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Motivic sheaves and intersection cohomology

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We propose a motivic refinement of a result in [BBFGK]. The formulation involves the notion of intersection Chow group, introduced by A. Corti and the author.

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§1. Intersection Chow groups and lifting theorems

We consider quasi-projective varieties over $k = \mathbb{C}$. For a quasi-projective variety Z, $\mathrm{CH}_s(Z)$ denotes the Chow group of s-cycles on Z tensored with $\mathbb{Q}$; if Z is smooth, $\mathrm{CH}^r(Z) := \mathrm{CH}_{\dim Z - r}(Z)$. We consider only constructible sheaves of $\mathbb{Q}$-vector spaces. The singular (co-)homology, Borel-Moore homology, and intersection cohomology are those with $\mathbb{Q}$-coefficients.

Relative canonical filtration.

The study of filtration on the Chow group of a smooth projective variety was started by Bloch and continued by several people; of most relevance to us now are the works of Beilinson, Murre and Shuji Saito. Beilinson explained the filtration in terms of the conjectural framework of mixed motives. Murre proposed a set of conjectures, Murre's conjectures, on a decomposition of the diagonal class in the Chow ring of self-correspondences; he relates the decomposition to the filtration of Chow groups.

For X a smooth projective variety, its Chow group of codimension r cycles $\mathrm{CH}^r(X)$ should have a filtration $F^\bullet$ such that $\mathrm{CH}^r(X) = F^0\,\mathrm{CH}^r(X)$, $F^1\,\mathrm{CH}^r(X)$ is the homologically trivial part, $F^2\,\mathrm{CH}^r(X)$ is perhaps the kernel of Abel-Jacobi map, and so on. The subquotient $Gr_F^\nu\,\mathrm{CH}^r(X)$ should in some way be determined by cohomology $H^{2r-\nu}(X, \mathbb{Q})$.

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A candidate for the filtration was proposed by S. Saito, see [Sa 1] [Sa 2]. We extend his definition as follows. If $S = \operatorname{Spec} k$, it coincides with Saito's filtration.

Let S be a quasi-projective variety, and X a smooth variety with a projective map $p\colon X \to S$. For another smooth variety W with a projective map $q\colon W \to S$, an element $\Gamma \in \mathrm{CH}_{\dim X - s}(W \times_S X)$ induces a map $\Gamma_*\colon \mathrm{CH}^{r-s}(W) \to \mathrm{CH}^r(X)$, see [CH]. The cycle class of Γ in Borel-Moore homology gives a map $\Gamma_*\colon Rq_*\mathbb{Q}_W[-2s] \to Rp_*\mathbb{Q}_X$; passing to perverse cohomology one has a map (for each ν)

$$^p\mathcal{H}^{2r-\nu}\Gamma_*\colon {}^p\mathcal{H}^{2r-2s-\nu}Rq_*\mathbb{Q}_W \to {}^p\mathcal{H}^{2r-\nu}Rp_*\mathbb{Q}_X.$$

(Here $^p\mathcal{H}^*$ stands for perverse cohomology.)

We define a filtration $F_S^\bullet$ on $\mathrm{CH}^r(X)$ as follows. Let $\mathrm{CH}^r(X) = F_S^{-\dim S}\,\mathrm{CH}^r(X)$. Assume F_S^ν has been defined. Define

$$F_S^{\nu+1}\,\mathrm{CH}^r(X) := \sum \mathrm{Image}[\Gamma_*\colon F_S^\nu\,\mathrm{CH}^{r-s}(W) \to \mathrm{CH}^r(X)\,]$$

where the sum is over $(q\colon W \to S,\ \Gamma \in \mathrm{CH}_{\dim X - s}(W \times_S X))$ satisfying the following condition: the map $^p\mathcal{H}^{2r-\nu}\Gamma_*\colon {}^p\mathcal{H}^{2r-2s-\nu}Rq_*\mathbb{Q}_W \to {}^p\mathcal{H}^{2r-\nu}Rp_*\mathbb{Q}_X$ is zero. One can show:

