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Moduli of regular holonomic $\mathcal{D}_X$-modules with natural parabolic stability

Nitin Nitsure

Dedicated to Professor Masaki Maruyama on his 60th birthday

Abstract.

In this paper we re-visit the moduli problem for regular holonomic $\mathcal{D}$-modules with normal crossing singularities, and give a new definition of semistability, more general than the older notion, construct the moduli scheme, and describe its points. Independently of this, we introduce another natural parabolic notion of stability for such $\mathcal{D}$-modules, and construct the moduli in a special case.

§1. Introduction

Let X be a smooth complex projective variety. In his path-breaking paper [13], Simpson constructed a moduli scheme for pairs (E, ∇) consisting of an algebraic vector bundle E on X with an algebraic integrable connection ∇. These objects (E, ∇) are exactly the $\mathcal{O}_X$-coherent regular holonomic $\mathcal{D}_X$-modules, where $\mathcal{D}_X$ is the sheaf of linear partial differential operators on X in the algebraic category. Using the methods introduced by Simpson, the present author embarked on a program to solve the moduli problem for general regular holonomic $\mathcal{D}_X$-modules on X. To begin with, the general case differs from the case considered by Simpson in the following two important respects.

(1) Firstly, such a $\mathcal{D}_X$-module need not be $\mathcal{O}_X$-coherent, so we have to replace it by an $\mathcal{O}_X$-coherent description in order to apply the usual Hilbert schemes and GIT-based machinery of moduli theory.

(2) Secondly, the resulting $\mathcal{O}_X$-coherent objects (which now replace the original $\mathcal{D}_X$-modules) need to have a reasonable intrinsic notion of stability (or semi-stability) defined on them, which will correspond to GIT

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stability (or semi-stability) in the course of moduli construction. Simpson could do without explicitly introducing any notion of semi-stability in the case of an integrable connection (E, ∇), as the Hilbert polynomial of the underlying vector bundle E automatically equals that of a trivial vector bundle of the same rank.

In a sequence of three papers [10], [12] (jointly with Claude Sabbah) and [11], the above programme was carried out for regular holonomic $\mathcal{D}_X$-modules with singularities over a normal crossing divisor Y in X. We fix the normal-crossing divisor Y in X, and take the resulting stratification of X by locally closed smooth subvarieties $Z_i \subset X$ defined as the locus where exactly $d - i$ branches of Y intersect for $i < d$, and with $Z_d = X - Y$. This gives rise to a conical Lagrangian subvariety $C_{Y,X}$ of T^*X which is the union of the co-normal bundles $N^*_{Z_i,X} \subset T^*X$ over $i \leq d$. We require that the characteristic variety $car(M)$ of our regular holonomic $\mathcal{D}_X$-modules M should be contained in $C_{Y,X}$. A moduli for the corresponding perverse sheaves (which are exactly those perverse sheaves on X which are cohomologically constructible with respect to the stratification Z_i) was also constructed in the above papers, and the Riemann-Hilbert correspondence was shown to define an analytic morphism at the level of moduli.

However, the results so far had the following drawbacks.

(A) Instead of a moduli for the $\mathcal{D}_X$-modules themselves, the above constructions give a moduli for so-called 'pre-$\mathcal{D}$-modules' which are $\mathcal{O}_X$-coherent avatars of $\mathcal{D}_X$-modules (can be thought of as $\mathcal{D}_X$-modules with level structures). For example, instead of meromorphic connections we make a moduli for logarithmic connections. There is more than one pre-$\mathcal{D}$-module structure on any $\mathcal{D}$-module.

(B) The semistability condition on the pre-$\mathcal{D}$-modules in [12] and [11] is too restrictive, which though an open condition, leaves out a large class of $\mathcal{D}$-modules (which should ideally form some other irreducible components in a larger moduli).

(C) The link between meromorphic connections and Higgs bundles, established by Biquard [2] as a generalisation of the Narasimhan-Seshadri-Donaldson-Hitchin-Corlette-Simpson correspondence is neglected in the earlier moduli construction. Recall that the theorem of Biquard [2] makes a correspondence between parabolic stable logarithmic connections and parabolic stable Higgs bundles, when singular set Y is a smooth divisor.

The present article presents an improved moduli construction, to take care of the above points. We succeed in fully overcoming drawbacks (A) and (B) of the earlier constructions, and partly overcome (C).

The improvement in points (A) and (B) is made by paying attention to the relationship between the topology of the normal bundles of the components of the divisor on one hand and residual eigenvalues on the other hand. This has nothing to do with parabolic stability. Independently of this, to take care of point (C) we present another variation on the moduli construction for regular meromorphic connections via a natural parabolic definition of semi-stability for Deligne lattices (which explains the title of this article).

This article is arranged as follows. In Section 2, we set up preliminaries involving $\mathcal{D}$-modules and some topological properties.

In Section 3, we define the natural parabolic structure on a Deligne connection (which is the logarithmic connection corresponding to a regular meromorphic connection with real parts of residual eigenvalues in $[0,1)$, which was constructed by Deligne in early 1970's), and in section 4 we give the construction of a moduli for parabolic stable Deligne connections. Our construction presently works only under an extra assumption on the logarithmic tangent bundle of (X, Y), which we expect to remove in the future.

In Section 5, we treat the case of regular holonomic $\mathcal{D}$-modules whose singularity locus Y is a smooth divisor. We show how to improve our earlier constructions so as to overcome the limitations (A) and (B) discussed above. In Section 6, this is done in the general case where Y is normal crossing.

§2. Preliminaries

$\mathcal{D}$-modules

Let X be a nonsingular complex projective variety, with $\mathcal{D}_X$ the sheaf of linear partial differential operators acting on $\mathcal{O}_X$ (in the algebraic category).

Recall that $\mathcal{D}_X$ is generated as a sheaf of $\mathbb{C}$-algebras by $\mathcal{O}_X$ and T_X where $\mathcal{O}_X$ are the scalar operators on $\mathcal{O}_X$ while the tangent sheaf T_X acts on $\mathcal{O}_X$ by differentiation. We have the relations $\xi f - f\xi = \xi(f)$ and $\xi\eta - \eta\xi = [\xi, \eta]$ for $f \in \mathcal{O}_X$ and $\xi, \eta \in T_X$. The inclusion $\mathcal{O}_X \subset \mathcal{D}_X$ makes $\mathcal{D}_X$ a left-right $\mathcal{O}_X$-bimodule. For any $i \geq 0$, let $F^i\mathcal{D}_X \subset \mathcal{D}_X$

be the left $\mathcal{O}_X$-submodule generated by the image of $\otimes^i_{\mathbb{C}} T_X \to \mathcal{D}_X$. Then each $F^i\mathcal{D}_X$ a left-right sub-$\mathcal{O}_X$-bimodule, which is bi-coherent as an $\mathcal{O}_X$-module. We have $F^0\mathcal{D}_X = \mathcal{O}_X$, $F^1\mathcal{D}_X = \mathcal{O}_X \oplus T_X$ as left-$\mathcal{O}_X$-module, $F^i\mathcal{D}_X \cdot F^j\mathcal{D}_X = F^{i+j}\mathcal{D}_X$, and $\bigcup_{i\geq 0} F^j\mathcal{D}_X = \mathcal{D}_X$. Moreover, $[F^i\mathcal{D}_X, F^j\mathcal{D}_X] \subset F^{i+j-1}\mathcal{D}_X$, and the associated graded object is the graded $\mathcal{O}_X$-algebra $Sym^{\bullet}_{\mathcal{O}_X} T_X$.

By the phrase '$\mathcal{D}_X$-module' we will mean a left-$\mathcal{D}_X$-module which is $\mathcal{O}_X$-quasi-coherent, unless otherwise indicated.

A $\mathcal{D}_X$-module M is $\mathcal{D}_X$-coherent if and only if locally there exists a filtration F^iM by $\mathcal{O}_X$-coherent $\mathcal{O}_X$-submodules which is F^iD_X-good: $F^i\mathcal{D}_X F^jM \subset F^{i+j}M$, $F^iM = 0$ for $i << 0$ and locally $\exists\, k$ such that $F^i\mathcal{D}_X F^kM = F^{k+i}M$ for all $i \geq 0$.

The **characteristic variety** $car(M) \subset T^*X$ is the set-theoretic support of $gr(M)$ as a $Sym_{\mathcal{O}_X} T_X$-module. This is well-defined.

A $\mathcal{D}_X$-coherent module M is said to be **holonomic** if $dim(car(M)) = dim(X)$ or if $M = 0$, and **regular holonomic** if moreover local filtrations can be so chosen that $car(M)$ is a reduced subscheme.

Universal degree of line bundles

Definition 2.1. Let X be a path-connected topological space, and L a complex line bundle on X, with first Chern class $c_1(L) \in H^2(X;\mathbb{Z})$. Then we will call the non-negative integer d which generates the image of the group homomorphism $c_1(L) \cap - : H_2(X;\mathbb{Z}) \to H_0(X;\mathbb{Z}) = \mathbb{Z}$ as the **universal topological degree** of L.

The following lemma explains the above terminology.

Lemma 2.2. *The universal topological degree of L as defined above is the greatest common divisor of the degrees of all pull-backs of L to connected oriented compact hausdorff topological 2-manifolds under continuous maps.*

The above follows immediately from the following elementary lemma.

Lemma 2.3. *Let X be a topological space, and let $c \in H_2(X)$ be a singular cohomology element with coefficients $\mathbb{Z}$. Then there exist oriented connected compact hausdorff topological manifolds $Y_1, \ldots, Y_n$ and continuous maps $f_i : Y_i \to X$ such that $c = \sum_i f_{i*}[Y_i]$ where $[Y_i] \in H_2(Y_i)$ denotes the fundamental class of Y_i.* □

Lemma 2.4. *(1) Let S be a hausdorff, path connected topological space, and let N be a complex line bundle on S. Let $x \in S$, and let*

$\widehat{x} \in N_x - 0$ where N_x is the fiber of N over x. Let $N - S$ denote the complement of the zero section of N. Let $\tau \in \pi_1(N - S, \widehat{x})$ denote the image of the positive generator of $\pi_1(N_x - 0, \widehat{x})$ under the homomorphism induced by inclusion $N_x - 0 \hookrightarrow N - S$. Then τ lies in the center of the group $\pi_1(N - S, \widehat{x})$.

(2) Let C be a compact real oriented surface of genus $g \geq 0$, and let N be a complex line bundle on C of degree d. Let $x_0 \in C$, and let $\widehat{x_0}$ be a non-zero point in the fiber of N over x_0. If $d = 0$ then N is topologically trivial, and so $\pi_1(N - C, \widehat{x_0}) \cong \pi_1(C, x_0) \times \pi_1(\mathbb{C}^\times)$. More generally for arbitrary d, the fundamental group $\pi_1(N - C, \widehat{x_0})$ has the following description in terms of generators and relations. Let $\tau \in \pi_1(N - C, \widehat{x_0})$ denote the positive loop in $N_{x_0} - 0$ with base point $\widehat{x_0}$. Let $x_1 \neq x_0$ be another point on C, so that the fundamental group $\pi_1(C - x_1, x_0)$ is the free group $F\langle a_i, b_i\rangle$ on certain generators a_i, b_i with $1 \leq i \leq g$ where $g \geq 0$ is the genus of C, and $\pi_1(C, x_0)$ is the quotient of $F\langle a_i, b_i\rangle$ by the relation $\prod_i a_i b_i a_i^{-1} b_i^{-1} = 1$. Then $\pi_1(N - C, \widehat{x_0})$ is the quotient of the free group $F\langle a_i, b_i, \tau\rangle$ by the relation

$$\tau^d = \prod_i a_i b_i a_i^{-1} b_i^{-1}$$

(3) Let $\mu_d(\mathbb{C})$ denote the group of d th roots of 1 in $\mathbb{C}$ (in particular we take $\mu_0(\mathbb{C}) = \mathbb{C}^\times$). Then under any multiplicative character $\rho : \pi_1(N - C) \to \mathbb{C}^\times$, the image $\rho(\tau)$ lies in $\mu_d(\mathbb{C})$, moreover, the following sequence of abelian groups is short exact

$$1 \to Hom(\pi_1(C), \mathbb{C}^\times) \to Hom(\pi_1(N - C), \mathbb{C}^\times) \to \mu_d(\mathbb{C}) \to 1$$

where the first map is induced by the projection $N - C \to C$ and the second map is given by $\rho \mapsto \rho(\tau)$.

