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Maximal functions, Riesz potentials and Sobolev's inequality in generalized Lebesgue spaces

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Abstract.

Our aim in this paper is to deal with the boundedness of maximal functions in Lebesgue spaces with variable exponent. Our result extends the recent work of Diening [4], Cruz-Uribe, Fiorenza and Neugebauer [3] and the authors [8]. As an application of the boundedness of maximal functions, we show Sobolev's inequality for Riesz potentials with variable exponent.

§1. Introduction

Sobolev functions play a significant role in many fields of analysis. In recent years, the generalized Lebesgue spaces $L^{p(\cdot)}$ and the corresponding Sobolev spaces $W^{m,p(\cdot)}$ have attracted more and more attention, in connection with the study of elasticity, fluid mechanics and differential equations with $p(\cdot)$-growth; see Růžička [16]. One of the most important results for Sobolev functions is so called Sobolev's embedding theorem, and the corresponding result has been extended to Sobolev spaces of variable exponent by many authors; see for example [2, 5, 7, 8, 12, 17]. Our main task in this study is to obtain boundedness properties for Riesz potentials. For this purpose, the boundedness of maximal functions gives a crucial tool by a trick of Hedberg [11], which is originally based on the recent work by Diening [4].

Let Ω be an open set in $\mathbf{R}^n$. We use the notation $B(x,r)$ to denote the open ball centered at x of radius r. For a locally integrable function

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f on Ω, we consider the maximal function Mf defined by

$$Mf(x) = \sup_B \frac{1}{|B|} \int_{\Omega \cap B} |f(y)| dy,$$

where the supremum is taken over all balls $B = B(x, r)$ and $|B|$ denotes the volume of B. Let $p(\cdot)$ be a positive continuous function on Ω such that $p(x) > 1$ on Ω. Following Orlicz [15] and Kováčik and Rákosník [13], we define the $L^{p(\cdot)}(\Omega)$ norm by

$$\|f\|_{p(\cdot)} = \|f\|_{p(\cdot),\Omega} = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{f(y)}{\lambda}\right|^{p(y)} dy \le 1\right\}$$

and denote by $L^{p(\cdot)}(\Omega)$ the space of all measurable functions f on Ω with $\|f\|_{p(\cdot)} < \infty$.

In this paper we are concerned with $p(\cdot)$ satisfying a condition of the form :

$$|p(x) - p(y)| \le \frac{\log(\varphi(|x-y|))}{\log(1/|x-y|)}$$

whenever $x \in \Omega$, $y \in \Omega$ and $|x - y| < 1/2$, where φ is a positive nonincreasing function on $(0, \infty)$ of logarithmic type. Our typical example of φ is

$$\varphi(r) = a(\log(1/r))^b (\log\log(1/r)))^c$$

for small $r > 0$, where $a > 0$, $b > 0$ and $-\infty < c < \infty$. In case Ω is not bounded, we further assume that

$$|p(x) - p_\infty| \le \frac{C}{\log(e + |x|)} \qquad \text{whenever } x \in \Omega,$$

where $1 < p_\infty < \infty$.

Our first aim in this paper is to find a function $\Phi(t, x)$ on $\mathbf{R} \times \Omega$ such that

$$\int_\Omega \Phi(Mf(x), x) dx \le C \qquad \text{whenever } \|f\|_{p(\cdot)} \le 1$$

(in Theorems 2.7 and 4.7 below). If $\varphi(r) = a(\log(e + 1/r))^b$, then our result was proved by Diening [4] (when $b = 0$ and Ω is bounded), Cruz-Uribe, Fiorenza and Neugebauer [3, Theorem 1.5] (when $b = 0$ and Ω is not bounded), and the authors [8, Theorem 2.4] (when $b > 0$ and Ω is bounded).

We consider the Riesz potential of order α for a locally integrable function f on Ω, which is defined by

$$I_\alpha f(x) = \int_\Omega |x-y|^{\alpha-n} f(y)dy.$$

Here $0 < \alpha < n$. As an application of the boundedness of maximal functions, we give Sobolev's inequality for Riesz potentials with variable exponent. We in fact find a function $\Psi(t,x)$ on $\mathbf{R} \times \Omega$ such that

$$\int_\Omega \Psi(I_\alpha f(x), x)dx \le C \qquad \text{whenever } \|f\|_{p(\cdot)} \le 1$$

(see Theorems 3.5 and 5.6 below). In case $\varphi(r) = a(\log(e + 1/r))^b$, our result was proved by Samko [17] (when $b = 0$ and Ω is bounded), Diening [5] (when $b = 0$ and $p(\cdot)$ is constant outside of a large ball), Capone, Cruz-Uribe and Fiorenza [2, Theorem 1.6] (when $b = 0$ and Ω is not bounded), and the authors [8, Theorem 3.4] (when $b > 0$ and Ω is bounded).

For related results, see also Adams-Hedberg [1], Diening [5], Edmunds-Rákosník [6], Harjulehto-Hästö-Pere[10], Kokilshvili-Samko [12], Kováčik-Rákosník [13], Nekvinda [14], Růžička [16] and the authors [9].

§2. Maximal functions

Throughout this paper, let C denote various constants independent of the variables in question.

Consider a positive nonincreasing function φ on the interval $(0,\infty)$ of logarithmic type, which has the following properties:

(φ1) $\varphi(\infty) = \lim_{t\to\infty} \varphi(t) > 0$;

(φ2) $(\log(1/t))^{-\varepsilon_0}\varphi(t)$ is nondecreasing on $(0, r_0)$ for some $\varepsilon_0 > 0$ and $r_0 > 0$.

Remark 2.1. (i) By condition (φ2), we see that

$$C^{-1}\varphi(r) \le \varphi(r^2) \le C\varphi(r) \qquad \text{whenever } r > 0,$$

which implies the doubling condition on φ.

(ii) We see from (φ2) that for each $\delta > 0$, $t^\delta\varphi(t)$ is nondecreasing on some interval $(0,T)$, $T = T(\delta) > 0$.

(iii) Our typical example of φ is of the form

$$\varphi(t) = a(\log(1/t))^b(\log(\log(1/t)))^c$$

for small $t > 0$, where $a > 0$, $b > 0$ and $c \in \mathbf{R}$.

In this section, let Ω be an open set in $\mathbf{R}^n$. Let $p(\cdot)$ be a positive continuous function on Ω satisfying

(p1) $1 < p_-(\Omega) = \inf_\Omega p(x) \le \sup_\Omega p(x) = p_+(\Omega) < \infty$;

(p2) $|p(x)-p(y)| \le \log(\varphi(|x-y|))/\log(1/|x-y|)$ whenever $|x-y| < 1/2$, $x \in \Omega$ and $y \in \Omega$.

Lemma 2.2. *If* $0 < r_0 < 1$ *and* $\log\varphi(r_0) > \varepsilon_0$, *then* $\log\varphi(r)/\log(1/r)$ *is nondecreasing on* $(0, r_0)$.

