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Integral representation for space-time excessive functions

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Abstract.

We study space-time excessive functions with respect to a basic submarkovian semigroup $\mathbb{P}$. It is shown that under some regularity assumptions many space-time excessive functions on a half-space have a Choquet-type integral represention by suitably choosen densities of the adjoint semigroup $\mathbb{P}^*$. If $\mathbb{P}$ is a convolution semigroup which is absolutely continuous with respect to the Haar measure, then all space-time excessive functions admit such an integral representation.

§1. Introduction

Let $\triangle := \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ denote the Laplace operator on $E := \mathbb{R}^n$. We consider the heat operator $\frac{1}{2} \triangle - \frac{\partial}{\partial t}$ on the half space $E \times]0, \infty[$. It is well known that the positive solutions v of

$$\frac{1}{2} \triangle v - \frac{\partial v}{\partial t} \leq 0$$

on $E \times]0, \infty[$ (called supercaloric functions) admit a Choquet-type integral representation by minimal supercaloric functions (c.f. [15]). Moreover, these minimal supercaloric functions are just the densities of the Gaussian semigroup $\mathbb{P} = (P_t)_{t>0}$ which has the generator $\frac{1}{2}\triangle$.

It is a remarkable fact that all this remains true also in the degenerate case $n = 0$ (where $E = \{0\}$ is just a one-point set); in this case the above integral representation is exactly the standard correspondence between distribution functions v on $]0, \infty[$ and measures ρ on $[0, \infty[$ given

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by

$$v(s) = \rho([0, s[\quad \text{for } s > 0,$$

or, written in a fancier way,

$$v(s) = \int 1_{]s_0,\infty[}(s) d\rho(s_0) \quad \text{for } s > 0,$$

where $\{1_{]s_0,\infty[} : s_0 \geq 0\}$ is the set of normalized minimal supercaloric functions.

In this paper we show that a similar result holds in great generality: We replace the above Gaussian semigroup by a general basic semigroup $\mathbb{P}$ (i.e. there exists some measure μ such that $\varepsilon_x P_t$ is absolutely continuous with respect to μ for all x and t). Under some regularity assumptions concerning the adjoint semigroup $\mathbb{P}^*$ the appropriately choosen densities of $\mathbb{P}^*$ turn out to be minimal space-time excessive functions, which then give a Choquet-type integral representation of a large class of space-time excessive functions.

In the special setting of convolution semigroups which are absolutely continuous with respect to the Haar measure, all space-time excessive functions on a half-space are represented in this way. In particular, all excessive functions of the parabolic operator of order α (c.f. [7]) on the upper halpf plane admit this Choquet-type integral representation.

§2. Notations and Preliminaries

In the following we fix the central potential theoretic notions which will be used througout. As basic references we use [5] or [3] and [6]. $(E, \mathcal{E})$ will always denote a *standard Borel measurable space*, i.e. E may be identified with a Borel subset of a completely metrizable separable space equipped with it's Borel field $\mathcal{E}$.

We denote by $p\mathcal{E}$ the convex cone of positive numerical $\mathcal{E}$-measurable functions on E.

Remember that a *kernel* P on $(E, \mathcal{E})$ is a family $(P(x, \cdot))_{x \in E}$ of measures on $(E, \mathcal{E})$ such that for f in $p\mathcal{E}$ the function

$$Pf(x) = \int f(y) P(x, dy), \quad x \in E$$

is in $p\mathcal{E}$.

Then, for every measure μ on $(E, \mathcal{E})$ the measure μP satisfies

$$\int f d(\mu P) = \int P f d\mu \quad \text{for } f \in p\mathcal{E}.$$

We assume to be given a *measurable semigroup* $\mathbb{P} = (P_t)_{t>0}$ of *substochastic* kernels on $(E, \mathcal{E})$ (i.e. we have $P_s P_t = P_{s+t}, P_t 1 \leq 1$, and $(x,t) \to P_t f(x)$ is $\mathcal{E} \otimes \mathcal{B}(]0, \infty[)$ - measurable for every f in $p\mathcal{E}$).