The above sequence admits a splitting, hence for all d there is an isomorphism

$$Hom(\pi_1(N - C), \mathbb{C}^\times) \cong Hom(\pi_1(C), \mathbb{C}^\times) \oplus \mu_d(\mathbb{C})$$

Proof. (1) See Lemma 2.2.(1) in [11].

(2) This follows by considering the projection $p : N - C \to C$ and applying the Van Kampen theorem to the union $N - C = p^{-1}(U) \cup p^{-1}(C - x_1)$ where U is an open disc around x_1 which contains x_0. Note that N is trivial over U and over $V = C - x_1$, and with respect to any choice of trivializations, the transition function $g_{U,V} : U - x_1 \to \mathbb{C}^\times$ has winding number equal to d. This fact gives rise to the relation $\tau^d = \prod_i a_i b_i a_i^{-1} b_i^{-1}$.

(3) This follows by the description of $\pi_1(N - C)$ given in (2). Another proof can be given by a combination of the Mayer-Vietoris sequence and the universal coefficient theorem for coefficients $\mathbb{C}^\times$. Yet another proof follows from the Gysin sequence. □

§3. Parabolic connections

Seshadri's original introduction of parabolic structures on vector bundles on curves, starting from a unitary monodromy representation of the fundamental group of the complement of a finite set of points, should rightly be viewed as a precursor of the general construction of a Deligne lattice. Following this, it has been a common observation (independently due to many mathematicians) that there is a **natural parabolic structure** on any Deligne construction where parabolic weights are real parts of residual eigenvalues, and for this parabolic structure the parabolic degree is zero. In particular, any such parabolic logarithmic connection is automatically par-μ-semistable.

Drawing inspiration from this, we take parabolic Gieseker semi-stability as the condition for moduli construction. This is somewhat more restrictive than par-μ-semistability, but better suited for GIT methods. We combine the moduli construction methods invented by Simpson [13] and Maruyama-Yokogawa [9] to construct a moduli space.

Price to pay

In comparison with the moduli construction of [10], the parabolic construction given here has the restriction that now we have to fix eigenvalues of residue. However, this is not so severe a restriction, for this is automatic along all those components Y_a for which the universal topological degree (see Definition 2.1) $\delta(N_a)$ of the normal bundle $N_a = N_{Y_a,X}$ is non-zero, as the local loop τ_a around Y_a maps to $\delta(N_a)$ th roots of 1 under any $\pi_1(X - Y) \to \mathbb{C}^\times$ as a consequence of Lemmas 2.4 and 2.2.

Next, note that the moduli construction is for connections that are parabolic Gieseker semistable, which need not coincide with par-μ-semistable connections. However, in case all components of the divisor have self-intersection zero (for example, when X is a curve) or if the residue is nilpotent, then par-Gieseker semistability coincides with par-μ semistability.

So far, our construction works only for parabolic stable bundles in the special case where the vector bundle $T_X\langle \log Y\rangle$ is generated by global sections. We would like to see this restriction removed in the future.[1]

Parabolic sheaves and parabolic bundles on (X, Y)

Let X be a non-singular variety and let $Y \subset X$ be a normal crossing divisors, whose irreducible components Y_a are smooth. Our notion of a parabolic bundles on (X, Y) and parabolic Hilbert polynomials is a slight generalisation of the notions introduced by Maruyama and Yokogawa. The difference is that for us the parabolic structure lives on the normalisation of Y, rather than on Y itself as originally in [9].

Definition 3.1. Let X and Y be as above. A **parabolic sheaf** on (X, Y) consists of the following data.

(1) A coherent sheaf of $\mathcal{O}_X$-modules $\mathcal{E}$ on X, called the underlying $\mathcal{O}_X$-module of the parabolic sheaf.

(2) For each irreducible component Y_a a strictly decreasing filtration (of length $p(a)$ which depends on a) of the restriction $\mathcal{E}|_{Y_a}$ by coherent subsheaves

$$\mathcal{E}|_{Y_a} = F_{a,1}(\mathcal{E}) \supset \ldots \supset F_{a,p(a)}(\mathcal{E}) \supset 0$$

These filtrations are called the quasi-parabolic structure on $\mathcal{E}$.

(3) For each component Y_a a sequence of real numbers $0 \leq \alpha_{a,1} < \ldots < \alpha_{a,p(a)} < 1$, called the parabolic weights.

For simplicity of notation, a parabolic sheaf will be denoted just by $\mathcal{E}$.

If the underlying $\mathcal{O}_X$-module $\mathcal{E}$ is locally free and moreover if each $F_{a,i}(\mathcal{E})$ is a vector subbundle of $\mathcal{E}|_{Y_a}$, then $\mathcal{E}$ is called a **parabolic bundle** on (X, Y).

Remark 3.2. It is usual to combine (2) and (3) and define a decreasing filtration of $\mathcal{E}|_{Y_a}$ by subsheaves $F_{a,\alpha}(\mathcal{E})$ indexed by $\alpha \in [0, 1)$ which is left-continuous and has finitely many jumps which take place at the $\alpha_{a,i}$.

Definition 3.3. Let $\mathcal{E}$ be a parabolic sheaf and let $\mathcal{E}' \subset \mathcal{E}$ be a coherent subsheaf. The **induced parabolic structure** on $\mathcal{E}'$ is defined as follows. For any Y_a and $\alpha \in [0, 1)$ we define $F_{a,\alpha}(E')$ to be the inverse image of $F_{a,\alpha}(E)$ under the map $\mathcal{E}'|_{Y_a} \to \mathcal{E}|_{Y_a}$ which is induced

[1] In their preprint [6], Inaba, Iwasaki and Saito have independently given a moduli construction for parabolic connections when X a curve, without any such restriction.

by $\mathcal{E}' \hookrightarrow \mathcal{E}$. By Remark 3.2, this indeed defines a parabolic structure on $\mathcal{E}'$.

Parabolic Hilbert polynomial

Let H be a very ample divisor on X, and $\mathcal{O}_X(H)$ the corresponding line bundle. For any coherent sheaf $\mathcal{F}$ on X, by $\chi(\mathcal{F}, m)$ we mean the Euler characteristic of $\mathcal{F}(mH)$ on X. Inspired by Maruyama-Yokogawa (but with the difference that our formulation requires Y to be normal crossing, and pays attention to individual components Y_a), we define the parabolic Hilbert polynomial of a parabolic sheaf as follows.

Definition 3.4. Let $\mathcal{E}$ be a parabolic sheaf. The **parabolic Hilbert polynomial** of $\mathcal{E}$ is defined by the formula

$$par\,\chi(\mathcal{E}, m) = \chi(\mathcal{E}(-Y), m) + \sum_a \sum_i \alpha_{a,i}\, \chi(F_{a,i}(\mathcal{E})/F_{a,i+1}(\mathcal{E}), m)$$

When there is only one parabolic weight equal to 0, then note that $par\,\chi(\mathcal{E}, m) = \chi(\mathcal{E}(-Y), m)$.

Remark 3.5. If we index the filtration by α varying over $[0, 1)$ and define the graded object $gr_{a,\alpha}(\mathcal{E})$ as

$$gr_{a,\alpha}(\mathcal{E}) = F_{a,\alpha}(\mathcal{E})/F_{a,\alpha+\epsilon}(\mathcal{E})$$

where $\epsilon > 0$ is sufficiently small, then the above formula becomes

$$par\,\chi(\mathcal{E}, m) = \chi(\mathcal{E}(-Y), m) + \sum_a \int_0^1 \chi(gr_{a,\alpha}(\mathcal{E}), m)\, \alpha\, d\alpha$$

where $\chi(gr_{a,\alpha}(\mathcal{E}), m)$ is regarded as a distribution based at the point α.

Residues and Chern classes for logarithmic connection

Let $(x_1, \ldots, x_d)$ be local coordinates on X, with Y locally defined by $x_1 x_2 \cdots x_m = 0$. Then $\Omega^1_X\langle\log Y\rangle$ is locally free with basis $dx_1/x_1, \ldots$, dx_m/x_m, $dx_{m+1}, \ldots, dx_d$. Let $\widetilde{Y} \to Y$ denote the normalisation of Y. The Poincaré residue map

$$\mathrm{res} : \Omega^1_X\langle\log Y\rangle|_Y \to \mathcal{O}_{\widetilde{Y}}$$

is defined by $dx_a/x_a \mapsto 1$ for $a \leq m$ and $dx_b \mapsto 0$.

The link between residues and Chern classes originates from the following basic fact.

Lemma 3.6. *The Poincare residue map fits in a short exact sequence*

$$0 \to \Omega^1_X \hookrightarrow \Omega^1_X\langle \log Y\rangle \stackrel{\mathrm{res}}{\to} \mathcal{O}_{\widetilde{Y}} \to 0$$

Under the connecting map $H^0(\widetilde{Y}, \mathcal{O}_{\widetilde{Y}}) \to H^1(X, \Omega^1_X)$ we have $1 \mapsto -[Y]$.

The composite map $E \stackrel{\nabla}{\to} \Omega^1_X\langle \log Y\rangle \otimes E|_Y \stackrel{\mathrm{res}}{\to} E|_{\widetilde{Y}}$, is $\mathcal{O}_X$-linear for any a logarithmic connection (E, ∇) (even though ∇ is not so). Pullback to $\widetilde{Y}$ defines a section $\mathrm{res}(\nabla) \in End(E|_{\widetilde{Y}})$, called the **residue** of (E, ∇). This has the following description in terms of local coordinates. Let e_i be a local basis for E, and Y locally defined by $x_1 x_2 \cdots x_m = 0$ as above. Then we can write

$$\nabla(e_i) = \sum_j \left(\sum_{a \le m} R^j_{i,a} \frac{dx_a}{x_a} + \sum_{b > m} \Gamma^j_{i,b} dx_b \right) \otimes e_j$$

where $R^j_{i,a}$ and $\Gamma^j_{i,b}$ are local sections of $\mathcal{O}_X$. The matrices $(R^j_{i,a}|_{Y_a})$ define $\mathrm{res}(\nabla)$ on $E|_{Y_a}$, where Y_a is locally defined by $x_a = 0$.

Lemma 3.7. *Let X be a non-singular variety (not necessarily compact), and let $Y \subset X$ be a normal-crossing divisor on X whose irreducible components Y_a are smooth. (E, ∇) a vector bundle with a logarithmic connection on (X, Y). Then the following holds:*

(1) For each component Y_a, the corresponding residue $R_a \in End(E|_{Y_a})$ has a constant conjugacy class, in particular, the characteristic polynomial of R_a has constant coefficients.

