Advanced Studies in Pure Mathematics 46, 2007
Singularities in Geometry and Topology 2004
pp. 299–329

Characteristic classes of (pro)algebraic varieties

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Dedicated to Jean-Paul Brasselet on the occasion of his sixtieth birthday

§1. Introduction

Various characteristic classes of singular varieties have been introduced and studied. One of them is the so-called Chern–Schwartz–MacPherson class. Its unique existence was conjectured by P. Deligne and A. Grothendieck and it was affirmatively solved by R. MacPherson. This characteristic class is a fundamental and important characteristic class from the viewpoint of investigation of other characteristic classes.

In this paper, in the first half we make a quick survey on three interesting characteristic classes of singular varieties with a naïve motivation of constructing a "singular version" of the so-called generalized Hirzebruch–Riemann–Roch theorem behind, and state a "unification" theorem concerning these three characteristic classes and its bivariant-theoretic version. And in the latter half we make a quick survey on characrteristic classes of proalgebraic varieties, which are very much related to motivic measure and motivic integration.

§2. Hirzebruch–Riemann–Roch and Grothendieck–Riemann–Roch

A characteristic class of a vector bundle over a topological space X is defined to be a map from the set of isomorphism classes of vector bundles over X to the cohomology group (ring) $H^*(X;\Lambda)$ with a coefficient ring

Received January 29, 2005
Revised June 2, 2005
(*) Partially supported by Grant-in-Aid for Scientific Research (C) (No. 15540086 + No. 17540088), the Ministry of Education, Culture, Sports, Science and Technology (MEXT), Japan.
1991 *Mathematics Subject Classification.* 14C17, 14F99, 55N35.

Λ, which is supposed to be compatible with the pullback of vector bundle and cohomology group for a continuous map. Namely, it is an assignment $c\ell$: $\mathrm{Vect}(X) \to H^*(X;\Lambda)$ which satisfies that for a continuous map $f: X \to Y$ the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Vect}(Y) & \xrightarrow{cl} & H^*(Y;\Lambda) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ \mathrm{Vect}(X) & \xrightarrow[cl]{} & H^*(X;\Lambda). \end{array}$$

Here $\mathrm{Vect}(W)$ is the set of isomorphism classes of vector bundles over W. In this paper we only deal with complex vector bundles.

If cl is multiplicative, i.e., cl satisfies the Whitney sum condition

$$cl(E \oplus F) = cl(E)cl(F),$$

then the contravariant functor Vect can be replaced by the Grothendieck K-theory:

$$\begin{array}{ccc} \mathbf{K}(Y) & \xrightarrow{cl} & H^*(Y;\Lambda) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ \mathbf{K}(X) & \xrightarrow[cl]{} & H^*(X;\Lambda). \end{array}$$

For complex vector bundles, the Chern class is essential in the sense that any characteristic class is expressed as a polynomial of Chern classes. And furthermore any multiplicative characteristic class can be described via Hirzebruch's multiplicative sequence of Chern classes [Hir1].

For a complex manifold M its complex tangent bundle T_M is available and thus we can define a characteristic class $cl(T_M)$, which is called a *characteristic class* $cl(M)$ *of the manifold* M.

Let X be a non-singular complex projective variety and E a holomorphic vector bundle over X. Let

$$\chi(X, E) = \sum_{i \geq 0} (-1)^i \dim_{\mathbb{C}} H^i(X;\Omega(E))$$

be the Euler–Poincaré characteristic, where $\Omega(E)$ is the coherent sheaf of germs of sections of E. J.-P. Serre conjectured (in his letter to Kodaira and Spencer, dated September 29, 1953): *There exists a polynomial*

$P(X, E)$ of Chern classes of the base variety X and the vector bundle E such that

$$\chi(X, E) = \int_X P(X, E) \cap [X].$$

Within three months (December 9, 1953) F. Hirzebruch solved this conjecture: the above looked-for polynomial $P(X, E)$ can be expressed as

$$P(X, E) = ch(E) \cup td(X)$$

where $ch(E)$ is the total Chern character of E and $td(T_X)$ is the total Todd class of the tangent bundle T_X of X. For the sake of later use , we recall that for a complex vector bundle V the total cohomology classes $ch(V)$ and $td(V)$ are defined as follows:

$$ch(V) = \sum_{i=1}^{\operatorname{rank} V} e^{\alpha_i}$$

and

$$td(V) = \prod_{i=1}^{\operatorname{rank} V} \frac{\alpha_i}{1 - e^{-\alpha_i}}$$

where α_i's are the Chern roots of V. Namely, we have the following celebrated theorem of Hirzebruch:

Theorem (2.1) (Hirzebruch–Riemann–Roch)(**HRR**).

$$\chi(X, E) = T(X, E) := \int_X (ch(E) \cup td(X)) \cap [X].$$

$T(X, E)$ is called the T-characteristic ([Hir1]). For a more detailed historical aspect of **HRR**, see [Hir2].

A. Grothendieck (cf. [BoSe]) generalized **HRR** for non-singular quasi-projective algebraic varieties over any field and proper morphisms with Chow cohomology ring theory instead of ordinary cohomology theory. For the complex case we can still take the ordinary cohomology theory (or the homology theory by the Poincaré duality). Here we stick ourselves to complex projective algebraic varieties for the sake of simplicity. For a variety X, let $\mathbf{G}_0(X)$ denote the Grothendieck group of algebraic coherent sheaves on X and for a morphism $f: X \to Y$ the pushforward $f_!: \mathbf{G}_0(X) \to \mathbf{G}_0(Y)$ is defined by

$$f_!(\mathcal{F}) := \sum_{i \geq 0} (-1)^i \mathbf{R}^i f_* \mathcal{F},$$

where $\mathbf{R}^i f_* \mathcal{F}$ is (the class of) the higher direct image sheaf of $\mathcal{F}$. Then $\mathbf{G}_0$ is a covariant functor with the above pushforward (see [Grot1] and [Man]). Then Grothendieck showed the existence of a natural transformation from the covariant functor $\mathbf{G}_0$ to the $\mathbb{Q}$-homology covariant functor $H_*(\ ;\mathbb{Q})$ (see [BoSe]):

Theorem (2.2) (Grothendieck–Riemann–Roch)(**GRR**). *Let the transformation $\tau: \mathbf{G}_0(\ ) \to H_*(\ ;\mathbb{Q})$ be defined by $\tau(\mathcal{F}) = td(X)ch(\mathcal{F}) \cap [X]$ for any smooth variety X. Then τ is actually natural, i.e., for any morphism $f: X \to Y$ the following diagram commutes:*

$$\begin{tikzcd} \mathbf{G}_0(X) \arrow[r, "\tau"] \arrow[d, "f_!"'] & H_*(X;\mathbb{Q}) \arrow[d, "f_*"] \\ \mathbf{G}_0(Y) \arrow[r, "\tau"'] & H_*(Y;\mathbb{Q}) \end{tikzcd}$$

i.e.,

$$td(T_Y)ch(f_!\mathcal{F}) \cap [Y] = f_*(td(T_X)ch(\mathcal{F}) \cap [X]).$$

Clearly **HRR** is induced from **GRR** by considering a map from X to a point.

Note that the target of the transformation of the original **GRR** is the cohomology $H^*(\ ;\mathbb{Q})$ with the Gysin homomorphism instead of the homology $H_*(\ ;\mathbb{Q})$, but, by the definition of the Gysin homomorphism the original **GRR** can be put in as above.

§3. The Generalized Hirzebruch–Riemann–Roch

In Hirzebruch's book [Hir1, §12.1 and §15.5] he has generalized the characteristics $\chi(X, E)$ and $T(X, E)$ to the so-called χ_y-characteristic $\chi_y(X, E)$ and T_y-characteristic $T_y(X, E)$ as follows, using a parameter y (see also [HBJ, Chapter 5]).

