Advanced Studies in Pure Mathematics 43, 2006
Singularity Theory and Its Applications
pp. 401–419

On the topology of symmetry sets of smooth submanifolds in $\mathbb{R}^k$

Vyacheslav D. Sedykh

Abstract.

We study topology of symmetry sets, conflict sets and medial axes in the case when they have only stable singularities of corank 1. Singularities of these sets satisfy various conditions of coexistence. For example, isolated singularities and singularities forming smooth non-closed curves define a graph. If this graph is finite, then there is the following incidence relation: the sum of the local degrees of vertices of the graph is equal to the doubled number of its edges (the local degree of a vertex is the number of edges that are incident to this vertex; loops are counted twice). We give many-dimensional generalizations of this relation for sets mentioned above. These generalizations follow from some general facts on coexistence of wave front singularities found recently by the author.

Let M be a C^∞-smooth closed (compact without boundary) submanifold in the k-dimensional Euclidean space $\mathbb{R}^k$. The manifold M can have several connected components, perhaps of different dimensions (including isolated points). We will assume everywhere below that M lies neither in any hyperplane nor in any hypersphere in $\mathbb{R}^k$.

Let us equip the space of all embeddings $M \to \mathbb{R}^k$ by the C^∞-topology. Submanifolds in $\mathbb{R}^k$ corresponding to embeddings from an open dense subset in this space are called *generic*.

DEFINITION. The *symmetry set* of the manifold M is the closure of the set of centers of hyperspheres S^{k-1} in $\mathbb{R}^k$ that are tangent to M at two different points.

The symmetry set is a complicated singular subset in the ambient space. This set and various of its subsets (conflict sets, medial axes, etc)

Received February 4, 2004.
Revised February 3, 2005.
Supported in part by the grants RFBR 05-01-00104 and Support of Scientific Schools 4719.2006.1.

found applications in computer vision and other applied fields (see, for example, [3],[5],[6],[15]).

The symmetry set can be noncompact. Moreover, some of its points can be centers of several hyperspheres (of different radii) every of which is tangent to the manifold M at two or more points. Therefore, we will consider more simple object.

Let $\mathcal{H}(\mathbb{R}^k)$ be the set of all hyperspheres and hyperplanes in $\mathbb{R}^k$. This is a smooth $(k+1)$-dimensional manifold. Hyperplanes in $\mathbb{R}^k$ form a smooth hypersurface $\mathcal{P}(\mathbb{R}^k)$ in $\mathcal{H}(\mathbb{R}^k)$. The complement to $\mathcal{P}(\mathbb{R}^k)$ is the total space of a smooth fiber bundle

$$\varrho : \mathcal{H}(\mathbb{R}^k) \setminus \mathcal{P}(\mathbb{R}^k) \to \mathbb{R}^k,$$

that takes each hypersphere to its center. The fiber of this (trivial) bundle is the set of positive real numbers (the radius of a hypersphere).

DEFINITION. *Generalized symmetry set* $\Sigma(M)$ of the manifold M is the closure in $\mathcal{H}(\mathbb{R}^k)$ of the set of hyperspheres in $\mathbb{R}^k$ which are tangent to M at two different points.

The set $\Sigma(M)$ is a compact subset in $\mathcal{H}(\mathbb{R}^k)$. The image of the set $\Sigma(M)\setminus\mathcal{P}(\mathbb{R}^k)$ with respect to the projection ϱ is the symmetry set of the manifold M. Moreover, the intersection of the tangent cone to $\Sigma(M)$ at any point from $\Sigma(M) \setminus \mathcal{P}(\mathbb{R}^k)$ and the tangent space to the fiber of the bundle ϱ at this point is 0.