(2) Around any point $x \in X$ which lies on a k-fold intersection $Y_1 \bigcap \ldots \bigcap Y_k$ of components of Y, there exists a holomorphic trivialization of E with respect to which the all corresponding residues $R_1, \ldots R_k$ have constant matrices, and these commute. In particular, for any non-negative integers $q_1, \ldots, q_k$, the function $Tr(R_1^{q_1} \cdots R_k^{q_k})$ is locally constant on $Y_1 \bigcap \ldots \bigcap Y_k$.

Proof. (1) Let Y'_a be the open subscheme of Y_a defined as $Y_a - \bigcup_{b \neq a} Y_b$. Let $(x_1, \ldots, x_d)$ be étale local coordinates on a neighbourhood V in X, with Y_a locally defined by $x_1 = 0$, and Y locally defined by $x_1 \cdots x_m = 0$, so $T_X\langle \log Y\rangle$ has local basis $x_1\partial_1, \ldots, x_m\partial_m, \partial_{m+1} \ldots, \partial_d$ where $\partial_i = \partial/\partial x_i$. The restriction $E|_{Y'_a \cap V}$ has an integrable connection D defined by $D_{\partial_i} u = \nabla_{\partial_i} u$ for all $i \ge 2$. It follows from the facts that ∇ is integrable and $[x_1\partial_1, \partial_i] = 0$, that $[\nabla_{x_1\partial_1}, \nabla_{\partial_i}] = 0$ for all $i \ge 2$.

Hence with respect to the connection on $End(E)|_{Y'_a \cap V}$ induced by the connection D on $E|_{Y'_a \cap V}$, the residue is a flat section of $End(E)|_{Y'_a \cap V}$. The result follows as Y'_a is dense in Y_a.

(2) This follows by an argument similar to the above by considering a flat holomorphic basis for the integrable connection induced on the restriction of E to $Y_1 \bigcap \ldots \bigcap Y_k \cap V$. Q.E.D.

Connection with Newton classes $N_p(E)$

It is a well-known fact (see [Esnault-Viehweg 1986]) that for any logarithmic connection, Residue $\mapsto$ Atiyah obstruction $\mapsto$ Newton classes.

For $p \geq 0$, by definition the p th complexified Newton class of a vector bundle E is the element of $H^{2p}(X^{an}; \mathbb{C})$ given by $N_p(E) = \sum_{1 \leq i \leq r} (\gamma_i)^p$ where $r = rank(E)$ and γ_i are the complexified Chern roots of E.

Let $Y = Y_1 \cup \ldots \cup Y_m$ with irreducible components Y_a smooth, crossing normally. Let

$$R_a = \mathrm{res}(\nabla)|_{Y_a} \in H^0(Y_a, End(E|_{Y_a}))$$

Then as a consequence of Lemma 3.6, we have

$$N_p(E) = (-1)^p \sum_{q_1 + \ldots + q_m = p} Tr(R_1^{q_1} \cdots R_m^{q_m})[Y_1]^{q_1} \cdots [Y_m]^{q_m}$$

where $[Y_a] = c_1(\mathcal{O}_X(Y_a)) \in H^2(X^{an}; \mathbb{C})$. Consequently the Chern character $ch(E) = r + N_1 + N_2/2 + N_3/3! + \ldots$ is determined by the residue.

Natural parabolic structure on logarithmic connections

Definition 3.8. We will say that a logarithmic connection (E, ∇) on (X, Y) is a **Deligne connection** if the real parts of all eigenvalues of the residue $R_a \in End(E|_{Y_a})$ over each irreducible component Y_a of Y all lie in the interval $[0, 1)$. Let $0 \leq \alpha_{a,1} < \ldots < \alpha_{a,p} < 1$ be the distinct real parts of the residual eigenvalues along Y_a (it is possible that two distinct eigenvalues have the same real part). Then $E|_{Y_a}$ gets a direct sum decomposition

$$E|_{Y_a} = E_{a,1} \oplus \ldots \oplus E_{a,p}$$

where $E_{a,i}$ is the direct sum of all generalised eigensubbundles $V_\lambda \subset E|_{Y_a}$ of R_a corresponding to all eigenvalues λ with real part $\mathbf{Re}(\lambda) = \alpha_{a,i}$. This in particular allows us to define a decreasing filtration

$$E|_{Y_a} = F_{a,1} \supset \ldots \supset F_{a,p} \supset 0$$

where $F_{a,i} = \oplus_{j\geq i} E_{a,j}$ which is a vector subbundle of $E|_{Y_a}$. The filtration $F_{a,i}$ with weights $\alpha_{a,i}$ is the **natural parabolic structure on a Deligne connection** E.

Parabolic Hilbert polynomial of a Deligne connection

Let H be a very ample divisor on X, and $\mathcal{O}_X(H)$ the corresponding line bundle. For any coherent sheaf $\mathcal{F}$ on X, by $\chi(\mathcal{F}, m)$ we mean the Euler characteristic of $\mathcal{F}(mH)$ on X. Generalising Maruyama-Yokogawa, we define the parabolic Hilbert polynomial of a Deligne connection as follows.

Proposition 3.9. *Let E be a Deligne connection and let $E|_{Y_a} = E_{a,1} \oplus \ldots \oplus E_{a,p}$ be the direct sum decomposition as in Definition 3.8 indexed by the real parts of eigenvalues of residues. Then the parabolic Hilbert polynomial of E satisfies the following equality.*

$$par\,\chi(E, m) = \chi(E(-Y), m) + \sum \alpha_{a,i}\chi(E_{a,i}, m)$$

Proposition 3.10. *The parabolic Hilbert polynomial of a Deligne connection has the form*

$$par\,\chi(E, m) = \frac{r(E)[H]^d}{d!}m^d + \frac{r(E)(c_1(X)/2 - [Y])[H]^{d-1}}{(d-1)!}m^{d-1} + \ldots$$

where the remaining terms are of degrees $\leq d-2$ in m.

In particular, the parabolic degree of any such E is zero and so any such E is necessarily parabolic μ-semistable.

Proof. By Riemann-Roch theorem, we get $\chi(E(-Y), m) =$

$$\frac{r(E)[H]^d}{d!}m^d + \frac{(r(E)c_1(X)/2 + c_1(E(-Y)))[H]^{d-1}}{(d-1)!}m^{d-1} + \ldots$$

(lower order terms in m), and

$$\sum \alpha_{a,i}\chi(E_{a,i}, m) = \sum \alpha_{a,i}\frac{r(E_{a,i})[Y_a][H]^{d-1}}{(d-1)!}m^{d-1} + \ldots$$

By the relationship between residue and Chern classes described earlier,

$$\sum \alpha_{a,i}r(E_{a,i})[Y_a] = \sum tr(R_a)[Y_a] = -c_1(E)$$

Substituting in the above equation gives

$$\sum \alpha_{a,i}\chi(E_{a,i}, m) = -\frac{c_1(E)[H]^{d-1}}{(d-1)!}m^{d-1} + \ldots$$

Hence we get the desired equality for $par\,\chi(E, m)$. Q.E.D.

Remark 3.11. In contrast to the coefficients of degrees d and $d-1$, the coefficients of $par\,\chi(E,m)$ in degrees $\leq d-2$ can depend on data involving residual eigenvalues and their intersection multiplicities. So parabolic Gieseker semistability is not automatic except in special cases – say when each $[Y_a][Y_b]=0$ or when along each Y_a there is exactly one parabolic weight $\mathbf{Re}(\lambda_{a,i})$. In particular if all residues R_a are nilpotent, then $par\,\chi(E,m) = r(E)\cdot\chi(\mathcal{O}_X(-Y),m)$ and so parabolic Gieseker semistability is automatic when all local monodromies are unipotent.

Definition 3.12. We say that a Deligne connection E is **parabolic semistable** if for each nonzero ∇-invariant vector subbundle F with $0\neq F\neq E$, we have

$$par\,\chi(F,m)/r(F)\leq par\,\chi(E,m)/r(E)$$

where $r(F)$ and $r(E)$ are the ranks of the respective vector bundles, and F is given the induced parabolic structure. If strict inequality always holds, we say that E is **parabolic stable**.

Strong local freeness

Lemma 3.13. *Let E be a Deligne connection on (X,Y), and let $\mathcal{F}\subset E$ be an $\mathcal{O}_X$-coherent sub-$\mathcal{D}_X\langle\log Y\rangle$-module. If $E/\mathcal{F}$ is torsion-free then $E/\mathcal{F}$ is locally free, that is, $\mathcal{F}$ is a vector subbundle of E.*

Proof. Let $\mathcal{M}$ be the local system on $X-Y$ defined by $E|_{X-Y}$. As $\mathcal{F}|_{X-Y}$ is an $\mathcal{O}_{X-Y}$-coherent $\mathcal{D}_{X-Y}$-module, it is locally free, and its local integrable sections define a sub local system $\mathcal{L}\subset\mathcal{M}$. The Deligne construction applied to this inclusion of local systems gives a vector subbundle $V\subset E$, with $V|_{X-Y}=\mathcal{F}|_{X-Y}$. The composite $\mathcal{F}\to E\to E/V$ is zero on $X-Y$ hence identically zero, as E/V is locally free hence torsion-free. So $\mathcal{F}\subset V$. As $V|_{X-Y}=\mathcal{F}|_{X-Y}$, it follows that $V/\mathcal{F}$ is a torsion sheaf, while the inclusion $V/\mathcal{F}\subset E/\mathcal{F}$ into $E/\mathcal{F}$ together with the hypothesis that $E/\mathcal{F}$ is torsion-free, shows that $V/\mathcal{F}$ is torsion-free. It follows that $V/\mathcal{F}=0$, and so $\mathcal{F}=V$. Q.E.D.

Because of the following lemma, we do not need to impose any condition on ∇-invariant $\mathcal{O}$-coherent subsheaves other than vector subbundles.

Lemma 3.14. *Let E be a Deligne connection on (X,Y), and let $\mathcal{F}\subset E$ be a non-zero $\mathcal{O}_X$-coherent sub-$\mathcal{D}_X\langle\log Y\rangle$-module. Let $\overline{\mathcal{F}}\subset E$ be the inverse image of the torsion subsheaf $(E/\mathcal{F})_{tors}$ of $E/\mathcal{F}$ under the quotient map $E\to E/\mathcal{F}$. Then $\overline{\mathcal{F}}$ is a vector subbundle of E which is invariant under ∇, and given the induced parabolic structures on $\mathcal{F}$*

and $\overline{\mathcal{F}}$, the normalised parabolic Hilbert polynomials satisfy

$$par\,\chi(\mathcal{F},m)/r(\mathcal{F}) \leq par\,\chi(\overline{\mathcal{F}},m)/r(\overline{\mathcal{F}})$$

Equality holds (if and) only if $\mathcal{F}=\overline{\mathcal{F}}$, that is, (if and) only if $\mathcal{F}\subset E$ is a vector subbundle.

Proof. By Lemma 3.13, $\overline{\mathcal{F}}$ is a vector subbundle of E which is invariant under ∇. Moreover, $\mathcal{F}|_{X-Y} = \overline{\mathcal{F}}|_{X-Y}$ which in particular means $r(\mathcal{F}) = r(\overline{\mathcal{F}})$. The parabolic filtration on $\mathcal{F}|_{Y_a}$ is induced from that on $\overline{\mathcal{F}}|_{Y_a}$, which implies that we have inclusion of corresponding graded pieces $gr_{a,\alpha}(\mathcal{F}) \subset gr_{a,\alpha}(\overline{\mathcal{F}})$. These inclusions, along with the inclusion $\mathcal{F}\subset\overline{\mathcal{F}}$ give the inequality between parabolic Hilbert polynomials $par\,\chi(\mathcal{F},m) \leq par\,\chi(\overline{\mathcal{F}},m)$. Now the result follows by dividing by $r(\mathcal{F}) = r(\overline{\mathcal{F}})$. $\square$

Remark 3.15. Sub-connections of E have only **finitely many** possible Hilbert polynomials and parabolic Hilbert polynomials. This is because the residual eigenvalues (with multiplicities) of a sub-connections of E come from that of E, and these determine the Hilbert polynomials and parabolic Hilbert polynomials.