Examples 2.1. i) The trivial example is given by the one-point set $E = \{0\}$ and the semigroup $P_t(0, \cdot) = \varepsilon_0$ for $t > 0$.
ii) The standard example is the *Gaussian semigroup* on $E := \mathbb{R}^n$ which is given by the Lebesgue densites $p_t(x,y) = q_t(x-y)$ with

$$q_t(x) = \frac{1}{\sqrt{2\pi t}^n} \exp(-\frac{|x|^2}{2t}), \qquad t > 0, x, y \in \mathbb{R}^n.$$

iii) More general examples are given by semigroups associated with second order linear parabolic or elliptic differential operators on a domain of $\mathbb{R}^n$ (c.f. [2]) or suitable pseudo-differential operators (c.f. [8]). In particular, absolutely continouous convolution semigroups on $\mathbb{R}^n$ fit into this setting (c.f. [1]).

In the general setting we denote by $\mathbb{V} = (V_\lambda)_{\lambda \geq 0}$ the associated *resolvent* defined by

$$V_\lambda f(x) := \int_0^\infty e^{-\lambda t} P_t f(x) dt, \qquad x \in E, f \in p\mathcal{E}, \lambda \geq 0.$$

$V := V_0$ is called the *potential kernel* of $\mathbb{P}$. The resolvent $\mathbb{V}$ determines the semigroup $\mathbb{P}$ uniquely.

Remember that a set $N \in \mathcal{E}$ is called a *set of potential zero* if $V 1_N = 0$. We say that some property holds V-a.e. if this property holds except on a set of potential zero.

Remember that $v \in p\mathcal{E}$ is called an *excessive function* (with respect to the given semigroup $\mathbb{P}$) if $\sup_{t>0} P_t v = v$, or equivalently, $P_t v \uparrow v$ for $t \downarrow 0$. For f in $p\mathcal{E}$ the *potential* Vf generated by f is an excessive function.

We denote $\mathcal{S} := \mathcal{S}(\mathbb{P}) := \{v : v \text{ is excessive}, v < \infty \text{ V-a.e.}\}$.
A σ-finite measure η on $(E, \mathcal{E})$ is called an *excessive measure* if $\eta P_t \uparrow \eta$ for $t \downarrow 0$. *Exc* $:=$ *Exc*$(\mathbb{P})$ denotes the convex cone of all excessive measures.

The set of *potential measures*

$$Pot := Pot(\mathbb{P}) := \{\mu V : \mu \text{ is a measure such that } \mu V \text{ is } \sigma\text{-finite }\}$$

is a convex subcone of Exc.

In this paper we study space-time excessive functions. Therefore we associate with the given semigroup $\mathbb{P}$ on E the *space-time semigroup* $\mathbb{Q} = (Q_t)_{t>0}$ on $E\times]0,\infty[$ defined by

$$\varepsilon_{(x,s)}Q_t = 1_{]t,\infty[}(s)(\varepsilon_x P_t \otimes \varepsilon_{s-t}), \qquad x \in E, s > 0$$

for $t > 0$. $\mathbb{Q}$ is again a measurable semigroup of substochastic kernels. We denote by $\mathbb{W} = (W_\lambda)_{\lambda\geq 0}$ the resolvent associated with $\mathbb{Q}$.

Of course, there are variants of these space-time semigroups, some of them will appear later.

Examples 2.2. i) In the trivial example $E = \{0\}$ and $P_t = \varepsilon_0$ for $t > 0$ it is easily seen that a positive function v belongs to $\mathcal{S}(\mathbb{Q})$ if and only if v is finite, increasing, and left continuous (or: lower semicontinuous) on $]0,\infty[$. A measure η belongs to Exc $(\mathbb{Q})$ if and only if η has a finite, decreasing, and right continuous (or: lower semicontinouous) Lebesgue density v with respect to Lebesgue measure $\lambda\!\!\lambda$ on $]0,\infty[$.
ii) For the Gaussian semigroup the cone of space-time excessive functions is just the cone of supercaloric functions mentioned in the introduction.
iii) In general, for f in $p\mathcal{E}$ the function

$$v(x,s) := P_s f(x), \qquad x \in E, s > 0$$

is excessive with respect to $\mathbb{Q}$, since

$$Q_t v(x,s) = 1_{]t,\infty[}(s) P_t P_{s-t} f(x) = 1_{]t,\infty[}(s) v(x,s) \uparrow v(x,s)$$

for $t > 0, t \downarrow 0$.