Definition (3.1).

$$\chi_y(X,\, E) := \sum_{p\geq 0}\left(\sum_{q\geq 0}(-1)^q \dim_{\mathbb{C}} H^q(X,\, \Omega(E)\otimes \Lambda^P T_X^{\vee})\right) y^p$$
$$= \sum_{p\geq 0}\chi(X,\, E\otimes \Lambda^P T_X^{\vee}))y^p$$

where $T_X^{\vee}$is the dual of the tangent bundle T_X, i.e., the cotangent bundle of X.

$$T_y(X,\, E) := \int_X \widetilde{td_{(y)}}(T_X)ch_{(1+y)}(E)\cap [X],$$
$$\widetilde{td_{(y)}}(T_X) := \prod_{i=1}^{\dim X}\left(\frac{\alpha_i(1+y)}{1-e^{-\alpha_i(1+y)}} - \alpha_i y\right),$$
$$ch_{(1+y)}(E) := \sum_{j=1}^{\operatorname{rank} E} e^{\beta_j(1+y)},$$

where $\alpha_i{}'s$ are the Chern roots of T_X and $\beta_j{}'s$ are the Chern roots of E.

F. Hirzebruch [Hir1, §21.3] showed the following theorem:

Theorem (3.2) (The generalized Hirzebruch–Riemann–Roch)(**g-HRR**).

$$\chi_y(X,\, E) = T_y(X,\, E).$$

The above modified Todd class $\widetilde{td_{(y)}}(T_X)$ defined above unifies the following three important characteristic cohomology classes:
$(y=-1)$ the total Chern class

$$\widetilde{td_{(-1)}}(T_X) = c(T_X),$$

$(y=0)$ the total Todd class

$$\widetilde{td_{(0)}}(T_X) = td(T_X),$$

$(y=1)$ the total Thom–Hirzebruch L-class

$$\widetilde{td_{(1)}}(T_X) = L(T_X).$$

In particular, for E=the trivial line bundle, for these special values $y=-1,\, 0,\, 1$ the **g-HRR** reads as follows:

$(y = -1)$ *Gauss–Bonnet–Chern Theorem*:

$$e(X) = \int_X c(T_X) \cap [X],$$

$(y = 0)$ *Riemann–Roch*:

$$\chi(X) = \int_X td(T_X) \cap [X],$$

$(y = 1)$ *Hirzebruch's Signature Theorem*:

$$\sigma(X) = \int_X L(T_X) \cap [X].$$

§4. Characteristic classes of singular varieties

In the following we consider only compact spaces.

For a singular complex algebraic or analytic variety X its tangent bundle is not available any longer because of the existence of singularities, thus one cannot define its characteristic class $cl(X)$ as in the above case of manifolds, although a "tangent-like" bundle such as Zariski tangents is available. A main theme for defining reasonable characteristic classes for singular varieties is that reasonable ones should be interesting enough; for example, they must be geometrically or topologically interesting, and they should be quite well related to other well-known interesting invariants of varieties (see [Mac3]).

The theory of characteristic classes of vector bundles is nothing but saying that the assignment $c\ell\colon \mathrm{Vect}(X) \to H^*(X;\Lambda)$ is *a natural transformation from the contravariant functor* Vect *to the contravariant cohomology functor* $H^*(\ ;\Lambda)$. This *naturality* is a *key* for various theories of characteristic classes for singular varieties.

The first example of a characteristic class formulated as a natural transformation was the Stiefel–Whitney class transformation due to Dennis Sullivan [Sull] (also see [Fu-Mc]). And the complex version of the Stiefel–Whitney class, i.e., the first characteristic class of singular complex varieties formulated as a natural transformation is MacPherson's Chern class transformation [Mac2].

Let $F(X)$ be the abelian group of constructible functions on a variety X. Then the assignment $F\colon \mathcal{V} \to \mathcal{A}$ is a contravariant functor (from the category of varieties to the category of abelian groups) by the usual functional pullback: for a morphism $f\colon X \to Y$

$$f^*\colon F(Y) \to F(X) \quad \text{defined by} \quad f^*(\alpha) := \alpha \circ f.$$

For a constructible set $Z \subset X$, we define

$$\chi(Z;\alpha) := \sum_{n\in\mathbb{Z}} n\chi(Z\cap\alpha^{-1}(n)).$$

Then it turns out that the assignment $F\colon \mathcal{V} \to \mathcal{A}$ also becomes a covariant functor by the following pushforward:

$$f_*\colon F(X) \to F(Y) \quad \text{defined by} \quad f_*(\alpha)(y) := \chi(f^{-1}(y);\alpha).$$

To show this requires a stratification theory (see [Mac2]).

P. Deligne and A. Grothendieck conjectured (around 1969) and R. MacPherson [Mac2] solved the following:

Theorem (4.1). *There exists a unique natural transformation*

$$c_*\colon F \to H_*$$

from the constructible function covariant functor F to the homology covaraint functor H_* satisfying the "normalization" that the value of the characteristic function $1\!\!1_X$ of a smooth complex algebraic variety X is the Poincaré dual of the total Chern cohomology class:

$$c_*(1\!\!1_X) = c(T_X)\cap[X].$$

The main ingredients are Chern–Mather classes, local Euler obstructions (also see [Br3], [Gon] and [Sa]) and "graph construction" (also see [Mac1]). The uniqueness follows from the resolution of singularities. For recent investigations on local Euler obstruction, e.g. see [BLS], [BMPS] and [STV1, STV2], etc.

J.-P. Brasselet and M.-H. Schwartz [BrSc] showed that the distinguished value $c_*(1\!\!1_X)$ of the characteristic function of a variety embedded into a complex manifold is isomorphic under this transformation to the Schwartz class [Sc1, Sc2] via the Alexander duality. Thus, for a complex algebraic variety X, singular or nonsingular, $c_*(1\!\!1_X)$ is called the total *Chern–Schwartz–MacPherson class* of X and denoted simply by $c_*(X)$. By considering mapping X to a point, one can get

$$e(X) = \int_X c_*(X)$$

which is a singular version of the Gauss–Bonnet–Chern theorem.

Motivated by the formulation of MacPherson's Chern class transformation, P. Baum, W. Fulton and R. MacPherson [BFM] have extended **GRR** to singular varieties, by introducing the so-called *localized Chern* *character* $ch_X^M(\mathcal{F})$ of a coherent sheaf $\mathcal{F}$ with X embedded into a nonsingular quasi-projective variety M, as a substitute of $ch(F) \cap [X]$ in the above **GRR**. Note that if X is smooth $ch_X^X(\mathcal{F}) = ch(F) \cap [X]$. In [BFM] they showed the following theorem:

Theorem (4.2) (Baum–Fulton–MacPherson's Riemann–Roch) (**BFM-RR**). (i) *$td_*(\mathcal{F}) := td(i_M^* T_M) \cap ch_X^M(\mathcal{F})$ is independent of the embedding $i_M \colon X \to M$.*
(ii) *Let the transformation $td_* \colon \mathbf{G}_0(\ ) \to H_*(\ ;\mathbb{Q})$ be defined by*

$$td_*(\mathcal{F}) = td(i_M^* T_M) \cap ch_X^M(\mathcal{F})$$

for any variety X. Then td_* is actually natural, i.e., for any morphism $f \colon X \to Y$ the following diagram commutes:

$$\begin{array}{ccc} \mathbf{G}_0(X) & \xrightarrow{td_*} & H_*(X;\mathbb{Q}) \\ {\scriptstyle f_!}\downarrow & & \downarrow{\scriptstyle f_*} \\ \mathbf{G}_0(Y) & \xrightarrow[td_*]{} & H_*(Y;\mathbb{Q}) \end{array}$$

i.e., for any embeddings $i_M \colon X \to M$ and $i_N \colon Y \to N$

$$td(i_N^* T_N) \cap ch_Y^N(f_! \mathcal{F}) = f_*(td(i_M^* T_M) \cap ch_X^M(\mathcal{F})).$$

For a complex algebraic variety X, singular or nonsingular, $td_*(X) := c_*(\mathcal{O}_X)$ is called the Baum–Fulton–MacPherson's Todd homology class of X. And we get

$$\chi(X) = \int_X td_*(X)$$

which is a singular version of the Riemann–Roch.