Proposition 1.1. *The filtration $F_S^\bullet$ on $\mathrm{CH}^r(X)$ has the following properties.*

(1) $\mathrm{CH}^r(X) = F_S^{-\dim S}\,\mathrm{CH}^r(X)$. *For any $\Gamma \in \mathrm{CH}_{\dim X - s}(W \times_S X)$, the induced map $\Gamma_*\colon \mathrm{CH}^{r-s}(W) \to \mathrm{CH}^r(X)$ respects $F_S^\bullet$.*

(2) *If $^p\mathcal{H}^{2r-\nu}\Gamma_*\colon {}^p\mathcal{H}^{2r-2s-\nu}Rq_*\mathbb{Q}_W \to {}^p\mathcal{H}^{2r-\nu}Rp_*\mathbb{Q}_X$ is zero, then Γ_* sends $F_S^\nu\,\mathrm{CH}^{r-s}(W)$ to $F_S^{\nu+1}\,\mathrm{CH}^r(X)$.*

(3) *The filtration is the smallest one with properties* (1) *and* (2).

Intersection Chow group.

We refer to a forthcoming paper with A. Corti for details on intersection Chow groups.

Let S be a quasi-projective variety, X a smooth variety, and $p\colon X \to S$ a projective map. There is an algebraic Whitney stratification

$$S = S_0 \supset S_1 \supset \cdots \supset S_\alpha \supset \cdots \supset S_{\dim S}$$

of S, so that $S_\alpha - S_{\alpha+1}$ is smooth of codimension α, satisfying the following condition.

(i) p is smooth projective over $S^0 := S - S_1$, and

(ii) there is an algebraic stratification of X such that p is a stratified fiber bundle over each stratum $S_\alpha^0 := S_\alpha - S_{\alpha+1}$.

We then say $p\colon X \to S$ is a stratified map with respect to $\{S_\alpha\}$. The stratification can be chosen to satisfy a stronger condition as follows.

Let $X_\alpha = p^{-1}S_\alpha$. There exist resolutions $\tilde{X}_\alpha \to X_\alpha$ (with $\tilde{X}_0 = X$) such that

(i) the induced map $\tilde{p}_\alpha \colon \tilde{X}_\alpha \to S_\alpha$ is smooth over S_α^0, and

(ii) there is a stratificaton on $\tilde{X}_\alpha$ such that $\tilde{p}_\alpha$ is a stratified fiber bundle over S_β^0 for $\beta \geq \alpha$. (In other words, $\tilde{p}_\alpha$ is a stratified map with respect to $\{S_\beta\}_{\beta\geq\alpha}$.)

In this case we say the data $(p\colon X \to S, \{\tilde{X}_\alpha \to X_\alpha\})$ is stratified with respect to $\{S_\alpha\}$.

$$\begin{array}{ccc} & & \tilde{X}_\alpha \\ & \swarrow & \downarrow \\ X & \hookleftarrow & X_\alpha \\ {\scriptstyle p}\downarrow & & \downarrow \\ S & \hookleftarrow & S_\alpha \end{array}$$

Let $\iota_\alpha \colon \tilde{X}_\alpha \to X$ be the induced map.

We now restrict ourselves to the birational case: let S be a quasi-projective variety and $p\colon X \to S$ a resolution of singularities. One has maps ($d = \dim S$)

$$\mathrm{CH}_{d-r}(\tilde{X}_\alpha) \xrightarrow{\iota_{\alpha *}} \mathrm{CH}^r(X) \xrightarrow{\iota_\alpha{}^*} \mathrm{CH}^r(\tilde{X}_\alpha)$$

Each group has filtration $F_S^\bullet$.

Define the *intersection Chow group* as a subquotient of the Chow group of X given by:

$$\mathrm{ICH}^r(S) := \frac{\bigcap_{\alpha\geq 1}(\iota_\alpha^*)^{-1} F_S^{2r-d+1}\,\mathrm{CH}^r(\tilde{X}_\alpha)}{\sum_{\alpha\geq 1} \iota_{\alpha *} F_S^{2r-d+1}\,\mathrm{CH}_{d-r}(\tilde{X}_\alpha)}$$

Theorem 1.2. $\mathrm{ICH}^r(S)$ *is well-defined (independent of the choice of stratification and resolution).*

Denote by $IH^i(S)$ the intersection cohomology with middle perversity and with $\mathbb{Q}$-coefficients.