In this paper we are interested in Choquet-type integral representations for *space-time excessive* functions, i.e for functions which are excessive with respect to $\mathbb{Q}$.

Remark 2.1. The following general results on Choquet-type integral representations of excessive measures and functions are known:
i) Under very general assumptions every excessive measure η has a unique representation as a mixture of minimal excessive measures, i.e.

$$\eta(A) = \int_F \nu(A) d\rho(\nu) \qquad \text{for } A \in \mathcal{E},$$

where ρ is a measure on the space F of suitably normalized minimal excessive measures.
Here, $\nu \epsilon$ Exc is called *minimal* if $\nu = \nu_1 + \nu_2$ for $\nu_1, \nu_2 \in$ Exc can only hold if ν_1 and ν_2 are proportional to ν (c.f. [17]).
ii) If the potential kernel V is proper, then the corresponding integral representation of every excessive function by minimal excessive functions holds if and only if V is basic, i.e. $\varepsilon_x V \ll \mu$ for all x for some σ-finite measure μ (c.f. [3] and [10]).

Consequently, a Choquet-type integral representation for all space-time excessive functions exists if and only if the potential kernel W is basic, i.e. for some σ-finite measure m on $E \times]0,\infty[$ all the measures $\varepsilon_{x,s} W$ are absoluteley continuous with respect to m. It is well known that under this assumption there exists a σ-finite measure μ on $(E,\mathcal{E})$ such that $\varepsilon_x P_t \ll \mu$ for all $x \in E, t > 0$, hence the following Assumption 3.1 is quite natural.

§3. Choquet-type integral representation of space-time excessive functions

To obtain the wanted integral representation we need the existence of a nice dual semigroup $\mathbb{P}^*$.

Assumption 3.1. $\mathbb{P}$ and $\mathbb{P}^*$ are substochastic measurable semigroups on a standard Borel measurable space $(E,\mathcal{E})$ which are in duality and absolutely continuous with respect to some σ-finite measure μ, i.e.

$$\varepsilon_x P_t \ll \mu, \quad \varepsilon_x P_t^* \ll \mu \qquad \text{for all } x \in E, t > 0 \text{ and}$$

$$\int P_t f \cdot g d\mu = \int f \cdot P_t^* g d\mu \qquad \text{for all } t > 0, f, g \in p\mathcal{E}.$$

From [19] we know that we can choose very nice densities for the associated space-time potential kernels W and W^*:

Theorem 3.1. *There exists a unique* $\mathcal{B}(]0,\infty[) \otimes \mathcal{E} \otimes \mathcal{E}$*-measurable function* $p :]0,\infty[\times E \times E \to \bar{\mathbb{R}}_+$ *such that for* $s, t > 0$, $x, y \in E$ *and* $f \in p\mathcal{E}$ *the following is true:*
i) $P_t f(x) = \int f(z) p_t(x,z) d\mu(z)$
ii) $P_t^* f(x) = \int f(z) p_t(z,x) d\mu(z)$
iii) $p_{s+t}(x,y) = \int p_s(x,z) p_t(z,y) d\mu(z)$