Denote by $\mathcal{F}(M)$ the set of hyperspheres and hyperplanes in $\mathbb{R}^k$ that are tangent to the submanifold M. This is the front of some Legendre mapping into $\mathcal{H}(\mathbb{R}^k)$ (see [9]). Its simplest singularities are *stable singularities of corank* 1. These singularities are classified by (nonzero) elements $\mathcal{A} = A_{\mu_1} + \cdots + A_{\mu_p}$ of the free additive Abelian semigroup $\mathbb{A}$ whose generators are the symbols $A_1, A_2, \ldots, A_\mu, \ldots$

Namely a generic front $\mathcal{F}$ in a smooth n-dimensional manifold V has a singularity of type A_μ at a given point $v \in V$ if its germ $(\mathcal{F}, v)$ at this point is diffeomorphic to a germ at zero of the hypersurface in $\mathbb{R}^n = \{(\lambda_0, \ldots, \lambda_{n-1})\}$ formed by points, where the polynomial $t^{\mu+1} + \lambda_{\mu-1}t^{\mu-1} + \cdots + \lambda_1 t + \lambda_0$ has a multiple real root. The front $\mathcal{F}$ has a singularity of type $A_{\mu_1} + \cdots + A_{\mu_p}$ at a point v if the germ $(\mathcal{F}, v)$ consists of p irreducible components having (as germs of fronts) singularities of types $A_{\mu_1}, \ldots, A_{\mu_p}$ at the point v, and if, moreover, germs of the manifolds of these singularities on the corresponding components intersect transversally at the point v. The numbers $\mu_1, \ldots, \mu_p$ are called multiplicities of a singularity of type $A_{\mu_1} + \cdots + A_{\mu_p}$.

Let us suppose that the front $\mathcal{F}(M)$ has only stable singularities of corank 1. Then for a generic manifold M, the points of the front $\mathcal{F}(M)$, where it has singularities of type $\mathcal{A} \in \mathbb{A}$, are in one to one correspondence with tangent $\mathcal{A}$-spheres and $\mathcal{A}$-planes of the manifold M. The latter are defined by the following way.

DEFINITION. A hypersphere (hyperplane) π in $\mathbb{R}^k$ is said to be *tangent* $\mathcal{A} = A_{\mu_1} + \cdots + A_{\mu_p}$*-sphere* (*plane*) of the manifold M if

1) it is tangent to M exactly at p points $x_1, \ldots, x_p$ that are the vertices of an $(p-1)$-dimensional simplex;

2) for any $i = 1, \ldots, p$ and for any smooth function in $\mathbb{R}^k$ equal to 0 on π and having noncritical value at x_i, a germ at x_i of the restriction of this function onto M is given by the formula

$$\pm t_1^{\mu_i+1} \pm t_2^2 \pm \cdots \pm t_{m_i}^2$$

in suitable local coordinates $t_1, \ldots, t_{m_i}$ on M.

REMARK. By definition, any hypersphere (hyperplane) in $\mathbb{R}^k$ passing through a given point is a tangent A_1-sphere (plane) for this point.

Tangent $\mathcal{A}$-spheres (planes) of the manifold M are called its *tangent hyperspheres* (*hyperplanes*) *of corank* 1. The number $c(\mathcal{A}) = \mu_1 + \cdots + \mu_p$ is called the *codimension* of tangency of an $\mathcal{A}$-sphere (plane) with the manifold M. The number $d(\mathcal{A}) = c(\mathcal{A}) + p$ is called the *degree* of this tangency.

Let M be a generic manifold. Then $c(\mathcal{A}) \leq k+1$ for any its tangent $\mathcal{A}$-sphere and $c(\mathcal{A}) \leq k$ for any tangent $\mathcal{A}$-plane. Moreover, for any fixed $\mathcal{A} \in \mathbb{A}$ the set $\mathcal{A}_M$ consisting of all tangent $\mathcal{A}$-spheres and $\mathcal{A}$-planes of the manifold M is a smooth submanifold (generally speaking, nonclosed) of codimension $c(\mathcal{A})$ in $\mathcal{H}(\mathbb{R}^k)$. The restriction of the projection ϱ onto the manifold $\mathcal{A}_M \setminus \mathcal{P}(\mathbb{R}^k)$ is a smooth immersion.

Now, if $\mathcal{A} = 2A_1$ or $c(\mathcal{A}) > 2$, then the manifold $\mathcal{A}_M$ belongs to the set $\Sigma(M)$ and is called the manifold of singularities of type $\mathcal{A}$ of this set. The numbers $c(\mathcal{A})$ and $d(\mathcal{A})$ are called the codimension and the degree of these singularities, respectively.