Lemma 3.16. *In any family E of Deligne connections parametrised by a scheme S, the conditions of parabolic semistability and parabolic stability define open subschemes of S.*

Proof. Let $\pi : Q \to S$ be the relative quot scheme of $\mathcal{O}$-coherent quotients of E having any one of the possible Hilbert polynomials for quotients modulo sub-connections. There are only finitely many such polynomials by Remark 3.15, so Q is proper over S. The condition that the kernel of the quotient is ∇-invariant is a closed condition, defining a closed subscheme $Q' \subset Q$. Note that Q' has a closed subschemes $Q_1 \subset Q_2 \subset Q'$ such that $S-\pi(Q_2)$ is the stable locus and $S-\pi(Q_1)$ is the semi-stable locus in S, hence these are open in S. Q.E.D.

Lemma 3.17. *Let $Y\subset X$ be a smooth divisor such that $[Y]^2 = 0 \in H^4(X^{an};\mathbb{C})$. Then for any logarithmic connection E which is a Deligne extension with natural parabolic structure, the parabolic Hilbert polynomial equals $r(E)\chi(\mathcal{O}_X(-Y),m)$.*

Proof. Note that $[Y] = \sum_a [Y_a]$ and by assumption, $[Y_a][Y_b] = 0$ for all a,b. Hence we have $ch(\mathcal{O}_X(-Y_a)) = 1-[Y_a]$, and so

$$ch(\mathcal{O}_{Y_a}) = ch(\mathcal{O}_X) - ch(\mathcal{O}_X(-Y_a)) = 1 - ch(\mathcal{O}_X(-Y_a)) = [Y_a]$$

Note that $c_1(E) = -\sum_a trace(R_a)[Y_a]$ where $R_a = trace(\mathrm{res}_{Y_a}(E))$. As $[Y]^2 = 0$, the complex Newton classes $N_p(E)$ vanish for $p \geq 2$. Hence the Chern character of $E(-Y)$ is given by

$$ch(E(-Y)) = r(E) - \sum_a (r(E) + trace(R_a))[Y_a]$$

Consider any piece $E_{a,i}$ in the direct sum decomposition $E|_{Y_a} = E_{a,1} \oplus \ldots \oplus E_{a,p}$. Again by $[Y_a]^2 = 0$, the Newton classes $N_p(E_{a,i})$ vanish for $p \geq 1$, so $E_{a,i}$ has Chern character

$$ch(E_{a,i}) = r(E_{a,i})ch(\mathcal{O}_{Y_a}) = r(E_{a,i})[Y_a]$$

Moreover, note that $trace(R_a) = \sum_i r(E_{a,i})\,\alpha_{a,i}$. As $\sum_i r(E_{a,i}) = r(E)$, this gives

$$ch(E(-Y)) + \sum_{a,i} \alpha_{a,i} ch(E_{a,i}) = r(E)(1 - [Y]) = r(E)ch(\mathcal{O}_X(-Y))$$

Multiplying both sides of the above equation by $ch(\mathcal{O}_X(mH))td(X)$ (where $td(X)$ denotes the Todd class of X) and integrating over X, the result follows by the Hirzebruch-Riemann-Roch theorem. Q.E.D.

§4. Moduli of parabolic stable connections

Given (X, Y) with $Y = \cup_a Y_a$, and rank n, we fix

(1) along each Y_a, eigenvalues (with multiplicities) for residues R_a with real parts in $[0, 1)$ (these are constant by Lemma 3.7),

(2) along each connected component of intersection $Y_{a_1} \cap \ldots \cap Y_{a_r}$, fix ranks of intersections of the generalised eigen-subbundles of residues R_{a_i} (these ranks are constant by Lemma 3.7).

As explained earlier, this fixes all the Newton classes of any E with the above data. Let $P(m)$ denote the resulting Hilbert polynomial.

Proposition 4.1. *Let the rank r and residue data be fixed. There exists an integer N_0 such that for any Deligne connection of rank r with the above data, the following holds for any $N \geq N_0$:*

(1) The bundle $E(N)$ is generated by global sections, all its higher cohomologies vanish, and all the higher cohomologies of the line bundle $\det(E(N))$ *vanish.*

(2) The higher cohomologies of the restrictions $E|_{Y_a}$ also vanish.

Combining Simpson's method for Λ-modules with that of Bhosle, Maruyama, Yokogawa for parabolic bundles converts the moduli problem into a quotient problem and into a GIT problem.

By boundedness there exists N such that $E(m)$ has all higher cohomologies zero for $m \geq N$, and is generated by global sections.

Locally universal family

Fix the rank r and the residual eigenvalues with multiplicities, together with ranks of intersections of generalised eigen-subbundles of the Deligne connection, as explained above. Let $\mathcal{D}_X\langle\log Y\rangle \subset \mathcal{D}_X$ consists of operators which preserve $I_Y \subset \mathcal{O}_X$. By intersection with the filtration on $\mathcal{D}_X$, this acquires an exhaustive filtration

$$0 \subset \mathcal{O}_X = F^0\mathcal{D}_X\langle\log Y\rangle \subset F^1\mathcal{D}_X\langle\log Y\rangle \subset \ldots$$

where each $F^i\mathcal{D}_X\langle\log Y\rangle$ is an $\mathcal{O}_X$-bimodule (with commutating left and right structures), which is $\mathcal{O}_X$-bi-coherent. A logarithmic connections is the same as a left $\mathcal{D}_X\langle\log Y\rangle$-module which is coherent and locally free over $\mathcal{O}_X$. For any integer i, we denote $F^i\mathcal{D}_X\langle\log Y\rangle$ simply by F^i. To keep clear whether the left or the right $\mathcal{O}_X$-module structure is used for the tensor product, we will use the notation

$$F^i(m,n) = \mathcal{O}_X(m) \otimes_{\mathcal{O}_X} F^i\mathcal{D}_X\langle\log Y\rangle \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$$

Following ideas of Simpson, we construct a locally universal family for Deligne connections of rank r with given residual data, parametrised by a scheme C, defined as a locally closed subscheme of a certain Quot scheme Q. We define Q as the Quot scheme which parametrises left-$\mathcal{O}_X$-linear epimorphisms

$$q : F^{r+1}(0,-N)^{\oplus P(N)} = F^{r+1}\mathcal{D}_X\langle\log Y\rangle \otimes_{\mathcal{O}_X} \mathcal{O}_X(-N)^{\oplus P(N)} \to \mathcal{F}$$

such that $\mathcal{F}$ is $\mathcal{O}_X$-coherent with Hilbert polynomial $P(m)$, and where N is so chosen (using Proposition 4.1) that for $m \geq N$, the sheaves $E(m)$ and $E_{a,\alpha}(m)$ are generated by global sections and their higher H^i's vanish, and also the same holds for I_Y times or I_x times the sheaves for $x \in X$ and the same holds for the $\mathcal{D}_X\langle\log Y\rangle$-subquotient bundles of E and $E_{a,\alpha}$.

Let $C \subset Q = Quot^{P(m)}_{F^{r+1}(0,-N)^{\oplus n}/X}$ be the locally closed subscheme, defined by the following conditions (where $n = P(N)$):

(i) $\mathcal{F}$ is locally free (this is an open condition on Q).

(ii) Consider the bi-$\mathcal{O}_X$-module homomorphism

$$\mathcal{O}_X(-N)^{\oplus n} \to F^{r+1}(0,-N)^{\oplus n} : a \mapsto 1 \otimes a.$$

We impose the condition that the composite map

$$p : \mathcal{O}_X(-N)^{\oplus n} \to F^{r+1}(0,-N)^{\oplus n} \to \mathcal{F}$$

is surjective, and on applying $\mathcal{O}_X(N) \otimes_{\mathcal{O}_X} -$ followed by $H^0(X,-)$ it induces an isomorphism $\mathbb{C}^n \to H^0(\mathcal{F}(N))$ (this is an open condition).

(iii) $F^{r+1} \otimes \mathcal{O}_X(-N)^n \to \mathcal{F}$ factors via the surjection $1_{F^{r+1}} \otimes p : F^{r+1} \otimes \mathcal{O}_X(-N)^n \to F^{r+1} \otimes \mathcal{F}$, giving a (uniquely determined) map $\mu : F^{r+1} \otimes \mathcal{F} \to \mathcal{F}$, and the product $F^i \otimes F^j \to F^{i+j}$ for $i+j \leq r+1$ is respected (closed condition, makes $\mathcal{F}$ a $\mathcal{D}_X\langle \log Y\rangle$-module.)

(iv) Residual eigenvalues are correct (closed condition).

The scheme C in invariant under the action of SL_n and a good quotient $C//SL_n$, **if it exists**, is the coarse moduli for Deligne connections with given rank and residual eigenvalues. Let $C^{par\, ss} \subset C$ be the open subscheme consisting of parabolic semistable bundles. Then we show that a good quotient $C^{par\, ss}//SL_n$ exists.

Parabolic polarisation and GIT quotient

We have a natural embedding of C into a product of Quot schemes $Q_0 \times \prod_{a,i} Q_{a,i}$ for quotients of the type $F^{r+1}(0,-N)^{\oplus n} \to E$ and of the type $F^{r+1}(0,-N)^{\oplus n} \to E|_Y/F_{a,i+1}$ where $F_{a,i} = \oplus_{j\geq i} E_{a,i}$ is the parabolic filtration.

We now construct an analog of the Gieseker space (originally due to Gieseker, defined in [Ge]), for our extra requirement that we need to encode not just a vector bundle E but the logarithmic connection on it.

The space Z we construct will be the total space of a projective fibration $Z \to A$, where A is a union of certain finitely many components of the Picard scheme of X, to define which we first need the following lemma.

Lemma 4.2. *There exists an integer $c \geq 0$ such that for each $0 \leq i \leq r+1$ and for each N, the sheaf $F^i(N+c,-N)$ is generated by its global sections.*

Proof. The graded pieces of of the filtration

$$0 \subset \mathcal{O}_X = F^0(N+c,-N) \subset F^1(N+c,-N) \subset \ldots \subset F^{r+1}(N+c,-N)$$

are $Sym^i(T_X\langle\log Y\rangle)(c)$, which are independent of N. The result follows.
Q.E.D.

