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Non-selfadjoint perturbation of Schrödinger and wave equations

Can one completely determine the behavior of solutions by only spectra ?

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Abstract.

Non-selfadjoint perturbation of Schrödinger and wave equations is considered. In some cases, we can determine the behavior of solutions (energy decay or scattering) by only its spectral structure.

Chapter I. Motivations and Problems.

Consider the following problem:

$$\frac{dv}{dt}(t) = Av(t) \quad (t > 0), \qquad v(0) = v_0, \tag{1.1}$$

where $v(t) = (v_1(t), v_2(t), \cdots, v_n(t))$ is the vector of unknown n-functions $v_j(t)$ $(j = 1, 2, \cdots, n)$, $\frac{dv}{dt}(t)$ is time derivative of vector $v(t)$ and A denotes an $n \times n$ matrix with constant coefficients.

For the sake of simplicity, we assume that A has distinct n-eigenvalues $\lambda_1, \lambda_2, \cdots \lambda_n \in \mathbb{C}$. Then as is well-known in linear algebra, A has spectral decomposition

$$A = \sum_{j=1}^{n} \lambda_j P_{\lambda_j},$$

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where P_{λ_j} denotes projection onto the eigenspace with respect to eigenvalue λ_j. Thus, the solution of (1.1) is given by

$$v(t) = e^{tA}v_0 = \left(\sum_{j=1}^{n} e^{\lambda_j t} P_{\lambda_j}\right) v_0. \tag{1.2}$$

In particular we find from (1.2) that the spectra of A determine completely the behavior of solutions $v(t)$.

Our aim is to construct the general theory of above problem (1.1) in the case of A is a non-selfadjoint partial differential operator. But, at the moment, general theory on spectral decomposition for non-selfadjoint operator does not exist. So, we restrict equations to typical ones in mathematical physics — Schrödinger equations and wave equations, and we aim to pursue relative fact towards establishment of general theory as above.

Chapter II. Schrödinger equations.

§1. Problem and Result.

As first example, we state affirmative results for above problem on Schrödinger equations with non-selfadjoint rank one perturbation.

Consider the following equations:

$$\begin{cases} \frac{dv}{dt}(t) = -iH_\alpha v(t) \quad (t>0), \qquad v(0) = v_0, \\ H_\alpha = -\frac{d^2}{dx^2} + \alpha\langle\cdot,\delta\rangle\delta, \end{cases}$$

where δ denotes the Dirac delta function, and coupling constant α is assumed to be $\alpha = \alpha_1 + i\alpha_2$ for non-positive α_1, α_2. We treat the operator H_α in the space $\mathcal{H} = L^2(\mathbb{R}^1)$.

The domain of H_α, $\mathfrak{D}(H_\alpha)$ is given by

$$\begin{aligned}\mathfrak{D}(H_\alpha) =& \{\, U = u + aH_0(H_0^2+1)^{-1}\delta \mid u \in \mathcal{H}^2,\ a \in \mathbb{C}, \\ & \langle u, \delta\rangle = -a(\alpha^{-1} + \langle \delta, H_0(H_0^2+1)^{-1}\delta\rangle)\}(\alpha \neq 0),\end{aligned}$$

where $H_0 = -\frac{d^2}{dx^2}$ with $\mathfrak{D}(H_0) = H^2(\mathbb{R}^1)$ (the Sobolev space of order 2).

Then it follows that H_α with $\alpha_2 < 0$ is maximal dissipative (the case $\alpha_2 = 0$ is self-adjoint), i.e., H_α with $\alpha_2 < 0$ generates a contraction semi-group $\{e^{-itH_\alpha}\}_{t\geq 0}$ (the case $\alpha_2 = 0$ generates a unitary group $\{e^{-itH_{\alpha_1}}\}_{t\in\mathbb{R}}$). So, we can define the wave operator $W(\alpha) =$

$\text{s-}\lim_{t\to+\infty} e^{itH_0}e^{-itH_\alpha}$ as a non-trivial operator from $\mathcal{H}$ to $\mathcal{H}$, where $\alpha_1 \le 0$ and $\alpha_2 < 0$. Moreover, for each $f \in \mathcal{H}$, we obtain a unique decomposition : $f = f_s + f_d$, where $f_s \equiv f - P_{-\alpha^2/4}f \in \mathrm{Ker}P_{-\frac{\alpha^2}{4}}$ (Kernel of projection with respect to eigenvalue $-\frac{\alpha^2}{4}$) and $f_d \equiv P_{-\alpha^2/4}f \in \mathrm{Range}P_{-\frac{\alpha^2}{4}}$ (range of $P_{-\frac{\alpha^2}{4}}$). In above suffix and following notations, s, (S) and d, (D) means *scattering* and *decay* states, respectively.

Before describing our result, we prepare some notations. The point, discrete, residual, continuous and essential spectrum is denoted by $\sigma_p(A)$, $\sigma_d(A)$, $\sigma_r(A)$, $\sigma_c(A)$ and $\sigma_{ess}(A)$, respectively.

Theorem 2.1 (Kadowaki-Nakazawa-Watanabe [11]).

(1) (*Spectral structure of* H_α) *Let* $\alpha = \alpha_1 + i\alpha_2$ *with* $\alpha_1 \le 0$, $\alpha_2 \le 0$. *Then the spectrum of* H_α *is given by* $\sigma(H_\alpha) = [0,\infty) \cup \{-\frac{\alpha^2}{4}\}\,(\alpha_1 < 0), = [0,\infty)\,(\alpha_1 = 0)$. *Exact classification of the spectrum* $\sigma(H_\alpha)$ *is* $\sigma_{ess}(H_\alpha) = \sigma_c(H_\alpha) = [0,\infty)$, $\sigma_r(H_\alpha) = \emptyset$ *and* $\sigma_p(H_\alpha) = \sigma_d(H_\alpha) = \{-\frac{\alpha^2}{4}\}\,(\alpha_1 < 0), = \emptyset\,(\alpha_1 = 0)$. *Moreover the projection with respect to* $-\frac{\alpha^2}{4}(\alpha_1 \neq 0)$ *is given by* $P_{-\alpha^2/4}f = -\alpha/2\langle f, e^{(\overline{\alpha}|\cdot|)/2}\rangle e^{(\alpha|x|)/2}$.

(2) (*Kernel of* $W(\alpha)$) (*i*) *Assume that* $\alpha_1 < 0$ *and* $\alpha_2 < 0$. *Then* $\mathrm{Ker}W(\alpha) = \mathrm{Range}P_{-\frac{\alpha^2}{4}}$. (*ii*) *Assume that* $\alpha_1 = 0$ *and* $\alpha_2 < 0$. *Then* $\mathrm{Ker}W(i\alpha_2) = \{0\}$.