Denote by $\chi(\mathcal{A}_M)$ the topological Euler characteristic of the manifold $\mathcal{A}_M$ (the alternated sum of the Betti numbers of the homology groups with compact supports). Sometimes, to simplify notations, we omit the subscript M and write $\chi(\mathcal{A})$.

DEFINITION. The *index* $I_{\mathcal{A}}(X)$ of a singularity of type $X = A_{\nu_1} + \cdots + A_{\nu_q} \in \mathbb{A}$ with respect to a singularity of type $\mathcal{A} = A_{\mu_1} + \cdots + A_{\mu_p}$ is the nonnegative integer defined recursively by the following conditions:

1) if $\mu^* = \max\{\mu_1, \ldots, \mu_p\} > \nu^* = \max\{\nu_1, \ldots, \nu_q\}$, then $I_{\mathcal{A}}(X)$ is equal to 0;

2) if $\mu^* \leq \nu^*$, then $I_{\mathcal{A}}(X)$ is equal to

$$\sum_{\nu_i=\mu^*,\mu^*+1} I_{\mathcal{A}-A_{\mu^*}}(X - A_{\nu_i}) + \sum_{\nu_i>\mu^*+1} I_{\mathcal{A}-A_{\mu^*}}(X - A_{\nu_i} + A_{\nu_i-\mu^*-1}),$$

where $I_{\emptyset}(Y) = 1$ for any Y (here $\emptyset$ is the zero of the semigroup $\mathbb{A}$).

THEOREM 1. *Let M be a smooth closed submanifold in $\mathbb{R}^k$. Suppose that all tangent hyperspheres (hyperplanes) of M are tangent hyperspheres (hyperplanes) of corank 1. Then for a generic M the following statements are valid.*

1) *If the manifold $\mathcal{A}_M$ of singularities of type $\mathcal{A} \in \mathbb{A}$ of the generalized symmetry set $\Sigma(M)$ of the manifold M has an odd dimension, then its Euler characteristic $\chi(\mathcal{A}_M)$ is a linear combination*

$$\chi(\mathcal{A}_M) = \sum_X K_{\mathcal{A}}(X)\chi(X_M) \tag{1}$$

of the Euler characteristics $\chi(X_M)$ of even-dimensional manifolds X_M of singularities of types $X \in \mathbb{A}$ where $c(X) > c(\mathcal{A})$. This combination is universal in the sense that every its coefficient $K_{\mathcal{A}}(X)$ depends only on $\mathcal{A}$ and X (that is, it does not depend on the topology of the manifold M). Namely,

$$K_{\mathcal{A}}(X) = \sum_{i=0}^{[c(X)-c(\mathcal{A})-1]/2} (-1)^i P_i(\mathcal{A}, X),$$

where $P_i(\mathcal{A}, X)$ is equal to the sum of the products of the form

$$\prod_{j=0}^{i} \frac{I_{Y_j}(Y_{j+1})}{I_{Y_j}(Y_j)}$$

by all ordered sets $(Y_0, Y_1, \ldots, Y_{i+1})$ of elements of the semigroup $\mathbb{A}$ such that $Y_0 = \mathcal{A}, Y_{i+1} = X$ and

$$c(\mathcal{A}) < c(Y_1) < \cdots < c(Y_i) < c(X),$$

$$c(Y_1) \equiv \cdots \equiv c(Y_i) \equiv c(\mathcal{A}) \pmod 2 .$$

2) *The list of the formulas (1) for singularities of codimension $c \leq 6$ is given in Table 1 for an even k and in Table 2 for an odd k.*

3) *The Euler characteristic $\chi(\Sigma(M))$ of the set $\Sigma(M)$ is a universal linear combination*

$$(2) \qquad \chi(\Sigma(M)) = \sum_X \mathcal{K}(X)\chi(X_M)$$

of the Euler characteristics $\chi(X_M)$ *of even-dimensional manifolds* X_M *of singularities of types* $X \in \mathbb{A} \setminus \{A_1, A_2\}$. *The coefficients of this combination are calculated by the formula*

$$\mathcal{K}(X) = 1 - \sum_{\mathcal{A}} K_{\mathcal{A}}(X),$$

where the summation is taken by all nonzero elements $\mathcal{A}$ *of the semigroup* $\mathbb{A}$ *such that* $\mathcal{A} \neq A_1$, $\mathcal{A} \neq A_2$, $c(\mathcal{A}) < c(X)$ *and* $c(\mathcal{A}) \equiv k \pmod 2$.