Note that any quotient $F^{r+1}(0,-N)^{\oplus P(N)} \to E$, which represents a $\mathbb{C}$-valued point of Q_0, gives a quotient

$$F^{r+1}(N+c,-N)^{\oplus P(N)} \to E(N+c)$$

Let $A \subset Pic_{X/S}$ be the open and closed subscheme which parametrises all line bundles on X whose first Chern class is equal to that of $\det(V)$ where V is any vector bundle on X with Hilbert polynomial $P(N+c)$. By choosing a $\mathbb{C}$-rational point $x \in X$ as base point, we get a unique Poincaré line bundle $\mathcal{L}$ on $X \times A$, trivialised on $x \times A$. Let $(p_A)_*(\mathcal{L})$ denote its direct image on A under the projection $p_A : X \times A \to A$. The sheaf $(p_A)_*(\mathcal{L})$ will be a vector bundle by our choice of A, because all the higher cohomologies $H^i(X,L)$ for $i \geq 1$ vanish for line bundles L represented by points of A. Let Z be the projective scheme over A defined as

$$\begin{aligned}
Z &= \mathbf{P}\left[\hom(H^0\left(X,\, \wedge^r[F^{r+1}(N+c,-N)^{\oplus P(N)}]\right) \otimes_{\mathbb{C}} \mathcal{O}_A,\, (p_A)_*\mathcal{L})^\vee\right] \\
&= \operatorname{Proj}\operatorname{Sym}_{\mathcal{O}_A}\left[H^0\left(X,\, \wedge^r[F^{r+1}(N+c,-N)^{\oplus P(N)}]\right) \otimes_{\mathbb{C}} (p_{A*}\mathcal{L})^\vee\right].
\end{aligned}$$

The $\mathbb{C}$-valued points of Z over a point $a \in A$ are represented by equivalence classes of pairs

$$\left(L,\, \phi : H^0(X, \wedge^r[F^{r+1}(N+c,-N)^{\oplus P(N)}]) \to H^0(X,L)\right)$$

where L is a line bundle representing the point a, and ϕ is a non-zero linear map.

There is a natural linear representation of SL_n, where $n = P(N)$, on the vector space $H^0(X, \wedge^r[F^{r+1}(N+c,-N)^{\oplus P(N)}])$, which induces an action of SL_n on the pair $(Z, \mathcal{O}_Z(1))$.

Recall that if $\mathcal{F}$ is a coherent torsion-free sheaf of rank 1 on a non-singular variety X, then there exists (up to isomorphism) a unique pair (L,i) consisting of an invertible sheaf L on X and a homomorphism $i : \mathcal{F} \to L$ which is an isomorphism outside a closed subset of codimension ≥ 2. The first Chern class of L is equal to that of $\mathcal{F}$.

We define a morphism $\psi : Q_0 \to Z$, by defining a natural transformation on functors of points as follows. For any $\mathbb{C}$-scheme S, we have a set map

$Q_0(S) \to Z(S)$ sending an S-valued point $F^{r+1}(0,-N)^{\oplus P(N)} \boxtimes \mathcal{O}_S \to E$ of Q_0 to the S-valued point of Z represented by a pair (L, ϕ) where L is the line bundle $\det(E(N+c))$ corresponding to the torsion free sheaf $\wedge^r(E(N+c))$ of generic rank 1, and ϕ is the composite map $H^0(X, \wedge^r[F^{r+1}(N+c,-N)^{\oplus P(N)}]) \otimes_{\mathbb{C}} \mathcal{O}_S \to \pi_{S*} \wedge^r (E(N+c))$
$\to \pi_{S*} \det(E(N+c))$.

This map is clearly SL_n-equivariant, where $n = P(N)$.

Proposition 4.3. *For all N sufficiently large, the morphism $\psi : C \to Z$ is an isomorphism of C with a locally closed subscheme of Z.*

Proof (Sketch) The scheme C represents the functor which associates to each S a pair (E, u) where E is a family of Deligne connection on $X \times S$ with prescribed residual eigenvalues data, and $u : \mathcal{O}_S^{P(N)} \to p_{X*}E$ is an isomorphism of vector bundles on S. By the use of flattening stratification, it can be seen that Z has a locally closed subscheme C' which represents this same functor, such that ψ induces natural isomorphism of h_C with $h_{C'}$. (Such a use of the flattening stratification was originally made by Grothendieck to construct Hilbert and Quot schemes as locally closed subschemes of Grassmannians. Also, Maruyama-Yokogawa use it to prove the analogous proposition in [9]). Now the result follows by Yoneda lemma. Q.E.D.

We now define certain Grassmannians $Gr_{a,i}$ as follows. Let $P_{a,i}(m)$ denote the Hilbert polynomial of $E|_Y/F_{a,i}(E)$. Let $Gr_{a,i}$ be the Grassmannian of quotients of $H^0(X, F^{r+1}(N+c,-N)^{\oplus P(N)})$ of dimension $P_{a,i}(N+c)$. We have morphisms $\psi_{a,i} : C \to Gr_{a,i}$ which at the level of functors of points sends a quotients $q_S : F^{r+1}(0,-N)^{\oplus P(N)} \boxtimes \mathcal{O}_S \to E$ to the quotient

$$\psi_{a,i}(q_S) : H^0(X, F^{r+1}(N+c,-N)^{\oplus P(N)}) \otimes_{\mathbb{C}} \mathcal{O}_S \to \pi_{S*}(E|_Y/F_{a,i}(E))(N+c)$$

Together, these define a morphism

$$\theta = (\psi, \psi_{a,i}) : C \to Z \times \prod_{a,i} Gr_{a,i}$$

which is a locally closed embedding, as its composition with the projection on Z is ψ which is a locally closed embedding by Proposition 4.3. (Even though the morphism $\psi : C \to Z$ by itself is a locally closed embedding, we need the other factors $Gr_{a,i}$ to get the polarisation right in order that parabolic semi-stability of a Deligne connection will correspond to GIT semistability of points of $Z \times \prod_{a,i} Gr_{a,i}$).

Remark 4.4. Unlike in the moduli construction of [1] or [9], we do not have properness of the morphism $C^{par\ ss} \to (Z \times \prod_{a,i} Gr_{a,i})^{ss}$.

Linearisation of action

For simplicity, we assume each $\alpha_{a,i}$ is rational. This entails no loss of generality, as originally shown in the paper of Mehta-Seshadri [M-S]. Let $L_{a,i}$ denote the positive generator of $Pic(Gr_{a,i})$. With respect to these line bundles on the factors, give $Z \times \prod_{a,i} Gr_{a,i}$ the polarisation

$$(P(N+c)/r,\ \epsilon_{a,i}/\delta)$$

where $\epsilon_{a,i} = \alpha_{a,i+1} - \alpha_{a,i}$ for $i = 1, \ldots \ell(a) - 1$ and $\epsilon_{a,\ell(a)} = 1 - \alpha_{a,\ell(a)}$, and $\delta = \dim H^0(X, D)$. (Except for the presence of the constants c and δ, the above is the same as the corresponding polarisation in Maruyama-Yokogawa [9] page 94.) In terms of line bundles, our choice of a very ample line bundle $\mathcal{L}$ on $Z \times \prod_{a,i} Gr_{a,i}$ is any line bundle of the form

$$\mathcal{L} = \pi_Z^* \mathcal{O}_Z(m \cdot P(N+c)/r) \otimes \left(\otimes_{a,i} \pi_{a,i}^* L_{a,i}^{m \cdot \epsilon_{a,i}/\delta}\right)$$

where $m \geq 1$ is any positive integer which clears all the denominators in the above formula.

The natural action of SL_n (where $n = P(N)$) on $Z \times \prod_{a,i} Gr_{a,i}$ lifts to the very ample line bundle $\mathcal{L}$. Hence we get open subschemes

$$(Z \times \prod_{a,i} Gr_{a,i})^s \subset (Z \times \prod_{a,i} Gr_{a,i})^{ss} \subset Z \times \prod_{a,i} Gr_{a,i}$$

of GIT-stable and GIT-semistable points.

At this point, we would ideally like to prove the following:

4.5. *When N is sufficiently large, a Deligne connection E is parabolic stable (or parabolic semistable) if and only if for any point $q : F^{r+1}(0, -N)^{\oplus n} \to E$ of C, the image $\theta(q)$ in $Z \times \prod_{a,i} Gr_{a,i}$ is GIT-stable (or GIT-semistable) with respect to the polarisation $(P(N+c)/r,\ \epsilon_{a,i}/\delta)$.*

However, so far we can not prove this in general, but only under the additional assumption that we can choose c to be 0, which is the case when $T_X\langle \log Y\rangle$ is itself generated by global sections. (We expect to be able to eventually remove this assumption. One possible method may be via the 'triples' introduced by Inaba, Iwasaki and Saito [6].)

To prove the above, we calculate the Mumford weights $\mu(\psi(q), \lambda, \mathcal{L})$ corresponding to limits of orbits under 1-parameter subgroups λ of SL_n.

Calculation of Mumford weights

Lemma 4.6. *Let E be a logarithmic connection on (X, Y) of rank r, and let $\mathcal{F} \subset E$ be a coherent sub-$\mathcal{O}_X$-module. Let $E' \subset E$ be the $\mathcal{O}_X$-saturation of the image of $F^{r+1} \otimes \mathcal{F} \to E$. Then E' is a sub $\mathcal{D}_X\langle \log Y\rangle$-module of E.*

Proof. This is just Simpson [13] Lemma 3.2. Q.E.D.

Remark 4.7. Let V be a finite dimensional vector space over $\mathbb{C}$ together with a given linear action of $SL(V)$ on a projective variety $(Y, \mathcal{O}_Y(1))$. Consider any subspace $V' \subset V$. and let $\dim(V') = n'$. Choose a direct sum decomposition $V = V' \oplus V''$. Let $\dim(V) = n$ and $\dim(V') = n'$ so $\dim(V'') = n - n'$. Let $\lambda : \mathbf{G}_m \to SL(V)$ be the 1-parameter sub group, defined by $\lambda(t) = (t^{n'-n} 1_{V'}, t^{n'} 1_{V''})$. Then any point $y \in Y$ the limit $y_0 = \lim_{t\to 0} \lambda(t)y$ is independent of the choice of the supplement V''. Moreover, the Mumford weight $\mu(y, \lambda, \mathcal{O}_Y(1))$ (which is by definition the weight of the character by which λ acts on the fiber of $\mathcal{O}_Y(1)$ at y_0) is also independent of the choice of the supplement V''.

Let $q : F^{r+1}(0, -N) \otimes V \to E$ be a point of C, where $V = \mathbb{C}^n$ with $n = P(N)$. Let $\psi(q) \in Z(\mathbb{C})$ be its image, which is represented by the linear map $H^0(X, \wedge^r(D\otimes V)) \to H^0(X, L)$ where $D = F^{r+1}(N+c, -N)$ and $L = \det(E(N + c))$. Let $M' \subset E$ be the $\mathcal{O}_X$-submodule which is the image of $F^{r+1}(0, -N) \otimes V' \to E$ under q, and let $E' \subset E$ be the $\mathcal{O}_X$-saturation of M'. By Lemmas 4.6 and 3.13, E' is a vector subbundle which is a sub-Deligne connection of E. Let $r' = \mathrm{rank}(E')$.