Proof. Let $0 < r_1 < r_2 < r_0 < 1$. By ($\varphi 2$), we have

$$\begin{aligned}\frac{\log\varphi(r_1)}{\log(1/r_1)} &\le \varepsilon_0 \frac{\log(\log(1/r_1)) - \log(\log(1/r_2))}{\log(1/r_1)} + \frac{\log\varphi(r_2)}{\log(1/r_1)} \\ &= \frac{\log\varphi(r_2)}{\log(1/r_2)} + \frac{1}{\log(1/r_1)}\left\{\varepsilon_0 \log\left(\frac{\log(1/r_1)}{\log(1/r_2)}\right)\right. \\ &\quad \left. + \frac{\log(r_1/r_2)}{\log(1/r_2)}\log\varphi(r_2)\right\}.\end{aligned}$$

Since $\log(1+t) < t$ for $t > 0$,

$$\log\left(\frac{\log(1/r_1)}{\log(1/r_2)}\right) \le \frac{\log(r_2/r_1)}{\log(1/r_2)},$$

so that

$$\begin{aligned}&\frac{\log\varphi(r_1)}{\log(1/r_1)} - \frac{\log\varphi(r_2)}{\log(1/r_2)} \\ \le\ &\frac{1}{\log(1/r_1)}\left(\frac{\log(r_2/r_1)}{\log(1/r_2)}\right)(\varepsilon_0 - \log\varphi(r_2)) < 0,\end{aligned}$$

as required. □

Let $1/p'(x) = 1 - 1/p(x)$. Then note that

$$\begin{aligned}p'(y) - p'(x) &= \frac{p(x) - p(y)}{(p(x)-1)(p(y)-1)} \\ &= \frac{p(x)-p(y)}{(p(x)-1)^2} + \frac{(p(x)-p(y))^2}{(p(x)-1)^2(p(y)-1)}.\end{aligned}$$

Hence, in view of ($\varphi 2$), we have the following result.

Lemma 2.3. *There exists a positive constant C such that*

$$|p'(x) - p'(y)| \le \omega(|x-y|) \qquad \text{whenever } x \in \Omega \text{ and } y \in \Omega,$$

where

$$\omega(r) = \omega(r; x, C) = \frac{1}{(p(x)-1)^2} \frac{\log(C\varphi(r))}{\log(1/r)}$$

for $0 < r \le r_0$ and $\omega(r) = \omega(r_0)$ for $r \ge r_0$.

In what follows, we may assume that $\omega(r)$ is nondecreasing as a function of $r \in (0, \infty)$. Moreover, if f is a function on Ω, then we set $f = 0$ outside Ω.

Lemma 2.4. *Let f be a nonnegative measurable function on Ω with $\|f\|_{p(\cdot)} \le 1$. Then*

$$\{Mf(x)\}^{p(x)} \le C\left\{Mg(x)(\varphi(Mg(x)^{-1}))^{n/p(x)} + 1\right\}$$

for all $x \in \Omega$, where $g(y) = f(y)^{p(y)}$.

Proof. For $0 < \mu \le 1$ and $r > 0$, we have by Lemma 2.3

$$\begin{aligned} f_B &\equiv \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)dy \\ &\le \mu \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} (1/\mu)^{p'(y)} dy + \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)^{p(y)} dy \right) \\ &\le \mu \left((1/\mu)^{p'(x)+\omega(r)} + F \right), \end{aligned}$$

where $F = |B(x,r)|^{-1} \int_{B(x,r)} f(y)^{p(y)} dy$. When F is bounded, say $F \le R_0$, by considering $\mu = 1$, we have

$$f_B \le C.$$

Hence it suffices to treat the case that $F \ge R_0 > r_0^{-1}$; in this case we may assume that $0 < r < r_0$ since $\|f\|_{p(\cdot)} \le 1$. By considering $\mu = F^{-1/\{p'(x)+\omega(r)\}}$ when $F > 1$, we find

$$f_B \le 2F^{1/p(x)} F^{\omega(r)/\{p'(x)(p'(x)+\omega(r))\}} \le 2F^{1/p(x)} F^{\omega(r)/p'(x)^2}.$$

If $r \le F^{-1} < r_0$, then we see from Lemma 2.2 that

$$f_B \le CF^{1/p(x)} (\varphi(F^{-1}))^{1/p(x)^2} \le CF^{1/p(x)} (\varphi(F^{-1}))^{n/p(x)^2}.$$

If $F^{-1} < r < r_0$, then

$$\begin{aligned} & F^{1/p(x)+\omega(r)/p'(x)^2} \\ \le \; & Cr^{-n/p(x)-n\omega(r)/p'(x)^2} \left(\int_{B(x,r)} f(y)^{p(y)} dy \right)^{1/p(x)+\omega(r)/p'(x)^2} . \end{aligned}$$

Since $r^{-n\omega(r)/p'(x)^2} \le C\varphi(r)^{n/p(x)^2}$ and $\int_{B(x,r)} f(y)^{p(y)} dy \le 1$ by our assumption, we obtain

$$\begin{aligned} & F^{1/p(x)+\omega(r)/p'(x)^2} \\ \le \; & Cr^{-n/p(x)} \varphi(r)^{n/p(x)^2} \left(\int_{B(x,r)} f(y)^{p(y)} dy \right)^{1/p(x)+\omega(r)/p'(x)^2} \\ \le \; & Cr^{-n/p(x)} \varphi(r)^{n/p(x)^2} \left(\int_{B(x,r)} f(y)^{p(y)} dy \right)^{1/p(x)} \\ \le \; & Cr^{-n/p(x)} \varphi(F^{-1})^{n/p(x)^2} \left(\int_{B(x,r)} f(y)^{p(y)} dy \right)^{1/p(x)} \\ \le \; & CF^{1/p(x)} \varphi(F^{-1})^{n/p(x)^2}. \end{aligned}$$

Now it follows that

$$f_B \le CF^{1/p(x)} \varphi(F^{-1})^{n/p(x)^2},$$

which completes the proof. □

Lemma 2.5. *For each $\delta > 0$, there exists $T_0 > e$ such that $s^\delta \varphi(s^{-1})^{-1}$ is nondecreasing on (T_0, ∞).*

Proof. By (φ2), it follows that $(\log s)^{\varepsilon_0} \varphi(s^{-1})^{-1}$ is nondecreasing on (T_1, ∞) for some $T_1 > e$. Since

$$s^\delta \varphi(s^{-1})^{-1} = s^\delta (\log s)^{-\varepsilon_0} \times (\log s)^{\varepsilon_0} \varphi(s^{-1})^{-1},$$

the present lemma is obtained. □

Lemma 2.6. *If $\|f\|_{p(\cdot)} \le 1$, then*

$$\left\{ Mf(x) (\varphi(Mf(x)^{-1}))^{-n/p(x)^2} \right\}^{p(x)} \le C \left(Mg(x) + 1 \right)$$

for $x \in \Omega$.

Proof. For simplicity, set $a = Mf(x)$ and $b = Mg(x)$. By Lemma 2.4, we have

$$a^p \le C\left(b\varphi(b^{-1})^{cp} + 1\right)$$

with $p = p(x)$ and $c = n/p^2$. We may assume that a is large enough, that is, $a > T_0 > 1$. Using Lemma 2.5, we find

$$\left\{a\varphi(a^{-1})^{-c}\right\}^p \le Cb\varphi(b^{-1})^{cp} \times \varphi(Cb^{-1/p}\varphi(b^{-1})^{-c})^{-cp}.$$

Note from (φ2) that

$$\varphi(Cb^{-1/p}\varphi(b^{-1})^{-c})^{-1} \le C\varphi(b^{-1})^{-1}.$$

Hence it follows that

$$\left\{a\varphi(a^{-1})^{-c}\right\}^p \le Cb$$

whenever $a > T_0$, which proves

$$\left\{a\varphi(a^{-1})^{-c}\right\}^p \le C(b+1),$$

as required. □

Theorem 2.7. *Let Ω be an open set in $\mathbf{R}^n$ such that $|\Omega| < \infty$. If $A(x) = a/p(x)^2$ with $a > n$, then*

$$\int_\Omega \left\{Mf(x)(\varphi(Mf(x)^{-1}))^{-A(x)}\right\}^{p(x)} dx \le C$$

whenever f is a measurable function on Ω with $\|f\|_{p(\cdot)} \le 1$.