(3) (*The classification of asymptotics by the initial data*) (*i*) *Under the same assumption for* α *as in* (2)-(*i*), *we have for each* $f \in \mathcal{H}$ *decomposed as above,*

$$\text{(S)}\qquad f_s \neq 0 \iff \begin{cases} \lim_{t\to\infty} \|e^{-itH_\alpha}f - e^{-itH_0}W(\alpha)f\| = 0, \\ W(\alpha)f \neq 0 \end{cases}$$

and

$$\text{(D)}\qquad f_s = 0 \iff \lim_{t\to\infty} \|e^{-itH_\alpha}f\| = 0 \quad (e^{-itH_\alpha}f = e^{i\frac{\alpha^2}{4}t}f_d).$$

(*ii*) *Under the same assumption for* α *as in* (2)–(*ii*), *we have*

$$f \in \mathcal{H} \quad \text{and} \quad f \neq 0 \iff \begin{cases} \lim_{t\to\infty} \|e^{-itH_{i\alpha_2}}f - e^{-itH_0}W(i\alpha_2)f\| = 0, \\ W(i\alpha_2)f \neq 0. \end{cases}$$

Remark 2.2. (3) is the corollary of (2).

§2. Outline of Proof of Theorem 2.1 (1).

We shall define the operator

$$\tilde{H}_\alpha = H_0 + \alpha\langle\cdot,\varphi\rangle\varphi$$

with the domain

$$\mathfrak{D}(\tilde{H}_\alpha) = \begin{cases} \{\, U = u + aH_0(H_0^2+1)^{-1}\varphi \,|\, u \in \mathcal{H}^2,\ a \in \mathbb{C}, \\ \qquad \langle u,\varphi\rangle = -a(\alpha^{-1} + \langle\varphi, H_0(H_0^2+1)^{-1}\varphi\rangle)\} \quad (\alpha \neq 0), \\ \mathcal{H}^2 \quad (\alpha = 0), \end{cases}$$

where $\alpha \in \mathbb{C}$, $\varphi \in \mathcal{H}^{-1} \setminus \mathcal{H}$.

For $\alpha \neq 0$, $U \in \mathfrak{D}(\tilde{H}_\alpha)$ means $\tilde{H}_\alpha U \in \mathcal{H}$ for any $U \in \mathfrak{D}(\tilde{H}_\alpha)$ since $\tilde{H}_\alpha U = H_0 u - a(H_0^2+1)^{-1}\varphi$. Put $\alpha = \alpha_1 + i\alpha_2$ with $\alpha_1 \leq 0$ and $\alpha_2 \leq 0$. Then $\tilde{H}_\alpha$ is dissipative, i.e., $\mathrm{Im}\langle\tilde{H}_\alpha U, U\rangle \leq 0$ for $U \in \mathfrak{D}(\tilde{H}_\alpha)$, and $\tilde{H}_{\overline{\alpha}}$ is accretive, i.e., $\mathrm{Im}\langle\tilde{H}_{\overline{\alpha}} V, V\rangle \geq 0$ for $V \in \mathfrak{D}(\tilde{H}_{\overline{\alpha}})$. Moreover we have the following: (i) $\tilde{H}_\alpha$ is a maximal dissipative operator, (ii) $\tilde{H}_{\overline{\alpha}}$ is a maximal accretive operator, (iii) $\tilde{H}_\alpha^* = \tilde{H}_{\overline{\alpha}}$.

Especially, H_α denotes the operator $\tilde{H}_\alpha$ defined by choosing $\varphi = \delta$ (Dirac delta) $\in \mathcal{H}^s (s < -1/2)$. We also denote by $\tilde{R}_\alpha(z)$ (resp. $R_\alpha(z)$) the resolvent $(\tilde{H}_\alpha - z)^{-1}$ (resp. $(H_\alpha - z)^{-1}$) of $\tilde{H}_\alpha$ (resp. H_α) for $z \in \rho(\tilde{H}_\alpha)$ (resp. $z \in \rho(H_\alpha)$), where $\rho(A)$ is the resolvent set of a closed operator A in $\mathcal{H}$.

First of all, we find

Lemma 2.3 (Representation of $\tilde{R}_\alpha(z)$). *Assume that $\alpha = \alpha_1 + i\alpha_2$ with $\alpha_1 \leq 0$ and $\alpha_2 \leq 0$. Then we have*

$$\tilde{R}_\alpha(z)f = R_0(z)f - \alpha\{1 + \alpha\langle R_0(z)\varphi,\varphi\rangle\}^{-1}\langle R_0(z)f,\varphi\rangle R_0(z)\varphi$$

for any $f \in \mathcal{H}$ and $z \in \rho(H_0) \cap \{\, z \in \mathbb{C} \,|\, 1 + \alpha\langle R_0(z)\varphi,\varphi\rangle \neq 0 \,\}$.

Putting $\varphi = \delta$ in above lemma, we have

Lemma 2.4 (Representation of $R_\alpha(z)$). *For any $f \in \mathcal{H}$,*

$$(R_\alpha(z)f)(x) = (R_0(z)f)(x) + \int_{\mathbb{R}^1} K(x,y;z)f(y)dy,$$

$$\textit{where} \quad K(x,y;z) = -\frac{\alpha}{2i\sqrt{z}(2i\sqrt{z}-\alpha)}e^{i\sqrt{z}(|x|+|y|)} \in L^2(\mathbb{R}^1_x \times \mathbb{R}^1_y)$$

with $\mathrm{Im}\sqrt{z} > 0$, where $z \in \rho(H_\alpha) = \mathbb{C} \setminus \left([0,\infty) \cup \{-\alpha^2/4\}\right)$ $(\alpha_1 < 0)$, $= \mathbb{C} \setminus [0,\infty)$ $(\alpha_1 = 0)$.

This implies the assertion on essential spectrum in Theorem 2.1 (1).

Next, since the point $-\frac{\alpha^2}{4}$ is isolated and simple, we obtain the conclusions on point spectrum and on the representation of projection by Lemma 2.4 and the residual theorem.