Conclusion 3.1. *i) For* $s_0 \geq 0$ *and* $x_0 \in E$ *the function*

$$w_{x_0,s_0}(x,s) := 1_{]s_0,\infty[}(s) p_{s-s_0}(x,x_0), \quad x \in E, s > 0$$

belongs to $\mathcal{S}(\mathbb{Q})$, since the Chapman-Kolmogorov equation, Theorem 3.1.iii, gives

$$Q_t w_{x_0,s_0} = 1_{]s_0+t,\infty[} w_{x_0,s_0}.$$

ii) By Fubini's theorem we conclude that for every σ-finite measure ρ on $E \times [0,\infty[$ the function

$$w^\rho(x,s) := \int 1_{]s_0,\infty[}(s) p_{s-s_0}(x,x_0) d\rho(x_0,s_0), \quad x \in E, s > 0$$

is space-time excessive.
Moreover, if ρ is concentrated on $E \times \{0\}$, then w^ρ is "invariant up to the exit from $E \times]0,\infty[$" for the space-time process, i.e. $Q_t w^\rho(x,s) = w^\rho(x,s)$ for all $0 < t < s, x \in E$.

For our main result we need an additional regularity hypothesis:

Assumption 3.2. $\mathbb{P}^*$ is a right semigroup on E, i.e. there exists an associated right Markov process (c.f. [14]).

Remark 3.1. If V^* is a proper kernel, then Assumption 3.2 is equivalent with the following potential theoretic properties of the convex cones $\mathcal{S}^*, Exc^*, Pot^*$ with respect to $\mathbb{P}^*$ (c.f. [16] for details):
i) $\mathcal{S}^*$ is inf-stable, $1 \in \mathcal{S}^*, \sigma(\mathcal{S}^*) = \mathcal{E}$,
ii) E is *semisaturated, i.e. Pot^* is hereditary in Exc^* (i.e. for $\eta \in Exc^*$ satisfying $\eta \leq \mu V^* \in Exc^*$ we have $\eta = \nu V^*$ for some measure ν).
If $\mathbb{P}^*$ induces a strong harmonic space in the sense of [4], or if $\mathbb{P}^*$ induces a balayage space in the sense of [2], then Assumption 3.2 is satisfied.

In the following result we use the functions w_{x_0,s_0} and w^ρ introduced in Conclusion 3.1.

Theorem 3.2. *We assume Assumption 3.1 and Assumption 3.2. Then the following is true:*
i) Let $v \in \mathcal{S}(\mathbb{Q})$ satisfy $v \leq w^{\rho_0} \in \mathcal{S}(\mathbb{Q})$ for some measure ρ_0 on $E \times [0,\infty[$.
Then there exists a unique measure ρ on $E \times [0,\infty[$ such that $v = w^\rho$, i.e.

$$v(x,s) = \int w_{x_0,s_0}(x,s) d\rho(x_0,s_0), \quad x \in E, s > 0.$$

For all $x_0 \in E$ and $s_0 \geq 0$ the function w_{x_0,s_0} is a minimal element of $\mathcal{S}(\mathbb{Q})$.
ii) Every $v \in \mathcal{S}(\mathbb{Q})$ decomposes uniquely into $v = w^\rho + v'$, where ρ is a unique measure on $E \times [0,\infty[$ and $v' \geq w^\tau$ holds only for the zero measure τ.

Proof. i) Let $m = \mu \otimes \lambda$. We denote by $\mathbb{Q}^*$ the semigroup on $E\times]0,\infty[$ given by

$$\varepsilon_{x,s} Q_t^* = \varepsilon_x P_t^* \otimes \varepsilon_{s+t}, \quad x \in E, s > 0, t > 0,$$

and we denote by $\mathbb{W}^*$ the associated resolvent. Then $\mathbb{W}$ is in strong duality with $\mathbb{W}^*$with respect to m, and

$$\Theta : v \to vm$$

is a bijection from $\mathcal{S}(Q)$ onto Exc $(\mathbb{Q}^*)$. Obviously, Exc $(\mathbb{Q}^*)$ = Exc $(\bar{\mathbb{Q}}^*)$ for the extended semigroup $\bar{\mathbb{Q}}^*$ on $E \times [0,\infty[$ given by

$$\varepsilon_{x,s} \bar{\mathbb{Q}}_t^* = \varepsilon_x P_t^* \otimes \varepsilon_{s+t}, \qquad x \in E, s \geq 0, t > 0$$