Proof. Let $p_0(x) = p(x)/p_0$ for $1 < p_0 < p_-(\Omega)$. Then Lemma 2.6 yields

$$\left\{Mf(x)(\varphi(Mf(x)^{-1}))^{-n/p_0(x)^2}\right\}^{p_0(x)} \le C\left\{Mg_0(x) + 1\right\}$$

for $x \in \Omega$, where $g_0(y) = f(y)^{p_0(y)}$. Choosing $p_0 > 1$ such that $np_0^2/p(x)^2 < A(x)$, we establish

$$\left\{Mf(x)(\varphi(Mf(x)^{-1}))^{-A(x)}\right\}^{p(x)} \le C\left\{Mg_0(x) + 1\right\}^{p_0}.$$

Since $g_0 \in L^{p_0}(\Omega)$, we deduce the required inequality by the boundedness of maximal functions in L^{p_0}. □

Remark 2.8. Set $\Phi(r,x) = \{r\varphi(r^{-1})^{-A(x)}\}^{p(x)}$ for $r \geq 0$ and $x \in \Omega$. Then Theorem 2.7 assures the existence of $C > 0$ such that

$$\int_\Omega \Phi(Mf(x)/C, x)dx \leq 1 \qquad \text{whenever } \|f\|_{p(\cdot)} \leq 1.$$

As in Edmunds and Rákosník [6], we define

$$\|f\|_\Phi = \|f\|_{\Phi,\Omega} = \inf\{\lambda > 0 : \int_\Omega \Phi(|f(x)|/\lambda, x)dx \leq 1\};$$

then it follows that

$$\|Mf\|_\Phi \leq C\|f\|_{p(\cdot)} \qquad \text{for } f \in L^{p(\cdot)}(\Omega).$$

If $\varphi(r) = a(\log(e+1/r))^b$, then Theorem 2.7 was proved by Diening [4] (when $b = 0$) and the authors [8, Theorem 2.4] (when b is general).

Remark 2.9. For $0 < r < 1/2$, let

$$G = \{x = (x_1, x_2) : 0 < x_1 < 1, -1 < x_2 < 1\}$$

and

$$G(r) = \{x = (x_1, x_2) : 0 < x_1 < r, r < x_2 < 2r\}.$$

For $p(0) = p_0 > 1$, define

$$p(x_1, x_2) = \begin{cases} p_0 - \log(\varphi(x_2))/\log(1/x_2) & \text{when } 0 < x_2 \leq r_0, \\ p_0 & \text{when } x_2 \leq 0; \end{cases}$$

set $p(x_1, x_2) = p(x_1, r_0)$ when $x_2 > r_0$. Here we take $r_0 > 0$ so small that $p(x_1, r_0) > 1$. Consider

$$f_r(y) = \chi_{G(r)}(y)$$

with χ_E denoting the characteristic function of a set E, and set $g_r = f_r/\|f_r\|_{p(\cdot),G}$. Then we insist for $0 < r < r_0$:

(i) $\|f_r\|_{p(\cdot),G} \leq C_1 r^{2/p_0}\varphi(r)^{-2/p_0^2}$;

(ii) $Mg_r(x) \geq C_2 r^{-2/p_0}\varphi(r)^{2/p_0^2}$ for $0 < x_1 < r$ and $-r < x_2 < 0$.

By integration of (ii) we see that

$$\int_G \left\{Mg_r(x)(\varphi(Mg_r(x)^{-1}))^{-2/p(x)^2}\right\}^{p(x)} dx \geq C_3,$$

which means that Theorem 2.7 does not hold for $A(x) < 2/p(x)^2$.

Remark 2.10. For $0 < r < 1/2$, let G and $G(r)$ be as above. Define

$$p(x_1, x_2) = \begin{cases} p_0 + \log(\varphi(x_2))/\log(1/x_2) & \text{when } 0 < x_2 \le r_0, \\ p_0 & \text{when } x_2 \le 0; \end{cases}$$

and $p(x_1, x_2) = p(x_1, r_0)$ when $x_2 > r_0$. Setting

$$G'(r) = \{x = (x_1, x_2) : 0 < x_1 < r, -r < x_2 < 0\},$$

we consider

$$f_r'(y) = \chi_{G'(r)}(y)$$

and set $g_r' = f_r'/\|f_r'\|_{p(\cdot),G}$. Then we insist for $0 < r < r_0/2$:

(i) $\|f_r'\|_{p(\cdot),G} = r^{2/p_0}$;

(ii) $Mg_r'(x) \ge C_1 r^{-2/p_0}$ for $0 < x_1 < r$ and $r < x_2 < 2r$;

(iii) $\int_G \left\{ Mg_r'(x)\varphi(Mg_r'(x)^{-1})^{-2/p(x)^2} \right\}^{p(x)} dx \ge C_2$,

as above.

§3. Sobolev's inequality

Let $p(\cdot)$ be a continuous function on Ω satisfying (p1) and (p2). Further, suppose

$$p_+ = p_+(\Omega) < n/\alpha$$

and set

$$\frac{1}{p^\sharp(x)} = \frac{1}{p(x)} - \frac{\alpha}{n}.$$

For $0 < \alpha < n$, we consider the Riesz potential $I_\alpha f$ of measurable functions $f \in L^{p(\cdot)}(\Omega)$, which is defined by

$$I_\alpha f(x) = \int |x - y|^{\alpha - n} f(y) dy;$$

recall that we set $f = 0$ outside Ω. Set

$$S_f = \{x \in \mathbf{R}^n : f(x) \neq 0\}.$$

In this section, we assume

$$|S_f| < \infty,$$

where $|E|$ denotes the n-dimensional measure of a measurable set E.

Lemma 3.1. *Let f be a nonnegative measurable function on Ω such that $\|f\|_{p(\cdot)} \le 1$ and $|S_f| \le 1$. Then*

$$\int_{\Omega \setminus B(x,\delta)} |x - y|^{\alpha - n} f(y) dy \le C\delta^{-n/p^\sharp(x)} \varphi(\delta)^{n/p(x)^2}$$

for $x \in \Omega$ *and* $\delta \in (0,1)$.

Proof. For $\mu > 0$, since $\|f\|_{p(\cdot)} \le 1$, we have

$$
\begin{aligned}
&\int_{\Omega\setminus B(x,\delta)} |x-y|^{\alpha-n} f(y)dy \\
\le\ & \mu\left(\int_{S_f\setminus B(x,\delta)} (|x-y|^{\alpha-n}/\mu)^{p'(y)}dy + \int_{S_f\setminus B(x,\delta)} f(y)^{p(y)}dy\right) \\
\le\ & \mu\left(\int_{S_f\setminus B(x,\delta)} (|x-y|^{\alpha-n}/\mu)^{p'(y)}dy + 1\right).
\end{aligned}
$$

Consider the set

$$E = \{y \in S_f : |x-y|^{\alpha-n} \ge \mu\} \cap \Omega.$$

Then we have

$$\int_{S_f\setminus\{E\cup B(x,\delta)\}} (|x-y|^{\alpha-n}/\mu)^{p'(y)}dy \le |S_f| \le 1$$

by our assumption. Further, since $p'(y) \le p'(x) + \omega(|x-y|)$ by Lemma 2.3, we have

$$
\begin{aligned}
&\int_{E\setminus B(x,\delta)} (|x-y|^{\alpha-n}/\mu)^{p'(y)}dy \\
\le\ & \int_{E\setminus B(x,\delta)} (|x-y|^{\alpha-n}/\mu)^{p'(x)+\omega(|x-y|)}dy.
\end{aligned}
$$