Using the notion of "perversity", M. Goresky and R. MacPherson [GM1, GM2] have introduced *Intersection Homology Theory*, in which almost all properties, such as the Poincaré duality, of the (co)homology of smooth manifolds are saisfied. Note that the intersection homology group is not a homotopy invariant unlike the (co)homology group. For the intersection homology theory, e.g., see also [Bor], [Br2] and [Kir].

In [GM1], they introduced a homology L-class $L_*^{\mathrm{GM}}(X)$ such that if X is nonsingular it becomes the Poincaré dual of the original Thom–Hirzebruch L-class:

$$L_*^{\mathrm{GM}}(X) = L(TX) \cap [X].$$

Later, S. Cappell and J. Shaneson [CS1] (see also [CS2] and [Sh]), using some topological aspects of perverse sheaves [BBD], introduced a homology L-class transformation L_*, which turns out to be a natural transformation from the abelian group Ω of cobordism classes of self-dual constructible complexes to the rational homology group [BSY2] (cf. [Y1]):

Theorem (4.3) (Cappell–Shaneson's homology L-class). *There exists a natural transformation*

$$L_* : \Omega \to H_*(\ ; \mathbb{Q})$$

such that for X smooth

$$L_*(\mathbb{Q}_X[2\dim X]) = L(TX) \cap [X].$$

Here $\mathbb{Q}_X$ is the constant sheaf (considered as a complex concentrated at degree 0) of X.

For a complex algebraic variety X, singular or nonsingular, the value $L_*(IC_X)$ of the middle intersection cohomology complex IC_X is the total Goresky-MacPherson's homology L-class $L_*^{\mathrm{GM}}(X)$ of X and simply denoted by $L_*(X)$. And we get

$$\sigma(X) = \int_X L_*(X)$$

which is a singular version of Hirzebruch's signature theorem. Here $\sigma(X)$ is defined by the pairing of the intersection homology group with middle perversity.

For a survey concerning characteristic classes of singular varieties other than MacPherson's survey article [Mac3], there are now various articles available, e.g., [Alu1], [Br4], [Pa] (also see [PP]), [Su3] (also see [Su1, Su2]), [Sch2] (also see [Sch4]), [SY] etc., and also consult various papers therein.

§5. A "unification" theorem

So far we have seen that the generalized Hirzebruch–Riemann–Roch **g-HRR** unifies the three important and distinguished characteristics (or genera):

($y = -1$) the topological Euler–Poincaré characteristic $e(X)$,
($y = 0$) the arithmetic genus $\chi(X)$,
($y = 1$) the signature $\sigma(X)$,

and that corresponding to these three invariants there are three distinguished natural transformations of characteristic homology classes of possibly singular varieties, which are respectively,

($y=-1$) MacPherson's Chern class transformation $c_*\colon F(\ )\to H_*(\ ;\mathbb{Z})$,

($y=0$) Baum–Fulton–MacPherson's Riemann–Roch $td_*\colon \mathbf{G}_0(\ )\to H_*(\ ;\mathbb{Q})$,

($y=1$) Cappell–Shaneson's homology L-class $L_*\colon \Omega(\ )\to H_*(\ ;\mathbb{Q})$.

It seems to be natural to pose the following naïve problem (cf. [Mac2] and [Y2]):

Problem (5.1). *Is there a theory of characteristic homology classes unifying the above three characteristic homology classes of possibly singular varieties ? A naïve question is whether or not there is a reasonable "singular version"* $\boxed{?}_y$ *of the generalized Hirzebruch–Riemann–Roch* ***g-HRR*** *such that*
($y=-1$) $\boxed{?}_{-1}$ *gives rise to the rationalized MacPherson's Chern class transformation* $c_*\otimes\mathbb{Q}$,
($y=0$) $\boxed{?}_0$ *gives rise to the Baum–Fulton–MacPherson's Riemann–Roch* td_*, *and*
($y=1$) $\boxed{?}_1$ *gives rise to the Cappell–Shaneson's homology L-class* L_*.

An obvious problem for this unification problem is that the source covariant functors of these three natural transformations are all different!

A "reasonable" answer for the above problem has been obtained [BSY2] (cf. [BSY3] and [SY]) via the so-called *relative Grothendieck ring of complex algebraic varieties over* X, denoted by $K_0(\mathcal{V}/X)$. This ring was introduced by E. Looijenga in [Lo] and further studied by F. Bittner in [Bit].

The relative Grothendieck group $K_0(\mathcal{V}/X)$ (of morphisms over a variety X) is the quotient of the free abelian group of isomorphism classes of morphisms to X (denoted by $[Y\to X]$ or $[Y\xrightarrow{h}X]$), modulo the following relation:

$$[Y\xrightarrow{h}X]=[Z\hookrightarrow Y\xrightarrow{h}X]+[Y\setminus Z\hookrightarrow Y\xrightarrow{h}X]$$

for $Z\subset Y$ a closed subvariety of Y. The ring structure is given by the fiber square: for $[Y\xrightarrow{f}X],\ [W\xrightarrow{g}X]\in K_0(\mathcal{V}/X)$

$$[Y\xrightarrow{f}X]\cdot[W\xrightarrow{g}X]:=[Y\times_X W\xrightarrow{f\times_X g}X].$$

Here $Y \times_X W \xrightarrow{f\times_X g} X$ is $g \circ f' = f \circ g'$ where f' and g' are as in the following diagram

$$\begin{tikzcd} Y \times_X W \arrow[r, "f'"] \arrow[d, "g'"'] & W' \arrow[d, "g"] \\ Y \arrow[r, "f"] & X. \end{tikzcd}$$

The relative Grothendieck ring $K_0(\mathcal{V}/X)$ has the unit $1_X := [X \xrightarrow{\mathrm{id}_X} X]$.

Note that when $X = pt$ is a point, the relative Grothendieck ring $K_0(\mathcal{V}/pt)$ is nothing but the usual Grothendieck ring $K_0(\mathcal{V})$ of $\mathcal{V}$, which is the free abelian group generated by the isomorphism classes of varieties modulo the subgroup generated by elements of the form $[V] - [V'] - [V \setminus V']$ for a subvariety $V' \subset V$, and the ring structure is given by the Cartesian product of varieties.

For a morphism $f\colon X' \to X$, the pushforward

$$f_*\colon K_0(\mathcal{V}/X') \to K_0(\mathcal{V}/X)$$

is defined by

$$f_*[Y \xrightarrow{h} X'] := [Y \xrightarrow{f\circ h} X].$$

With this pushforward, the assignement $X \longmapsto K_0(\mathcal{V}/X)$ is a covariant functor. The pullback

$$f^*\colon K_0(\mathcal{V}/X) \to K_0(\mathcal{V}/X')$$

is defined as follows: for a fiber square

$$\begin{tikzcd} Y' \arrow[r, "g'"] \arrow[d, "f'"'] & X' \arrow[d, "f"] \\ Y \arrow[r, "g"] & X \end{tikzcd}$$

the pullback $f^*[Y \xrightarrow{g} X] := [Y' \xrightarrow{g'} X']$. With this pullback, the assignement $X \longmapsto K_0(\mathcal{V}/X)$ is a contravariant functor.