Proposition 1.3. *There is a natural map*

$$\mathrm{ICH}^r(S) \to IH^{2r}(S).$$

The Conjectures.

We recall three well-known conjectures concerning cohomology, Chow group, and higher Chow group of a smooth projective variety over a field. In this paper we refer to them as Conjectures. The addition of the third conjecture is needed to prove the existence of a t-structure on the triangulated category of mixed motives. See [Ha].

1. Grothendieck's Standard conjecture.

This concerns the functorial behavior of cycle classes in (singular or étale) cohomology. It has two components, the Lefschetz type conjecture and the Hodge type conjecture. For $k = \mathbb{C}$, the latter holds true (Hodge index theorem). The Lefschetz type conjecture itself consists of three statements, Conjecture (A), (B) and (C). Conjecture (C) says: the Künneth components of the diagonal class of a smooth projective variety are algebraic.

The standard conjecture implies the semi-simplicity of the category of pure homological motives (Grothendieck).

2. Murre's conjecture (Bloch-Beilinson-Murre conjecture)

One of the formulation of the conjectural filtration on Chow group is due to Murre, and stated as the existence of a orthogonal decomposition to projectors of the diagonal class Δ_X in $\mathrm{CH}(X \times X)$. To be precise, the conjecture states:

(A) Let X be a smooth projective variety. There exists a decomposition $\Delta_X = \sum \Pi^i$ to orthogonal projectors in the Chow ring such that the cohomology class of Π^i is the Künneth component $\Delta(2\dim X - i,\, i)$. The decomposition is called the *Chow-Künneth decomposition.*

(B) Π^i with $i = 0, \dots, r-1$ or $i = 2d, \dots, 2r+1$ acts as zero on $\mathrm{CH}^r(X)$.

(C) Put $F^0 = \mathrm{CH}^r(X)$, $F^1 = \mathrm{Ker}\,\Pi^{2r}$, $F^2 = \mathrm{Ker}(\Pi^{2r-1}|F^1), \dots,$ $F^r = \mathrm{Ker}(\Pi^{r+1}|F^{r-1})$, $F^{r+1} = 0$. This is independent of the choice of the decomposition in (A).

(D) $F^1 = \mathrm{CH}^r(X)_{\mathrm{hom}}$, the homologically trivial part.

Note a Chow-Künneth decomposition gives a decomposition in the category of Chow motives over k $h(X) = \bigoplus h^i(X)$, where $h^i(X)$ carries cohomology in degree i only. For the category of Chow motives, see §2.

3. Variant of Beilinson-Soulé vanishing conjecture: Let $(X, 0, P)$ be an object of the category of Chow motives $\mathcal{CHM}(k)$ whose realization is of cohomological degree $\geq 2r - n$ if $n > 0$ and $> 2r$ if $n = 0$. Then $P_* \mathrm{CH}^r(X, n) = 0$.

When we give results that hold under the three Conjectures, we will always say so; some of them require only the first two. For example,

Proposition 1.4 (Under Conjectures). *$F_S^\nu \mathrm{CH}^r(X) = 0$ for ν large enough.*

We have:

Theorem 1.5 (Under Conjectures). *The map*

$$p_*\colon \mathrm{CH}^r(X) \to \mathrm{CH}_{d-r}(S)$$

induces a surjective map $\mathrm{ICH}^r(S) \to \mathrm{CH}_{d-r}(S)$.

Under Conjectures, one has (1.5), which immediately implies the following Theorem (1.6) in [BBFGK]. One has the cycle class map $cl\colon \mathrm{CH}_{d-r}(S) \to H^{BM}_{2(d-r)}(S)$ (the latter is the Borel-Moore homology). There is a natural map $IH^{2r}(S) \to H^{BM}_{2(d-r)}(S)$.