4) *The formula* (2) *for an even* k *has the form*

$$\begin{aligned}\chi(\Sigma(M)) = {} & \tfrac{1}{2}[\chi(A_3) - 4\chi(3A_1)] \\ & + \tfrac{1}{2}[\chi(A_5) + 3\chi(A_4 + A_1) + 2\chi(A_3 + A_2) + 6\chi(A_3 + 2A_1) \\ & \quad + 4\chi(2A_2 + A_1) + 12\chi(A_2 + 3A_1) + 32\chi(5A_1)] \\ & - \tfrac{1}{4}[27\chi(A_7) + 52\chi(A_6 + A_1) + 38\chi(A_5 + A_2) + 96\chi(A_5 + 2A_1) \\ & \quad + 41\chi(A_4 + A_3) + 70\chi(A_4 + A_2 + A_1) + 168\chi(A_4 + 3A_1) \\ & \quad + 74\chi(2A_3 + A_1) + 52\chi(A_3 + 2A_2) + 124\chi(A_3 + A_2 + 2A_1) \\ & \quad + 288\chi(A_3 + 4A_1) + 88\chi(3A_2 + A_1) + 208\chi(2A_2 + 3A_1) \\ & \quad + 480\chi(A_2 + 5A_1) + 1088\chi(7A_1)] + \dots\end{aligned}$$

(*the dots denote a universal linear combination of the Euler characteristics of manifolds of singularities of odd codimensions starting from the codimension* 9); *for an odd* k

$$\begin{aligned}\chi(\Sigma(M)) = {} & \chi(2A_1) \\ & - \tfrac{1}{2}[2\chi(A_4) + 3\chi(A_3 + A_1) + 2\chi(2A_2) \\ & \quad + 4\chi(A_2 + 2A_1) + 6\chi(4A_1)] \\ & + \tfrac{1}{2}[10\chi(A_6) + 14\chi(A_5 + A_1) + 11\chi(A_4 + A_2) \\ & \quad + 20\chi(A_4 + 2A_1) + 10\chi(2A_3) + 15\chi(A_3 + A_2 + A_1) \\ & \quad + 28\chi(A_3 + 3A_1) + 12\chi(3A_2) + 22\chi(2A_2 + 2A_1) \\ & \quad + 40\chi(A_2 + 4A_1) + 70\chi(6A_1)] + \dots\end{aligned}$$

(*the dots denote a universal linear combination of the Euler characteristics of manifolds of singularities of even codimensions starting from the codimension* 8).

PROOF. Let a generic front $\mathcal{F}$ in a smooth manifold V have only stable singularities of corank 1. Assume that the closure of the submanifold in V formed by singularities of type $2A_1$ of this front is compact. Then there are universal linear relations between the Euler characteristics of the manifolds of singularities of $\mathcal{F}$ lying in the mentioned closure. These relations are given in [12]. Theorem 1 is a corollary of these results in the case $\mathcal{F} = \mathcal{F}(M)$.

REMARK. Theorem 1 is valid for any generic curve in $\mathbb{R}^k$. In the case $\dim M > 1$, the main condition of Theorem 1 (all tangent hyperspheres (hyperplanes) of M are tangent hyperspheres (hyperplanes) of corank 1) is valid only for very special manifold M. For example, a smooth closed connected generic surface with such a property in $\mathbb{R}^3$ has no umbilic points. Therefore it is diffeomorphic to a torus (see [4]).