If $\mu > 1$, then we see that

$$
\begin{aligned}
&\int_{E\setminus B(x,\delta)} (|x-y|^{\alpha-n}/\mu)^{p'(x)+\omega(|x-y|)}dy \\
\le\ & \mu^{-p'(x)-\omega(\delta)} \int_{\mathbf{R}^n\setminus B(x,\delta)} |x-y|^{(\alpha-n)(p'(x)+\omega(|x-y|))}dy \\
\le\ & C\mu^{-p'(x)-\omega(\delta)}\delta^{(\alpha-n)(p'(x)+\omega(\delta))+n} \\
\le\ & C\mu^{-p'(x)-\omega(\delta)}\delta^{p'(x)(\alpha-n/p(x))}\varphi(\delta)^{(n-\alpha)/(p(x)-1)^2} \\
=\ & C\mu^{-p'(x)-\omega(\delta)}\delta^{-p'(x)n/p^\sharp(x)}\varphi(\delta)^{(n-\alpha)/(p(x)-1)^2}.
\end{aligned}
$$

Hence it follows that

$$
\begin{aligned}
&\int_{\Omega\setminus B(x,\delta)} |x-y|^{\alpha-n} f(y)dy \\
\le\ & C\mu\left(\mu^{-p'(x)-\omega(\delta)}\delta^{-p'(x)n/p^\sharp(x)}\varphi(\delta)^{(n-\alpha)/(p(x)-1)^2} + 1\right).
\end{aligned}
$$

Considering $\mu = \delta^{-n/p^\sharp(x)}\varphi(\delta)^{n/p(x)^2}$ when δ is small, we see that

$$\int_{\Omega\setminus B(x,\delta)} |x-y|^{\alpha-n} f(y)dy \le C\delta^{-n/p^\sharp(x)}\varphi(\delta)^{n/p(x)^2},$$

as required. □

Lemma 3.2. *Let f be a nonnegative measurable function on Ω such that $\|f\|_{p(\cdot)} \le 1$ and $|S_f| \le 1$. Then*

$$I_\alpha f(x) \le C\left[\{Mf(x)\}^{p(x)/p^\sharp(x)}\{\varphi(Mf(x)^{-1})\}^{\alpha/p(x)} + 1\right]$$

for $x \in \Omega$.

Proof. For $0 < \delta < 1$ we have by Lemma 3.1

$$\begin{aligned} I_\alpha f(x) &= \int_{B(x,\delta)} |x-y|^{\alpha-n} f(y)dy + \int_{\Omega\setminus B(x,\delta)} |x-y|^{\alpha-n} f(y)dy \\ &\le C\delta^\alpha Mf(x) + C\delta^{-n/p^\sharp(x)}\varphi(\delta)^{n/p(x)^2}. \end{aligned}$$

Considering $\delta = \{Mf(x)\}^{-p(x)/n}\{\varphi(Mf(x)^{-1})\}^{1/p(x)}$ when $Mf(x)$ is large enough, we see that

$$I_\alpha f(x) \le C\left[\{Mf(x)\}^{p(x)/p^\sharp(x)}\{\varphi(Mf(x)^{-1})\}^{\alpha/p(x)} + 1\right],$$

as required. □

Lemma 3.3. *Let $p > 1$ and $1/p^\sharp = 1/p - \alpha/n$. For $\beta > \alpha$, set $c = \beta/p$ and $d = \gamma/p^2$, where $\beta/\gamma = \alpha/n$. If $s > 0$, $t > 0$ and $s^{p^\sharp} \le C_1\left\{t^p\varphi(t^{-1})^{cp^\sharp} + 1\right\}$, then*

$$\left\{s\varphi(s^{-1})^{-d}\right\}^{p^\sharp} \le C_2\left\{t^p\varphi(t^{-1})^{-dp} + 1\right\},$$

where C_2 is a positive constant independent of s and t.

Proof. We may assume that t is large enough, that is, $t > T_0 > 1$. Using Lemma 2.5, we find

$$\left\{s\varphi(s^{-1})^{-d}\right\}^{p^\sharp} \le Ct^p\varphi(t^{-1})^{cp^\sharp} \times \varphi(t^{-p/p^\sharp}\varphi(t^{-1})^{-c})^{-dp^\sharp},$$

with $d = \gamma/p^2$. Note from (φ2) that

$$\varphi(t^{-p/p^\sharp}\varphi(t^{-1})^{-c})^{-1} \le C\varphi(t^{-1})^{-1}.$$

Hence it follows that

$$\left\{s\varphi(s^{-1})^{-d}\right\}^{p^\sharp} \le Ct^p\varphi(t^{-1})^{(c-d)p^\sharp} = Ct^p\varphi(t^{-1})^{-dp}$$

whenever $t > T_0$, which proves

$$\left\{s\varphi(s^{-1})^{-d}\right\}^{p^\sharp} \le C\left\{t^p\varphi(t^{-1})^{-dp} + 1\right\},$$

as required. □

By Lemmas 3.2 and 3.3, we have the following result.

Corollary 3.4. *Let f be a nonnegative measurable function on Ω such that $\|f\|_{p(\cdot)} \le 1$ and $|S_f| \le 1$. If $A(x) = a/p(x)^2$ with $a > n$, then*

$$\begin{aligned} &\left\{I_\alpha f(x)(\varphi(I_\alpha f(x)^{-1}))^{-A(x)}\right\}^{p^\sharp(x)} \\ \le\ & C\left[\left\{Mf(x)(\varphi(Mf(x)^{-1}))^{-A(x)}\right\}^{p(x)} + 1\right] \end{aligned}$$

for $x \in \Omega$.

Thus Theorem 2.7 and Corollary 3.4 yield the following Sobolev inequality for Riesz potentials.

Theorem 3.5. *Let Ω be an open set in $\mathbf{R}^n$ such that $|\Omega| < \infty$. Suppose $p_+(\Omega) < n/\alpha$. If $A(x) = a/p(x)^2$ with $a > n$, then*

$$\int_\Omega \left\{I_\alpha f(x)(\varphi(I_\alpha f(x)^{-1}))^{-A(x)}\right\}^{p^\sharp(x)} dx \le C$$

whenever f is a nonnegative measurable function on Ω with $\|f\|_{p(\cdot),\Omega} \le 1$.

Remark 3.6. If $\varphi(r) = a(\log(e + 1/r))^b$, then Theorem 3.5 was proved by the authors [8, Theorem 3.4]. See also Capone, Cruz-Uribe and Fiorenza [2, Theorem 1.6], Diening [4] and the authors [9, Theorem 3.3].

Remark 3.7. In Remark 2.9, we see that

$$I_\alpha g_r(x) \ge C_1 r^{-2/p^\sharp(x)}\varphi(r)^{2/p_0^2}$$

for $0 < x_1 < r$ and $-r < x_2 < 0$. Hence we have

$$\int_G \left\{ I_\alpha g_r(x)(\varphi(I_\alpha g_r(x)^{-1}))^{-2/p(x)^2} \right\}^{p^\sharp(x)} dx \geq C_2.$$

Remark 3.8. In Remark 2.10, we see that

$$I_\alpha g'_r(x) \geq C_1 r^{-2/p_0^\sharp}$$

for $0 < x_1 < r$ and $r < x_2 < 2r$. Hence we have

$$\int_G \left\{ I_\alpha g'_r(x)(\varphi(I_\alpha g'_r(x)^{-1}))^{-2/p(x)^2} \right\}^{p^\sharp(x)} dx \geq C_2.$$

In the next section, we treat the case when Ω might not be bounded, as in Cruz-Uribe, Fiorenza and Neugebauer [3].

§4. Maximal functions on general domains

In this section we treat the boundedness of maximal functions on general domains, which gives a generalization of the result by Cruz-Uribe, Fiorenza and Neugebauer [3].

