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Survey of admissible shock waves for 2×2 systems of conservation laws with an umbilic point

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Abstract.

In this survey note, we consider 2×2 systems of conservation laws with an umbilic point. At the umbilic point, the characteristic speeds coincide and the Jacobian matrix of the flux functions is diagonalizable. It is shown that if there are no undercompressive shocks with viscous shock profile then any compressive shock has a viscous shock profile and any overcompressive shock has a infinitely many viscous shock profiles.

§1. Introduction.

We consider one-dimensional flow of three immiscible fluid phases in a porous medium [25]. In the oil reservoir flow, the fluid is composed of water, oil and gas. The conservation of mass of water and oil and Darcy's Law are expressed by the systems of two equations

$$\frac{\partial}{\partial t}U + \frac{\partial}{\partial x}F(U) = \varepsilon \frac{\partial}{\partial x}\left[B(U)\frac{\partial}{\partial x}U\right]. \tag{1}$$

where $U = {}^t(u, v)$ and B denotes the effects of capillary pressure [5, 22, 30]. The systems (1) are a generalization of the classical Buckley-Leverett equation for two-phase flow [7]. The difficulty with this approach is that the systems (1) have an elliptic region Ω_E where the eigenvalues of the Jacobian matrix $F'(U)$ are not real:

$$\Omega_E = \left\{U : g_E(U) = \det\left[F'(U) - \frac{1}{2}\mathrm{tr}F'(U) \cdot I\right] > 0\right\}. \tag{2}$$

In the elliptic region Ω_E, the initial value problem of the systems with $B(U) \equiv 0$ is ill-posed. Majda and Pego find a sufficient condition for

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the linearized instability of the system (1) in terms of $B(U)$ and $F'(U)$ [21]. Let us define

$$(3)\quad \Omega = \left\{ U : g_{BT}(U) = \det\left[F'(U) - \frac{\mathrm{tr}\left[B(U)^{-1}F'(U)\right]}{\mathrm{tr}\left[B(U)^{-1}\right]} \cdot I \right] \geq 0 \right\}$$

to be the Majda-Pego instability region [5]. The Majda-Pego instability region Ω of the systems (1) contains strictly the elliptic region Ω_E. In fact

$$(4)\qquad g_{BT}(U) - g_E(U) = \left(\frac{\mathrm{tr}\left[B(U)^{-1}F'(U)\right]}{\mathrm{tr}\left[B(U)^{-1}\right]} - \frac{1}{2}\mathrm{tr}F'(U) \right)^2 .$$

In the Majda-Pego instability region Ω, the nonuniqueness [4, 6, 14] and the nonexistence [8, 9, 10] of solutions for Riemann problems can occur even in strictly hyperbolic region. In addition, we remark that

$$\begin{aligned} & \{U : g_E(U) = 0\} \\ = \ & \{U : \exists\sigma, \det\left[F'(U) - \sigma I\right] = 0 \text{ and } \mathrm{tr}\left[F'(U) - \sigma I\right]\} \end{aligned}$$

is the *coincidence locus* where the characteristic speeds coincide and that

$$\begin{aligned} & \{U : g_{BT}(U) = 0\} \\ = \ & \left\{U : \exists\sigma, \det\left[F'(U) - \sigma I\right] = 0 \text{ and } \mathrm{tr}\left\{B(U)^{-1}\left[F'(U) - \sigma I\right]\right\} = 0\right\} \end{aligned}$$

is the *Bogdanov-Takens locus* where the Bogdanov-Takens bifurcation occurs. Thus if the closure of the elliptic region $\overline{\Omega_E}$ shrink to be a point U^*, then the systems (1) are reduced to the systems having an isolated umbilic point. Thus the systems are strictly hyperbolic except at one point U^* and the point U^* is an umbilic point, namely the eigenvalues $\lambda_1(U)$, $\lambda_2(U)$ of the Jacobian matrix $F'(U)$ are real distinct at any point except the point U^* and $F'(U^*)$ has the multiple real eigenvalues. We have the Taylor expansion of $F(U)$ near $U = U^*$:

$$F(U) = F(U^*) + F'(U^*)(U - U^*) + Q(U - U^*) + O(1)|U - U^*|^3$$

where $Q : \boldsymbol{R}^2 \to \boldsymbol{R}^2$ is a homogeneous quadratic mapping. The following hypothesis describes the class of the systems to be studied in this note:

Hypothesis 1. *The closure of the elliptic region $\overline{\Omega_E}$ for the systems (1) shrink to an isolated umbilic point U^* satisfying*

(1) *$F'(U^*)$ is diagonalizable thus $F'(U^*) = \lambda^* I$ where $\lambda^* := \lambda_1(U^*) = \lambda_2(U^*)$,*

(2) $F(U) = F(U^*) + F'(U^*)(U - U^*) + Q(U - U^*)$,
(3) $B(U) \equiv I$.

After the Galilean change of variables: $x \to x - \lambda^* t$, $F(U) \to F(U) - F(U^*)$ and $U \to U + U^*$, we observe that the system of equations (1) is reduced to

$$U_t + Q(U)_x = \varepsilon U_{xx}, \quad (x,t) \in \boldsymbol{R} \times \boldsymbol{R}_+. \tag{5}$$

Now by a change of unknown functions $V = S^{-1}U$ with a regular constant matrix S, we have a new system of equations $V_t + P(V)_x = \varepsilon V_{xx}$ where $P(V) = S^{-1}Q(SV)$. Thus we come to

Definition 1. *Two quadratic mappings $Q_1(U)$ and $Q_2(U)$ are said to be* equivalent, *if there is a constant matrix $S \in GL_2(\boldsymbol{R})$ such that*

$$Q_2(U) = S^{-1}Q_1(SU) \quad \text{for all} \quad U \in \boldsymbol{R}^2. \tag{6}$$

A general quadratic mapping $Q(U)$ has six coefficients and $GL_2(\boldsymbol{R})$ is a four dimensional group. Thus by the above equivalence transformations, we can eliminate four parameters. These procedures are successfully carried out by Schaeffer and Shearer [26] and they obtained the following *normal forms*.

Let $Q(U)$ be a hyperbolic quadratic mapping with an isolated umbilic point $U = 0$, then there exist two real parameters a and b with $a \neq 1 + b^2$ such that $Q(U)$ is equivalent to $\frac{1}{2}\nabla C$ where $\nabla = {}^t(\partial_u, \partial_v)$ and

$$C(U) = \tfrac{1}{3}au^3 + bu^2v + uv^2. \tag{7}$$

Therefore, in this note, we confine ourselves to the systems

$$\begin{pmatrix} u \\ v \end{pmatrix}_t + \frac{1}{2}\begin{pmatrix} au^2 + 2buv + v^2 \\ bu^2 + 2uv \end{pmatrix}_x = \varepsilon \begin{pmatrix} u \\ v \end{pmatrix}_{xx} \quad (a \neq 1 + b^2). \tag{8}$$

We remark that if $a = 1+b^2$ then the coincidence locus consists of the line $bu + v = 0$. The classification of the systems (8) in [26] is the following: Case I is $a < \frac{3}{4}b^2$; Case II is $\frac{3}{4}b^2 < a < 1+b^2$; for $a > 1+b^2$, the boundary between Case III and Case IV is $4\{4b^2-3(a-2)\}^3 - \{16b^3+9(1-2a)b\}^2 = 0$. The drastic change across $a = 1 + b^2$ was recognized by Darboux [11] even in the 19th century. Appendix of [26] states, in collaboration with Marchesin and Paes-Leme, that the quadratic approximation of the flux functions for oil reservoir flow is either Case I or Case II, to which we shall confine ourselves in the following argument. The corresponding systems of conservation laws are

$$(9)\qquad \begin{pmatrix} u \\ v \end{pmatrix}_t + \frac{1}{2}\begin{pmatrix} au^2 + 2buv + v^2 \\ bu^2 + 2uv \end{pmatrix}_x = 0 \ (a \neq 1 + b^2).$$