Theorem 1.6. *For any* $z \in \mathrm{CH}_{d-r}(S)$, *its class* $cl(z) \in H^{BM}_{2(d-r)}(S)$ *can be (non-canonically) lifted to an element of intersection cohomology.*

Indeed, we can show (1.6) without assuming Conjectures, but still using the same ideas as for the proof of (1.5).

§2. Motivic categories and decompositions of motives

Theory of Chow motives.

Let S be a quasi-projective variety over $k = \mathbb{C}$. Let $\mathcal{CHM}(S)$ be the pseudo-abelian category of Chow motives over S. It has the following properties (for details see [CH]).

- An object of $\mathcal{CHM}(S)$ is of the form

$$(X, r, P) = (X/S, r, P)$$

where X is a smooth variety over k with a projective (not necessarily smooth) map $p\colon X \to S$, $r \in \mathbb{Z}$, and if X has connected components X_i,

$$P \in \textstyle\bigoplus_i \mathrm{CH}_{\dim X_i}(X \times_S X_i)$$

such that $P \circ P = P$. Here $\circ$ denotes composition of relative correspondences defined in [CH], which makes $\bigoplus_i \mathrm{CH}_{\dim X_i}(X \times_S X_i)$ a ring with the diagonal Δ_X as the identity element. If (Y, s, Q) is another object, Y_j the components of Y, then

$$\mathrm{Hom}((X, r, P), (Y, s, Q)) = Q \circ (\textstyle\bigoplus_j \mathrm{CH}_{\dim Y_j - s + r}(X \times_S Y_j)) \circ P.$$

Composition of morphisms is induced from the composition of relative correspondences.

- Let $h(X/S) = (X, 0, \mathrm{ip})$ and $h(X/S)(r) = (X, r, \mathrm{ip})$. More generally, Tate twist is defined to be the functor ($t \in \mathbb{Z}$)

$$K = (X, r, P) \mapsto K(t) = (X, r+t, P)$$

on objects.

- One has a functor

$$\mathrm{CH}^t\colon \mathit{CHM}(S) \to Vect_{\mathbb{Q}}, \quad \mathrm{CH}^t((X, r, P)) = P_* \mathrm{CH}^{r+t}(X).$$

Note $\mathrm{CH}^t(K) = \mathrm{CH}^0(K(t))$ and $\mathrm{CH}^r(h(X/S)) = \mathrm{CH}^0(h(X/S)(r)) = \mathrm{CH}^r(X)$.

- If X and Y are smooth varieties with projective maps to S and $f\colon X \to Y$ a map over S, there corresponds a morphism

$$f^*\colon h(Y/S) \to h(X/S).$$

If X, Y are equidimensional, there corresponds

$$f_*\colon h(X/S) \to h(Y/S)(\dim Y - \dim X).$$

It is of use to define the *homological motive* of X/S: if X has components X_i,

$$h'(X/S) := \bigoplus h(X_i/S)(\dim X_i).$$

Then a map $f\colon X \to Y$ induces a morphism $f_*\colon h'(X/S) \to h'(Y/S)$.

- Let $D^b_c(S) = D^b_c(S, \mathbb{Q})$ be the derived category of sheaves of $\mathbb{Q}$-vector spaces on S with constructible cohomology. There is the realization functor

$$\rho\colon \mathit{CHM}(S) \to D^b_c(S)$$

such that on objects

$$(X, r, P) \mapsto P_* Rp_* \mathbb{Q}_X[2r],$$

($P_* \in \mathrm{End}_{D^b_c(S)}(Rp_*\mathbb{Q}_X)$ is a projector, and $P_*Rp_*\mathbb{Q}_X$ is its image, which exists since $D^b_c(S)$ is pseudo-abelian.) Note $\rho(h(X/S)(r)) = Rp_*\mathbb{Q}_X[2r]$ and

$$\rho(h'(X/S)(r)) = Rp_* D_X[2r],$$

where D_X is the dualizing complex of X. Recall $D_X = \mathbb{Q}_X[2\dim X]$ if X is smooth.