The decomposition $V = V' \oplus V''$ gives a decomposition

$$\wedge^r(D \otimes V) = \oplus_{i+j=n} \wedge^i (D \otimes V') \otimes \wedge^j(D \otimes V'')$$

Then as in the theory of Gieseker, the limit $\lim_{t\to 0} \lambda(t)\psi(q)$ is represented by the point of Z given by the composite $\mathbb{C}$-linear map

$$\begin{aligned} H^0(X, \wedge^r(D \otimes V)) &\to H^0(X, \wedge^{r'}(D \otimes V') \otimes \wedge^{r-r'}(D \otimes V'')) \\ &\to H^0(X, \det(E'(N + c)) \otimes \det((E/E')(N + c))) \end{aligned}$$

As $\lambda(t)$ acts by $t^{n'-n}$ on V' and $t^{n'}$ on V'', it acts by the characters $t^{-r'(n-n')}$ and $t^{(r-r')n'}$ respectively on $\wedge^{r'}(D \otimes V')$ and $\wedge^{r'}(D \otimes V'')$, and hence it acts by the character $t^{r'(n-n')-(r-r')n'}$ on the vector space $H^0(X, \wedge^{r'}(D \otimes V') \otimes \wedge^{r-r'}(D \otimes V''))$. This implies the following:

4.8. Under the action of λ, the point $\psi(q)$ of Z has the Mumford weight

$$\mu(\psi(q),\ \lambda,\ \mathcal{O}_Z(1)) = r'n - rn'$$

Next, we consider the point $\theta(q) \in Z \times \prod_{a,i} Gr_{a,i}$. For any $\psi_{a,i} : C \to Gr_{a,i}$, recall that $\psi_{a,i}(q)$ is represented by the quotient

$$\psi_{a,i}(q) : H^0(X, F^{r+1}(N+c, -N) \otimes V) \to H^0(X, (E|_Y/F_{a,i}(E))(N+c))$$

where $V = \mathbb{C}^n$ where $n = P(N)$. Let $V = V' \oplus V''$ and λ the corresponding 1-parameter subgroup as above. Then we get $\lim_{t\to 0} \lambda(t)\psi_{a,i}(q)$ to be represented by the quotient which is the composite

$$\begin{aligned} H^0(X, D\otimes V) &\xrightarrow{\sim} H^0(X, D\otimes V') \oplus H^0(X, D \otimes V'') \\ \to \psi_{a,i}H^0(X, D\otimes V') &\oplus \frac{H^0(X, (E|_Y/F_{a,i}(E))(N+c))}{\psi_{a,i}H^0(X, D\otimes V')} \end{aligned}$$

This implies the following:

4.9. If $m'_{a,i} = \dim(\psi_{a,i}H^0(X, D \otimes V'))$, this gives the value for the Mumford weight

$$\mu(\psi_{a,i}(q),\ \lambda,\ L_{a,i}) = \delta \cdot (m'_{a,i}P(N) - P_{a,i}(N+c)n')$$

where $\delta = h^0(X, D)$ and $P_{a,i}$ is the Hilbert polynomial of $E|_Y/F_{a,i}(E)$.

Remark 4.10. In particular, if $E' \subset E$ is a sub-connection of E, and $V' = H^0(X, E'(N))$, then for any 1-parameter subgroup λ of SL_n defined by choosing a splitting $V = V' \oplus V''$ (where the choice of V'' is arbitrary), the corresponding Mumford weight is $P(N)$ times

$$\frac{r(E')}{r(E)} par\, \chi(E, N) - par\, \chi(E', N) - h^0(E'(N))\frac{P(N+c)}{P(N)} + h^0(E'(N+c))$$

In particular, when $c = 0$, the value of the Mumford weight takes the simple form

$$P(N)\left(\frac{r(E')}{r(E)} par\, \chi(E, N) - par\, \chi(E', N)\right)$$

Armed with the above calculations 4.8 and 4.9 of Mumford weights, *in the case where we can take* $c = 0$, the rest of the proof of the statement 4.5 is the same as that of the corresponding Proposition 3.4 of Maruyama-Yokogawa [9].

The open subscheme $(Z \times \prod_{a,i} Gr_{a,i})^s \subset Z \times \prod_{a,i} Gr_{a,i}$ has a geometric quotient $M = (Z \times \prod_{a,i} Gr_{a,i})^s/SL_n$. This is the moduli space for parabolic stable Deligne connections.

§5. Regular holonomic modules when Y is smooth

In this section we consider regular holonomic $\mathcal{D}$-modules whose characteristic variety is contained in $(X - Y) \bigcup N^*_{Y,X}$ where Y is a smooth divisor. These are not $\mathcal{O}_X$-coherent in general, so the concept of a pre-$\mathcal{D}$-module was introduced in the paper [12] to give an $\mathcal{O}_X$-coherent description of these $\mathcal{D}$-modules, much as logarithmic connections give an $\mathcal{O}_X$-coherent description of meromorphic connections. We now show how to re-define the concept of a pre-$\mathcal{D}$-module and its semi-stability, so as to take care of the relationship with the topology of $N_{Y,X}$, which makes the resulting notion of semi-stability (whether ordinary or parabolic) much more inclusive.

In this new construction, we have to fix the residual eigenvalues with their intersection multiplicities. However, by Lemmas 2.2 and 2.4, this is automatic whenever the universal topological degrees of the various normal bundles are non-zero, which is generally the case in higher dimensions. Hence this is a very mild restriction.

The sheaf of rings $\mathcal{D}_X\langle\log Y\rangle|_Y$

As $\mathcal{D}_X\langle\log Y\rangle$ preserves I_Y, we have $I_Y\mathcal{D}_X\langle\log Y\rangle = \mathcal{D}_X\langle\log Y\rangle I_Y$, which is therefore a 2-sided ideal. The quotient ring is the restriction $\mathcal{D}_X\langle\log Y\rangle|_Y$, which can be regarded as $\mathcal{O}$-module restriction of both the left and the right $\mathcal{O}_X$-module $\mathcal{D}_X\langle\log Y\rangle$ to Y.

Over Y, we have a short-exact sequence

$$0 \to \mathcal{O}_Y \to T_X\langle\log Y\rangle|_Y \to T_Y \to 0$$

The image of $1 \in H^0(Y, \mathcal{O}_Y)$ defines a section $\mathbf{e} \in H^0(Y, T_X\langle\log Y\rangle|_Y)$, which has the following description in local coordinates: if $(x_1, \ldots, x_d)$ are (analytic or étale) local coordinates on X, with Y locally defined by $x_1 = 0$, then $\mathbf{e}$ is locally defined as the restriction to Y of the logarithmic vector field $x_1(\partial/\partial x_1)$.

Definition 5.1. The **Euler operator** is the section of $\mathcal{D}_X\langle\log Y\rangle|_Y$ over Y which is the image of the above section $\mathbf{e} \in H^0(Y, T_X\langle\log Y\rangle|_Y)$ under the map induced by the inclusion $T_X\langle\log Y\rangle \hookrightarrow \mathcal{D}_X\langle\log Y\rangle$. We denote the Euler operator again by $\mathbf{e}$.

Remark 5.2. The Euler operator $\mathbf{e}$ is central in $\mathcal{D}_X\langle\log Y\rangle|_Y$. This fact is easy to verify, and it has the following topological analog: in the space $N_{Y,X} - Y$ (or more generally, in the open subset $L - Y$ of the total space any complex line bundle L on a base Y (complement of the zero section) the fundamental loop in a fiber defines a central element of $\pi_1(L - Y)$.

Modules over $\mathcal{D}_X\langle\log Y\rangle|_Y$

For any $\mathcal{D}_X\langle\log Y\rangle|_Y$-module F, the induced endomorphism $\mathbf{e} : F \to F$ will be called the **residue endomorphism** and it will be denoted by $\mathrm{res}(F)$.

Remark 5.3. (1) As $\mathbf{e}$ is central, $\mathrm{res}(F)$ is $\mathcal{D}_X\langle\log Y\rangle|_Y$-linear, so its eigenvalues are constants (even without compactness of Y). (2) If E is a logarithmic connection on (X, Y), then the restriction $F = E|_Y$ is naturally a $\mathcal{D}_X\langle\log Y\rangle|_Y$-module, and the corresponding residues are the same.

Lemma 5.4. *Suppose that Y is smooth, with connected components Y_a. Let F be an $\mathcal{O}_Y$-coherent $\mathcal{D}_X\langle\log Y\rangle|_Y$-module. Then the following holds.*

(1) F is locally free as an $\mathcal{O}_Y$-module (but the rank r_a of $F|_{Y_a}$ could vary from component to component).

(2) Let $\alpha_{a,i}$ be the eigenvalues of $\mathrm{res}(F)$, and let $F = \oplus F_{a,i}$ be the decomposition into generalised eigensubbundles, with ranks $r_{a,i}$. Then the Chern character of $F_{a,i}$ is given by

$$ch(F_{a,i}) = r_{a,i}\exp(-\alpha_{a,i}c_1(N_Y))$$

In particular, the Hilbert polynomial of $F_{a,i}$ is of the form $\chi(F_{a,i}, m) = r_{a,i}f_{a,i}(m)$ where the polynomial $f_{a,i}(m)$ is independent of $F_{a,i}$.

(3) Each $F_{a,i}$ is a semi-stable $\mathcal{D}_X\langle\log Y\rangle$-module (in both Gieseker and μ sense).

Proof. (1) Choose a local trivialization for N_Y. Then locally we get a flat connection on $F|_Y$, showing it is locally free since it is given to be $\mathcal{O}_Y$-coherent.

(2) The pull-back of F to the total space of N_Y becomes a logarithmic connection on (N_Y, Y). Now the earlier result about logarithmic connections applies to give the Chern character of the pull-back of F on the total space of N_Y, and the result follows by restricting to the zero section of N_Y.

(3) Any $\mathcal{O}_{Y_a}$-coherent $\mathcal{D}_X\langle\log Y\rangle$-submodule of $F_{a,i}$ will be a vector subbundle, with the same residual eigenvalue. Hence the result follows from the description of its Hilbert polynomial given by (2). Q.E.D.

Pre-$\mathcal{D}$-modules on (X, Y), semistability

The following is our modified definition of a pre-$\mathcal{D}$-module. Here the modification is minor, the main modification is in the definition of semistability (Definition 5.7).

Definition 5.5. Let Y be a smooth divisor on the A **pre-$\mathcal{D}$-module** on (X, Y) is a tuple (E, F, s, t) where

(i) E is a Deligne connection on (X, Y),

(ii) F is an $\mathcal{O}_Y$-coherent $\mathcal{D}_X\langle\log Y\rangle|_Y$-module on Y such that any eigenvalue of $\mathrm{res}(F)$ has real part in $[0, 1)$.

(iii) $s : F \to E|_Y$ and $t : E|_Y \to F$ are $\mathcal{D}_X\langle\log Y\rangle$-linear with $st = \mathrm{res}(E)$ and $ts = \mathrm{res}(F)$.

Remark 5.6. It follows from the above that the $\mathcal{D}_X\langle\log Y\rangle$-linear homomorphisms s and t are isomorphisms on generalised eigensubbundles of res for all eigenvalues $\lambda \neq 0$.

Definition 5.7. We call pre-$\mathcal{D}$-module (E, F, s, t) **semistable** (respectively, **parabolic semistable**) if the Deligne connection E is semistable (respectively, parabolic semistable) as a $\mathcal{D}_X\langle\log Y\rangle$-module as defined in [10] (respectively, with its natural parabolic structure as defined earlier).

Remark 5.8. The difference in this new definition and the old one of [12] is that now we do not put any semistability condition on the $\mathcal{D}_X\langle\log Y\rangle|_Y$-module F, while earlier we had demanded that F should be semistable as a $\mathcal{D}_X\langle\log Y\rangle|_Y$-module.

Relation with $\mathcal{D}$-modules

We now describe how a pre-$\mathcal{D}$-module gives rise to a $\mathcal{D}$-module. Let (E, F, s, t) be a pre-$\mathcal{D}$-module on (X, Y). Then we get a $\mathcal{D}_X\langle\log Y\rangle$-submodule

$$E \oplus_s F \subset E \oplus F$$

which by definition consists of all local sections (u, v) with $u|_Y = s(v)$. As s is $\mathcal{D}_X\langle\log Y\rangle|_Y$-linear, this condition indeed defines a $\mathcal{D}_X\langle\log Y\rangle$-submodule of $E \oplus F$.