Lemma 2.5. *Under the same assumption as in Lemma 2.3, we have for* $U, V \in \mathfrak{D}(\tilde{H}_\alpha)$,

$$\langle \tilde{H}_\alpha U, V\rangle - \langle U, \tilde{H}_\alpha V\rangle = \frac{2i\mathrm{Im}\alpha}{|\alpha|^2} a\bar{b},$$

where for some $u, v \in \mathcal{H}^2$,

$$a = -\frac{\langle u, \varphi\rangle}{\alpha^{-1} + \langle \varphi, H_0(H_0^2+1)^{-1}\varphi\rangle}$$

and

$$b = -\frac{\langle v, \varphi\rangle}{\alpha^{-1} + \langle \varphi, H_0(H_0^2+1)^{-1}\varphi\rangle}.$$

From this we have

Lemma 2.6. *Assume* $\alpha_1 \leq 0$ *and* $\alpha_2 < 0$. *Then we find*

$$\sigma_p(\tilde{H}_\alpha) \cap \mathbb{R} = \emptyset, \quad \sigma_p(\tilde{H}_{\overline{\alpha}}) \cap \mathbb{R} = \emptyset, \quad \sigma_r(\tilde{H}_\alpha) \cap \mathbb{R} = \emptyset.$$

Since the spectral theory for the self-adjoint operator implies $\sigma_r(H_{\alpha_1}) = \emptyset$, the proof of Theorem 2.1 (1) is complete. Q.E.D.

§3. Outline of Proof of Theorem 2.1 (2) with $\alpha_1 < 0$.

We define $K = (H_\alpha - i)^{-1} - (H_0 - i)^{-1}$. Then as is easily seen from Lemma 2.3, this operator K becomes a compact in $\mathcal{H}$. Moreover, we have from Mellin transforms estimates,

Lemma 2.7. *Let* P_+ *and* P_- *be the positive and negative spectral projections for the generator of dilation* $\frac{1}{2i}(x\frac{d}{dx} + \frac{d}{dx}x)$, *respectively. Then for the pair* $(A, B) = (K, P_+), (K^*, P_+)$ *and* (K^*, P_-), *we have*

$$\int_0^\infty \|Ae^{-itH_0}\psi(H_0)B\|_{B(\mathcal{H},\mathcal{H})} dt < \infty,$$

$$\operatorname*{w-lim}_{t\to+\infty} e^{itH_0}\psi(H_0)P_- f_t = 0$$

where $\psi \in C_0^\infty((0,\infty))$, $\|\cdot\|_{B(\mathcal{H},\mathcal{H})}$ *means the operator norm for bounded operators in* $\mathcal{H}$, $\{f_t\}_{t\in\mathbf{R}}$ *satisfies* $\sup_{t\in\mathbf{R}}\|f_t\| < \infty$ *and* w–lim *denotes weak limit.*

Noting this lemma and the first relation in Lemma 2.6 with $\varphi = \delta$, we have from theorem by Kadowaki [10] (see [11] Theorem B in Appendix B),

Proposition 2.8 (Existence of the Wave Operator). *Let* $\alpha = \alpha_1 + i\alpha_2$ *with* $\alpha_1 \le 0$, $\alpha_2 < 0$. *Then there exists*

$$W(\alpha) = \operatorname{s-lim}_{t\to+\infty} e^{itH_0}e^{-itH_\alpha}$$

as non-trivial operator from $\mathcal{H}$ *to* $\mathcal{H}$, *where* s–lim *denotes strong limit.*

From Theorem 2.1 (1) and Proposition 2.8, we find the following:

Corollary 2.9. *Let* $\alpha = \alpha_1 + i\alpha_2$ *with* $\alpha_1 < 0$ *and* $\alpha_2 < 0$. *Then we have*

$$\mathrm{Range}P_{-\frac{\alpha^2}{4}} \subset \mathrm{Ker}W(\alpha) = \left\{ f \;\middle|\; \lim_{t\to+\infty} \|e^{-itH_\alpha}f\| = 0 \right\}.$$

On the other hand, we can show the existence of the following wave operators by Cook-Kuroda method;

$$\Omega_-(\alpha) = \operatorname{s-lim}_{t\to+\infty} e^{-itH_\alpha}e^{itH_0} \quad \text{and} \quad \Omega_+(\alpha) = \operatorname{s-lim}_{t\to+\infty} e^{itH_{\overline{\alpha}}}e^{-itH_0}$$

in $\mathcal{H}$. So, we can define the scattering operator $S(\alpha)$ by

$$S(\alpha) = W(\alpha)\Omega_-(\alpha).$$

Next, we define generalized Fourier transform. The following follows from Kuroda [19], Chapter 5;

Proposition 2.10 (Generalized Fourier Transform for $\boldsymbol{H_\alpha}$). *Assume that* α *satisfies the same condition as in Proposition 2.8 and define*

$$\mathcal{F}_\alpha = \mathcal{F}_0 W(\alpha).$$

Then the representation of $\mathcal{F}_\alpha$ *is given by*

$$(\mathcal{F}_\alpha f)(k) = \lim_{R\to+\infty}\int_{|x|<R} \overline{\psi_\alpha(x,k)}f(x)dx \quad in \quad \mathcal{H}$$

for any $f \in \mathcal{H}$, where

$$\overline{\psi_\alpha(x,k)} = (2\pi)^{-1/2}\left(e^{-ixk} + \frac{\alpha}{(2i|k|-\alpha)}e^{i|x||k|}\right).$$

Furthermore, we obtain

$$(\mathcal{F}_\alpha H_\alpha f)(k) = |k|^2(\mathcal{F}_\alpha f)(k) \quad \text{for} \quad f \in \mathfrak{D}(H_\alpha).$$

Now, we shall state the generalized Parseval formula (cf., Pavlov [33], Theorem 2.1):

Lemma 2.11 (Generalized Parseval Formula for $\boldsymbol{\alpha_1 < 0}$). *For any $f, g \in \mathcal{H} \cap L^1(\mathbb{R}^1)$ and for $\alpha \in \{\alpha = \alpha_1 + i\alpha_2; \alpha_1 < 0\} \equiv D$ we have*

$$\langle \mathcal{F}_\alpha f, \mathcal{F}_{\bar{\alpha}} g\rangle = \langle f, g\rangle + \frac{\alpha}{2}\langle f, e^{(\bar{\alpha}|\cdot|)/2}\rangle\langle e^{(\alpha|\cdot|)/2}, g\rangle.$$

As for the proof of Theorem 2.1 (2) with $\alpha_1 < 0$, we have only to show $W(\alpha)f = 0 \Leftrightarrow f_s = 0$. This follows from Corollary 2.9, Lemma 2.11 and density argument. Q.E.D.

§4. Outline of Proof of Theorem 2.1 (2) with $\boldsymbol{\alpha_1 = 0}$.

