e. $t > 0$ and $\ell = 1, \ldots, M$, then (1.1) *with $T = \infty$ admits a unique solution*

$$(v, \theta) \in W_{q,p}^{2,1}(\Omega \times (0,\infty))^n \times L_p((0,\infty), W_q^1(\Omega)),$$

which satisfies the estimates:

$$\begin{aligned} &\|e^{\gamma t} v\|_{W_{q,p}^{2,1}(\Omega\times(0,\infty))} + \|e^{\gamma t}\theta\|_{L_p((0,\infty),W_q^1(\Omega))} \\ &\leqq C\,\{\|v_0\|_{\mathcal{D}_{q,p}(\Omega)} + \|e^{\gamma t} f\|_{L_p((0,\infty),L_q(\Omega))} + \|e^{\gamma t} g\|_{L_p((0,\infty),W_q^1(\Omega))} \\ &\qquad + \|e^{\gamma t}\tilde{g}\|_{W_p^1((0,\infty),L_q(\Omega))} + \|e^{\gamma t} h\|_{H_{q,p,0}^{1,1/2}(\Omega\times(0,\infty))}\} \end{aligned}$$

and the condition:

$$(v(\cdot,t), p_\ell)_\Omega = 0 \quad \textit{for } t \geqq 0 \textit{ and } \ell = 1, \ldots, M.$$

Remark 1.3. Let $B_{q,p}^{2(1-1/p)}(\Omega)$ denote the Besov space defined by

$$B_{q,p}^{2(1-1/p)}(\Omega) = [L_q(\Omega), W_q^2(\Omega)]_{1-1/p,p}.$$

From Steiger [9] and Triebel [10] we know that

$$\mathcal{D}_{q,p}(\Omega) = \begin{cases} \{v \in B^{2(1-1/p)}_{q,p}(\Omega) \mid \operatorname{div} v = 0 \text{ in } \Omega,\ S(v, K(v))\nu|_\Gamma = 0\} \\ \qquad\qquad\qquad\qquad \text{when } \ 2(1-1/p) > 1 + 1/q, \\ \{v \in B^{2(1-1/p)}_{q,p}(\Omega) \mid \operatorname{div} v = 0 \text{ in } \Omega\} \\ \qquad\qquad\qquad\qquad \text{when } \ 2(1-1/p) < 1 + 1/q. \end{cases}$$

The following theorem shows the L_p-L_q maximal regularity of solutions of (1.1) local in time.

Theorem 1.4. *Let $1 < p, q < \infty$ and $T > 0$. If initial data v_0 and right members f, g, $\tilde{g}$ and h for* (1.1) *satisfy the condition:*

$$v_0 \in \mathcal{D}_{q,p}(\Omega), \quad f \in L_p((0,T), L_q(\Omega))^n, \quad g \in L_{p,0}((0,T), W^1_q(\Omega)),$$
$$\tilde{g} \in W^1_{p,0}((0,T), L_q(\Omega))^n, \quad h \in H^{1,1/2}_{q,p,0}(\Omega \times (0,T))^n$$

then (1.1) *admits a unique solution*

$$(v, \theta) \in W^{2,1}_{q,p}(\Omega \times (0,T))^n \times L_p((0,T), W^1_q(\Omega))$$

which enjoys the estimate:

$$\begin{aligned} (1.5)\quad & \|v\|_{W^{2,1}_{q,p}(\Omega\times(0,T))} + \|\theta\|_{L_p((0,T),W^1_q(\Omega))} \\ & \leqq C(1+T)\{\|v_0\|_{\mathcal{D}_{q,p}(\Omega)} + \|f\|_{L_p((0,T),L_q(\Omega))} + \|g\|_{L_p((0,T),W^1_q(\Omega))} \\ & \qquad + \|\tilde{g}\|_{W^1_p((0,T),L_q(\Omega))} + \|h\|_{H^{1,1/2}_{q,p,0}(\Omega\times(0,T))}\}, \end{aligned}$$

where the constant C is independent of T, v, θ, f, g, $\tilde{g}$ and h.