Now let $M_0 = E$ and $M_1 = \mathcal{O}_X(Y) \otimes_{\mathcal{O}_X} (E \oplus_s F)$. As $\mathcal{O}_X(Y)$ is naturally a left $\mathcal{D}_X\langle \log Y\rangle$-module, the tensor product $M_1 = \mathcal{O}_X(Y) \otimes_{\mathcal{O}_X} (E \oplus_s F)$ has a left $\mathcal{D}_X\langle \log Y\rangle$-module structure given by putting

$$\xi(p \otimes q) = \xi(p) \otimes q + p \otimes \xi(q)$$

where ξ, p, and q are local sections respectively of $T_X\langle \log Y\rangle$, $\mathcal{O}_X(Y)$ and $E \oplus_s F$. We have a $\mathcal{D}_X\langle \log Y\rangle$-linear inclusion $M_0 \hookrightarrow M_1$ defined by

$$u \mapsto x^{-1} \otimes (xu, 0)$$

We now define a $\mathbb{C}$-linear sheaf homomorphism $\nabla : M_0 \to \Omega^1_X \otimes M_1$ by putting

$$\nabla_\eta(u) = x^{-1} \otimes (\,(x\eta)(u),\ \eta(x)t(u|Y)\,)$$

for any local section η of the tangent sheaf T_X. This is compatible with given $\mathcal{D}_X\langle \log Y\rangle$-structures, in the sense that if η is a section of $T_X\langle \log Y\rangle \subset T_X$, then $\nabla_\eta(u)$ equals the image of $\eta(u) \in M_0$ under the inclusion $M_0 \hookrightarrow M_1$.

Finally, we define M to be the left $\mathcal{D}_X$ module which is the quotient of $\mathcal{D}_X \otimes_{\mathcal{D}_X\langle \log Y\rangle} M_1$ by its submodule generated by all elements of the type $\eta \otimes u - 1 \otimes \eta(u)$.

Proposition 5.9. *Given any pre-$\mathcal{D}$-module (E, F, s, t), the associated $\mathcal{D}_X$-module M is regular holonomic, with characteristic variety contained in $C_{X,Y}$. The following properties hold.*

(i) The module M is a non-singular connection if and only if $F = 0$.

(ii) The module M is a meromorphic connection if and only if $E \neq 0$ and $s : F \to E|_Y$ is an isomorphism.

(iii) The module M is set-theoretically supported on Y if and only if $E = 0$.

Moreover, given any regular holonomic $\mathcal{D}_X$-module N with characteristic variety contained in $C_{X,Y}$, there exists (up to unique isomorphism) a unique pre-$\mathcal{D}$-module (E, F, s, t) such that the associated $\mathcal{D}_X$-module M is isomorphic to N.

Proof (Sketch, also see [11]) The module M has a filtration F^iM which is 'good' with respect to the filtration $F^i\mathcal{D}_X$ of $\mathcal{D}_X$, defined by $F^0M = M_0$, $F^1M = M_1$, and $F^iM = (F^{i-1}\mathcal{D}_X)M_1$ for $i \geq 2$. The associated graded module over $Sym^*(T_X)$ shows that M is regular holonomic with the characteristic variety contained in $C_{X,Y}$. The statements (i), (ii), (iii) are clear from the construction of M. The backward passage from a $\mathcal{D}_X$-module to a pre-$\mathcal{D}$-module is via the existence of a **V-filtration** (which due to Malgrange [7] and Verdier [V]) $\ldots V^i(M) \subset V^{i+1}(M) \ldots$ on M, which is a certain filtration by $\mathcal{O}_X$-coherent $\mathcal{D}_X\langle \log Y\rangle$-modules. In terms of V-filtration, we put $E = V^0(M)$, $F = V^1(M)/V^0(M)$, and define s and t as the maps locally induced by x and $\partial/\partial x$, where x is a local defining equation for Y. The construction of a V-filtration depends on the choice of a fundamental domain in $\mathbb{C}$ for the exponential map $z \mapsto exp(2\pi i z)$. If we define the fundamental domain by the condition that $0 \leq Re(z) < 1$, then both $E = V^0(M)$ and $F = V^1(M)/V^0(M)$ will have all real parts of residual eigenvalues in $[0, 1)$ as desired, showing that (E, F, s, t) is a pre-$\mathcal{D}$-module as in Definition 5.5. Q.E.D.

Remark 5.10 Infinitesimal rigidity As a consequence of infinitesimal rigidity for Deligne construction (see [10]), a pre-$\mathcal{D}$-module (E, F, s, t) does not admit any nontrivial infinitesimal deformation such that the associated $\mathcal{D}_X$-module is constant.

Moduli construction

A **family** $(E, F, s, t)_S$ of pre-$\mathcal{D}$-modules parametrised by a complex scheme S will consist of a family of Deligne connections E parametrised by S, a vector bundle F on $Y \times S$ equipped with a structure of a $\mathcal{D}_{X\times S/S}\langle Y \times S\rangle|_{Y\times S}$-module with all real parts of residual eigenvalues in $[0, 1)$, and $\mathcal{D}_{X\times S/S}\langle Y \times S\rangle|_{Y\times S}$-linear homomorphisms $s : F \to E|_{Y\times S}$ and $t : E|_{Y\times S} \to F$ with $st = \mathrm{res}(E)$ and $ts = \mathrm{res}(F)$.

We now fix the ranks of E and F, and the residual eigenvalues of E with their multiplicities (in other words, the characteristic polynomial of $res(E)$ is fixed). This automatically fixes the characteristic polynomial of $res(F)$ by Remark5.6. Note that by [10], there exists a quasi-projective scheme S over $\mathbb{C}$ parametrising a locally universal family of semistable logarithmic connections, together with the action of a reductive algebraic group G such that a good quotient $S//G$ exists in the sense of GIT, and is the moduli of semistable logarithmic connections. The scheme U has a closed subscheme R where the the characteristic polynomial of $res(E)$ is the given one, and this subscheme is invariant under the action G. Hence a good quotient $R//G$ exists, and it is the moduli of Deligne connections. Similarly, by the theory of Λ-modules developed in [13], there exists a quasi-projective scheme R' over $\mathbb{C}$ parametrising a locally universal family of semistable $\mathcal{D}_X\langle \log Y\rangle|_Y$-modules on which the action of the Euler operator **e** is nilpotent, together with the action of a reductive algebraic group G' such that a good quotient $R'//G$ exists in the sense of GIT, and is the moduli of semistable $\mathcal{D}_X\langle\log Y\rangle|_Y$-modules with given residual eigenvalues. The extra data (s,t) is parametrised by a scheme V which is affine over $R\times R'$, such that there is a natural lift of $G\times G'$-action to U. By Ramanathan's lemma, a good quotient $\mathcal{M}$ exists. By its construction, $\mathcal{M}$ is coarse moduli for semistable pre-$\mathcal{D}$-modules (E,F,s,t) with given ranks and residues. By construction, this is a quasi-projective scheme over $\mathbb{C}$.

If a moduli for parabolic semistable Deligne constructions is obtained as a good GIT quotient as above, then a similar construction will give the moduli for parabolic semistable pre-$\mathcal{D}$-modules (E,F,s,t) with given ranks and residues.

Points of the moduli

Proposition 5.11. *The closed points of the moduli scheme of semistable pre-$\mathcal{D}$-modules bijectively correspond to isomorphism classes of $\mathcal{D}_X$-modules of the form*

$$j_{!+}(E|_{X-Y})\oplus i_+(W)$$

where E is a polystable Deligne connection on (X,Y), $j_{!+}(E|_{X-Y})$ is the minimal prolongation of the $\mathcal{D}_{X-Y}$-module $E|_{X-Y}$ to a $\mathcal{D}_X$-module, $i:Y\to X$ is the closed embedding of Y in X, W is a semisimple nonsingular integrable connection on Y, and $i_+(W)$ is the D_X-module which is the direct image (in the sense of $\mathcal{D}$-modules) of the $\mathcal{D}_Y$-module W.

Remark 5.12. Let E' be kernel of $E\to coker(\mathrm{res}(E))$, which is an **elementary transform** of E in the sense of Maruyama [8]. Then $j_{!+}(E|_{X-Y})$ is the $\mathcal{D}_X$-sub-module of $j_*(E|_{X-Y})$ generated by $\mathcal{O}_X(Y)\otimes E'$.

Proof of 5.11: Any Deligne connection E gives rise to a pre-$\mathcal{D}$-module

$$P_E=(E,\mathrm{im}(\mathrm{res}(E)),\theta_E,\mathrm{res}(E))$$

where $\mathrm{im}(res(E))\subset E|_Y$ is naturally a sub-$\mathcal{D}_X\langle\log Y\rangle|_Y$-module with inclusion $\theta_E:\mathrm{im}(res(E))\hookrightarrow E|_Y$, and $\mathrm{res}(E):E|_Y\to\mathrm{im}(\mathrm{res}(E))$ is the map induced by $\mathrm{res}(E):E|_Y\to E|_Y$.

Similarly, an integrable connection W on Y corresponds to a pre-$\mathcal{D}$-module

$$Q_W=(0,W,0,0)$$

In terms of the correspondence between pre-$\mathcal{D}$-modules and $\mathcal{D}$-modules, to prove the above proposition we have to show that the closed points of $\mathcal{M}$ are in bijection with the isomorphism classes of pre-$\mathcal{D}$-modules of the type

$$P_E \oplus Q_W$$

where E is a polystable Deligne connection with the given rank and residual eigenvalues, and W is an integrable connection on Y.

Note that closed orbits in R and R' exactly correspond to polystable modules, which in the case of (non-singular) connections on Y means those which are semi-simple (monodromy is completely reducible). So the set of isomorphism classes of pre-$\mathcal{D}$-modules of the type $P_E \oplus Q_W$, where E is polystable and W semisimple, injects into the set of closed points of the moduli. It remains to show that these are all the points.

For this, given any pre-$\mathcal{D}$-module (E, F, s, t), let $W' = \ker(s)$ with inclusion $\alpha : W' \hookrightarrow F$ and $W'' = \mathrm{im}(s)/\mathrm{im}(\mathrm{res}(E))$ with quotient map $\beta : \mathrm{im}(s) \to W''$. These vector bundles on Y which are $\mathcal{D}_Y$-modules, and the maps α and β are $\mathcal{D}_X\langle\log Y\rangle|_Y$-linear. We have the following commutative diagram with short-exact columns.

$$\begin{array}{ccccc}
0 & \to & W' & \to & 0 \\
\downarrow & & \alpha \downarrow & & \downarrow \\
E|_Y & \stackrel{t}{\to} & F & \stackrel{t}{\to} & E|_Y \\
\| & & s \downarrow & & \| \\
E|_Y & \stackrel{\mathrm{res}(E)}{\to} & \mathrm{im}(s) & \hookrightarrow & E|_Y
\end{array}$$

This allows us to construct a family of pre-$\mathcal{D}$-modules parametrised by the affine line $\mathbf{A}^1_{\mathbb{C}}$, which outside the origin is the constant family corresponding to (E, F, s, t) and at origin restricts to $(0, W', 0, 0) \oplus (E, \mathrm{im}(s), \nu, \mathrm{res}(E))$ where $\nu : \mathrm{im}(s) \hookrightarrow E|_Y$ is the inclusion.

Next, we have the commutative diagram with short-exact columns

$$\begin{array}{ccccc}
E|_Y & \stackrel{\mathrm{res}(E)}{\to} & \mathrm{im}(\mathrm{res}(E)) & \hookrightarrow & E|_Y \\
\| & & \downarrow & & \downarrow \\
E|_Y & \stackrel{s}{\to} & \mathrm{im}(s) & \hookrightarrow & E|_Y \\
\downarrow & & \beta \downarrow & & \downarrow \\
0 & \to & W'' & \to & 0
\end{array}$$

Therefore there exists a family of pre-$\mathcal{D}$-modules parametrised by $\mathbf{A}^1_{\mathbb{C}}$, which outside the origin is the constant family $(E, \mathrm{im}(s), \nu, \mathrm{res}(E))$ and at origin restricts to $(E, \mathrm{im}(\mathrm{res}(E)), \theta_E, \mathrm{res}(E)) \oplus (0, W'', 0, 0)$ where by definition $\theta_E : \mathrm{im}(\mathrm{res}(E)) \hookrightarrow E|_Y$ is the inclusion.