since $m(E \times \{0\}) = 0$ and since all measures in Exc$(\bar{\mathbb{Q}}^*)$ are absolutely continuous w.r. to m (c.f. [3] for general details of this identification). Since $\mathbb{P}^*$ admits an associated right Markow process, this remains true for $\bar{\mathbb{Q}}^*$. Consequently, every $\bar{\mathbb{Q}}^*$-excessive measure $vm \leq w^{\rho_0} m = \rho_0 \bar{W}^*$ is of the form $vm = \rho \bar{W}^*$ for a suitable unique measure ρ. Inverting the mapping Θ shows that $v = w^\rho$.
Applying this to $w_{x_0,s_0} = w^\rho$ for $\rho := \varepsilon_{(x_0,s_0)}$ gives the minimality of w_{x_0,s_0} for $x_0 \in E, s_0 \geq 0$.
ii) For general $v \in \mathcal{S}(\mathbb{Q})$ the measure vm decomposes uniquely as $vm = \rho \bar{W}^* + v'm$, where $v'm \geq \tau \bar{W}^*$ holds only for $\tau = 0$ (i.e. $v'm$ is the harmonic part of vm with respect to $\bar{\mathbb{Q}}^*$ according to [6]). Transporting this decomposition by the inverse of Θ gives the stated result. □

Remark 3.2. Simple examples show that in general it is not true that every space-time excessive funtion admits a representation as in Theorem 3.2.i. A setting where this is true is described below in Application 3.1. Motivated by § 3 in [15] one might conjecture that it is sufficient that E be thermically closed, i.e. $f \leq P_t f$ for all $t > 0$ for every $\mathbb{P}$-subharmonic $f \in p\mathcal{E}$.
In the setting of uniformly elliptic differential operators in gradient form on a domain in $\mathbb{R}^n$ Murata gave sufficient conditions for the non-existence of a non-zero positive space-time harmonic function on $E\times]0,\infty[$ with boundary values 0 on $E \times \{0\}$ (c.f. Theorem 4.2 in [12]).

Remember that $u \in \mathcal{S}$ is called *quasibounded* iff u can be written as a countable sum of bounded elements of $\mathcal{S}$. It is well known that in classical potential theory associated with Laplace's equation a potential is quasibounded if and only if the associated Riesz measure does not charge polar sets. The same result is true for the potential theory associated with the heat equation according to [18].

Corollary 3.1. *Let $h \in \mathcal{S}(\mathbb{P})$ be invariant, i.e. $P_t h = h < \infty$ V-a.e. for all $t > 0$. Then every h-quasibounded $v \in \mathcal{S}(\mathbb{Q})$ admits a unique integral representation*

$$v(x,s) = \int w_{x_0,s_0}(x,s)d\rho(x_0,s_0), \quad x \in E, s > 0.$$

Here $v \in \mathcal{S}(\mathbb{Q})$ is called h-quasibounded if $v = \sum_{n\in\mathbb{N}} v_n$ for some sequence $(v_n) \subset \mathcal{S}(\mathbb{Q})$ such that $v_n \leq c_n h$ for suitable constants c_n for all n in $\mathbb{N}$.

Proof. For $\rho_0 := (h\mu) \otimes \varepsilon_0$ we have obviously $h = w^{\rho_0}$, hence the stated integral representation holds for every h-bounded v_n in $\mathcal{S}(\mathbb{Q})$. Summing these formulae for n in $\mathbb{N}$ gives the wanted result. □

Application 3.1. Let G be a locally compact abelian group with countable base of the topology, and let $(\mu_t)_{t>0}$ be a convolution semigroup of measures on G such that all measures μ_t are absolutely continuous with respect to the Haar measure. Let $(P_t)_{t>0}$ be the associated semigroup of convolution kernels on G (c.f. [1]). The reflected measures given by

$$\int f d\mu_t^* = \int f(-x)d\mu_t(x), \quad t > 0, f \in p\mathcal{E}$$

define a dual basic convolution semigroup. $(P_t)_{t>0}$ and $(P_t^*)_{t>0}$ are strong Feller kernels. Consequently, the Assumptions 3.1 and 3.2 are satisfied (in fact, the associated Markov processes are very nice Lévy Processes). The densities of $(P_t)_{t>0}$ according to Theorem 3.1 are of the form $p_t(x,y) = q_t(x-y)$ for $t > 0, x, y \in G$ for suitable densities q_t of μ_t with respect to the Haar measure on G.