The Riemann problem for (9) is the Cauchy problem with initial data of the form

$$(10)\qquad U(x,0) = \begin{cases} U_L & \text{for} \quad x < 0, \\ U_R & \text{for} \quad x > 0 \end{cases}$$

where U_L, U_R are constant states. The structure of the rarefaction waves is investigated in [11]. A jump discontinuity defined by

$$(11)\qquad U(x,t) = \begin{cases} U_L & \text{for} \quad x < st, \\ U_R & \text{for} \quad x > st \end{cases}$$

is a *shock wave*, piecewise constant weak solution to the Riemann problem, provided these quantities satisfy the *Rankine-Hugoniot condition*:

$$(12)\qquad s(U_R - U_L) = F(U_R) - F(U_L).$$

We say that the above discontinuity is a *j-compressive* shock wave ($j = 1, 2$) if it satisfies the *Lax entropy conditions* [19, 20]:

$$(13)\qquad \lambda_j(U_R) < s < \lambda_j(U_L), \quad \lambda_{j-1}(U_L) < s < \lambda_{j+1}(U_R).$$

Here we adopt the convention $\lambda_0 = -\infty$ and $\lambda_3 = \infty$. We shall also face with the *overcompressive* shock wave: a jump discontinuity satisfying

$$(14)\qquad \lambda_1(U_R) < s < \lambda_1(U_L), \quad \lambda_2(U_R) < s < \lambda_2(U_L)$$

and also the *undercompressive* shock wave satisfying

$$(15)\qquad \lambda_1(U_R) < s < \lambda_2(U_R), \quad \lambda_1(U_L) < s < \lambda_2(U_L).$$

A state U_R can be joined to a state U_L by a shock wave (11) if and only if $U_R \in H(U_L)$ where

$$(16)\qquad H(U_L) = \{U : \exists s, F(U_R) - F(U_L) - s(U_R - U_L) = 0\}$$

called *Hugoniot locus through* U_L. In [19], it is shown that the Hugoniot locus through U_L in a neighborhood of U_L consists of a union of two curves through U_L if U_L is not an umbilic point. Introducing a parameter ξ by

$$(17)\qquad v - v_L = \xi(u - u_L) \text{ where } U_L = {}^t(u_L, v_L),$$

we have a rational parametrization of the Hugoniot locus through U_L [2, 27]:

$$(18) \qquad u = \frac{(\xi^3 - a\xi + b)u_0 + 2(1 - b\xi - \xi^2)v_0}{\xi^3 + 2b\xi^2 + (a-2)\xi - b},$$

$$(19) \qquad v = \frac{2\{b\xi - (a-1)\xi^2 - b\xi^3\}u_0 + (-b + a\xi - \xi^3)v_0}{\xi^3 + 2b\xi^2 + (a-2)\xi - b},$$

$$(20) \qquad s = \frac{(\xi^2 + b\xi + b^2 - a)(u_0\xi - v_0)}{\xi^3 + 2b\xi^2 + (a-2)\xi - b}.$$

The denominator of the above parametrization $\Phi(\xi) = \xi^3 + 2b\xi^2 + (a-2)\xi - b$ has 3 distinct real zeros μ_1, μ_2, μ_3 in Case I & II [2]. The expression (18), (19) shows that the Hugoniot locus through U_L consists of 2 curves through U_L and a detached curve. In [2], we investigate shock waves for the systems of conservation laws (9) in Case I & II and it is shown that:

Suppose that U_L is not located on the medians $\{U = {}^t(u, v) : v = \mu_j u \ (j = 1, 2, 3)\}$. The Hugoniot locus consists of three components: 1-Hugoniot curve, 2-Hugoniot curve and detached curve. In Case I, the 1-Hugoniot curve ultimately consists of the 1-compressive as $s \to -\infty$ in each side; the compressive part of the 2-Hugoniot curve is bounded; the detached curve is ultimately 1-compressive as $s \to -\infty$ in one side. Overcompressive shock waves appear in Case II. Here, the 1-Hugoniot curve has the 1-compressive part ultimately as $s \to -\infty$ in one side and the overcompressive part of this curve is bounded; the compressive part of the 2-Hugoniot curve is bounded and the overcompressive part of this curve ultimately appears as $s \to -\infty$ in one side. The Hugoniot locus for $U_L = 0$ (the umbilic point) is also studied: each half of medians constitute a wave curve. Overcompressive waves do not appear in Case I.