In the case $\alpha_1 = 0$, the generalized Parseval formula has the following form:

Proposition 2.12 (Generalized Parseval formula for $\boldsymbol{\alpha_1 = 0}$). *For any $f, g \in \mathcal{H} \cap L^1(\mathbb{R}^1)$, we have*

$$\lim_{\varepsilon\to 0}\langle \mathcal{F}_{i\alpha_2} f, \chi_\varepsilon \mathcal{F}_{-i\alpha_2} g\rangle = \langle f, g\rangle + \frac{i\alpha_2}{4}\int_{\mathbb{R}^1} e^{\frac{i\alpha_2}{2}|x|} f(x)dx \int_{\mathbb{R}^1} e^{\frac{i\alpha_2}{2}|y|}\overline{g(y)}dy,$$

where χ_a is the characteristic function on $\{k \in \mathbb{R}; a \le ||k| + \alpha_2/2|\}$ for $a > 0$.

Define the space $\mathcal{E}$ as follows;

$$\mathcal{E} = \left\{g \in \mathcal{H} \cap L^1(\mathbb{R}^1) : \int_{\mathbb{R}^1} |y||g(y)|dy < \infty, \quad \int_{\mathbb{R}^1} e^{-\frac{i\alpha_2}{2}|y|} g(y)dy = 0\right\}.$$

Then we find

Lemma 2.13. (1) *$\mathcal{E}$ is dense in $\mathcal{H}$.* (2) *Let $f \in \mathcal{H}$ and $g \in \mathcal{E}$. Then it holds that $\langle \mathcal{F}_{i\alpha_2} f, \mathcal{F}_{-i\alpha_2} g \rangle = \langle f, g \rangle$.*

Now we shall show Theorem 2.1 (2) with $\alpha_1 = 0$. It suffices to show $W(i\alpha_2)f = 0 \Rightarrow f = 0$. Since $\mathcal{F}_\alpha = \mathcal{F}_0 W(\alpha)$, we assume $\mathcal{F}_{i\alpha_2} f = 0$. Then Lemma 2.13 (2) implies $\langle f, g \rangle = 0$ for any $g \in \mathcal{E}$. Thus we obtain $f = 0$ by Lemma 2.13 (1). Q.E.D.

§5. Related Results.

Non-selfadjoint perturbations of Schrödinger equations were previously studied by several authors. After the works of Naimark [32], Schwartz [36] and Pavlov [33] etc., Kato [16] established the concept of so-called Kato-smoothness in the context of small perturbation. Within the scope of argument by Ikebe [4] and Faddeev [3], Mochizuki [24], [25], (see also [27]) dealt with Schrödinger operators with short range complex potentials and showed eigenfunction expansion theorem and the uniqueness of scattering inverse problem at high energy. But in his papers, detailed argument on spectrum is not given. He also investigated the large perturbation for the Friedrichs model [23]. Saito [34], [35] showed the principle of limiting absorption. Kako-Yajima [14] established spectral and scattering theory. Kitada [18] generalized in a sense Mochizuki's result [25]. As recent results, Adamyan and Neidhardt [1] studied absolute continuous spectrum. Stepin [37] argued about spectral singularity. As for the fixed energy inverse scattering problem, Isozaki-Nakazawa-Uhlmann [6] studied optical model in which the operator involves energy dependent complex potentials. But the spectral structure for this operator is unclear.

Chapter III. Wave equations.

§1. Wave equations with some Coulomb dissipations.

As second example, we consider the wave equation of the form;

$$
(3.1) \quad \begin{cases} w_{tt} - \Delta w + b(x) w_t = 0, & x \in \mathbb{R}^N, \quad t > 0, \\ w(x,0) = w_1(x), \quad w_t(x,0) = w_2(x), & x \in \mathbb{R}^N, \end{cases}
$$

where we assume that the function $b(x) \in C^1(\mathbb{R}^N \setminus \{0\})$ is non-negative. Our result is given by

Theorem 3.1 (Kadowaki-Nakazawa-Watanabe [12]).
(1) (*Explicit solution*) *If* $b(x) = b_0(x)$ *is the following function:*

$$b_0(x) = \begin{cases} (3-N)|x|^{-1} & (N = 1, 2), \\ (N-1)|x|^{-1} & (N \geq 3), \end{cases} \tag{3.2}$$

then the explicit radial solution of (3.1) *with*

$$w_1(x) \equiv \begin{cases} |x| f(|x|), & (N = 1) \\ f(|x|), & (N \geq 2) \end{cases}, \quad w_2(x) = \partial_{|x|} \{w_0(|x|)\},$$

where $f(|x|) = e^{\beta|x|} g(|x|)$, $\beta < 0$ *and* $g \in \mathcal{S}'$, *is given by*

$$w(t, x) = \begin{cases} |x| f(|x| + t), & (N = 1) \\ f(|x| + t). & (N \geq 2) \end{cases}$$

Therefore if $g \in H^1$ (*the Sobolev space of order* 1) *then the total energy decays exponentially as* t *tends to infinity.*

(2) (*Spectral structure*) *Assume* $N \geq 3$, (3.2) *and put*

$$H_b = i \begin{pmatrix} 0 & 1 \\ \Delta & -b \end{pmatrix} \text{ with domain } \mathcal{D}(H_b) = \{ v = (v_1, v_2) \in E \mid H_b v \in E \},$$

where $E = \dot{H}^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ *is energy space.*
Then we have

$$\sigma_p(H_b) = \mathbb{C}_-, \qquad \sigma_r(H_b) = \emptyset, \qquad \sigma_c(H_b) = \mathbb{R}, \qquad \rho(H_b) = \mathbb{C}_+.$$

Remark 3.2. The solution obtained in Theorem 3.1 (1) is a concrete example of *disappearing solution* studied by A. Majda [20].

Remark 3.3. (In the case of $N = 3$ the following decomposition has been already shown by Kadowaki [8]). What the equation (3.1) has the progressive wave solution as in Theorem 3.1 (1) in the case $N \geq 2$ and $b(x)$ satisfies (3.2) follows from the following decomposition of (3.1). In the following, $r = |x|$. If $N = 2$, then

$$\begin{cases} \left(\partial_t + \partial_r + r^{-1}\right)(\partial_t - \partial_r)\, w(t, r) = 0, \\ r^{-1}(\partial_t + \partial_r)\, r\, (\partial_t - \partial_r)\, w(t, r) = 0, \end{cases}$$

and if $N \geq 3$, then

$$\left(\partial_t + \partial_r + (N-1) r^{-1}\right)(\partial_t - \partial_r)\, w(t, r) = 0.$$

On the other hand in the case $N = 1$, we have

$$r^{-1}(\partial_t + \partial_r) r \left(\partial_t - \partial_r + r^{-1}\right) w(t,r) = 0.$$

The key to the proof of Theorem 3.1 (1) is the following observation. As the solution of the stationary problem for (3.1);

$$(-\Delta - i\kappa b(x) - \kappa^2)u(x) = 0 \qquad (\kappa \in \mathbb{C}), \tag{3.3}$$

we consider the following one:

$$\begin{cases} u(x) &= e^{p(r)}, \\ p(r) &= -i\kappa r - \frac{(N-1)}{2}\log r + \frac{1}{2}\int_1^r b(s)ds \end{cases}$$

(cf., Kato [15], Kawashita-Nakazawa-Soga [17]). If $b(x)$ is given by (3.2), this solution coincides with $u(x) = re^{-i\kappa r}$ $(N = 1)$, $= e^{-i\kappa r}$ $(N \geq 2)$, where $\kappa = \alpha + i\beta$ $(\alpha \in \mathbb{R},\ \beta < 0)$. As for the construction of solutions for (3.1), we have only to superpose these solutions with respect to real spectral parameter α.