Remark 1.5. Solonnikov [8, Theorem 2] stated a maximal regularity theorem on a finite time interval $(0,T)$ corresponding to Theorem 1.4 under the condition that $p = q > 3$, replacing $H^{1/2}_p((0,T), L_q(\Omega))$ and the constant $C(1+T)$ in (1.5) by $W^{1/2}_p((0,T), L_q(\Omega))$ and some constant C_T which is a nondecreasing function of T, respectively.

§2. An idea of our proof of Theorems 1.2 and 1.4

Roughly speaking, we can show our maximal regularity result as follows. First of all, we show the L_p-L_q maximal regularity of solutions to the model problems in the whole space and in the half-space by applying the Weis operator valued Fourier multiplier theorem (Theorem 2.2, below) to the exact solution formulas, and therefore it is the key to show the $\mathcal{R}$ boundedness of the family of solution operators to the corresponding resolvent problem on $\mathcal{B}(L_q)$. Several techniques to show the $\mathcal{R}$ boundedness can be found in [2]. After such analysis for the model problems, using the usual localization procedure and estimating the perturbation terms by using the estimate: $\|e^{-A_q t}v_0\|_{W^1_q(\Omega)} \leqq Ct^{-1/2}e^{-ct}\|v_0\|_{L_q(\Omega)}$ (C, $c > 0$ and v_0 being orthogonal to $\mathcal{R}$), we obtain the L_p-L_q maximal regularity result for (1.1) with $g = \tilde{g} = h = 0$. By using the solution to the Laplace equation with the zero Dirichlet boundary condition, we reduce the non-zero divergence condition to the divergence free case. Finally, non-homogeneous Neumann condition case is treated by using the solution to the dual problem with the homogeneous Neumann condition.

In this section, we show an idea of our proof of the L_p-L_q maximal regularity of solutions to the whole space and the half-space model problems, which is one of the essential parts of our argument. In this paper, let us consider the heat equation instead of the Stokes equation for the sake of simplicity. The detail of the proof of Theorems 1.2 and 1.4 will be given in the forthcoming paper [6].

Let X and Y be Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. $\mathcal{B}(X,Y)$ denotes the set of all bounded linear operators from X into Y and $\mathcal{B}(X) = \mathcal{B}(X,X)$.

Definition 2.1. A family of operators $\mathcal{T} \subset \mathcal{B}(X,Y)$ is called $\mathcal{R}$-bounded, if there exists a constant $C > 0$ and $p \in [1,\infty)$ such that for each $m \in \mathbb{N}$, $\mathbb{N}$ being the set of all natural numbers, $T_j \in \mathcal{T}$, $x_j \in X$ and for all sequences $\{r_j(u)\}$ of independent, symmetric, $\{-1,1\}$-valued random variables on $[0,1]$ there holds the inequality:

$$\int_0^1 \|\sum_{j=1}^m r_j(u)T_j(x_j)\|_Y^p\,du \leqq C\int_0^1 \|\sum_{j=1}^m r_j(u)x_j\|_X^p\,du. \tag{2.1}$$

The smallest such C is called $\mathcal{R}$-bound of $\mathcal{T}$, which is denoted by $\mathcal{R}(\mathcal{T})$.

We shall give an operator-valued Fourier multiplier theorem due to Weis [11]. We denote by $\mathcal{D}(\mathbb{R},X)$ the space of X-valued C^∞-functions with compact support and by $\mathcal{D}'(\mathbb{R},X) = \mathcal{B}(\mathcal{D}(\mathbb{R}),X)$ the space of X-valued distributions. The X-valued Schwartz spaces $\mathcal{S}(\mathbb{R},X)$ and $\mathcal{S}'(\mathbb{R},X)$ are defined similarly. Given $M \in L_{1,\mathrm{loc}}(\mathbb{R},\mathcal{B}(X,Y))$, we may define an operator $T_M : \mathcal{F}^{-1}\mathcal{D}(\mathbb{R},X) \to \mathcal{S}'(\mathbb{R},Y)$ by means of

$$T_M\phi = \mathcal{F}^{-1}M\mathcal{F}\phi \tag{2.2}$$

for $\phi \in \mathcal{S}(\mathbb{R},X)$ such that $\mathcal{F}\phi \in \mathcal{D}(\mathbb{R},X)$. Since $\mathcal{F}^{-1}\mathcal{D}(\mathbb{R},X)$ is dense in $L_p(\mathbb{R},X)$, we see that T_M is a well-defined linear operator from a dense subset of $L_p(\mathbb{R}, X)$ to $\mathcal{S}'(\mathbb{R}, Y)$. Concerning the boundedness of the operator T_M, the following theorem was proved by Weis [11].