Although we have an extensive bibliography: ([13, 15, 16, 17, 18, 26, 27, 29, 31] etc.), the analysis of shock waves has been carried out mainly through numerical computations so far; thus rigorous mathematical study will be appreciated. Our aim of this note is to survey an idea of proof of the existence of a viscous shock profile for any compressive shock in Case I & II and infinitely many viscous shock profiles for any overcompressive shock in Case II. A more detailed proof will be given in [3]. We say that a shock wave (11) to the systems (9) has a viscous shock profile [12] if there exists a travelling wave solution $U = \widehat{U}\left(\frac{x - st}{\varepsilon}\right)$, called *viscous profile joining U_L and U_R*, to the systems (8) with the boundary conditions at the infinity

$$\widehat{U}(-\infty) = U_L \text{ and } \widehat{U}(+\infty) = U_R. \tag{21}$$

One can easily see that the shock wave (11) has a viscous profile(s) if and only if U_L and U_R are critical points joined by an orbit(s) of the vector field $X_s(U; U_L)$ where

$$X_s(U; U_L) = -s(U - U_L) + F(U) - F(U_L). \tag{22}$$

The characters of shock waves can be stated in terms of the critical points of the vector fields $X_s(U; U_L)$:

Proposition 1. *The shock wave (11) is*

- *1-compressive shock if and only if U_L is repeller and U_R is saddle.*
- *2-compressive shock if and only if U_L is saddle and U_R is attractor.*
- *overcompressive shock if and only if U_L is repeller and U_R is attractor.*
- *crossing shock if and only if U_L and U_R are saddles.*

Proof. The eigenvalues of $dX_s(U; U_L)$ are $-s + \lambda_j(U)(j = 1, 2)$.
Q.E.D.

We call, for example, an orbit from a repeller point to a saddle point *repeller-saddle connection.*

§2. Existence of admissible shock waves.

We begin with the classification of the critical points of the vector fields $X_s(U; U_L)$.

Lemma 1. *The vector field $X_s(U, U_L)$ has two, three or four critical points in the bounded region of $U-$plane.*

(i) If The vector field $X_s(U, U_L)$ has four critical points in the bounded region of $U-$plane, then the critical points are one node and three saddles in Case I; two nodes and two saddles in Case II.

(ii) If The vector field $X_s(U, U_L)$ has three critical points in the bounded region of $U-$plane, then the critical points are two saddles one saddle-node in Case I; one node, one saddle and one saddle-node in Case II.

(iii) If The vector field $X_s(U, U_L)$ has two critical points in the bounded region of $U-$plane, then the critical points are two saddles in Case I; one node and one saddle or two saddle-nodes in Case II.

Proof. The Poincaré transformation [1, 13] enables us to make a one-to-one correspondence from Uplane including the infinity to the unit sphere $S^2 = \{(x_1, x_2, x_3) \in \mathbf{R}^3;\ x_1^2 + x_2^2 + x_3^2 = 1\}$ by identifying two antipodal points. We study the critical points at the infinity which corresponds to the equator $\{x_3 = 0\} (\subseteq S^2)$. A use of Poincaré-Hopf theorem [23] gives the assertion. For a more detailed proof, we refer the reader to [3]. Q.E.D.

Our new result is the following:

Theorem 1. *Suppose that (a, b) belongs to Case I& II. Suppose that there exist four critical points of $X_s(U, U_L)$ and one of the critical point and U_L constitute a j-compressive $(j = 1, 2)$ shock wave with shock speed s. If there are no saddle-saddle connections, then*

(1) *in Case I, there exists a viscous shock profile connecting U_L and each of three saddle points.*

(2) *in Case II, two nodes consists of a attractor point and a repeller point. For each of two saddle points and each of two nodes, there exists a viscous profile connecting the saddle point and the node. There exist infinitely many viscous profiles connecting the repeller and the attractor.*