In this particular case we have the following

Result. For every $v \in \mathcal{S}(\mathbb{Q})$ there exists a unique measure ρ on $E \times [0,\infty[$ such that

$$v(s,x) = \int 1_{]s_0,\infty[}(s)q_{s-s_0}(x-x_0)d\rho(x_0,s_0), \quad x \in G, s > 0$$

Proof. Let $v \in \mathcal{S}(\mathbb{Q})$. We use the notations of the proof of Theorem 3.2. Then vm ist in Exc $(\mathbb{Q}^*)$. Obviously, vm is also an excessive measure with respect to the space-time convolution semigroup $(\mu_t^* \otimes \varepsilon_t)_{t>0}$ on the group $G \times \mathbb{R}$.
Let $\kappa^* := \int_0^\infty \mu_t^* \otimes \varepsilon_t dt$ denote the associated potential kernel measure on $G \times \mathbb{R}$. From Theorem 16.7 in [1] we know that vm decomposes uniquely

as $vm = \rho * \kappa^* + \rho_1$, where ρ_1 is invariant with respect to $(\mu_t^* \otimes \varepsilon_t)_{t>0}$. Since vm is supported by $E \times [0, \infty[$ we conclude $\rho_1 = 0$, and ρ is a measure supported by $E \times [0, \infty[$. Consequently, we have $vm = \rho * \kappa^* = w^\rho m$, and the stated integral representation $v = w^\rho$ follows. □

Examples 3.1. As examples of Application 3.1 we obtain an explicit integral representation of all space-time excessive functions in the following cases:
i) the Brownian semigroup with densities given in Example 2.1.ii.
ii) the symmetric stable semigroup of order 1 (the Cauchy semigroup) with densities

$$q_t(x) = \frac{at}{(t^2 + \mid x \mid^2)^{\frac{n+1}{2}}}, \qquad x \in \mathbb{R}^n, t > 0$$

for $a = \Gamma(\frac{n+1}{2})/\pi^{(n+1)/2}$.
Similar results hold for more general α-stable semigroups (c.f. [7] and [13]), except that there the densities are not elementary functions (only their Fourier transforms are explicitly given).

Remark 3.3. We formulated our results for a basic semigroup. The standard example for such a semigroup is determined by some second order elliptic linear partial differential operator L on a domain E in $\mathbb{R}^n$, where the coefficients of L depend on the space variables in E and have to be reasonably nice and not to degenerate. More generally, one may consider coefficients which are also time dependent. This leads to transition families $\mathbb{P} = (P_{s,t})_{s<t}$ which are no longer time homogeneous. Nevertheless, our reasoning carries over to this more general setting due to the fact that in [19] the existence of nice densities of such non-homogeneous families $\mathbb{P}$ has been proven.
Some examples of harmonic spaces associated with such time dependant differential operators appear in [9].

Remark 3.4. In [11] Murata proved for a large class of uniformly elliptic differential operators L in gradient form on a domain E in $\mathbb{R}^n$ that the Martin boundary for the associated heat operator is given by $(E \times \{0\}) \cup (\partial \times]0, \infty[)$, where ∂ is the Martin boundary of E with respect to L. It should be true in our setting that the Martin-Poisson space associated with the space-time semigroup $\mathbb{Q}$ is given by $(E \cup \partial) \times [0, \infty[)$, where ∂ denotes the set of suitably normalized minimal $\mathbb{P}$-harmonic functions (c.f. [3] for details).

Remark 3.5. It is easily verified, that the integral representation of Corollary 3.1 for $v \in \mathcal{S}(\mathbb{Q})$ holds already, if v is only w^{ρ_0}-quasi-bounded for some general $w^{\rho_0} \in \mathcal{S}(\mathbb{Q})$. The proof is the same as that of Corollary 3.1.

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