Theorem 2.2. *Suppose that X and Y are UMD Banach spaces and let $1 < p < \infty$. Let M be a function in $C^1(\mathbb{R} \setminus \{0\}, \mathcal{B}(X, Y))$ such that the following conditions are satisfied:*

$$\mathcal{R}(\{M(\tau) \mid \tau \in \mathbb{R} \setminus \{0\}\}) = a_0 < \infty,$$
$$\mathcal{R}(\{\tau M'(\tau) \mid \tau \in \mathbb{R} \setminus \{0\}\}) = a_1 < \infty.$$

Then, the operator T_M defined by (2.2) is extended to a bounded linear operator from $L_p(\mathbb{R}, X)$ into $L_p(\mathbb{R}, Y)$ with norm

$$\|T_M\|_{\mathcal{B}(L_p(\mathbb{R},X),L_p(\mathbb{R},Y))} \leqq C(a_0 + a_1),$$

where $C > 0$ depends only on p, X and Y.

First we consider a model problem in the whole space. Let us consider the heat equation:

$$u_t - \Delta u = f \quad \text{in } \mathbb{R}^n \times \mathbb{R}. \tag{2.3}$$

Let $1 < p, q < \infty$. Suppose that

$$f \in L_{p,0}(\mathbb{R}_+, L_q(\mathbb{R}^n)).$$

We would like to show that the solution u of (2.3) satisfies the L_p-L_q maximal regularity estimate:

(2.4)
$$\|e^{-\gamma t} u_t\|_{L_p(\mathbb{R}, L_q(\mathbb{R}^n))} + \|e^{-\gamma t}\nabla^2 u\|_{L_p(\mathbb{R}, L_q(\mathbb{R}^n))} \leqq C\|e^{-\gamma t} f\|_{L_p(\mathbb{R}, L_q(\mathbb{R}^n))}$$

for any $\gamma \geqq 0$. We may assume that $f \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}_+)$, because $C_0^\infty(\mathbb{R}^n \times \mathbb{R}_+)$ is dense in $L_{p,0}(\mathbb{R}_+, L_q(\mathbb{R}^n))$. We have the solution formula:

$$u(x,t) = \mathcal{L}^{-1}\Big[\frac{\mathcal{L}[f](\xi,\lambda)}{\lambda + |\xi|^2}\Big](x,t),$$

where $\mathcal{L}$ and $\mathcal{L}^{-1}$ denote the Fourier-Laplace transform and its inverse defined by

$$[\mathcal{L}f](\xi,\lambda) = \iint_{\mathbb{R}^{n+1}} e^{-\lambda t - ix\cdot\xi} f(x,t)\,dxdt = \mathcal{F}[e^{-\gamma t} f](\xi,\tau),$$
$$[\mathcal{L}^{-1}g](x,t) = \frac{1}{(2\pi)^{n+1}} \iint_{\mathbb{R}^{n+1}} e^{\lambda t + ix\cdot\xi}\, g(\xi,\lambda)\,d\xi d\tau$$
$$= e^{\gamma t}\mathcal{F}^{-1}[g(\xi, \gamma + i\tau)](x,t), \quad \lambda = \gamma + i\tau,$$

respectively. In particular, u_t is given by the formula:

$$u_t(x,t) = \mathcal{L}^{-1}\Big[\frac{\lambda\mathcal{L}[f](\xi,\lambda)}{\lambda+|\xi|^2}\Big](x,t).$$

Set

$$k_\gamma(\tau,x) = \mathcal{F}^{-1}[\lambda(\lambda+|\xi|^2)^{-1}](x), \quad \lambda=\gamma+i\tau,$$
$$[K_\gamma(\tau)g](x) = \int_{\mathbb{R}^n} k_\gamma(\tau,x-y)g(y)\,dy.$$