On the other hand, the proof of Theorem 3.1 (2) is as follows. By the above observation with the proof of Theorem 3.1 (1), it holds that $\mathbb{C}_- \subset \sigma_p(H_b)$. If $\kappa \in \mathbb{C}_+$, then we obtain $(\mathrm{Im}\,\kappa)||v||_E \leq ||(H_b - \kappa)v||_E$ from (3.3). From this, it follows that $\sigma_p(H_b) \cap \mathbb{C}_+ = \emptyset$. Assume $\kappa \in \mathbb{R}$. Multiplying $\overline{u}$ on (3.3) and integrating by parts, we have $||\sqrt{b(\cdot)}u||^2 = 0$ and from this, it holds $\sigma_p(H_b) \cap \mathbb{R} = \emptyset$. These arguments show $\sigma_p(H_b) = \mathbb{C}_-$. Since $H_b^* = H_{-b}$, we have $\sigma_p(H_b^*) = \mathbb{C}_+$. Therefore we have $\sigma_r(H_b) = \emptyset$ if we note the relation $\kappa \in \sigma_r(H_b) \Leftrightarrow \overline{\kappa} \in \sigma_p(H_b^*)$, $\kappa \notin \sigma_p(H_b)$. Since the operator H_b is maximal dissipative and Range$(H_b - i) = E$, therefore the operator H_b generates the contraction semi-group in energy space E under $N \geq 3$, we have the relation $\mathbb{C}_+ \subset \rho(H_b) \subset \mathbb{R} \cup \mathbb{C}_+$ on resolvent set. Since the resolvent set is open in $\mathbb{C}$, we obtain $\rho(H_b) = \mathbb{C}_+$. Thus $\sigma_c(H_b) = \mathbb{R}$ holds. Q.E.D.

As for equation (3.1), we do not know whether we can obtain affirmative result or not for our problem.

Finally, we state a result on exponential decaying solution without dissipation in equation (3.1);

$$\begin{cases} w_{tt}(x,t) - \Delta w(x,t) = 0, & x \in \mathbb{R}^3, \quad t > 0, \\ i\sqrt{\sigma}w(0,t) - w_r(0,t) = 0, & t > 0, \\ w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x), & x \in \mathbb{R}^3, \end{cases} \tag{3.4}$$

where $\sigma \in \mathbb{C}$.

Theorem 3.4 (Kadowaki-Nakazawa-Watanabe [12]). *Assume that*

$$w_0(x) = f(r) \equiv e^{i\sqrt{\sigma}r}, \quad w_1(x) = 0,$$

where $\sigma \in \mathbb{C}$, $Im\,\sigma < 0$, $Im\sqrt{\sigma} > 0$. *Then the solution of* (3.4) *is given by* $w(x,t) = f(r+t)$, *and the total energy decays exponentially as* t *goes to infinity.*

Remark 3.5. This example can be regarded as the following equation, formally :

$$w_{tt}(x,t) + (-\Delta + \sigma\delta(x))w(x,t) = f(x,t),$$

where $\delta(x)$ is the delta-function on $\mathbb{R}^3$. (cf. [11]).

§2. Wave equations with some rank one perturbations.

As final example, we state related problem to which we can obtain positive result under more restrictive conditions.

Consider the following equation:

$$\text{(3.5)} \qquad w_{tt} - \partial_x^2 w + \langle w_t, \varphi\rangle_0 \varphi(x) = 0, \quad x \in \mathbb{R}^1, \quad t > 0,$$

where $\langle \cdot, \cdot \rangle_0$ denotes L^2-inner product and φ is assumed to be

$$\text{(3.6)} \quad (i) \quad \varphi \in L^2_{s+1}\ (s > 1/2), \qquad (ii) \quad \Phi(\lambda) \leq \Phi(\mu)\ (0 \leq \mu \leq \lambda),$$

where L^2_s is well-known weighted L^2-spaces,

$$\Phi(\lambda) = |\hat{\varphi}(\lambda)|^2 + |\hat{\varphi}(-\lambda)|^2\ (\lambda \geq 0),$$

$\hat{\varphi}$ denotes the Fourier transform of φ and C_1, C_2 are some positive constants.

We deal with (3.5) as a perturbed system of

$$\text{(3.7)} \qquad \partial_t^2 u(x,t) - \partial_x^2 u(x,t) = 0, \quad (x,t) \in \mathbb{R} \times \mathbb{R}.$$

To do so, we introduce the Hilbert space $\mathcal{H}$ with the inner product:

$$\langle f, g\rangle = \int_{\mathbb{R}} (\partial_x f_1(x)\overline{\partial_x g_1(x)} + f_2(x)\overline{g_2(x)})dx,$$

for $f = {}^t(f_1, f_2)$ and $g = {}^t(g_1, g_2)$ (the norm in $\mathcal{H}$ is denoted by $\|\cdot\|$).

Putting $f(t) = {}^t(u(x,t), \partial_t u(x,t))$, we then rewrite (3.5) and (3.7) as

$$\partial_t f(t) = -iAf(t) \qquad \text{and} \qquad \partial_t f(t) = -iA_0 f(t),$$

where

$$A = i\begin{pmatrix} 0 & 1 \\ \partial_x^2 & -\langle \cdot, \varphi\rangle_0 \varphi \end{pmatrix} \qquad \text{and} \qquad A_0 = i\begin{pmatrix} 0 & 1 \\ \partial_x^2 & 0 \end{pmatrix}$$

with domain

$$D(A) = D(A_0) = \{f = {}^t(f_1, f_2) \in \mathcal{H}; \partial_x^2 f_1 \in L^2(\mathbb{R}), f_2 \in H^1(\mathbb{R})\},$$

respectively.