Idea of proof. We begin by remarking that the vector fields $X_s(U; U_L)$ have a potential. Namely, by setting

$$\phi_s(U; U_L) = \{F'(U_L) - sI\}[U - U_L] + C(U - U_L) \tag{23}$$

where $\{F'(U_L) - sI\}[V] = {}^tV\{F'(U_L) - sI\}V$, the vector fields $X_s(U; U_L)$ can be represented by the gradient of $\frac{1}{2}\phi_s(U; U_L)$:

$$X_s(U; U_L) = \frac{1}{2}\nabla\phi_s(U; U_L). \tag{24}$$

Then the viscous shock profile $U = U(\xi)$ joining U_L and U_R is a solution to

$$\frac{dU}{d\xi} = \frac{1}{2}\nabla\phi_s(U; U_L) \tag{25}$$

with the boundary condition at the infinity $U(-\infty) = U_L$ and $U(+\infty) = U_R$. This is the equation of finding the *steepest ascent path* on the graph of $\phi_s(U; U_L)$. We notice that U is a *local minimum point*, a *local maximum point* or a *saddle point* of $\phi_s(U; U_L)$, respectively, according as it is a repeller point, an attractor point or a saddle point of the vector field $X_s(U, U_L)$, respectively. We suppose that (a, b) belongs to Case I and that U_L and one of the other critical point constitute a $1-$compressive shock wave. It can be shown that the level set $\{U : \phi_s(U;U_L) = 0\}$ consists of the point $U = U_L$ and three unbounded regular curves. For ε be a positive small constant, the level set $\{U : \phi_s(U;U_L) = \epsilon\}$ is composed of a small closed curve enclosing U_L and three unbounded regular curves because U_L is a local minimum point of $\phi_s(U;U_L)$. Suppose that a critical point U_1 exists on the level set $\{\phi_s(U;U_L) = p_1\}$ such that there are no critical points in $\{\epsilon \le \phi_s(U;U_L) \le p_1 - \epsilon\}$. We can show that $\phi_s^{-1}[\epsilon, p_1 - \epsilon]$ is a disjoint union of integral curves. On the other hand, we have the usual configuration of integral curves around a saddle point in a small neighborhood of U_1. Hence by choosing a sufficiently small ϵ, the level set $\{\phi_s^{-1}(p_1 - \epsilon)\}$ passes through such a neighborhood. Then the above configuration shows that there exists a unique integral curve connecting U_L and U_1. If there are two or three critical points on the level set $\{\phi_s(U;U_L) = p_1\}$, the argument is the same. Next we consider level sets $\{\phi_s(U;U_L) = p\}$ for $p \in (p_1 - \epsilon, p_1 + \epsilon)$ with a positive small constant ϵ. The configuration of level sets near a saddle point says that, as p varies from $p_1 - \epsilon$ to $p_1 + \epsilon$, the level set changes: {a closed curve and three unbounded regular curves} $\rightarrow$ {an unbounded curve with a node and two unbounded regular curves} $\rightarrow$ {three unbounded regular curves}. Let U_2 be the second critical point contained in the level set $\{\phi_s(U;U_L) = p_2\}$. By the same argument, $\phi_s^{-1}[p_1 + \epsilon, p_2 - \epsilon]$ is a disjoint union of integral curves and there exists a unique integral curve from either U_L or U_1 to U_2. By the assumption, there are no integral curves connecting two saddle points U_1 and U_2. Therefore there exists a unique integral curve connecting U_L and U_2. As p varies from $p_2 - \epsilon$ to $p_2 + \epsilon$, the level set changes: {three unbounded regular curves} $\rightarrow$ {an unbounded curve and two unbounded curves with an intersection} $\rightarrow$ {three unbounded regular curves}. Let U_3 be the third critical point contained in the level set $\{\phi_s(U;U_L) = p_3\}$. Repeating the same argument between p_2 and p_3, we conclude that the point U_3 is connected to U_L by integral curve. Here we make a use of a generalization of Morse theory to non-compact level sets [24, 28]. Thus the theorem is proved. For a more detailed proof and a discussion in Case II, we refer the reader to [3].

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