Let Ω be an open set in $\mathbf{R}^n$. Consider a positive continuous function $p(\cdot)$ on Ω such that

(p1) $1 < p_-(\Omega) = \inf_\Omega p(x) \leq \sup_\Omega p(x) = p_+(\Omega) < \infty$;

(p2) $|p(x) - p(y)| \leq \log(\varphi(|x-y|))/\log(1/|x-y|)$ whenever $|x-y| < 1/2$, $x \in \Omega$ and $y \in \Omega$;

(p3) $|p(x) - p(y)| \leq C/\log(e + |x|)$ whenever $x \in \Omega$, $y \in \Omega$ and $|y| \geq |x|$.

If (p3) holds, then p has a finite limit p_∞ at infinity and

$$|p(x) - p_\infty| \leq \frac{C}{\log(e + |x|)} \qquad \text{for all } x \in \Omega. \tag{p3$'$}$$

For a nonnegative measurable function f on Ω, set

$$F(x) = \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)^{p(y)} dy,$$

as before. If $\|f\|_{p(\cdot)} \leq 1$ and $F(x) \geq 1$, then we have by the proof of Lemma 2.4

$$f_B = \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y) dy \leq C F(x)^{1/p(x)} \varphi(F(x)^{-1})^{n/p(x)^2},$$

so that

$$\left\{ f_B(\varphi(f_B^{-1}))^{-n/p(x)^2} \right\}^{p(x)} \le CF(x). \tag{4.1}$$

Lemma 4.1. *Let f be a nonnegative measurable function on Ω. If $F(x) \le 1$ and $f(y) \ge 1$ or $f(y) = 0$ for $y \in \Omega$, then*

$$(f_B)^{p(x)} \le F(x).$$

Proof. If $f(y) \ge 1$ or $f(y) = 0$ for $y \in \Omega$, then

$$f_B = \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)dy \le \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)^{p(y)} dy = F(x).$$

Since $F(x) \le 1$, $f_B \le 1$, so that

$$(f_B)^{p(x)} \le f_B \le F(x),$$

as required. □

By (4.1) and Lemma 4.1 we have the following result.

Corollary 4.2. *Let f be a nonnegative measurable function on Ω such that $\|f\|_{p(\cdot)} \le 1$. If $f(y) \ge 1$ or $f(y) = 0$ for $y \in \Omega$, then*

$$\left\{ Mf(x)(\varphi(Mf(x)^{-1}))^{-n/p(x)^2} \right\}^{p(x)} \le CMg(x),$$

where $g(y) = f(y)^{p(y)}$.

For a function f on $\mathbf{R}^n$, we define the Hardy operator H by

$$Hf(x) = \frac{1}{|B(0,|x|)|} \int_{B(0,|x|)} |f(y)| dy$$

for $x \in \mathbf{R}^n \setminus \{0\}$ and $Hf(0) = 0$.

Lemma 4.3. *Let f be a nonnegative measurable function on Ω. If $f \le 1$ on Ω, then*

$$(f_B)^{p(x)} \le C\left[F(x) + e(x) + \{Hf(x)\}^{p(x)} \right], \tag{4.2}$$

where $e(x) = (e + |x|)^{-n}$.

Proof. Note that $F(x) \leq 1$ since $f \leq 1$ on Ω. If $x \in \Omega \cap B(0,1)$, then

$$f_B \leq 1,$$

which proves (4.2).

For every subset E of Ω, we set $p_+(E) = \sup_E p(x)$ and $p_-(E) = \inf_E p(x)$. Fix $x \in \Omega \setminus B(0,1)$, and take a ball $B = B(x,r)$. We will consider two cases.

Case 1: $r \geq |x|/2$. Since $p_+ = p_+(\Omega) < \infty$, we have

$$\begin{aligned} (f_B)^{p(x)} &\leq 2^{p_+}\left(\frac{1}{|B|}\int_{B\cap B(0,|x|)} f(y)dy\right)^{p(x)} \\ &\quad +2^{p_+}\left(\frac{1}{|B|}\int_{B\setminus B(0,|x|)} f(y)dy\right)^{p(x)}. \end{aligned}$$

Then, since $r \geq |x|/2$, we see that

$$\frac{1}{|B|}\int_{B\cap B(0,|x|)} f(y)dy \leq CHf(x).$$

We set $E = (B \setminus B(0,|x|)) \cap \Omega$ and

$$D = \{y : f(y) \geq e(x)\}.$$

By Hölder's inequality, we have

$$\frac{1}{|B|}\int_E f(y)dy \leq \left(\frac{1}{|B|}\int_{E\cap D} f(y)^{p_-(E)}dy\right)^{1/p_-(E)} + e(x).$$

By assumption (p3), if $y \in E$, then

$$0 \leq p(y) - p_-(E) \leq p_+(E) - p_-(E) \leq \frac{C}{\log(e+|x|)}.$$

Therefore, if $y \in E \cap D$, then

$$\begin{aligned} f(y)^{p_-(E)} &= f(y)^{p(y)} f(y)^{p_-(E)-p(y)} \\ &\leq f(y)^{p(y)} e(x)^{-C/\log(e+|x|)} \leq Cf(y)^{p(y)}, \end{aligned}$$

so that

$$
\begin{aligned}
& \left(\frac{1}{|B|} \int_E f(y)dy \right)^{p(x)} \\
\leq \ & C \left(\frac{1}{|B|} \int_{E \cap D} f(y)^{p(y)} dy \right)^{p(x)/p_-(E)} + Ce(x)^{p(x)} \\
\leq \ & CF(x)^{p(x)/p_-(E)} + Ce(x)^{p(x)}.
\end{aligned}
$$

Since $F(x) \leq 1$ by our assumption and $e(x) \leq 1$, we obtain

$$
\left(\frac{1}{|B|} \int_E f(y)dy \right)^{p(x)} \leq CF(x) + Ce(x),
$$

which proves (4.2).

Case 2: $0 < r \leq |x|/2$. In this case, we see as before that

$$
0 \leq p(y) - p_-(B \cap \Omega) \leq p_+(B \cap \Omega) - p_-(B \cap \Omega) \leq \frac{C}{\log(e + |x|)}
$$

for $y \in B \cap \Omega$. Hence it follows as above that

$$
\begin{aligned}
& \left(\frac{1}{|B|} \int_B f(y)dy \right)^{p(x)} \\
\leq \ & C \left(\frac{1}{|B|} \int_B f(y)^{p(y)} dy \right)^{p(x)/p_-(B \cap \Omega)} + Ce(x)^{p(x)} \\
\leq \ & CF(x) + Ce(x),
\end{aligned}
$$

as required. □

Lemma 4.4. *Let f be a nonnegative measurable function on Ω such that $f \leq 1$ on Ω. Then*

$$
\{Hf(x)\}^{p(x)} \leq CHg(x) + Ce(x),
$$

where $g(y) = f(y)^{p(y)}$.

Proof. Let f be a nonnegative measurable function on Ω such that $f \leq 1$ on Ω. Then, since $0 \leq f \leq 1$ on Ω, we see that

$$
H(f\chi_{B(0,r_0)})(x) \leq Ce(x) \qquad \text{on } \Omega.
$$

Hence we may assume that $f = 0$ on $B(0, r_0)$.