Then as is easily seen, A and A_0 generate contraction semi-group $\{e^{-itA}\}_{t\geq 0}$ and unitary group $\{e^{-itA_0}\}_{t\in\mathbb{R}}$, respectively. We denote by $R(z)$ (resp. $R_0(z)$) the resolvent of A, $(A-z)^{-1}$ (resp. $(A_0 - z)^{-1}$) for $z \in \rho(A)$ (resp. $z \in \rho(A_0)$).

Applying the proof of Mochizuki [26] or Kadowaki [9], under the assumption $\varphi \in L_s^2$ $(s > 1/2)$, we find that A has no real eigenvalues and the wave operator

$$W = \text{s-}\lim_{t\to\infty} e^{itA_0} e^{-itA}$$

exists as a non-trivial operator from $\mathcal{H}$ to $\mathcal{H}$.

Secondly, we define an operator P as follows; for $f, g \in \mathcal{H}$

$$\langle Pf, g\rangle = \frac{-1}{2\pi i}\int_\Gamma \langle R(z)f, g\rangle dz,$$

where Γ $(\subset \mathbb{C}_-)$ is a closed curve enclosed Σ_- defined by

$$\Sigma_- = \{z \in \mathbb{C}_- \,;\, \Gamma(z) = 0\}$$

with $\Gamma(z) = 1 - iz\langle(-\partial_x^2 - z^2)^{-1}\varphi, \varphi\rangle_0$. Then, $P^2 = P$ holds, thus, we may call P generalized eigen-projector.

Theorem 3.6 (Kadowaki-Nakazawa-Watanabe [13]). *Assume that* (3.6). *Then we find the following assertions:*

(1) (*Kernel of W and decay*) *It follows that $KerW = RangeP$. This is equivalent to*

$$f \in RangeP \iff \lim_{t\to\infty} \|e^{-itA}f\| = 0.$$

(2) (*Non decay and scattering*) *For $f \in \mathcal{H}$, it follows that*

$$f - Pf \neq 0 \iff Wf \neq 0$$

and

$$\lim_{t\to\infty} \|e^{-itA}f - e^{-itA_0}Wf\| = 0.$$

This is equivalent to

$$f - Pf \neq 0 \quad \Longleftrightarrow \quad \lim_{t\to\infty} \|e^{-itA} f\| \neq 0.$$

§3. Outline of Proof of Theorem 3.6.

We shall start from the result corresponding to Lemma 2.3. To do so, we define

$$\Gamma(z) = 1 - iz\langle r_0(z)\varphi, \varphi\rangle_0, \quad \Gamma(\lambda \pm i0) = 1 - i\lambda\langle r_0(\lambda \pm i0)\varphi, \varphi\rangle_0,$$
$$\Sigma_\pm = \{z \in \mathbb{C}_\pm; \Gamma(z) = 0\}, \quad \Sigma_\pm^0 = \{\lambda \in \mathbb{R}; \Gamma(\lambda \pm i0) = 0\},$$

where $r_0(z) = (-\partial_x^2 - z^2)^{-1}$.

Lemma 3.7 (Property of $\Gamma(z)$). *The following inequalities hold:*

$$(1) \quad \inf_{\mathrm{Im} z \geq 0} Re\Gamma(z) \geq 1, \qquad (2) \quad \liminf_{|z|\to\infty, \mathrm{Im} z\leq 0} |Re\Gamma(z)| \geq C$$

for some positive constant C.

From the principle of limiting absorption for $R_0(z)$ (which follows from that of $r_0(z)$), we have

Lemma 3.8 (Representation of $R(z)$). *$z \notin \Sigma_+$ (resp. Σ_-) if and only if $z \in \rho(A) \cap \mathbb{C}_+$ (resp. $z \in \rho(A) \cap \mathbb{C}_-$) and for any $f =^t (f_1, f_2) \in \mathcal{H}$, it holds*

$$R(z)f = R_0(z)f + \frac{i\langle f, v(\overline{z})\rangle}{\Gamma(z)} v(z).$$

In order to analyze the singularities (eigenvalues and spectral singularities) of A, we have to characterize $\Sigma_+, \Sigma_-, \Sigma_+^0$ and Σ_-^0.

Lemma 3.9 (Structure of Σ_+ and Σ_+^0). $\Sigma_+ = \Sigma_+^0 = \emptyset$.

We shall denote the set of bounded operator from X to Y by $\mathcal{B}(X, Y)$. Then we have from the above lemma,

Proposition 3.10 (The principle of limiting absorption and representation of $R(\lambda + i0)$). *Let $s > 1/2$. Then for every $\lambda \in \mathbb{R}$, the limit $R(\lambda + i0) = \lim_{\kappa\downarrow 0} R(\lambda + i\kappa)$ exists in the uniformly operator topology of $\mathcal{B}(\mathcal{H}_s, \mathcal{H}_{-s})$ and has the form*

$$R(\lambda + i0)f = R_0(\lambda + i0)f + \frac{i\langle f, v(\lambda - i0)\rangle}{\Gamma(\lambda + i0)} v(\lambda + i0),$$

for any $f = {}^t(f_1, f_2) \in \mathcal{H}_s$, *where* $\mathcal{H}_s$ *is defined by*

$$\mathcal{H}_s = \left\{ f = {}^t(f_1, f_2); \int_{\mathbb{R}} (1 + |x|^2)^s (|\partial_x f_1(x)|^2 + |f_2(x)|^2) dx < \infty \right\}.$$

Next, we shall characterize Σ_- and Σ_-^0 as a set of spectral singularities and eigenvalues of A, respectively.

Lemma 3.11 (Structure of Σ_- and Σ_-^0).
(1) *Assume* (3.6) *(ii). Then* Σ_- *consists of pure imaginary numbers at most.*
(2) *Assume* (3.6). *Then one has*

$$\Sigma_-^0 = \begin{cases} \{0\} & (\text{if } \ 1 - \frac{1}{2}\left|\int_{\mathbb{R}} \varphi(x)dx\right|^2 = 0) \\ \emptyset & (\text{if } \ 1 - \frac{1}{2}\left|\int_{\mathbb{R}} \varphi(x)dx\right|^2 \neq 0) \end{cases}$$

Especially, in the case of $1 - \frac{1}{2}\left|\int_{\mathbb{R}} \varphi(x)dx\right|^2 = 0$, *the multiplicity of* $0\,(\in \Sigma_-^0)$ *is one.*

The above assertion (1) follows from the integral representation for $\Gamma(z)$. On the other hand, (2) follows from (1).
Corresponding to Proposition 3.10, we have