EXAMPLE. Let M be a smooth closed curve (not necessary connected) in the plane $\mathbb{R}^2$. Then its generalized symmetry set $\Sigma(M)$ is the union of a graph and a smooth closed one-dimensional manifold. The edges of the graph are simply connected components of the manifold $(2A_1)_M$. The vertices are singularities $3A_1, A_2 + A_1, A_3$. The incidence relation in this graph has the form: $2\chi(2A_1) = 6\chi(3A_1) + 2\chi(A_2 + A_1) + \chi(A_3)$. This is the first formula of Table 1 (in the case $k = 2$).

Consider *supporting* hyperspheres and hyperplanes of the manifold M. They are tangent hyperspheres and hyperplanes such that M lies on one side of them. If a supporting hypersphere (hyperplane) is a tangent $A_{\mu_1} + \cdots + A_{\mu_p}$-sphere (plane), then $\mu_1, \ldots, \mu_p$ are odd numbers.

Let $\mathbb{A}_{odd} \subset \mathbb{A}$ be the free additive Abelian semigroup whose generators are the symbols $A_1, A_3, \ldots, A_{2l+1}, \ldots$ Then for $k \leq 6$ the set of supporting hyperspheres (hyperplanes) of a smooth closed generic submanifold M in $\mathbb{R}^k$ consists of supporting $\mathcal{A}$-spheres (planes) where $\mathcal{A} \in \mathbb{A}_{odd}$ and $c(\mathcal{A}) \leq k+1$ ($c(\mathcal{A}) \leq k$ for hyperplanes). If the dimension of each connected component of the manifold M is at most 1, then this is true for any k (see [1],[16]).

Supporting hyperspheres and hyperplanes of the manifold M that lie in its generalized symmetry set $\Sigma(M)$ are called *singular*. Singular supporting hyperspheres and hyperplanes of the manifold M form a compact subset $\Sigma_{sup}(M)$ in $\mathcal{H}(\mathbb{R}^k)$. This subset is the closure in $\mathcal{H}(\mathbb{R}^k)$ of the set of supporting hyperspheres that are tangent to M at two different points.

The set of singular supporting $\mathcal{A}$-spheres and $\mathcal{A}$-planes of a generic manifold M (in dimensions mentioned above) is a smooth submanifold of codimension $c(\mathcal{A})$ in $\mathcal{H}(\mathbb{R}^k)$ for any $\mathcal{A} \in \mathbb{A}_{odd}$. It is called the manifold of singularities of type $\mathcal{A}$ of the set $\Sigma = \Sigma_{sup}(M)$ and is denoted by $\mathcal{A}_\Sigma$. The numbers $c(\mathcal{A})$ and $d(\mathcal{A})$ are called the codimension and the degree of these singularities, respectively.

THEOREM 2. *Let M be a smooth closed submanifold in $\mathbb{R}^k$. Assume that $k \leq 6$ or the dimension of each connected component of the manifold*

M is at most 1. Then for a generic M the statements 1 – 5 below are valid for $\Sigma = \Sigma_{sup}(M)$ and $\chi_0 = (-1)^k[1 - \chi(M)] + 1$.

1) *If the manifold $\mathcal{A}_\Sigma$ of singularities of type $\mathcal{A} \in \mathbb{A}_{odd}$ of the set Σ has an odd dimension, then its topological Euler characteristic $\chi(\mathcal{A}_\Sigma)$ is a linear combination*

$$\chi(\mathcal{A}_\Sigma) = \sum_X K_{\mathcal{A}}^{odd}(X)\,\chi(X_\Sigma) \tag{3}$$

of the Euler characteristics $\chi(X_\Sigma)$ of even-dimensional manifolds X_Σ of singularities of types $X \in \mathbb{A}_{odd}$ where $c(X) > c(\mathcal{A})$. This combination is universal in the sense that every its coefficient $K_{\mathcal{A}}^{odd}(X)$ depends only on $\mathcal{A}$ and X (that is, it does not depend on the topology of the manifold M). Namely,

$$K_{\mathcal{A}}^{odd}(X) = \sum_{i=0}^{[c(X)-c(\mathcal{A})-1]/2} (-1)^i \widetilde{P}_i(\mathcal{A}, X),$$

where $\widetilde{P}_i(\mathcal{A}, X)$ is equal to the sum of the products of the form