Theorem (5.2). *Let $K_0(\mathcal{V}/X)$ be the Grothendieck group of morphisms over X. Then there exists a unique natural transformation*

$$T_y\colon K_0(\mathcal{V}/\ \ ) \to H_*(\ \ ) \otimes \mathbb{Q}[y]$$

such that for X nonsingular

$$T_y([X \xrightarrow{\mathrm{id}} X]) = \widetilde{td_{(y)}}(X) \cap [X].$$

And we have the following theorem:

Theorem (5.3). ($y = -1$) *There exists a unique natural transformation* $\epsilon\colon K_0(\mathcal{V}/\ \ ) \to F(\ \ )$ *such that for* X *nonsingular* $\epsilon([X \xrightarrow{\mathrm{id}} X]) = 1\!\!1_X$. *And the following diagram commutes*

$$\begin{tikzcd}
K_0(\mathcal{V}/X) \arrow[rr, "\epsilon"] \arrow[dr, "T_{-1}"'] & & F(X) \arrow[dl, "c_*"] \\
& H_*(X) \otimes \mathbb{Q} &
\end{tikzcd}$$

($y = 0$) *There exists a unique natural transformation* $\gamma\colon K_0(\mathcal{V}/\ \ ) \to \mathbf{G}_0(\ \ )$ *such that for* X *nonsingular* $\gamma([X \xrightarrow{\mathrm{id}} X]) = [\mathcal{O}_X]$. *And the following diagram commutes*

$$\begin{tikzcd}
K_0(\mathcal{V}/X) \arrow[rr, "\gamma"] \arrow[dr, "T_0"'] & & \mathbf{G}_0(X) \arrow[dl, "td_*"] \\
& H_*(X) \otimes \mathbb{Q} &
\end{tikzcd}$$

($y = 1$) *There exists a unique natural transformation* $\omega\colon K_0(\mathcal{V}/\ \ ) \to \Omega(\ \ )$ *such that for* X *nonsingular* $\omega([X \xrightarrow{\mathrm{id}} X]) = \big[\mathbb{Q}_X[2\dim X]\big]$. *And the following diagram commutes*

$$\begin{tikzcd}
K_0(\mathcal{V}/X) \arrow[rr, "\omega"] \arrow[dr, "T_1"'] & & \Omega(X) \arrow[dl, "L_*"] \\
& H_*(X) \otimes \mathbb{Q} &
\end{tikzcd}$$

An original proof of Theorem (5.2) uses Saito's theory of mixed Hodge modules [Sai] and it turns out that it can be also proved without it and, instead, via a Bittner-Looijenga's theorem about the relative Grothendieck group [Bit].

§6. Bivariant Theories

In [FM] W. Fulton and R. MacPherson introduced the notion of *Bivariant Theory*, which is a simultaneous generalization of a pair of covariant and contravariant functors. Most pairs of covariant and contravariant theories, e.g., such as homology theory, K-theory, etc, extend to bivariant theories. They also introduced the *operational bivariant theory* (also see [Fu]), which can be always constructed from any covariant functor.

A bivariant theory $\mathbb{B}$ on a category $\mathcal{C}$ with values in the category of abelian groups is an assignment to each morphism $X \xrightarrow{f} Y$ in the category $\mathcal{C}$ a graded abelian group $\mathbb{B}(X \xrightarrow{f} Y)$, which is equipped with the following three basic operations:
(Product operations): For morphisms $f: X \to Y$ and $g: Y \to Z$, the product operation

$$\bullet: \mathbb{B}(X \xrightarrow{f} Y) \otimes \mathbb{B}(Y \xrightarrow{g} Z) \to \mathbb{B}(X \xrightarrow{gf} Z)$$

is defined.
(Pushforward operations): For morphisms $f: X \to Y$ and $g: Y \to Z$ with f proper, the pushforward operation

$$f_\star: \mathbb{B}(X \xrightarrow{gf} Z) \to \mathbb{B}(Y \xrightarrow{g} Z)$$

is defined.
(Pullback operations): For a fiber square

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y, \end{tikzcd}$$

the pullback operation

$$g^\star: \mathbb{B}(X \xrightarrow{f} Y) \to \mathbb{B}(X' \xrightarrow{f'} Y')$$

is defined. And these three operations are required to satisfy the seven compatibility axioms (see [FM, Part I, §2.2] for details).

Let $\mathbb{B}$, $\mathbb{B}'$ be two bivariant theories on a category $\mathcal{C}$. Then a *Grothendieck transformation* from $\mathbb{B}$ to $\mathbb{B}'$

$$\gamma: \mathbb{B} \to \mathbb{B}'$$

is a collection of homomorphisms

$$\mathbb{B}(X \to Y) \to \mathbb{B}'(X \to Y)$$

for a morphism $X \to Y$ in the category $\mathcal{C}$, which preserves the above three basic operations:

(i) $\gamma(\alpha \bullet_{\mathbb{B}} \beta) = \gamma(\alpha) \bullet_{\mathbb{B}'} \gamma(\beta)$,
(ii) $\gamma(f_\bigstar \alpha) = f_\bigstar \gamma(\alpha)$, and
(iii) $\gamma(g^\bigstar \alpha) = g^\bigstar \gamma(\alpha)$.

$B_*(X) := \mathbb{B}(X \to pt)$ and $B^*(X) := \mathbb{B}(X \xrightarrow{\mathrm{id}} X)$ become a covariant functor and a contravariant functor, respectively. And a Grothendieck transformation $\gamma\colon \mathbb{B} \to \mathbb{B}'$ induces natural transformations $\gamma_*\colon B_* \to B'_*$ and $\gamma^*\colon B^* \to B'^*$. If we have a Grothendieck transformation $\gamma\colon \mathbb{B} \to \mathbb{B}'$, then via a bivariant class $b \in \mathbb{B}(X \xrightarrow{f} Y)$ we get the commutative diagram

$$\begin{tikzcd} B_*(Y) \arrow[r, "\gamma_*"] \arrow[d, "b\bullet"'] & B'_*(Y) \arrow[d, "\gamma(b)\bullet"] \\ B_*(X) \arrow[r, "\gamma_*"'] & B'_*(X). \end{tikzcd}$$

This is called *the Verdier-type Riemann–Roch formula associated to the bivariant class b.*

Fulton–MacPherson's bivariant group $\mathbb{F}(X \xrightarrow{f} Y)$ of constructible functions consists of all the constructible functions on X which satisfy the local Euler condition with respect to f. Here a constructible function $\alpha \in F(X)$ is said to satisfy the *local Euler condition with respect to f* if for any point $x \in X$ and for any local embedding $(X, x) \to (\mathbf{C}^N, 0)$ the equality $\alpha(x) = \chi\left(B_\epsilon \cap f^{-1}(z); \alpha\right)$ holds, where B_ϵ is a sufficiently small open ball of the origin 0 with radius ϵ and z is any point close to $f(x)$ (cf. [Br1], [Sa]). In particular, if $1\!\!1_f := 1\!\!1_X$ belongs to the bivariant group $\mathbb{F}(X \xrightarrow{f} Y)$, then the morphism $f\colon X \to Y$ is called *an Euler morphism.* For example, a holomorphic submersion between complex spaces is an Euler morphism.

The three operations on $\mathbb{F}$ are defined as follows:

(i) the product operation $\bullet\colon \mathbb{F}(X \xrightarrow{f} Y) \otimes \mathbb{F}(Y \xrightarrow{g} Z) \to \mathbb{F}(X \xrightarrow{gf} Z)$ is defined by

$$\alpha \bullet \beta := \alpha \cdot f^*\beta,$$

(ii) the pushforward operation $f_\bigstar\colon \mathbb{F}(X \xrightarrow{gf} Z) \to \mathbb{F}(Y \xrightarrow{g} Z)$ is the usual pushforward f_*, i.e.,

$$f_\bigstar(\alpha)(y) := \int c_*(\alpha|_{f^{-1}}),$$

(iii) for a fiber square

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y, \end{tikzcd}$$

the pullback operation $g^\bigstar\colon \mathbb{F}(X \xrightarrow{f} Y) \to \mathbb{F}(X' \xrightarrow{f'} Y')$ is the functional pullback ${g'}^*$, i.e..,

$$g^\bigstar(\alpha)(x') := \alpha(g'(x')).$$

Note that for any bivariant constructible function $\alpha \in \mathbb{F}(X \xrightarrow{f} Y)$, the Euler–Poincaré characteristic $\chi(f^{-1}(y);\alpha) = \int c_*(\alpha|_{f^{-1}(y)})$ of α restricted to each fiber $f^{-1}(y)$ is locally constant, i.e., constant along connected components of the base variety Y; in particular, if $f\colon X \to Y$ is an Euler morphism, then the Euler–Poincaré characteristic of the fibers are locally constant.