Proposition 3.12 (Structure of Σ_-, the principle of limiting absorption and representation of $R(\lambda - i0)$). *Assume* (3.6). *Then*
(1) Σ_- *consists of finite isolated points on the imaginary axis at most. Moreover the multiplicity of each point is finite.*
(2) *For every*

$$\lambda \in \begin{cases} \mathbb{R} \setminus \{0\} & (\text{if } \ 1 - \frac{1}{2}\left|\int_{\mathbb{R}} \varphi(x)dx\right|^2 = 0) \\ \mathbb{R} & (\text{if } \ 1 - \frac{1}{2}\left|\int_{\mathbb{R}} \varphi(x)dx\right|^2 \neq 0) \end{cases}$$

and $s > 1/2$, *the limit* $R(\lambda - i0) = \lim_{\kappa\downarrow 0} R(\lambda - i\kappa)$ *exists in the uniformly operator topology of* $\mathcal{B}(\mathcal{H}_s, \mathcal{H}_{-s})$ *and for any* $f = {}^t(f_1, f_2) \in \mathcal{H}_s$, *it holds*

$$R(\lambda - i0)f = R_0(\lambda - i0)f + \frac{i\langle f, v(\lambda + i0)\rangle}{\Gamma(\lambda - i0)} v(\lambda - i0).$$

This follows from Lemma 3.11 and Lemma 3.7 (2). From Lemma 3.11 and Proposition 3.12, we can put

$$\Sigma_- = \{i\kappa_j\}_{j=1,2,\cdots m}$$

for some $m \in \mathbb{N}$ and $\kappa_j < 0$. Therefore putting

$$v(z) = \begin{pmatrix} ir_0(z)\varphi \\ zr_0(z)\varphi \end{pmatrix},$$

we have from Lemma 3.8

$$\langle Pf, g\rangle = \frac{-1}{2\pi i}\sum_{j=1}^{m}\int_{\gamma_j}\frac{i\langle f, v(\overline{z})\rangle}{\Gamma(z)}\langle v(z), g\rangle dz,$$

where $\gamma_j(\subset \mathbb{C}_-)$ is a closed curve enclosed z_j and $\gamma_j \cap \gamma_i = \emptyset$ $(i \neq j)$ (as for P, see the preceding part of Theorem 3.6). Thus it follows from Jordan form that

$$\|e^{-itA}Pf\| \leq Ce^{-\delta t}\|f\|,$$

for some $C, \delta > 0$. This concludes

Proposition 3.13. *Assume* (3.6). *Then $RangeP \subset KerW$ holds.*

Next using the above results we construct the the spectral representation of A. First of all, we state on that of A_0.

For $f = {}^t(f_1, f_2) \in \mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R})$, where $\mathcal{S}(\mathbb{R})$ is Schwartz space, we define an operator $\mathfrak{F}_0$:

$$(\mathfrak{F}_0 f)(\lambda) = {}^t\left(\frac{\pm\lambda\hat{f}_1(\pm\lambda) \pm i\hat{f}_2(\pm\lambda)}{\sqrt{2}}, \frac{\pm\lambda\hat{f}_1(\mp\lambda) \pm i\hat{f}_2(\mp\lambda)}{\sqrt{2}}\right), \quad (\pm\lambda > 0).$$

Then we have the following well-known lemma;

Lemma 3.14 (Spectral representation of A_0). *$\mathfrak{F}_0$ is extended to a unitary operator from $\mathcal{H}$ onto $L^2(\mathbb{R};\mathbb{C}^2)$. Moreover it holds that for any $f \in D(A_0)$ and $g \in \mathcal{H}$,*

$$\langle A_0 f, g\rangle = \int_{-\infty}^{\infty}\lambda\langle(\mathfrak{F}_0 f)(\lambda), (\mathfrak{F}_0 g)(\lambda)\rangle_{\mathbb{C}^2}d\lambda,$$

where $\langle\cdot,\cdot\rangle_{\mathbb{C}^2}$ is usual inner-product of $\mathbb{C}^2$.

On the basis of the above lemma, we define the spectral representation of A as follows.

Firstly, by the standard argument in the stationary scattering theory (e.g., Kuroda [19]) and Lemma 3.8, we obtain

$$\langle Wf, g\rangle = \int_{-\infty}^{\infty}\langle(\mathfrak{F}f)(\lambda), (\mathfrak{F}_0 g)(\lambda)\rangle_{\mathbb{C}^2}d\lambda,$$

where $\mathfrak{F}$ is defined by

$$(\mathfrak{F}f)(\lambda) = (\mathfrak{F}_0 f)(\lambda) + \frac{i\langle f, v(\lambda - i0)\rangle}{\Gamma(\lambda + i0)}(\mathfrak{F}_0 \begin{pmatrix} 0 \\ \varphi \end{pmatrix})(\lambda, \cdot).$$

Then the following holds:

Proposition 3.15. *$\mathfrak{F}$ is extended to a bounded operator from $\mathcal{H}$ to $L^2(\mathbb{R}; \mathbb{C}^2)$ and satisfies $\mathfrak{F} = \mathfrak{F}_0 W$.*

Noting the intertwining property of W we have for $f \in D(A)$ and $g \in \mathcal{H}$,

$$\int_{-\infty}^{\infty} \langle (\mathfrak{F}Af)(\lambda), (\mathfrak{F}_0 g)(\lambda)\rangle_{\mathbb{C}^2} d\lambda = \int_{-\infty}^{\infty} \lambda \langle (\mathfrak{F}f)(\lambda), (\mathfrak{F}_0 g)(\lambda)\rangle_{\mathbb{C}^2} d\lambda.$$

So we call an operator $\mathfrak{F}$ the spectral representation of A.

In the self-adjoint case, only $\mathfrak{F}$ is enough to analyze. But in our case, we need to construct the spectral representation of adjoint operator of A. To do so, let $\mathfrak{G}$ be an operator defined by

$$(\mathfrak{G}g)(\lambda) = (\mathfrak{F}_0 g)(\lambda) - \frac{i\langle g, v(\lambda - i0)\rangle}{1 + i\lambda\langle r_0(\lambda + i0)\varphi, \varphi\rangle_0}(\mathfrak{F}_0 \begin{pmatrix} 0 \\ \varphi \end{pmatrix})(\lambda)$$

for $g = {}^t(g_1, g_2) \in \mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R})$ and $\lambda \notin \Sigma_0^-$. The key equality for $\mathfrak{F}$ and $\mathfrak{G}$ is given by