Hence by the separatedness of the quasi-projective moduli scheme $\mathcal{M}$, it follows that the original semistable pre-$\mathcal{D}$-module (E,F,s,t) and the semistable pre-$\mathcal{D}$-module $(E,\mathrm{im}(\mathrm{res}(E)),\theta_E,\mathrm{res}(E))\oplus(0,W'\oplus W'',0,0)$ are represented by the same point of $\mathcal{M}$.

Next, let $0\subset E_1\subset\ldots\subset E_\ell=E$ be an S-filtration of E, that is, the E_i are sub Deligne connections which have the same reduced Hilbert polynomial as E. Let $\mathrm{Gr}(E)$ be the associated graded Deligne connection. Hence there exists a 1-parameter family of logarithmic connections which is generically E and restricts to $\mathrm{Gr}(E)$ at the origin. This gives a 1-parameter family of pre-$\mathcal{D}$-modules which is generically $(E,\mathrm{im}(\mathrm{res}(E)),\theta,\mathrm{res}(E))$ and restricts at the origin to the object $(\mathrm{Gr}(E),\mathrm{Gr}(\mathrm{im}(\mathrm{res}(E)),\mathrm{Gr}(\theta_E),\mathrm{Gr}(\mathrm{res}(E)))$. In turn the latter deforms to a pre-$\mathcal{D}$-module

$$(\mathrm{Gr}(E),\mathrm{im}(\mathrm{res}(\mathrm{Gr}(E))),\theta_{\mathrm{Gr}(E)},\mathrm{res}(\mathrm{Gr}(E)))\oplus(0,W''',0,0)$$

where by definition $\theta_{\mathrm{Gr}(E)}:\mathrm{im}(\mathrm{res}(\mathrm{Gr}(E)))\hookrightarrow\mathrm{Gr}(E)|_Y$ is the inclusion (which does not equal $\mathrm{Gr}(\theta_E)$ in general).

If W is the semisimplification of $W'\oplus W''\oplus W'''$, then it follows that (E,F,s,t) is represented by the closed point corresponding to the polystable pre-$\mathcal{D}$-module $P_{\mathrm{Gr}(E)}\oplus Q_W$. This completes the proof of the proposition. Q.E.D.

§6. Case of a normal crossing divisor

In this last section, we consider the general case where Y is a normal crossing divisor. To carry out the general theory, it is not necessary to assume that the irreducible components of Y are smooth. We can in that case set up a system of finite étale Galois covers of the normalisations of the closures of the strata defined by Y as in [11], and carry out the discussion below. However, just to keep the notation simple in this article, we make the assumption that Y is the union of two smooth divisors $Y=Y_1\bigcup Y_2$, which intersect transversely along $Z=Y_1\bigcap Y_2$ which is a smooth connected codimension 2 subvariety of X. This simplified situation is already adequate to exhibit the changes we make in the definition of pre-$\mathcal{D}$-modules and in the definition of their semistability.

Let $\mathbf{e}_1\in H^0(Y_1,\mathcal{D}_X\langle\log Y\rangle)|_{Y_1}$ and $\mathbf{e}_2\in H^0(Y_2,\mathcal{D}_X\langle\log Y\rangle)|_{Y_2}$ denote the respective Euler operators. Any point of Z has an analytic or étale coordinate neighbourhood in X with coordinates $x_1,\ldots,x_d$ ($d=\dim(X)$) in which Y is defined by $x_1x_2=0$. Then $\mathbf{e}_1$ is locally defined by $x_1(\partial/\partial x_1)$ and $\mathbf{e}_2$ is defined by $x_2(\partial/\partial x_2)$.

We are now ready to state the new definition of a pre-$\mathcal{D}$-module and its semi-stability, in the above simple set-up.

Definition 6.1. A **pre-$\mathcal{D}$-module** on (X, Y) consists of the following data, satisfying certain conditions.

(1) A $\mathcal{D}_X\langle\log Y\rangle|_Y$-module E on (X, Y), $\mathcal{D}_X\langle\log Y\rangle$-modules F_1 and F_2 on Y_1 and Y_2, and a $\mathcal{D}_X\langle\log Y\rangle|_Y$-module G on (X, Y) such that E is a vector bundle on X, F_1 and F_2 are vector bundles on Y_1 and Y_2, and G is a vector bundle on Z.

(2) For $a = 1, 2$, we are given $\mathcal{D}_X\langle\log Y\rangle)|_{Y_a}$-linear maps $s_a : F_a \to E|_{Y_a}$ and $t_a : E|_{Y_a} \to F_a$.

(3) For $a = 1, 2$, we are given $\mathcal{D}_X\langle\log Y\rangle)|_Z$-linear maps $s'_b : G \to F_a|_Z$ and $t'_b : F_a|_Z \to G$.

The above data should satisfy the following conditions:

(4) Consider the endomorphism $\mathrm{res}_a(E)$ of $E|_{Y_a}$ induced by $\mathbf{e}_a$, the endomorphism $\mathrm{res}_a(F_a)$ of F_a induced by $\mathbf{e}_a$, the endomorphism $\mathrm{res}_b(F_a)$ of $F_a|_Z$ induced by $\mathbf{e}_b$ for $a \neq b$, and the endomorphism of $\mathrm{res}_a(G)$ of G induced by $\mathbf{e}_a$, for $a = 1, 2$. All the eigenvalues of these generalised residue endomorphisms should have their real parts in $[0, 1)$. (In particular, E is a Deligne connection.)

(5) We should have $s_a t_a = \mathrm{res}_a(E)$, $t_a s_a = \mathrm{res}_a(F_a)$, $s'_b t'_b = \mathrm{res}_b(F_a)$ for $a \neq b$, and $t'_b s'_b = \mathrm{res}_b(G)$.

(6) The following commutativity conditions should hold over Z for $a \neq b$:

$$
\begin{aligned}
&(1) \qquad & (s_b|_Z) \circ s'_a &= (s_a|_Z) \circ s'_b : G \to E|_Z \\
&(2) \qquad & t'_b \circ (t_a|_Z) &= t'_a \circ (t_b|_Z) : E|_Z \to G \\
&(3) \qquad & s'_a \circ t'_b &= (t_b|_Z) \circ (s_a|_Z) : F_a|_Z \to F_b|_Z
\end{aligned}
$$

We say that a pre-$\mathcal{D}$-module as defined above is **semi-stable** if the following two conditions hold:

(1) The logarithmic connection E is semistable.

(2) The generalised eigensubbundle $F_{a,0} \subset F_a$ for eigenvalue 0 of $\mathrm{res}_a(F_a)$ is a semistable $\mathcal{D}_X\langle\log Y\rangle|_{Y_a}$-module for $a = 1, 2$.

Remark 6.2. The intersection G_0 of the generalised eigensubbundles of G for eigenvalue 0 under $\mathrm{res}_a(G)$ and $\mathrm{res}_b(G)$ (which is indeed a vector subbundle by integrability) is automatically a semistable $\mathcal{D}_X\langle\log Y\rangle|_Z$-module.

Relation with $\mathcal{D}$-modules

A regular holonomic $\mathcal{D}$-module M on X whose characteristic variety is contained in $C_{X,Y}$ corresponds uniquely to a pre-$\mathcal{D}$-module as defined above. Over a polydisk, this is essentially the content of the main theorem of the paper of Galligo, Granger, Maisonobe [5]. This is best expressed in terms of a V-filtration on M (see Section 4.3, page 16 [11]), which actually produces a pre-$\mathcal{D}$-module as defined above (only the new definition of a pre-$\mathcal{D}$-module was missing earlier!).

Again, this bijective correspondence is infinitesimally rigid as shown in [11].

Construction of moduli

We fix the ranks of E, F_a, G and also the characteristic polynomials of the various residue endomorphisms $\mathrm{res}_a(E)$, $\mathrm{res}_a(F_a)$, $\mathrm{res}_b(F_a)$, and $\mathrm{res}_a(G)$, and the dimensions of intersections of their generalised eigensubbundles. In particular, this fixes the Hilbert polynomials of the bundles E, F_a, G, and those of the generalised eigensubbundles of the various residues.

By the theory of Simpson of moduli for Λ-modules together with the Lemma 3.13 above, there exist the following:

(1) A quasi-projective scheme R together with action of a reductive algebraic group H and a locally universal family of $\mathcal{O}_X$-locally free semistable $\mathcal{D}_X\langle\log Y\rangle$-modules E with prescribed residual eigenvalues for $\mathrm{res}_a(E)$ over Y_a and prescribed ranks for intersection of generalised eigensubbundles of $\mathrm{res}_a(E)$ and $\mathrm{res}_b(E)$ over Z, with lift of H-action to the family, such that a good quotient $R//H$ exists, which is the moduli of semistable Deligne connections on (X,Y) for the prescribed residual data.

(2) For $a = 1, 2$, a quasi-projective scheme R'_a with action of a reductive algebraic group H'_a, with a locally family of $\mathcal{O}_{Y_a}$-locally free semistable $\mathcal{D}_X\langle\log Y\rangle$-modules E with $\mathrm{res}_a(F_a)$ nilpotent, and prescribed residual eigenvalues for $\mathrm{res}_b(F_a)$ over Z and prescribed ranks for intersection of generalised eigensubbundles of $\mathrm{res}_a(F_{b,0})$ and $\mathrm{res}_b(F_{a,0})$ over Z, with lift of H-action to the family, such that a good quotient $R'_a//H_a$ exists, which is the moduli of $\mathcal{O}_{Y_a}$-locally free semistable $\mathcal{D}_X\langle\log Y\rangle$-modules for the given residual data.

(3) A quasi-projective scheme R'' together with action of reductive algebraic groups H'' and a locally universal family of $\mathcal{O}_Z$-locally free semistable $\mathcal{D}_X\langle\log Y\rangle$-modules G_0 such that $\mathrm{res}_a(G)$ is nilpotent for $a = 1, 2$, with lift of H''-action to the family, such that a good quotient $R''//H''$

exists, which is the moduli of $\mathcal{O}_Z$-locally free semistable $\mathcal{D}_X\langle \log Y\rangle$-modules G_0 for the given residual data.

Over the product $U = R \times R_1' \times R_2' \times R''$, we have an scheme V with an affine morphism to U and a natural lift of the action of $K = H \times H_1' \times H_2' \times H''$, such that V parametrises a locally universal family of semistable pre-$\mathcal{D}$-modules with the prescribed residual data. Then $V//K$ (a good quotient which exists by Ramanathan's lemma) is the desired moduli scheme $\mathcal{M}$ for semi-stable pre-$\mathcal{D}$-modules. It is quasi-projective over $\mathbb{C}$ by its construction.

A description of the points of $\mathcal{M}$, analogous to that of Proposition 5.11 is possible also in the general case, and a description of the tangent space to the moduli functor in terms of certain hypercohomologies can be given.

Remark 6.3. As over any perverse sheaf we have at most one semi-stable pre-$\mathcal{D}$-module in the new sense, which moreover is infinitesimally rigid as before, the Riemann-Hilbert morphism from $\mathcal{M}$ to the moduli of perverse sheaf is an open embedding. It can fail to be surjective as semistability is not automatic for pre-$\mathcal{D}$-modules.

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School of Mathematics
Tata Institute of Fundamental Research
Homi Bhabha Road
Mumbai 400 005
India