The correspondence $\mathbb{F}^s(X \to Y) := F(X)$ assigning to a morphism $f\colon X \to Y$ the abelian group $F(X)$ of the source variety X, whatever the morphism f is, becomes a bivariant theory with the same operations above. This bivariant theory is called the *simple* bivariant theory of constructible functions (see [Y3] and [Sch3]). In passing, what we need to do to show that the Fulton–MacPherson's group of constructible functions satisfying the local Euler condition with respect to a morphism is a bivariant theory is to show that the local Euler condition with respect to a morphism is preserved by each of the above three operations.

Let $\mathbb{H}$ be Fulton–MacPherson's bivariant homology theory, constructed from the cohomology theory [FM, §3.1]. W. Fulton and R. MacPherson conjectured or posed as a question the existence of a so-called bivariant Chern class and J.-P. Brasselet [Br1] solved it:

Theorem (6.1) (J.-P. Brasselet). *For the category of embeddable complex analytic varieties with cellular morphisms, there exists a Grothendieck transformation*

$$\gamma\colon \mathbb{F} \to \mathbb{H}$$

such that for a morphism $f\colon X \to pt$ from a nonsingular variety X to a point pt and the bivariant constructible function $1\!\!1_f := 1\!\!1_X$ the following normalization condition holds:

$$\gamma(1\!\!1_f) = c(TX) \cap [X].$$

In [Z1, Z2] J. Zhou showed that the bivariant Chern classes constructed by J.-P. Brasselet [Br1] and by C. Sabbah [Sa] are identical in the case when the target variety is a nonsingular curve. And the present author showed the following uniqueness theorem of bivariant Chern classes for morphisms whose target varieties are nonsingular and *of any dimension*:

Theorem (6.2) ([Y4, Theorem (3.7)]). *If there exists a bivariant Chern class $\gamma\colon \mathbb{F} \to \mathbb{H}$, then it is unique when restricted to morphisms whose target varieties are nonsingular; explicitly, for a morphism $f\colon X \to Y$ with Y nonsingular and for any bivariant constructible function $\alpha \in \mathbb{F}(X \xrightarrow{f} Y)$ the bivariant Chern class $\gamma(\alpha)$ is expressed by*

$$\gamma(\alpha) = f^* s(TY) \cap c_*(\alpha)$$

where $s(TY) := c(TY)^{-1}$ is the Segre class of the tangent bundle.

See [Sch3] and [Y5, Y6, Y7, Y8, Y9] for other related results.

And in [BSY1] (see also [BSY4]) the above theorem is furthermore generalized to the case when the target variety can be singular but is "like a manifold":

Theorem (6.3). *Let Y be a complex analytic variety which is an oriented A-homology manifold. If there exists a bivariant Chern class $\gamma\colon \mathbb{F} \to \mathbb{H}$, then for any morphism $f\colon X \to Y$ the bivariant Chern class $\gamma_f\colon \mathbb{F}(X \xrightarrow{f} Y) \otimes A \to \mathbb{H}(X \xrightarrow{f} Y) \otimes A$ is uniquely determined and it is described by*

$$\gamma_f(\alpha) = f^* c^*(Y)^{-1} \cap c_*(\alpha).$$

Here $c^*(Y)$ is the unique cohomology class such that $c_*(1\!\!1_Y) = c^*(Y) \cap [Y]$. (Note that $c^*(Y)$ is invertible.)

Notice that $c^*(Y) = c(TY)$ for Y smooth and thus Theorem (6.3) indeed generalizes Theorem (6.2).

Remark (6.4). As to the uniqueness of operational bivariant Chern class [EY1, EY2] and operational bivariant Riemann–Roch [FM], one can also use a result due to S.-I. Kimura [Kim1] (also see [Kim2]).

Remark (6.5). In [BSY1] we have also shown that *a natural transformation of covariant theories extends uniquely to a Grothendieck transformation of suitable bivariant subtheories associated to them, provided that the given transformation commutes with exterior products.* This gives in a sense a positive solution to [FM, §10.9 Uniqueness questions]. For more details of this result and other results, see [BSY1].

Hence it follows from this general result that our natural transformation $T_y \colon K_0(\mathcal{V}/\ \ ) \to H_*(\ \ ) \otimes \mathbb{Q}[y]$ can be extended to a suitable bivariant version. Here, to get the suitable bivariant subtheories, the bivariant theories associated to the covariant functors which we consider are respectively the *simple* bivariant theory $\mathbb{K}_0^s(\mathcal{V}/X \xrightarrow{f} Y) := K_0(\mathcal{V}/X)$, just like the simple bivariant theory $\mathbb{F}^s$ of constructible functions as above, and the Fulton–MacPherson's bivariant homology theory $\mathbb{H}$ described above.

§7. Proconstructible functions and Euler–Poincaré characteristics of proalgebraic varieties

Let I be a directed set and let $\mathcal{C}$ be a given category. Then a projective system is, by definition, a system

$$\{X_i,\, \pi_{ii'} \colon X_{i'} \to X_i (i < i'),\, I\}$$

consisting of objects $X_i \in \mathrm{Obj}(\mathcal{C})$, morphisms $\pi_{ii'} \colon X_{i'} \to X_i \in \mathrm{Mor}(\mathcal{C})$ for each $i < i'$ and the index set I. The object X_i is called a *term* and the morphism $\pi_{ii'} \colon X_{i'} \to X_i$ a *bonding morphism* or *structure morphism* ([MS]). The projective system

$$\{X_i,\, \pi_{ii'} \colon X_{i'} \to X_i (i < i'),\, I\}$$

is sometimes simply denoted by $\{X_i\}_{i \in I}$.

Given a category $\mathcal{C}$, Pro-$\mathcal{C}$ is the category whose objects are projective systems $X = \{X_i\}_{i \in I}$ in $\mathcal{C}$ and whose set of morphisms from $X = \{X_i\}_{i \in I}$ to $Y = \{Y_j\}_{j \in J}$ is

$$\text{Pro-}\mathcal{C}(X, Y) := \varprojlim_J (\varinjlim_I \mathcal{C}(X_i, Y_j)).$$

Note that given a projective system $X = \{X_i\}_{i \in I} \in \text{Pro-}\mathcal{C}$, the projective limit $X_\infty := \varprojlim X_i$ may not exist or may not belong to the source category $\mathcal{C}$; for a certain sufficient condition for the existence of the projective limit in the category $\mathcal{C}$, see [MS] for example.

An object in Pro-$\mathcal{C}$ is called a *pro-object.* A projective system of algebraic varieties is called a *pro-algebraic variety* or simply *pro-variety* and its projective limit is called a *proalgebraic variety* or simply *provariety,* which may not be an algebraic variety but simply a topological space.

Remark (7.1). In Etale Homotopy Theory [AM] and Shape Theory (e.g., see [Boru], [Ed], [MS]) they stay in the pro-category and do not consider limits and colimits, because doing so throw away some geometric informations (also see [Grot2]).