For $\mu \geq 1$ and $r = |x| > r_0$, we have

$$\frac{1}{|B(0,r)|}\int_{B(0,r)} f(y)dy \ \leq \ \mu\left(\frac{1}{|B(0,r)|}\int_{B(0,r)} (1/\mu)^{p'(y)}dy + G\right),$$

where $G = Hg(x)$ with $g(y) = f(y)^{p(y)}$. Then note from (p3) that

$$-p'(y) \leq -p'(x) + \omega(|y|) \qquad \text{for } y \in B(0,r),$$

where $\omega(t) = C/\log(e+t)$. If $\log \mu \leq c_1 \log r$ and $0 < m < n$, then we can find $r_1 > e$ such that

$$\mu^{-p'(y)}t^m \leq C\mu^{-p'(x)+\omega(r)}r^m$$

whenever $r_1 \leq t = |y| < r = |x|$, which yields

$$Hf(x) \ \leq \ \mu\left(C\mu^{-p'(x)+\omega(r)} + G\right).$$

First assume $r^{-n} < G \leq 1$. Then we set $\mu = G^{-1/\{p'(x)-\omega(r)\}}$ and, noting that $\mu \leq Cr^n$, we have

$$Hf(x) \ \leq \ CG^{1/p(x)}G^{-\omega(r)/\{p'(x)(p'(x)-\omega(r))\}} \leq CG^{1/p(x)}.$$

Next, if $G \leq r^{-n}$, then we set $\mu = r^{n/p'(x)}$ and obtain

$$Hf(x) \leq Ce(x)^{1/p(x)} + G^{1/p(x)} \leq Ce(x)^{1/p(x)}.$$

If $|x| \leq r_1$, then

$$Hf(x) \leq 1 \leq Ce(x),$$

which completes the proof. □

Combining Lemma 4.3 with Lemma 4.4, we obtain the following result.

Corollary 4.5. *Let f be a nonnegative measurable function on Ω. If $f \leq 1$ on Ω, then*

$$\{Mf(x)\}^{p(x)} \leq C\{Mg(x) + e(x) + Hg(x)\},$$

where $e(x) = (e+|x|)^{-n}$ and $g(y) = f(y)^{p(y)}$.

By Hardy's inequality we can prove the following inequality (cf. Lemma 5.4).

Lemma 4.6. *Let g be a nonnegative measurable function on $\mathbf{R}^n$ such that $\|g\|_{p_0} \le 1$, $1 < p_0 < \infty$. Then*

$$\int \{Hg(x)\}^{p_0} dx \le C.$$

Now, as in Cruz-Uribe, Fiorenza and Neugebauer [3], we can prove the following result.

Theorem 4.7. *If $A(x) = a/p(x)^2$ with $a > n$, then*

$$\int_\Omega \left\{ Mf(x)(\varphi(Mf(x)^{-1}))^{-A(x)} \right\}^{p(x)} dx \le C$$

whenever f is a measurable function on Ω with $\|f\|_{p(\cdot)} \le 1$.

Proof. For $p_0 > 1$, set $p_0(x) = p(x)/p_0$ and $g_0(y) = f(y)^{p_0(y)}$. Then we have by Corollaries 4.2 and 4.5

$$\left\{ Mf(x)(\varphi(Mf(x)^{-1}))^{-n/p_0(x)^2} \right\}^{p_0(x)} \le C \left\{ Mg_0(x) + e(x) + Hg_0(x) \right\}.$$

If $a > np_0^2$, then

$$\begin{aligned} & \left\{ Mf(x)(\varphi(Mf(x)^{-1}))^{-A(x)} \right\}^{p(x)} \\ \le \; & C \left\{ Mg_0(x) + e(x) + Hg_0(x) \right\}^{p_0} \\ \le \; & CMg_0(x)^{p_0} + Ce(x)^{p_0} + C\{Hg_0(x)\}^{p_0}. \end{aligned}$$

Since $p_0 > 1$, M is bounded on $L^{p_0}(\Omega)$ and $e(x) \in L^{p_0}(\mathbf{R}^n)$, we find

$$\int_\Omega \left\{ Mf(x)(\varphi(Mf(x)^{-1}))^{-A(x)} \right\}^{p(x)} dx \le C + C \int_{\mathbf{R}^n} \{Hg_0(x)\}^{p_0} dx.$$

Thus Lemma 4.6 yields the required inequality. □

§5. Sobolev's inequality for general domains

In this section we extend Sobolev's inequality to general domains Ω. Consider a positive continuous function $p(\cdot)$ on Ω satisfying

(p1′) $1 < p_- = p_-(\Omega) \le p_+(\Omega) = p_+ < n/\alpha$;

(p2) $|p(x) - p(y)| \le \log(\varphi(|x-y|))/\log(1/|x-y|)$ whenever $x \in \Omega$, $y \in \Omega$ and $|x-y| < 1/2$;

(p3) $|p(x) - p(y)| \leq C/\log(e + |x|)$ whenever $x \in \Omega$, $y \in \Omega$ and $|y| \geq |x|$.

By (p3) or (p3′) we can find $R_0 > 1$ such that

$$p(x) \leq p_\infty + \frac{C}{\log(e + |x|)} < \frac{n}{\alpha} \tag{3}$$

for $x \in \Omega \setminus B(0, R_0/2)$.

Lemma 5.1. *If $A(x) = a/p(x)^2$ with $a > n$, then*

$$\int_\Omega \left\{ I_\alpha f(x)(\varphi(I_\alpha f(x)^{-1}))^{-A(x)} \right\}^{p^\sharp(x)} dx \leq C$$

whenever f is a nonnegative measurable function on Ω such that $f = 0$ outside $B(0, R_0)$ and $\|f\|_{p(\cdot)} \leq 1$.

Proof. Let f be a nonnegative measurable function on Ω such that $f = 0$ on $\mathbf{R}^n \setminus B(0, R_0)$ and $\|f\|_{p(\cdot)} \leq 1$. In view of Theorem 3.5, we have

$$\int_{B(0,2R_0)} \left\{ I_\alpha f(x)(\varphi(I_\alpha f(x)^{-1}))^{-A(x)} \right\}^{p^\sharp(x)} dx \leq C.$$

If $x \in \mathbf{R}^n \setminus B(0, 2R_0)$, then

$$\begin{aligned} I_\alpha f(x) &\leq (|x|/2)^{\alpha-n} \int_{B(0,R_0)} f(y)dy \\ &\leq (|x|/2)^{\alpha-n} \int_{B(0,R_0)} \{1 + f(y)^{p(y)}\}dy \leq C|x|^{\alpha-n}, \end{aligned}$$

so that

$$\int_{\Omega \setminus B(0,2R_0)} I_\alpha f(x)^{q_0} dx \leq C$$

whenever $q_0(\alpha - n) + n < 0$. Now it follows that

$$\int_\Omega \left\{ I_\alpha f(x)(\varphi(I_\alpha f(x)^{-1}))^{-A(x)} \right\}^{p^\sharp(x)} dx \leq C,$$

as required. □

Lemma 5.2. *If f is a nonnegative measurable function on Ω such that $\|f\|_{p(\cdot)} \leq 1$ and $f = 0$ on $B(0, R_0)$, then*

$$\int_{\Omega \setminus \{B(0,|x|/2) \cup B(x,\delta)\}} |x - y|^{\alpha-n} f(y)dy \leq C\delta^{\alpha-n/p(x)}$$

for $x \in \Omega \setminus B(0, R_0)$ *and* $\delta \geq 1$.