Lemma 3.16. *For $f = {}^t(f_1, f_2)$, $g = {}^t(g_1, g_2) \in \mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R})$ and $\lambda \notin \Sigma_-^0$, one has*

$$\begin{aligned}\langle (\mathfrak{F}f)(\lambda), (\mathfrak{G}g)(\lambda)\rangle_{\mathbb{C}^2} &= \langle (\mathfrak{F}_0 f)(\lambda), (\mathfrak{F}_0 g)(\lambda)\rangle_{\mathbb{C}^2} \\ &+ \frac{1}{2\pi}\frac{\langle f, v(\lambda - i0)\rangle\langle v(\lambda + i0), g\rangle}{\Gamma(\lambda + i0)} - \frac{1}{2\pi}\frac{\langle f, v(\lambda + i0)\rangle\langle v(\lambda - i0), g\rangle}{\Gamma(\lambda - i0)}.\end{aligned}$$

Remark 3.17. This also means for $f, g \in \mathcal{S}(\mathbb{R})$ and $\lambda \notin \Sigma_-^0$,

$$\langle (\mathfrak{F}f)(\lambda), (\mathfrak{G}g)(\lambda)\rangle_{\mathbb{C}^2} = \frac{1}{2\pi i}\langle (R(\lambda + i0) - R(\lambda - i0))f, g\rangle.$$

For the case of $\Sigma_-^0 \neq \emptyset \Longleftrightarrow 1 - |\int_{\mathbb{R}} \varphi(x)dx|^2 = 0$, we put

$$\mathcal{E} = \left\{ g \in \mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R}); \lim_{\lambda \to 0} \langle v(\lambda - i0), g\rangle = 0 \right\}.$$

Remark 3.18. As is easily seen, we have for $g = {}^t(g_1, g_2) \in \mathcal{E}$,

$$\lim_{\lambda \to 0} \langle v(\lambda - i0), g \rangle = 0 \Longleftrightarrow \langle \varphi, g_1 \rangle_0 - \frac{1}{2} \langle \varphi, 1 \rangle_0 \langle 1, g_2 \rangle_0 = 0.$$

Moreover we find

Proposition 3.19. *Assume* (3.6). *Then for any* $f \in \mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R})$ *and* $g \in \mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R})$ *(if* $1 - \frac{1}{2} \left| \int_{\mathbb{R}} \varphi(x) dx \right|^2 \neq 0$*),* $\in \mathcal{E}$ *(if* $1 - \frac{1}{2} \left| \int_{\mathbb{R}} \varphi(x) dx \right|^2 = 0$*), it is true that*

$$\int_{-\infty}^{\infty} \frac{1}{2\pi} \frac{\langle f, v(\lambda + i0) \rangle}{\Gamma(\lambda - i0)} \langle v(\lambda - i0), g \rangle d\lambda = \langle Pf, g \rangle,$$
$$\int_{-\infty}^{\infty} \frac{1}{2\pi} \frac{\langle f, v(\lambda - i0) \rangle}{\Gamma(\lambda + i0)} \langle v(\lambda + i0), g \rangle d\lambda = 0.$$

From Lemma 3.16 and Proposition 3.19, we obtain

Proposition 3.20 (Parseval formula). *Assume* (3.6). *Then for any* $f \in \mathcal{H}$ *and* $g \in \mathcal{H}$ *(if* $1 - \frac{1}{2} \left| \int_{\mathbb{R}} \varphi(x) dx \right|^2 \neq 0$*),* $\in \mathcal{E}$ *(if* $1 - \frac{1}{2} \left| \int_{\mathbb{R}} \varphi(x) dx \right|^2 = 0$*), it holds that*

$$\langle f, g \rangle = \int_{-\infty}^{\infty} \langle (\mathfrak{F} f)(\lambda), (\mathfrak{G} g)(\lambda) \rangle_{\mathbb{C}^2} d\lambda + \langle Pf, g \rangle.$$

As a corollary of Proposition 3.20, we also have the spectral decomposition for A:

Corollary 3.21 (Spectral decomposition for $\boldsymbol{A}$). *Assume* (3.6). *Then for any* $f \in D(A)$ *and* $g \in \mathcal{H}$ *(if* $1 - \frac{1}{2} \left| \int_{\mathbb{R}} \varphi(x) dx \right|^2 \neq 0$*),* $\in \mathcal{E}$ *(if* $1 - \frac{1}{2} \left| \int_{\mathbb{R}} \varphi(x) dx \right|^2 = 0$*), it holds that*

$$\langle Af, g \rangle = \int_{-\infty}^{\infty} \lambda \langle (\mathfrak{F} f)(\lambda), (\mathfrak{G} g)(\lambda) \rangle_{\mathbb{C}^2} d\lambda + \sum_{j=1}^{m} i\kappa_j \langle P_j f, g \rangle,$$

where P_j *is the eigen-projector of eigenvalue* $i\kappa_j$.

We have to check the denseness of $\mathcal{E}$ in $\mathcal{H}$.

Lemma 3.22. *Assume* (3.6) *and*

$$1-\frac{1}{2}\left|\int_{\mathbb{R}}\varphi(x)dx\right|^2=0.$$

Then $\mathcal{E}$ is dense in $\mathcal{H}$.

Now we shall prove Theorem 3.6. By Proposition 3.13, we have only to show $Wf=0 \Rightarrow f=Pf$. Consider the case $1-\frac{1}{2}\left|\int_{\mathbb{R}}\varphi(x)dx\right|^2=0$ only. The other case is proved in a similar way. Suppose $Wf=0$. Since $\mathfrak{F}_0$ is unitary, we may assume $\mathfrak{F}f=0$. Then it follows that $\langle f,g\rangle = \langle Pf,g\rangle$ for any $g\in\mathcal{E}$ from Proposition 3.20. Then Lemma 3.22 implies $f=Pf$. So we conclude (1). (2) is a corollary of (1). Q.E.D.

§4. Related Results

In response to an investigation by Eĭdus [2], Mizohata-Mochizuki [22] and Iwasaki [7] proved principle of limiting amplitude for dissipative wave equations under the short range dissipation. As already noted, Majda [20] studied disappearing solutions. Existence of scattering states was shown by Mochizuki [26] under the assumption $N\neq 2$. Energy decay was shown by Matsumura [21] for compactly supported data, and by Mochizuki [27] without such a condition. After that these results were improved by Mochizuki-Nakazawa [29] (energy decay and existence of scattering states), by Mochizuki [28] (spectral representations, scattering for $N\neq 2$ and integrable dissipation), and by Nakazawa [31] (scattering for $N\geq 2$ and the same dissipation). Principle of limiting absorption was shown by Nakazawa [30] under $N\neq 2$ and small and short range dissipation. The spectral structure is also disclosed in this paper. Fixed energy inverse scattering problem was solved by Mochizuki [28] for $N\geq 3$ under small and exponential decaying dissipation and by Watanabe [38] for $N=2$ under the compactly supported dissipation. But the spectral structure has not been uncovered in these two works. Finally we should quote the result by Ikebe [5] in which he studied the spectral structure.

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