Then we have

$$e^{-\gamma t}u_t = \mathcal{F}_\tau^{-1}[K_\gamma(\tau)\mathcal{F}_t[e^{-\gamma t}f](\tau)](t). \tag{2.5}$$

If we can apply Theorem 2.2 to (2.5), we obtain

$$\|e^{-\gamma t}u_t\|_{L_p(\mathbb{R},L_q(\mathbb{R}^n))} \leqq C\|e^{-\gamma t}f\|_{L_p(\mathbb{R},L_q(\mathbb{R}^n))}, \quad \forall\gamma \geqq 0. \tag{2.6}$$

What we have to do to obtain (2.6) is that the $\mathcal{R}$-boundedness of the families $\{K_\gamma(\tau) \mid \tau\in\mathbb{R}\setminus\{0\}\}$ and $\{\tau\partial_\tau K_\gamma(\tau) \mid \tau\in\mathbb{R}\setminus\{0\}\}$ on $\mathcal{B}(L_q(\mathbb{R}^n))$ for $1<q<\infty$. To do this, we shall use the following proposition.

Proposition 2.3. *Let $1<q<\infty$ and $\{k_s(x) \mid s\in\mathbb{R}\setminus\{0\}\}$ be a family of $L_{1,loc}(\mathbb{R}^n)$ functions. Set*

$$K_s g(x) = \int_{\mathbb{R}^n} k_s(x-y)g(y)\,dy, \ \ s\in\mathbb{R}\setminus\{0\}.$$

Suppose that there exists a constant $C>0$ independent of $s\in\mathbb{R}\setminus\{0\}$ such that

$$\|K_s g\|_{L_2(\mathbb{R}^n)} \leqq C\|g\|_{L_2(\mathbb{R}^n)}, \quad \forall g\in L_2(\mathbb{R}^n),$$
$$\sum_{|\beta|=1}|\partial_x^\beta k_s(x)| \leqq C|x|^{-(n+1)}, \quad \forall x\in\mathbb{R}^n\setminus\{0\} \tag{2.7}$$

for all $s\in\mathbb{R}\setminus\{0\}$. Then, $\{K_s \mid s\in\mathbb{R}\setminus\{0\}\}$ is $\mathcal{R}$-bounded on $\mathcal{B}(L_q(\mathbb{R}^n))$ and its $\mathcal{R}$-bound is less than or equal to $C_{n,q}C$ with some constant $C_{n,q}$.

Proposition 2.3 follows from the Benedek, Calderón and Panzone theorem [1].

Using the inequality:

$$|\lambda+|\xi|^2| \geqq c(|\lambda|+|\xi|^2), \quad \mathrm{Re}\,\lambda\geqq 0, \ \xi\in\mathbb{R}^n$$

with some positive constant c and Plancherel's formula, we have

$$\|K_\gamma(\tau)g\|_{L_2(\mathbb{R}^n)} \leqq C\|g\|_{L_2(\mathbb{R}^n)}, \quad \forall\gamma \geqq 0.$$

To check the condition (2.7), we use the following lemma ([4, Theorem 2.3]).

Lemma 2.4. *Let X be a Banach space and $\|\cdot\|_X$ its norm. Let a be a number $> -n$ and set $a = N + \sigma - n$, where $N \geqq 0$ is an integer and $0 < \sigma \leqq 1$. Let $f(\xi)$ be a function in $C^\infty(\mathbb{R}^n \setminus \{0\}, X)$ such that*

$$\partial_\xi^\alpha f(\xi) \in L_1(\mathbb{R}^n, X), \quad \forall\,|\alpha| \leqq N,$$
$$\|\partial_\xi^\alpha f(\xi)\|_X \leqq C_\alpha|\xi|^{a-|\alpha|}, \quad \forall\,\xi \neq 0,\ \forall\,\alpha \in \mathbb{N}_0^n.$$

Then we have

$$\|\mathcal{F}^{-1}[f](x)\|_X \leqq C_{n,a}(\max_{|\alpha|\leqq N+2} C_\alpha)|x|^{-(n+a)}, \quad \forall x \neq 0.$$