Proof. For $x \in \Omega \setminus B(0, R_0)$ and $\mu > 0$, since $\|f\|_{p(\cdot)} \leq 1$, we have

$$\begin{aligned}
&\int_{\Omega \setminus \{B(0,|x|/2) \cup B(x,\delta)\}} |x-y|^{\alpha-n} f(y)dy \\
\leq\ & \mu \left(\int_{\Omega \setminus \{B(0,|x|/2) \cup B(x,\delta)\}} (|x-y|^{\alpha-n}/\mu)^{p'(y)} dy \right. \\
&\left. + \int_{\Omega \setminus \{B(0,|x|/2) \cup B(x,\delta)\}} f(y)^{p(y)} dy \right) \\
\leq\ & \mu \left(\int_{\Omega \setminus \{B(0,|x|/2) \cup B(x,\delta)\}} (|x-y|^{\alpha-n}/\mu)^{p'(y)} dy + 1 \right).
\end{aligned}$$

First consider the case $1 \leq \delta \leq 2|x|$. Let $E = \{y \in \Omega \setminus B(0, |x|/2) : |x-y|^{\alpha-n}/\mu > 1\}$. If we set

$$\beta_1 \equiv \beta_1(x) = p'(x) - \frac{C}{\log(e + |x|)},$$

then it follows from (3) that

$$p'(y) \geq \beta_1 > \frac{n}{n-\alpha} \qquad \text{for } y \in \Omega \setminus B(0, |x|/2).$$

Hence we obtain

$$\begin{aligned}
&\int_{\Omega \setminus \{B(0,|x|/2) \cup B(x,\delta) \cup E\}} (|x-y|^{\alpha-n}/\mu)^{p'(y)} dy \\
\leq\ & \int_{\Omega \setminus \{B(0,|x|/2) \cup B(x,\delta) \cup E\}} (|x-y|^{\alpha-n}/\mu)^{\beta_1} dy \\
\leq\ & \mu^{-\beta_1} \int_{\Omega \setminus B(x,\delta)} |x-y|^{(\alpha-n)\beta_1} dy \\
\leq\ & C\mu^{-\beta_1} \delta^{(\alpha-n)\beta_1 + n}.
\end{aligned}$$

Considering $\mu = \delta^{\alpha - n + n/\beta_1}$, we see that

$$\int_{\Omega \setminus \{B(0,|x|/2) \cup B(x,\delta) \cup E\}} (|x-y|^{\alpha-n}/\mu)^{p'(y)} dy \leq C,$$

so that

$$\int_{\Omega \setminus \{B(0,|x|/2) \cup B(x,\delta) \cup E\}} |x-y|^{\alpha-n} f(y) dy \leq C\delta^{\alpha-n+n/\beta_1}.$$

Similarly, if we set

$$\beta_2 \equiv \beta_2(x) = p'(x) + \frac{C}{\log(e+|x|)},$$

then it follows from (3) that

$$p'(y) \le \beta_2 \qquad \text{for } y \in \Omega \setminus B(0,|x|/2).$$

Note here that

$$\begin{aligned}
\int_{E\setminus B(x,\delta)} (|x-y|^{\alpha-n}/\mu)^{p'(y)} dy &\le \int_{E\setminus B(x,\delta)} (|x-y|^{\alpha-n}/\mu)^{\beta_2} dy \\
&\le \mu^{-\beta_2} \int_{\mathbf{R}^n\setminus B(x,\delta)} |x-y|^{(\alpha-n)\beta_2} dy \\
&\le C\mu^{-\beta_2}\delta^{(\alpha-n)\beta_2+n}.
\end{aligned}$$

Since $\mu = \delta^{\alpha-n+n/\beta_1}$ and $\delta \ge 1$, we see that

$$\int_{E\setminus B(x,\delta)} (|x-y|^{\alpha-n}/\mu)^{p'(y)} dy \le C\delta^{n(1-\beta_2/\beta_1)} \le C,$$

so that

$$\int_{E\setminus B(x,\delta)} |x-y|^{\alpha-n} f(y) dy \le C\delta^{\alpha-n+n/\beta_1}.$$

Therefore

$$\int_{\Omega\setminus\{B(0,|x|/2)\cup B(x,\delta)\}} |x-y|^{\alpha-n} f(y) dy \le C\delta^{\alpha-n+n/\beta_1}.$$

Since $1 \le \delta \le 2|x|$,

$$\delta^{\alpha-n+n/\beta_1} \le C\delta^{\alpha-n+n/p'(x)} = C\delta^{\alpha-n/p(x)},$$

so that

$$\int_{\Omega\setminus\{B(0,|x|/2)\cup B(x,\delta)\}} |x-y|^{\alpha-n} f(y) dy \le C\delta^{\alpha-n/p(x)}.$$

Next consider the case $\delta > 2|x| \ge 2R_0$. Then

$$\int_{\Omega\setminus B(x,\delta)} |x-y|^{\alpha-n} f(y) dy \le C \int_{\Omega\setminus B(X_\delta,\delta/2)} |X_\delta - y|^{\alpha-n} f(y) dy,$$

where $X_\delta = (\delta/4, 0, ..., 0) \in \mathbf{R}^n$. Hence the above considerations yield

$$\int_{\Omega\setminus B(x,\delta)} |x-y|^{\alpha-n} f(y) dy \le C\delta^{\alpha-n/p(x)}.$$

Thus the proof is completed. □

For a measurable function f on $\mathbf{R}^n$, we define the operator H_α by

$$H_\alpha f(x) = |x|^{\alpha-n} \int_{B(0,|x|)} |f(y)|dy$$

for $x \in \mathbf{R}^n \setminus \{0\}$ and $H_\alpha f(0) = 0$.

Lemma 5.3. *Let f be a nonnegative measurable function on Ω with $\|f\|_{p(\cdot)} \le 1$. If $x \in \Omega$ and $Mf(x) \le 1$, then*

$$\{I_\alpha f(x)\}^{p^\sharp(x)} \le C\{Mf(x)\}^{p(x)} + C\{H_\alpha f(x)\}^{p^\sharp(x)}.$$

Proof. Let f be a nonnegative measurable function on Ω with $\|f\|_{p(\cdot)} \le 1$. For $\delta \ge 1$ we have by Lemma 5.2

$$\begin{aligned} I_\alpha f(x) &= \int_{B(x,\delta)} |x-y|^{\alpha-n} f(y)dy \\ &\quad + \int_{\Omega\setminus\{B(0,|x|/2)\cup B(x,\delta)\}} |x-y|^{\alpha-n} f(y)dy \\ &\quad + \int_{B(0,|x|/2)} |x-y|^{\alpha-n} f(y)dy \\ &\le C\delta^\alpha Mf(x) + C\delta^{\alpha-n/p(x)} + CH_\alpha f(x) \end{aligned}$$

for $x \in \mathbf{R}^n$. If we set $\delta = \{Mf(x)\}^{-p(x)/n}$, then it follows that

$$I_\alpha f(x) \le C\{Mf(x)\}^{p(x)/p^\sharp(x)} + CH_\alpha f(x),$$

which yields the required inequality. □

Lemma 5.4. *Let $1 < p_1 < n/\beta$ and $1/q_1 = 1/p_1 - \beta/n$. Then*

$$\|H_\beta f\|_{q_1} \le C\|f\|_{p_1}.$$

This is a consequence of the usual Sobolev's inequality; see e.g. the book by Adams and Hedberg [1].

Lemma 5.5. *If f is a nonnegative measurable function on Ω such that $\|f\|_{p(\cdot)} \le 1$ and $f = 0$ on $B(0, R_0)$, then*

$$\int_\Omega \{H_\alpha f(x)\}^{p^\sharp(x)} dx \le C.$$

Proof. Let f be a nonnegative measurable function on Ω such that $\|f\|_{p(\cdot)} \le 1$ and $f = 0$ on $B(0, R_0)$. Write

$$f = f_1 + f_2,$$

where $f_1 = f\chi_{\{y: f(y) \ge 1\}}$ and $f_2 = f\chi_{\{y: f(y) < 1\}}$. Then we see that

$$H_\alpha f_1(x) \le |x|^{\alpha-n} \int_{B(0,|x|)} f_1(y)^{p(y)} dy \le |x|^{\alpha-n}$$

for $|x| \ge R_0$, so that

$$\int_\Omega \{H_\alpha f_1(x)\}^{p^\sharp(x)} dx \le C.$$

Thus we may assume that $f = f_2 \le 1$ on Ω.