A pro-morphism between two pro-objects is quite complicated. However, it follows from [MS] that the pro-morphism can be described more naturally as a so-called *level preserving pro-morphism.* Suppose that we have two pro-algebraic varieties $X = \{X_\gamma\}_{\gamma \in \Gamma}$ and $Y = \{Y_\lambda\}_{\lambda \in \Lambda}$. Then a pro-algebraic morphism $\Phi = \{f_\lambda\}_{\lambda \in \Lambda} \colon X \to Y$ is described as follows: there is an order-preserving map $\xi \colon \Lambda \to \Gamma$, i.e., $\xi(\lambda) < \xi(\mu)$ for $\lambda < \mu$, and for each $\lambda \in \Lambda$ there is a morphism $f_\lambda \colon X_{\xi(\lambda)} \to Y_\lambda$ such that for $\lambda < \mu$ the following diagram commutes:

$$\begin{array}{ccc} X_{\xi(\mu)} & \xrightarrow{f_\mu} & Y_\mu \\ {\scriptstyle \rho_{\xi(\lambda)\xi(\mu)}}\downarrow & & \downarrow{\scriptstyle \pi_{\lambda\mu}} \\ X_{\xi(\lambda)} & \xrightarrow[f_\lambda]{} & Y_\lambda, \end{array}$$

Then, the projective limit of the system $\{f_\lambda\}$ is a morphism from the provariety $X_\infty = \varprojlim_{\lambda \in \Lambda} X_\lambda$ to the provariety $Y_\infty = \varprojlim_{\gamma \in \Gamma} Y_\lambda$. It is called a promorphism and denoted by $f_\infty \colon X_\infty \to Y_\infty$.

From here on, for the sake of simplicity, we only deal with the case when the directed set Λ is the natural numbers $\mathbb{N}$ and a pro-morphism $\{f_n\}$ of two pro-varieties $\{X_n\}$ and $\{Y_n\}$ is such that for each n the following diagram commutes:

$$\begin{array}{ccc} X_{n+1} & \xrightarrow{f_{n+1}} & Y_{n+1} \\ {\scriptstyle \rho_{n(n+1)}}\downarrow & & \downarrow{\scriptstyle \pi_{n(n+1)}} \\ X_n & \xrightarrow[f_n]{} & Y_n. \end{array}$$

The projective system $\{X_n\}$ induces the projective system of abelian groups of constructible functions:

$$\{F(X_n),\ \pi_{nm}* \colon F(X_m) \to F(X_n)(n < m)\}.$$

And a system of morphims $f_n \colon X_n \to Y_n$ induces the system of homomorphisms

$$f_{n*} \colon F(X_n) \to F(Y_n).$$

Thus the system of commutative diagrams

$$\begin{tikzcd} F(X_m) \arrow[r, "f_{m*}"] \arrow[d, "\rho_{nm*}"'] & F(Y_m) \arrow[d, "\pi_{nm*}"] \\ F(X_n) \arrow[r, "f_{n*}"'] & F(Y_n), \end{tikzcd}$$

induces the homomorphism

$$f_{*\infty}\colon \varprojlim_n F(X_n) \to \varprojlim_n F(Y_n).$$

Similarly we get the homomorphism of the projective limits of homology groups

$$f_{*\infty}\colon \varprojlim_n H_*(X_n) \to \varprojlim_n H_*(Y_n).$$

Furthermore the commutative diagram of Chern–Schwartz–MacPherson class homomorphisms

$$\begin{tikzcd} F(X_m) \arrow[r, "c_*"] \arrow[d, "\pi_{nm*}"'] & H_*(X_m) \arrow[d, "\pi_{nm*}"] \\ F(X_n) \arrow[r, "c_*"'] & H_*(X_n), \end{tikzcd}$$

induces the projective limit of MacPherson's Chern class transformations:

$$c_{*\infty}\colon \varprojlim_n F(X_n) \to \varprojlim_n H_*(X_n)$$

So, we define, for the proalgebraic variety $X_\infty = \varprojlim_{\lambda\in\Lambda} X_\lambda$,

$$\operatorname{pro} F(X_\infty) := \varprojlim_n F(X_n) \quad \text{and} \quad \operatorname{pro} H_*(X_\infty) := \varprojlim_n H_*(X_n).$$

If we define $\operatorname{pro} c_*\colon \operatorname{pro} F \to \operatorname{pro} H_*$ to be the above $c_{*\infty}$ and define $f_{\infty *}$ to be the above $f_{*\infty}$, then we have a naïve proalgebraic version of MacPherson's Chern class transformation

$$\operatorname{pro} c_*\colon \operatorname{pro} F \to \operatorname{pro} H_*,$$

i.e., for a proalgebraic morphism $f_\infty\colon X_\infty \to Y_\infty$ we have the commutative diagram

$$\begin{array}{ccc} \operatorname{pro} F(X_\infty) & \xrightarrow{\operatorname{pro} c_*} & \operatorname{pro} H_*(X_\infty) \\ {\scriptstyle f_{\infty *}}\big\downarrow & & \big\downarrow{\scriptstyle f_{\infty *}} \\ \operatorname{pro} F(Y_\infty) & \xrightarrow[\operatorname{pro} c_*]{} & \operatorname{pro} H_*(Y_\infty). \end{array}$$

Although the above construction by taking the projective limits is quite easy, the structure of the progroup $\operatorname{pro} F(X^\infty)$ is not so obvious and also it is not obvious how to capture an element of $\varprojlim_n F(X_n)$ as a function on the proalgebraic variety $X_\infty = \varprojlim_n X_n$.

Remark (7.2). In [Alu3] P. Aluffi considered the above projective limit for a certain special projective system of morphisms called *modification system*, which is more precisely a projective system of birational morphisms.

So, we consider the inductive limits:

Definition (7.3). For a proalgebraic variety $X_\infty = \varprojlim_n X_n$, the inductive limit of the inductive system $\{F(X_n),\ \rho^*_{nm}\colon F(X_n) \to F(X_m)\ (n < m)\}$ is denoted by $F^{\mathrm{pro}}(X_\infty)$;

$$F^{\mathrm{pro}}(X_\infty) := \varinjlim_n F(X_n) = \bigcup_n \rho^n\left(F(X_n)\right)$$

where $\rho^n\colon F(X_n) \to \varinjlim_n F(X_n)$ is the homomorphism sending α_n to its equivalence class $[\alpha_n]$ of α_n. An element of the group $F^{\mathrm{pro}}(X_\infty)$ is called a *proconstructible* function on the proalgebraic variety X_∞. As a function on X_∞, the value of $[\alpha_n]$ at a point $(x_m) \in X_\infty$ is defined by

$$[\alpha_n]\big((x_m)\big) := \alpha_n(x_n).$$

The terminology *proconstructible* is used in [Grom1] (cf. [Grom2]), but its definition is not given there.

Lemma (7.4). *For each positive integer n, let $G_n = \mathbb{Z}$ be the integers and $\pi_{n,n+1}\colon G_n \to G_{n+1}$ be the homomorphism defined by multiplication by a non-zero integer p_n, i.e., $\pi_{n,n+1}(m) = mp_n$. Then there exists a unique (injective) homomorphism*

$$\Psi\colon \varinjlim_n G_n \to \mathbb{Q}$$

such that the following diagram commutes

$$\begin{tikzcd} & G_n \arrow[dl, "\rho^n"'] \arrow[dr, "\times \frac{1}{p_0 p_1 \cdots p_{n-1}}"] & \\ \varinjlim_n G_n \arrow[rr, "\Psi"'] & & \mathbb{Q}. \end{tikzcd}$$

Here we set $p_0 := 1$.

Using this lemma we can show the following theorem:

Theorem (7.5). *Let* $X_\infty = \varprojlim_{n \in \mathbb{N}} X_n$ *be a provariety such that for each* n *the structure morphism* $\pi_{n(n+1)} \colon X_{n+1} \to X_n$ *satisfies the condition that the Euler–Poincaré characteristics of the fibers of* $\pi_{n,\, n+1}$ *are non-zero (which implies the surjectivity of the morphism* $\pi_{n(n+1)}$*) and the same; for example,* $\pi_{n(n+1)} \colon X_{n+1} \to X_n$ *is a locally trivial fiber bundle with fiber variety being* F_n *and* $\chi(F_n) \neq 0$. *Let us denote the constant Euler–Poincaré characteristic of the fibers of the morphism* $\pi_{n(n+1)} \colon X_{n+1} \to X_n$ *by* χ_n *and we set* $\chi_0 := 1$.