$$\prod_{j=0}^{i} \left(\frac{1}{2}\right)^{\operatorname{sign}[d(Y_{j+1})-d(Y_j)]} \frac{I_{Y_j}(Y_{j+1})}{I_{Y_j}(Y_j)}$$

by all ordered sets $(Y_0, Y_1, \dots, Y_{i+1})$ of elements of the semigroup $\mathbb{A}_{odd}$ such that $Y_0 = \mathcal{A}, Y_{i+1} = X$ and

$$c(\mathcal{A}) < c(Y_1) < \cdots < c(Y_i) < c(X),$$

$$c(Y_1) \equiv \cdots \equiv c(Y_i) \equiv c(\mathcal{A}) \pmod 2 .$$

2) *The list of the formulas (3) for singularities of codimension $c \le 8$ is given in Table 3 for an even k and in Table 4 for an odd k.*

3) *The Euler characteristic $\chi(\Sigma)$ of the set Σ is equal to χ_0. From the other side it is a universal linear combination*

$$\chi(\Sigma) = \sum_X \mathcal{K}^{odd}(X)\chi(X_\Sigma) \tag{4}$$

of the Euler characteristics $\chi(X_\Sigma)$ of even-dimensional manifolds X_Σ of singularities of types $X \in \mathbb{A}_{odd} \setminus \{A_1\}$. The coefficients of this combination are calculated by the formula

$$\mathcal{K}^{odd}(X) = 1 - \sum_{\mathcal{A}} K_{\mathcal{A}}^{odd}(X),$$

where the summation is taken by all nonzero elements $\mathcal{A}$ of the semigroup $\mathbb{A}_{odd}$ such that $\mathcal{A} \neq A_1$, $c(\mathcal{A}) < c(X)$ and $c(\mathcal{A}) \equiv k \pmod 2$.

4) *The formula* (4) *for an even k has the form*

$$\begin{aligned}\chi(\Sigma) = &\tfrac{1}{2}\left[\chi(A_3) - \chi(3A_1)\right] + \tfrac{1}{2}\left[\chi(A_5) + 2\chi(5A_1)\right] + \\ &+ \tfrac{1}{4}[\chi(A_7) - 2\chi(A_5 + 2A_1) - \chi(2A_3 + A_1) \\ &\quad - 3\chi(A_3 + 4A_1) - 17\chi(7A_1)] \\ &+ \tfrac{1}{4}[3\chi(A_9) + 4\chi(A_7 + 2A_1) + 3\chi(A_5 + A_3 + A_1) \\ &\quad + 13\chi(A_5 + 4A_1) + 2\chi(3A_3) + 8\chi(2A_3 + 3A_1) \\ &\quad + 30\chi(A_3 + 6A_1) + 124\chi(9A_1)] \\ &- \tfrac{1}{4}[7\chi(A_{11}) + 22\chi(A_9 + 2A_1) + 16\chi(A_7 + A_3 + A_1) \\ &\quad + 56\chi(A_7 + 4A_1) + 17\chi(2A_5 + A_1) + 12\chi(A_5 + 2A_3) \\ &\quad + 42\chi(A_5 + A_3 + 3A_1) + 152\chi(A_5 + 6A_1) \\ &\quad + 30\chi(3A_3 + 2A_1) + 106\chi(2A_3 + 5A_1) \\ &\quad + 378\chi(A_3 + 8A_1) + 1382\chi(11A_1)] + \dots\end{aligned}$$

(*the dots denote a universal linear combination of the Euler characteristics of manifolds of singularities of odd codimensions starting from the codimension* 13)*; for an odd k,*

$$\begin{aligned}\chi(\Sigma) = &\chi(2A_1) - \chi(4A_1) \\ &+ \tfrac{1}{2}[6\chi(6A_1) + \chi(A_3 + 3A_1) + \chi(A_5 + A_1)] \\ &- \tfrac{1}{2}[34\chi(8A_1) + 8\chi(A_3 + 5A_1) + 2\chi(2A_3 + 2A_1) \\ &\quad + 4\chi(A_5 + 3A_1) + \chi(A_5 + A_3) + \chi(A_7 + A_1)] \\ &+ \tfrac{1}{4}[620\chi(10A_1) + 167\chi(A_3 + 7A_1) + 46\chi(2A_3 + 4A_1) \\ &\quad + 13\chi(3A_3 + A_1) + 71\chi(A_5 + 5A_1) + 19\chi(A_5 + A_3 + 2A_1) \\ &\quad + 8\chi(2A_5) + 26\chi(A_7 + 3A_1) + 7\chi(A_7 + A_3) \\ &\quad + 11\chi(A_9 + A_1)] + \dots\end{aligned}$$