Theory of Grothendieck motives.

We also have the pseudo-abelian category of Grothendieck motives over S. The main properties are the following.

Denote by $\mathrm{Perv}(S)$ be the abelian category of perverse sheaves of $\mathbb{Q}$-vector spaces on S. There is a canonical full functor cano: $\mathit{CHM}(S) \to$

$\mathcal{M}(S)$ and a faithful realization functor $\rho\colon \mathcal{M}(S) \to \mathrm{Perv}(S)$. The following diagram commutes.

$$\begin{array}{ccc} CH\mathcal{M}(S) & \xrightarrow{\text{cano}} & \mathcal{M}(S) \\ \rho\downarrow & & \downarrow\rho \\ D^b_c(S) & \xrightarrow{{}^p\mathcal{H}^*} & Perv(S) \end{array}$$

Here ${}^p\mathcal{H}^* = \bigoplus_i {}^p\mathcal{H}^i$ is the total perverse cohomology functor.

Relative decomposition of motives.

The following is in [CH] (for this, we only need the first two of the three Conjectures). This is a motivic analogue of the decomposition theorem for the total direct image in [BBD].

Theorem 2.1 (Under Conjectures). *Let $p\colon X \to S$ be as before. Let $\{S_\alpha\}$ be a Whitney stratification of S, and $\tilde{X}_\alpha \to X_\alpha$ resolutions such that ($p\colon X \to S$, $\{\tilde{X}_\alpha \to X_\alpha\}$) is stratified with respect to $\{S_\alpha\}$. Then:*

(1) *There are local systems $\mathcal{V}^j_\alpha$ on $S_\alpha - S_{\alpha+1}$, non-canonical direct sum decomposition in $CH\mathcal{M}(S)$*

$$h(X/S) = \bigoplus_{j,\,\alpha} h^j_\alpha(X/S)$$

and isomorphisms

$$\rho(h^j_\alpha(X/S)) \cong IC_{S_\alpha}(\mathcal{V}^j_\alpha)[-j + \dim S_\alpha]$$

in $D^b_c(S)$.

(2) *For each i, the sum $\bigoplus_{j\le i,\,\alpha} h^j_\alpha(X/S)$ is a well-defined subobject of $h(X/S)$ (independent of the decomposition).*

(3) *The category $\mathcal{M}(S)$ is semi-simple abelian, and the functor $\rho\colon \mathcal{M}(S) \to \mathrm{Perv}(S)$ is exact and faithful.*

Relative canonical filtration and motives.

For a projective map $p\colon X \to S$ with X smooth, the filtration on $\mathrm{CH}^r(X)$ can be interpreted in terms of motives as follows. Keeping the notation in the above theorem, define subobjects of $h(X/S)$ by

$${}^p\tau_{\le i} h(X/S) := \bigoplus_{j\le i,\,\alpha} h^j_\alpha(X/S)$$

(the sum over (j, α) with $j \le i$) and subquotients

$${}^p\mathcal{H}^i h(X/S) := \bigoplus_\alpha h^i_\alpha(X/S).$$

More generally for $r \in \mathbb{Z}$, subobjects of $h(X/S)(r)$

$${}^p\tau_{\le i}\big(h(X/S)(r)\big) := \bigoplus_{j\le i+2r,\,\alpha} h^j_\alpha(X/S)(r)$$

and subquotients

$$^p\mathcal{H}^i\big(h(X/S)(r)\big) := \bigoplus_\alpha h_\alpha^{i+2r}(X/S)(r)$$

are defined. Then we have

$$\begin{aligned}\mathrm{CH}^r(X) &= \mathrm{CH}^0(h(X/S)(r)) \\ &= \mathrm{CH}^0(\textstyle\bigoplus_{\alpha,\,\nu} h_\alpha^{2r-\nu}(X/S)(r)), \\ F_S^\nu \mathrm{CH}^r(X) &= \mathrm{CH}^0\big({}^p\tau_{\leq -\nu}(h(X/S)(r))\big) \\ &= \mathrm{CH}^0\big(\textstyle\bigoplus_{\mu\leq 2r-\nu,\,\alpha} h_\alpha^\mu(X/S)(r)\big),\end{aligned}$$

and

$$\begin{aligned}\mathrm{Gr}_{F_S}^\nu \mathrm{CH}^r(X) &= \mathrm{CH}^0\big({}^p\mathcal{H}^{-\nu}(h(X/S)(r))\big) \\ &= \mathrm{CH}^0\big(\textstyle\bigoplus h_\alpha^{2r-\nu}(X/S)(r)\big).\end{aligned}$$