(i) *The canonical Euler–Poincaré (pro)characteristic homomorphism, i.e., a "canonical realization" of the inductive limit of the Euler–Poincaré characteristic homomorphisms* $\{\chi \colon F(X_n) \to \mathbb{Z}\}_{n \in \mathbb{N}}$, *is described as the homomorphism*

$$\chi^{\mathrm{pro}} \colon F^{\mathrm{pro}}(X_\infty) \to \mathbb{Q}$$

defined by

$$\chi^{\mathrm{pro}}([\alpha_n]) = \frac{\chi(\alpha_n)}{\chi_0 \cdot \chi_1 \cdot \chi_2 \cdots \chi_{n-1}}.$$

(Here "canonical realization" means "through the injective homomorphism in the above lemma".)

(ii) *In particular, if the Euler–Poincaré characteristics* χ_n *are all the same, say* $\chi_n = \chi$ *for any* n, *then the canonical Euler–Poincaré (pro)characteristic homomorphism* $\chi^{\mathrm{pro}} \colon F^{\mathrm{pro}}(X_\infty) \to \mathbb{Q}$ *is described by*

$$\chi^{\mathrm{pro}}([\alpha_n]) = \frac{\chi(\alpha_n)}{\chi^{n-1}}.$$

In this special case, the target ring $\mathbb{Q}$ *can be replaced by the ring* $\mathbb{Z}[1/\chi]$.

In a more special case, the target ring $\mathbb{Q}$ in the above theorem can be replaced by the Grothendieck ring of varieties.

Let $K_0(\mathcal{V}_{\mathbb{C}})$ be the Grothendieck ring of algebraic varieties, i.e., the free abelian group generated by the isomorphism classes of varieties modulo the subgroup generated by elements of the form $[V]-[V']-[V\setminus V']$ for a closed subset $V'\subset V$ with the ring structure $[V]\cdot[W]:=[V\times W]$. There are distinguished elements in $K_0(\mathcal{V}_{\mathbb{C}})$: $1\!\!1$ is the class $[p]$ of a point p and $\mathbb{L}$ is the Tate class $[\mathbb{C}]$ of the affine line $\mathbb{C}$. From this definition, we can see that any constructible set of a variety determines an element in the Grothendieck ring $K_0(\mathcal{V}_{\mathbb{C}})$. Provisionally the element $[V]$ in the Grothendieck ring $K_0(\mathcal{V}_{\mathbb{C}})$ is called the *Grothendieck "motivic" class of* V and let us denote it by $\Gamma(V)$. Hence we get the following homomorphism, called the *Grothendieck "motivic" class homomorphism*: for any variety X

$$\Gamma\colon F(X)\to K_0(\mathcal{V}_{\mathbb{C}}),$$

which is defined by

$$\Gamma(\alpha)=\sum_{n\in\mathbb{Z}} n\left[\alpha^{-1}(n)\right].$$

Or $\Gamma\left(\sum a_V 1\!\!1_V\right):=\sum a_V[V]$ where V is a constructible set in X and $a_V\in\mathbb{Z}$. From now on, we somtimes write $[\alpha]$ for $\Gamma(\alpha)$ for a constructible function α.

This Grothendieck "motivic" class homomorphism is *tautological* and its more "geometric" one is the Euler–Poincaré characteristic homomorphsim $\chi\colon F(X)\to\mathbb{Z}$. The above theorem is about extending the Euler–Poincaré characteristic homomorphsim $\chi\colon F(X)\to\mathbb{Z}$ to the category of proalgebraic varieties. Thus a very natural problem is to generalize the Grothendieck "motivic" class homomorphism $\Gamma\colon F(X)\to K_0(\mathcal{V}_{\mathbb{C}})$ to the category of proalgebraic varieties. Here one should be a bit careful; the Grothendieck ring $K_0(\mathcal{V}_{\mathbb{C}})$ is not a domain unlike the ring $\mathbb{Z}$ of integers as shown recently by B. Poonen [Po, Theorem 1].

Theorem (7.6). *Let $X_\infty=\varprojlim_{n\in\mathbb{N}}X_n$ be a proalgebraic variety such that each structure morphism $\pi_{n(n+1)}\colon X_{n+1}\to X_n$ satisfies the condition that for each n there exists a $\gamma_n\in K_0(\mathcal{V}_{\mathbb{C}})$ such that $[{\pi_{n(n+1)}}^{-1}(S_n)]=\gamma_n\cdot[S_n]$ for any constructible set $S_n\subset X_n$, for example, $\pi_{n(n+1)}\colon X_{n+1}\to X_n$ is a Zariski locally trivial fiber bundle with fiber variety being F_n (in which case $\gamma_n=[F_n]$).*

(i) *The canonical Grothendieck "motivic" proclass homomorphism,*

$$\Gamma^{\mathrm{pro}}\colon F^{\mathrm{pro}}(X_\infty)\to K_0(\mathcal{V}_{\mathbb{C}})_{\mathcal{G}}$$

is described by

$$\Gamma^{\mathrm{pro}}([\alpha_n]) = \frac{[\alpha_n]}{\gamma_0 \cdot \gamma_1 \cdot \gamma_2 \cdots \gamma_{n-1}}.$$

Here $\gamma_0 := \mathbb{1}$ *and* $K_0(\mathcal{V}_{\mathbb{C}})_{\mathcal{G}}$ *is the localization of* $K_0(\mathcal{V}_{\mathbb{C}})$ *with respect to the multiplicative set consisting of all the finite products of* $\gamma_j^{m_j}$, *i.e,*

$$\mathcal{G} := \left\{\gamma_{j_1}^{m_1}\gamma_{j_2}^{m_2}\cdots\gamma_{j_s}^{m_s} | j_i \in \mathbb{N}, m_i \in \mathbb{N}\right\}.$$

(ii) *In particular, if all the fibers are the same, say* $\gamma_n = \gamma$ *for any* n, *then the canonical Grothendieck "motivic" (ind)class homomorphism*

$$\Gamma^{\mathrm{pro}} \colon F^{\mathrm{pro}}(X_\infty) \to K_0(\mathcal{V}_{\mathbb{C}})_{\mathcal{G}}$$

is described by

$$\Gamma^{\mathrm{pro}}([\alpha_n]) = \frac{[\alpha_n]}{\gamma^{n-1}}.$$

In this special case the quotient ring $K_0(\mathcal{V}_{\mathbb{C}})_{\mathcal{G}}$ *shall be simply denoted by* $K_0(\mathcal{V}_{\mathbb{C}})_\gamma$.

Thus one can see that the so-called motivic measure (e.g., see [Bit], [Cr], [DL1, DL2], [Kon], [Loo], [Ve], etc., and also see [Na]) is a natural and reasonable object from the viewpoint of proconstructible functions. For a more general case when $\pi_{n(n+1)} \colon X_{n+1} \to X_n$ is not necessarily a Zariski locally trivial fiber bundle, see [Y10]. In this sense, our definition of proconstructible function is quite reasonable.

§8. Characteristic classes of proalgebraic varieties

In this section we make a quick review of the author's recent work on characteristic classes of proalgebraic varities (for more details see [Y10, Y11]).

Theorem (7.5) can be extended to a class version c_*^{pro} via the Bivariant Theory, in particular a bivariant Chern class [Br1]. Note that for a morphism $f \colon X \to pt$ from a variety X to a point pt, $\gamma \colon \mathbb{F}(X \to pt) \to \mathbb{H}(X \to pt)$ is nothing but the original MacPherson's Chern class transformation $c_* \colon F(X) \to H_*(X)$.