Since

$$|\partial_\xi^\alpha \lambda(\lambda + |\xi|^2)^{-1}(i\xi)^\beta| \leqq C|\xi|^{1-|\alpha|}$$

for any $\beta \in \mathbb{N}_0^n$ with $|\beta| = 1$ and $\alpha \in \mathbb{N}_0^n$, by Lemma 2.4 we have

$$\sum_{|\beta|=1} |\partial_x^\beta k_\gamma(\tau, x)| \leqq C|x|^{-(n+1)}, \quad \forall x \in \mathbb{R}^n \setminus \{0\},$$

where C is a constant independent of τ, γ and x. Therefore, by Proposition 2.3 we see that $\{K_\gamma(\tau) \mid \tau \in \mathbb{R}\setminus\{0\}\}$ is $\mathcal{R}$-bounded, whose $\mathcal{R}$-bound is independent of $\gamma \geqq 0$. We also see that $\{\tau\partial_\tau K_\gamma(\tau) \mid \tau \in \mathbb{R} \setminus \{0\}\}$ is $\mathcal{R}$-bounded, whose $\mathcal{R}$-bound is independent of $\gamma \geqq 0$. Therefore we can apply Theorem 2.2 to (2.5), and we have (2.6).

Employing the same arguments as above, we can also show that

$$\|e^{-\gamma t}\nabla^2 u\|_{L_p(\mathbb{R}, L_q(\mathbb{R}^n))} \leqq C\|e^{-\gamma t} f\|_{L_p(\mathbb{R}, L_q(\mathbb{R}^n))}, \quad \forall\gamma \geqq 0. \tag{2.8}$$

Combining (2.6) with (2.8), we obtain (2.4).

Next we consider a model problem in the half-space. Let us consider the Neumann problem:

$$u_t - \Delta u = 0 \quad \text{in } \mathbb{R}_+^n \times \mathbb{R}, \quad \partial_n u|_{x_n=0} = h|_{x_n=0}, \tag{2.9}$$

where $\mathbb{R}_+^n = \{x = (x_1, \ldots, x_n) \mid x_n > 0\}$. Let $1 < p, q < \infty$. Suppose that

$$h \in H_{q,p,0}^{1,1/2}(\mathbb{R}_+^n \times \mathbb{R}_+).$$

We would like to show that the solution u of (2.9) satisfies the L_p-L_q maximal regularity estimate:

(2.10)
$$\|e^{-\gamma t}u_t\|_{L_p(\mathbb{R},L_q(\mathbb{R}^n))} + \|e^{-\gamma t}\nabla^2 u\|_{L_p(\mathbb{R},L_q(\mathbb{R}^n))} \leqq C\|e^{-\gamma t}h\|_{H^{1,1/2}_{q,p}(\mathbb{R}^n_+\times\mathbb{R})}$$

for any $\gamma \geqq 0$. We may assume that $h \in C_0^\infty(\mathbb{R}^n_+ \times \mathbb{R}_+)$, because $C_0^\infty(\mathbb{R}^n_+ \times \mathbb{R}_+)$ is dense in $H^{1,1/2}_{q,p,0}(\mathbb{R}^n_+ \times \mathbb{R}_+)$.

Set $x' = (x_1, \dots, x_{n-1})$. We shall use the partial Laplace-Fourier transform with respect to (x', t) and its inverse defined by

$$\begin{aligned}
\mathcal{L}_{x',t}[f](\xi', x_n, \lambda) &= \iint_{\mathbb{R}^n} e^{-\lambda t - ix'\cdot\xi'} f(x', x_n, t)\, dx' dt \\
&= \mathcal{F}_{x',t}[e^{-\gamma t} f](\xi', x_n, \tau), \quad \lambda = \gamma + i\tau, \\
\mathcal{L}^{-1}_{\xi',\lambda}[g](x', x_n, t) &= \frac{1}{(2\pi)^n} \iint_{\mathbb{R}^n} e^{\lambda t + ix'\cdot\xi'} g(\xi', x_n, \lambda)\, d\xi' d\tau \\
&= e^{\gamma t}\mathcal{F}^{-1}_{\xi',\tau}[g(\xi', x_n, \gamma + i\tau)](x', t),
\end{aligned}$$

where $\mathcal{F}_{x',t}$ and $\mathcal{F}^{-1}_{\xi',\tau}$ denote the Fourier transform and its inverse with respect to (x', t) and (ξ', τ), respectively.