§3. Outline of the proof of (1.5)

We start with a result on perverse cohomology. Let X be smooth, $p\colon X \to S$ a projective map, and assume $(p\colon X \to S, \{\tilde{X}_\alpha \to X_\alpha\})$ is stratified with respect to $\{S_\alpha\}$. There are local systems $\mathcal{V}_\alpha^i$ on S_α^0 such that $Rp_*\mathbb{Q}_X \cong \bigoplus IC_{S_\alpha}(\mathcal{V}_\alpha^j)[-j + \dim S_\alpha]$. Let $d = \dim X$.

Proposition 3.1. (1) *Let $\iota_\alpha^*\colon Rp_*\mathbb{Q} \to i_{\alpha *}R\tilde{p}_{\alpha *}\mathbb{Q}_{\tilde{X}_\alpha}$ be the natural map ι_α induces, and*

$${}^p\mathcal{H}^i(\iota_\alpha^*)\colon {}^pR^ip_*\mathbb{Q} \to i_{\alpha *}{}^pR^i\tilde{p}_{\alpha *}\mathbb{Q}_{\tilde{X}_\alpha}$$

the induced map on perverse cohomology of degree i. The restriction to the direct summand $IC_{S_\alpha}(\mathcal{V}_\alpha^i)[\dim S_\alpha]$

$${}^p\mathcal{H}^i(\iota_\alpha^*)\colon IC_{S_\alpha}(\mathcal{V}_\alpha^i)[\dim S_\alpha] \to i_{\alpha *}{}^pR^i\tilde{p}_{\alpha *}\mathbb{Q}_{\tilde{X}_\alpha}$$

is a split injection.

(2) *Let $\iota_{\alpha *}\colon \iota_{\alpha *}R\tilde{p}_{\alpha *}D_{\tilde{X}_\alpha}(-d)[-2d] \to Rp_*\mathbb{Q}$ be the natural map, and*

$${}^p\mathcal{H}^i\iota_{\alpha *}\colon \iota_{\alpha *}{}^p\mathcal{H}^iR\tilde{p}_{\alpha *}D_{\tilde{X}_\alpha}(-d)[-2d] \to {}^pR^ip_*\mathbb{Q}$$

the induced map on perverse cohomology; here $D_{\tilde{X}_\alpha}$ is the dualizing complex. This map factors through a split surjection

$${}^p\mathcal{H}^i\iota_{\alpha *}\colon \iota_{\alpha *}{}^p\mathcal{H}^iR\tilde{p}_{\alpha *}D_{\tilde{X}_\alpha}(-d)[-2d] \to IC_{S_\alpha}(\mathcal{V}_\alpha^i)[\dim S_\alpha]$$

to the direct summand of the target.

We can extend the definition of the filtration $F_S^\bullet$ as follows. For any quasi-projective (possibly singular) variety Z with a quasi-projective map to S, one can define a filtration $F_S^\bullet$ on the Chow group $\mathrm{CH}_s(Z)$. This was done in [CH, §5] in the case $S = \operatorname{Spec} k$, and the general case is similar. For a projective map of varieties over S, $f\colon X \to Y$, the induced map $f_*\colon \mathrm{CH}_s(X) \to \mathrm{CH}_s(Y)$ respects the filtrations $F_S^\bullet$. If $S \to S'$ is a closed immersion, and $Z \to S$, then the filtrations $F_S^\bullet$ and $F_{S'}^\bullet$ on $\mathrm{CH}_s(Z)$ coincide.