Let $1/q_\infty = 1/p_\infty - \alpha/n$ and $1/p^\sharp(x) = 1/p(x) - \alpha/n$. For $1 < p_1 < p_-$, set $p_1(y) = p(y)/p_1$. Then for $r = |x| \ge R_0$ we have by Lemma 4.4

$$\begin{aligned} &\left(r^{\alpha-n} \int_{B(0,r)} f(y) dy\right)^{p^\sharp(x)} \\ \le\ & C\left(r^{\alpha p_\infty/p_1 - n} \int_{B(0,r)} f(y)^{p_1(y)} dy\right)^{p^\sharp(x) p_1/p(x)} + C r^{q_\infty(\alpha - n p_1/p_\infty)}. \end{aligned}$$

If $\int_{B(0,r)} f(y)^{p_1(y)} dy \le 1$, then the right hand side is dominated by

$$C r^{q_\infty(\alpha - n p_1/p_\infty)}.$$

Next suppose $\int_{B(0,r)} f(y)^{p_1(y)} dy > 1$. If $p^\sharp(x) p_1/p(r) \le q_\infty p_1/p_\infty$, then

$$\begin{aligned} &\left(r^{\alpha p_\infty/p_1 - n} \int_{B(0,r)} f(y)^{p_1(y)} dy\right)^{p^\sharp(x) p_1/p(r)} \\ \le\ & C\left(r^{\alpha p_\infty/p_1 - n} \int_{B(0,r)} f(y)^{p_1(y)} dy\right)^{q_\infty p_1/p_\infty}; \end{aligned}$$

if $p^\sharp(x) p_1/p(r) > q_\infty p_1/p_\infty$, then, since $r^{-n} \int_{B(0,r)} f(y)^{p_1(y)} dy \le C$, the above inequality is also true. Hence it follows that

$$\begin{aligned} &\left(r^{\alpha-n} \int_{B(0,r)} f(y) dy\right)^{p^\sharp(x)} \\ \le\ & C\left(r^{\alpha p_\infty/p_1 - n} \int_{B(0,r)} f(y)^{p_1(y)} dy\right)^{q_\infty p_1/p_\infty} + C r^{q_\infty(\alpha - n p_1/p_\infty)}. \end{aligned}$$

Since $1/(q_\infty p_1/p_\infty) = 1/p_1 - (\alpha p_\infty/p_1)/n$, it follows from Lemma 5.4 that

$$\int_\Omega \{H_\alpha f(x)\}^{p^\sharp(x)} dx \le C,$$

which yields the required inequality. □

Our final goal is to establish Sobolev's inequality of Riesz potentials defined in general domains, which gives an extension of Capone, Cruz-Uribe and Fiorenza [2, Theorem 1.6].

Theorem 5.6. *Suppose $p_+(\Omega) < n/\alpha$. If $A(x) = a/p(x)^2$ with $a > n$, then*

$$\int_\Omega \left\{ I_\alpha f(x)(\varphi(I_\alpha f(x)^{-1}))^{-A(x)} \right\}^{p^\sharp(x)} dx \le C$$

whenever f is a nonnegative measurable function on Ω with $\|f\|_{p(\cdot)} \le 1$.

Proof. Let f be a nonnegative measurable function on Ω with $\|f\|_{p(\cdot)} \le 1$. In view of Lemma 5.1, it suffices to treat the case when $f = 0$ on $B(0, R_0)$. Set

$$f = f_1 + f_2,$$

where $f_1 = f\chi_{\{y: f(y) \ge 1\}}$ and $f_2 = f\chi_{\{y: f(y) < 1\}}$. If $Mf_1(x) \ge 1$, then Corollary 3.4 gives

$$\begin{aligned} & \left\{ I_\alpha f_1(x)(\varphi(I_\alpha f_1(x)^{-1}))^{-A(x)} \right\}^{p^\sharp(x)} \\ \le\ & C \left\{ Mf_1(x)(\varphi(Mf_1(x)^{-1}))^{-A(x)} \right\}^{p(x)}, \end{aligned}$$

and if $Mf_1(x) < 1$, then Lemma 5.3 gives

$$\{I_\alpha f_1(x)\}^{p^\sharp(x)} \le C\{Mf_1(x)\}^{p(x)} + C\{H_\alpha f_1(x)\}^{p^\sharp(x)},$$

so that

$$\begin{aligned} & \left\{ I_\alpha f_1(x)(\varphi(I_\alpha f_1(x)^{-1}))^{-A(x)} \right\}^{p^\sharp(x)} \\ \le\ & C \left\{ Mf_1(x)(\varphi(Mf_1(x)^{-1}))^{-A(x)} \right\}^{p(x)} + C\{H_\alpha f_1(x)\}^{p^\sharp(x)}. \end{aligned}$$

Further we have by Lemma 5.3

$$\{I_\alpha f_2(x)\}^{p^\sharp(x)} \le C\{Mf_2(x)\}^{p(x)} + C\{H_\alpha f_2(x)\}^{p^\sharp(x)},$$

which proves

$$\left\{I_\alpha f_2(x)(\varphi(I_\alpha f_2(x)^{-1}))^{-A(x)}\right\}^{p^\sharp(x)}$$
$$\leq \ C\left\{Mf_2(x)(\varphi(Mf_2(x)^{-1}))^{-A(x)}\right\}^{p(x)} + C\{H_\alpha f_2(x)\}^{p^\sharp(x)}.$$

Now Theorem 4.7 and Lemma 5.5 give the required inequality. □

Remark 5.7. As in Remark 2.10, we consider p of the form:

$$p(y) = \begin{cases} p_\infty & \text{when } y_n \leq 0, \\ p_\infty + \dfrac{1}{\log(e+|y|)} \dfrac{a\log(\log(1/y_n))}{\log(1/y_n)} & \text{when } 0 < y_n \leq r_0, \\ p_\infty + \dfrac{1}{\log(e+|y|)} \dfrac{a\log(\log(1/r_0))}{\log(1/r_0)} & \text{when } y_n > r_0, \end{cases}$$

where $y = (y', y_n)$, $1 < p_\infty < n/\alpha$, $a > 0$ and $0 < r_0 < 1/e$. Let $B(R,r) = B(e(R),r)$ for $0 < r \leq r_0$, $R > 1$ and $e(R) = (R,0,...,0) \in \mathbf{R}^n$. Then Theorem 5.6 (or Theorem 3.5) implies that in case $a' > a/\log(e+R)$, we have

$$\int_{B(R,r_0)} \left\{I_\alpha f(x)(\log(e+I_\alpha f(x)))^{-a'n/p_\infty^2}\right\}^{p^\sharp(x)} dx \leq C \qquad (4)$$

whenever f is a nonnegative measurable function on $B(R,r_0)$ with $\|f\|_{p(\cdot)} \leq 1$.

We show that this is sharp. For this purpose, consider

$$f_r = \chi_{B_-(R,r)} \qquad (B_-(R,r) = B(R,r) \setminus H),$$

where $H = \{x = (x', x_n) \in \mathbf{R}^{n-1} \times \mathbf{R} : x_n > 0\}$. Then note that

$$\|f_r\|_{p(\cdot)} = Cr^{n/p_\infty}.$$

Setting $g_r = f_r/\|f_r\|_{p(\cdot)}$, we find

$$I_\alpha g_r(x) \geq Cr^{\alpha - n/p_\infty}$$

for $x \in B(R,r)$, so that

$$\int_{B(R,r)} \left\{I_\alpha g_r(x)(\log(e+I_\alpha g_r(x)))^{-an/\{p_\infty^2 \log(e+R)\}}\right\}^{p^\sharp(x)} dx \geq C.$$

This implies that (4) does not hold when $a' < a/\log(e+R)$.

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