Setting $B = \sqrt{\lambda + |\xi'|^2}$ with $\operatorname{Re} B > 0$, we have the solution formula:

$$\begin{aligned}
u(x,t) &= -\mathcal{L}^{-1}_{\xi',\lambda}[B^{-1}e^{-Bx_n}\mathcal{L}_{x',t}[h](\xi', 0, \lambda)](x', t) \\
&= \int_0^\infty \partial y_n \mathcal{L}^{-1}_{\xi',\lambda}[B^{-1}e^{-B(x_n+y_n)}\mathcal{L}_{x',t}[h](\xi', y_n, \lambda)](x', t)\, dy_n \\
&= \int_0^\infty \mathcal{L}^{-1}_{\xi',\lambda}[B^{-1}e^{-B(x_n+y_n)}(-B\mathcal{L}_{x',t}[h](\xi', y_n, \lambda) \\
&\qquad\qquad + \mathcal{L}_{x',t}[\partial_n h](\xi', y_n, \lambda))](x', t)\, dy_n.
\end{aligned}$$

In particular, u_t is given by the formula:

$$\begin{aligned}
u_t(x,t) = \int_0^\infty & e^{\gamma t}\mathcal{F}^{-1}_{\xi',\tau}[\lambda B^{-1}e^{-B(x_n+y_n)}(-B\mathcal{L}_{x',t}[h](\xi', y_n, \lambda) \\
&\qquad + \mathcal{L}_{x',t}[\partial_n h](\xi', y_n, \lambda))](x', t)\, dy_n.
\end{aligned}$$

If we set

$$\begin{aligned}
k_\gamma(\tau, x) &= \mathcal{F}^{-1}_{\xi'}[\lambda B^{-1}e^{-Bx_n}](x'), \quad \lambda = \gamma + i\tau, \\
[K_\gamma(\tau)f](x) &= \int_{\mathbb{R}^n_+} k_\gamma(\tau, x' - y', x_n + y_n) f(y)\, dy,
\end{aligned}$$

then we have

$$(2.11)\qquad e^{-\gamma t}u_t = \mathcal{F}_\tau^{-1}[K_\gamma(\tau)\mathcal{F}_t[h^\gamma](\tau)](t),$$
$$h^\gamma(x,t) = \mathcal{F}_{\xi',\tau}^{-1}[-B\mathcal{L}_{x',t}[h](\xi',x_n,\lambda) + \mathcal{L}_{x',t}[\partial_n h](\xi',x_n,\lambda)](x',t).$$

To show the $\mathcal{R}$-boundedness of $\{K_\gamma(\tau) \mid \tau \in \mathbb{R}\setminus\{0\}\}$, we use the following proposition.

Proposition 2.5. *Let $1 < q < \infty$. Let G be a domain in $\mathbb{R}^n$ and $\mathcal{T} = \{T_\mu \mid \mu \in \mathcal{M}\} \subset \mathcal{B}(L_q(G))$ be a family of the kernel operators:*

$$T_\mu f(x) = \int_G k_\mu(x,y)f(y)\,dy$$

for $x \in G$ and $f \in L_q(G)$. Suppose that there exists a $k_0(x,y)$ such that

$$|k_\mu(x,y)| \leqq k_0(x,y)$$

for almost all x, $y \in G$ and any $\mu \in \mathcal{M}$. Set

$$T_0 f(x) = \int_G k_0(x,y)f(y)\,dy.$$

If $T_0 \in \mathcal{B}(L_q(G))$, then $\mathcal{T}$ is $\mathcal{R}$-bounded on $\mathcal{B}(L_q(G))$, whose $\mathcal{R}$-bound is less than or equal to $C_{n,q,G}\|T_0\|_{\mathcal{B}(L_q(G))}$.

Proposition 2.5 can be proved following ideas due to Denk, Hieber and Prüss [2].