For a quasi-projective variety T, viewing it as a variety over T, one has filtration $F_T^\bullet$ on $\mathrm{CH}_s(T)$. For this filtration, one has the following result. The proof uses the triangulated category of mixed motives over a base, the perverse t-structure on it, and the interpretation of the filtration $F_S^\bullet$ on $\mathrm{CH}_s(Z)$ in terms of the perverse truncation (similar to the interpretation in §2). See [Ha] for the case where the base is $\operatorname{Spec} k$.

Lemma 3.2 (Under Conjectures). *For an irreducible quasi-projective variety T, $F_T^{-2s+\dim T+1}\,\mathrm{CH}_s(T) = 0$.*

From now on we assume the Conjectures throughout.

Let $p\colon X \to S$ be a desingularization. We have a decomposition $h(X/S) = \bigoplus h_\alpha^j(X/S)$ as in (2.1). In this case $h_0^\nu = 0$ for $\nu \neq d$, and it can be shown $CH^r(h_0^d) = \mathrm{ICH}^r(S)$ as a subquotient of $\mathrm{CH}^r(X)$.

Lemma (3.2) implies that $p_*\colon \mathrm{CH}^r(X) \to \mathrm{CH}_{d-r}(S)$ passes to a map $\mathrm{ICH}^r(S) \to \mathrm{CH}_{d-r}(S)$.

For the surjectivity we must show: For any $a \in \mathrm{CH}_{d-r}(S)$, there is an element $b \in \mathrm{CH}^r(X)$ such that

(i) $p_*(b) = a$, and

(ii) $\iota_\alpha^*(b) \in F_S^{2r-d+1}\,\mathrm{CH}^r(\tilde{X}_\alpha)$ for each $\alpha \geq 1$.

Let $a \in \mathrm{CH}_{d-r}(S)$ and $\nu \leq 2r-d+1$. Consider the following Claim $(I)_\nu$.

Claim $(I)_\nu$.

(1) (Case $\nu \leq 2r-d$) there is an element $b^\nu \in \mathrm{CH}^r(X)$ with (i) $p_*(b^\nu) = a$, and (ii) $b^\nu \in F_S^\nu\,\mathrm{CH}^r(X)$.

(2) (Case $\nu = 2r-d+1$) there is an element $b^{2r-d+1} \in \mathrm{CH}^r(X)$ satisfying the following (let $b = b^{2r-d+1}$ for short): (i) $p_*(b) = a$, and (ii) $b \in F_S^{2r-d}\,\mathrm{CH}^r(X)$ (not $F_S^{2r-d+1}\,\mathrm{CH}^r(X)$!), and $b \mod F_S^{2r-d+1} \in \mathrm{Gr}_{F_S}^{2r-d}\,\mathrm{CH}^r(X) = \bigoplus_{\alpha\geq 0}\mathrm{CH}^r(h_\alpha^d(X/S))$ is contained in the first summand $ICH^r(S) = \mathrm{CH}^r(h_0^d(X/S))$.

For ν small enough $(I)_\nu$ obviously holds: one can take any element satisfying (i). The larger ν is, the stronger $(I)_\nu$ is. What we must show is $(I)_{2r-d+1}$.

Proposition 3.3. *Let $\nu \leq 2r-d$. We have $(I)_\nu \Rightarrow (I)_{\nu+1}$.*

The proof of Proposition (3.3) is achieved by an argument that uses Proposition (3.1), the motivic interpretation of the filtration in §2, the two Lemmas (3.2) and (3.4), and semi-simplicity of the category $\mathcal{M}(S)$.

Lemma 3.4. *If $\nu < 2r - 2\dim \tilde{X}_\alpha + \dim S_\alpha$, then $h_\alpha^{2r-\nu}(X/S)$ is zero.*

Indeed using (3.1) one shows the realization of $h_\alpha^{2r-\nu}(X/S)$ is zero. Since $\rho\colon \mathcal{M}(S) \to \mathrm{Perv}(S)$ is exact and faithful, it follows $h_\alpha^{2r-\nu}(X/S)$ itself is zero.

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