Theorem (8.1)(Verdier-type Riemann–Roch formula for Chern classes) *For a bivariant constructible function* $\alpha \in \mathbb{F}(X \xrightarrow{f} Y)$ *we* *have the following commutative diagram:*

$$\begin{array}{ccc} F(Y) & \xrightarrow{c_*} & H_*(Y) \\ {\scriptstyle \alpha\bullet_{\mathbb{F}}=\alpha\cdot f^*}\downarrow & & \downarrow{\scriptstyle \gamma(\alpha)\bullet_{\mathbb{H}}} \\ F(X) & \xrightarrow[c_*]{} & H_*(X). \end{array}$$

In particular, for an Euler morphism we have the following diagram:

$$\begin{array}{ccc} F(Y) & \xrightarrow{c_*} & H_*(Y) \\ {\scriptstyle \mathbf{1}_f\bullet_{\mathbb{F}}=f^*}\downarrow & & \downarrow{\scriptstyle \gamma(\mathbf{1}_f)\bullet_{\mathbb{H}}} \\ F(X) & \xrightarrow[c_*]{} & H_*(X). \end{array}$$

(*The homomorphism $\gamma(1\!\!1_f)\bullet_{\mathbb{H}}$ shall be denoted by f^{**}.*)

For example, for a holomorphic submersion $f\colon X \to Y$ of complex varieties one gets $\gamma(1_f)\bullet_{\mathbb{H}} = c(T_f) \cap f^*$, where f^* is the smooth pullback in homology and T_f is the relative tangent bundle of the morphism f.

Using this Verdier–Riemann–Roch for Chern class (also see [FM] and [Sch1]), we can get the following theorem:

Theorem (8.2). *Let $X_\infty = \varprojlim_n X_n$ be a proalgebraic variety such that for each $n < m$ the structure morphism $\pi_{nm}\colon X_m \to X_n$ is an Euler proper morphism (hence surjective) of topologically connected algebraic varieties. Let $H^{\mathrm{pro}}_{**}(X_\infty)$ be the inductive limit of the inductive system $\{\pi^{**}_{nm}\colon H_*(X_n) \to H_*(X_m)\}$. Then there exists a proalgebraic MacPherson's Chern class homomorphism*

$$c^{\mathrm{pro}}_*\colon F^{\mathrm{pro}}(X_\infty) \to H^{\mathrm{pro}}_{**}(X_\infty) \quad \text{defined by} \quad c^{\mathrm{pro}}_*([\alpha_n]) = \rho^n(c_*(\alpha_n)).$$

What we have done so far is the proalgebraic Chern–Schwartz–MacPherson class homomorphism, and our eventual problem is whether one can capture this homomorphism as *a natural transformation* as in the original MacPherson's Chern class transformation.

If the commutative diagram

$$\begin{array}{ccc} Y_m & \xrightarrow{f_m} & X_m \\ {\scriptstyle \rho_{nm}}\downarrow & & \downarrow{\scriptstyle \pi_{nm}} \\ Y_n & \xrightarrow[f_n]{} & X_n \end{array}$$

is a fiber square, then we call the pro-morphism $\{f_n\colon Y_n \to X_n\}$ a *fiber-square pro-morphism*, abusing words.

Theorem (8.3). *Let $\{f_n \colon Y_n \to X_n\}$ be a fiber-square pro-morphism between two pro-algebraic varieties with structure morphisms being Euler morphisms. Then we have the following commutative diagram:*

$$\begin{array}{ccc}
F^{\mathrm{pro}}(Y_\infty) & \xrightarrow{c_*^{\mathrm{pro}}} & H_{**}^{\mathrm{pro}}(Y_\infty) \\
{\scriptstyle f_{\infty *}}\downarrow & & \downarrow{\scriptstyle f_{\infty *}} \\
F^{\mathrm{pro}}(X_\infty) & \xrightarrow[c_*^{\mathrm{pro}}]{} & H_{**}^{\mathrm{pro}}(X_\infty).
\end{array}$$

This can be furthermore generalized. First we introduce the following notion. For a morphism $f \colon X \to Y$ and a bivariant class $b \in \mathbb{B}(X \xrightarrow{f} Y)$, the pair $(f; b)$ is called a *bivariant-class-equipped morphism* and we just express $(f; b) \colon X \to Y$. If a system $\{b_{nm}\}$ of bivariant classes satisfies that

$$b_{nm} \bullet b_{ln} = b_{lm} \quad (l < n < m),$$

then we call the system *a projective system of bivariant classes*, abusing words. If $\{\pi_{nm} \colon X_m \to X_n\}$ and $\{b_{nm}\}$ are projective systems, then the system $\{(\pi_{nm}; b_{nm}) \colon X_m \to X_n\}$ shall be called *a projective system of bivariant-class-equipped morphisms.*

For a bivariant theory $\mathbb{B}$ on the category $\mathcal{C}$ and for a projective system $\{(\pi_{\lambda\mu}; b_{\lambda\mu}) \colon X_\mu \to X_\lambda\}$ of bivariant-class-equipped morphisms, the inductive limit

$$\varinjlim_n \{B_*(X_n),\, b_{nm}\bullet \colon B_*(X_n) \to B_*(X_m)\}$$

shall be denoted by

$$B_*^{\mathrm{pro}}(X_\infty; \{b_{nm}\})$$

emphasizing the projective system $\{b_{nm}\}$ of bivariant classes, because the above inductive limit surely depends on the choice of it. For example, in Theorem (7.4) we have that

$$F^{\mathrm{pro}}(X_\infty) = F_*^{\mathrm{pro}}(X_\infty; \{1\!\!1_{\pi_{nm}}\}).$$

Our more general theorem is the following

Theorem (8.4). (i) *Let $\gamma \colon \mathbb{B} \to \mathbb{B}'$ be a Grothendieck transformation between two bivariant theories $\mathbb{B}$, $\mathbb{B}' \colon \mathcal{C} \to \mathcal{C}'$ and let*

$$\{(\pi_{nm}; b_{nm}) \colon X_m \to X_n)\}$$

be a projective system of bivariant-class-equipped morphisms. Then we get the following pro-version of the natural transformation $\gamma_*\colon B_* \to B'_*$:

$$\gamma_*^{\mathrm{pro}}\colon B_*^{\mathrm{pro}}\big(X_\infty;\{b_{nm}\}\big) \to {B'}_*^{\mathrm{pro}}\big(X_\infty;\{\gamma(b_{nm})\}\big).$$

(ii) *Let* $\{f_n\colon Y_n \to X_n\}$ *be a fiber-square pro-morphism between two projective systems of bivariant-class-equipped morphisms such that* $b_{nm} = f_n^{\bigstar} b_{nm}$. *Then we have the following commutative diagram*:

$$\begin{tikzcd} B_*^{\mathrm{pro}}(Y_\infty) \arrow[r, "\gamma_*^{\mathrm{pro}}"] \arrow[d, "f_{\infty *}"'] & {B'}_*^{\mathrm{pro}}(Y_\infty) \arrow[d, "f_{\infty *}"] \\ B_*^{\mathrm{pro}}(X_\infty) \arrow[r, "\gamma_*^{\mathrm{pro}}"'] & {B'}_*^{\mathrm{pro}}(X_\infty). \end{tikzcd}$$

As remarked in Remark (6.5), the "motivic" characteristic class $T_y\colon K_0(\mathcal{V}/\ \ ) \to H_*(\ \ )\otimes\mathbb{Q}[y]$ can be extended to a Grothendieck transformation of suitable bivariant theories. Therefore it follows from the above general Theorem (8.4) that the "motivic" characteristic class $T_y\colon K_0(\mathcal{V}/\ \ ) \to H_*(\ \ )\otimes\mathbb{Q}[y]$ can be extended in a suitable way to a category of provarieties. More details and some other related work will be done in a different paper.

We hope to do further investigations on (motivic) characteristic classes of proalgebraic varieties and some applications of them. (Also, see recent articles [Alu2, Alu3], [dFLNU], [O1, O2], [PM], [To], [Ve] etc.)

Acknowledgements. The Third Franco-Japanese Symposium and School on Singularities "Singularités en Géométrie et Topologie" was held at the University of Hokkaido during September 13–19, 2004. I would like to thank the organizers, Jean-Paul Brasselet and Tatsuo Suwa, for inviting me to give a talk at the symposium. This paper is an extended version of the talk. I would like to thank Jörg Schürmann and the referee for their comments and suggestions.

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