Since we can show that

$$|\partial_{\xi'}^{\alpha'}[\lambda B^{-1}e^{-Bx_n}]| \leqq C_{\alpha'}|\lambda|(|\lambda|^{\frac{1}{2}}+|\xi'|)^{-1-|\alpha'|}e^{-d(|\lambda|^{1/2}+|\xi'|)x_n}, \forall\alpha' \in \mathbb{N}_0^{n-1}$$

with some positive constant $d > 0$, by Lemma 2.4 and the change of variable: $x_n\xi' = \eta'$, we have

$$|k_\gamma(\tau,x)| \leqq C|x|^{-n},$$

and therefore if we set

$$K_0 g(x) = \int_{\mathbb{R}^n_+} \frac{Cg(y)}{(|x'-y'|^2+(x_n+y_n)^2)^{n/2}}\,dy,$$

then we have

$$\|K_0 g\|_{L_q(\mathbb{R}^n_+)} \leqq C_q\|g\|_{L_q(\mathbb{R}^n_+)}, \quad 1 < q < \infty.$$

Therefore, applying Proposition 2.5, we see that $\{K_\gamma(\tau) \mid \tau \in \mathbb{R} \setminus \{0\}\}$ is $\mathcal{R}$-bounded on $\mathcal{B}(L_q(\mathbb{R}^n_+))$, whose $\mathcal{R}$-bound is independent of $\gamma \geqq 0$. In the same manner, we also see that $\{\tau\partial_\tau K_\gamma(\tau) \mid \tau \in \mathbb{R} \setminus \{0\}\}$ is $\mathcal{R}$-bounded on $\mathcal{B}(L_q(\mathbb{R}^n_+))$, whose $\mathcal{R}$-bound is independent of $\gamma \geqq 0$. Applying Theorem 2.2 to (2.11), we have

$$\|e^{-\gamma t}u_t\|_{L_p(\mathbb{R}, L_q(\mathbb{R}^n_+))} \leqq C\|h^\gamma\|_{L_p(\mathbb{R}, L_q(\mathbb{R}^n_+))}, \quad \forall\gamma \geqq 0. \tag{2.12}$$

Since

$$h^\gamma(x,t) = \mathcal{F}^{-1}_{\xi',\tau}[-\sqrt{\lambda + |\xi'|^2}\mathcal{F}_{x',t}[e^{-\gamma t}h](\xi', x_n, \tau) \\ + \mathcal{F}_{x',t}[e^{-\gamma t}\partial_n h](\xi', x_n, \tau)](x', t),$$

we obtain

(2.13)

$$\|h^\gamma\|_{L_p(\mathbb{R}, L_q(\mathbb{R}^n_+))} \leqq C\{\|e^{-\gamma t}h\|_{H^{1/2}_p(\mathbb{R}, L_q(\mathbb{R}^n_+))} + \|e^{-\gamma t}h\|_{L_p(\mathbb{R}, W^1_q(\mathbb{R}^n_+))}\} \\ \leqq C\|e^{-\gamma t}h\|_{H^{1,1/2}_{q,p}(\mathbb{R}^n_+ \times \mathbb{R})}.$$

Combining (2.12) with (2.13) we obtain

$$\|e^{-\gamma t}u_t\|_{L_p(\mathbb{R}, L_q(\mathbb{R}^n_+))} \leqq C\|e^{-\gamma t}h\|_{H^{1,1/2}_{q,p}(\mathbb{R}^n_+ \times \mathbb{R})}, \quad \forall\gamma \geqq 0. \tag{2.14}$$

Employing the same argument as above, we can also show that

$$\|e^{-\gamma t}\nabla^2 u\|_{L_p(\mathbb{R}, L_q(\mathbb{R}^n_+))} \leqq C\|e^{-\gamma t}h\|_{H^{1,1/2}_{q,p}(\mathbb{R}^n_+ \times \mathbb{R})}, \quad \forall\gamma \geqq 0. \tag{2.15}$$

Combining (2.14) with (2.15) we obtain (2.10).

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Yoshihiro Shibata
Department of Mathematical Sciences, School of Science and Engineering
Waseda University
Ohkubo 3-4-1, Shinjuku-ku, Tokyo 169-8555
Japan

Senjo Shimizu
Faculty of Engineering
Shizuoka University
Hamamatsu, Shizuoka 432-8561
Japan