(*the dots denote a universal linear combination of the Euler characteristics of manifolds of singularities of even codimensions starting from the codimension* 12)*;*

5) *Let k be even* ($k \leq 16$) *and* $\chi(\mathcal{A}_\Sigma) = 0$ *for any* $\mathcal{A} \in \mathbb{A}_{odd} \setminus \{A_1\}$ *such that* $d(\mathcal{A}) \leq k$. *Then*

$$\sum_{\mathcal{A} \in \mathbb{A}_{odd}\colon c(\mathcal{A}) = k+1} w(\mathcal{A})\chi(\mathcal{A}_\Sigma) = (1 + k/2)\, w(A_{k+1})\, \chi_0, \tag{5}$$

where

$$w(A_{\mu_1} + \dots + A_{\mu_p}) = (-1)^{[p/2]} w(A_{\mu_1}) \dots w(A_{\mu_p}),$$

$$w(A_{2l+1}) = \frac{1}{2l+1}\binom{2l+1}{l},$$

and $[x]$ *is the integral part of the number* x. *The number* $w(A_{2l+1})$ *is the* l*'th Catalan number. The formulas* (5) *for* $k \leq 12$ *are given in Table* 5 *below.*

PROOF. The subset $\Sigma_{sup}(M) \subset \mathcal{H}(\mathbb{R}^k)$ is diffeomorphic to the set of singular points of the boundary Γ of a connected component of the complement to some generic front $\mathcal{F}$ in a $(k+1)$-dimensional space (see [10]; the closure of this component is a $(k+1)$-dimensional C^0-manifold and Γ is its boundary). For any $\mathcal{A} \in \mathbb{A}_{odd}$, the manifold $\mathcal{A}_\Sigma$ is diffeomorphic to the manifold $\mathcal{A}_\Gamma$ of singularities of type $\mathcal{A}$ of the hypersurface Γ (that is the set of singularities of type $\mathcal{A}$ of the front $\mathcal{F}$ at points of Γ). Therefore Theorem 2 follows from [11] (see also [14]) where we found universal linear relations between the Euler characteristics of the manifolds of singularities on the boundary of a connected component of the complement to a generic front under the condition that this boundary has only stable corank 1 singularities with odd multiplicities.

REMARK. The restriction $k \leq 16$ in the condition of the statement 5 of Theorem 2 is pure technical. I think that it can be omitted.

EXAMPLE. The formula (5) is valid for any smooth closed convex generic curve M in the even-dimensional space $\mathbb{R}^k$ (see [13]; recall that a curve in $\mathbb{R}^k$ is convex if it intersects any hyperplane at most at k points taking multiplicities into account).

REMARK. One can prove that the formula (5) is valid for any smooth closed generic curve M in $\mathbb{R}^k$ (k is even) such that for any k of its points (taking multiplicities into account) there is a hypersphere that passes through these points and does not intersect the curve at other points.

Consider the set of supporting hyperspheres of the manifold M. A supporting hypersphere is called *externally* (*internally*) supporting if M lies outside (inside) the ball bounded by this hypersphere.

Let $\Sigma_{ext}(M)$ ($\Sigma_{int}(M)$) be the set of singular externally-supporting (internally-supporting, respectively) hyperspheres of the manifold M. The sets $\Sigma_{ext}(M)$ and $\Sigma_{int}(M)$ are disjoint (recall, we assume that M does not lie in any hypersphere and in any hyperplane in $\mathbb{R}^k$). The union of these sets is equal to $\Sigma_{sup}(M) \setminus \mathcal{P}(\mathbb{R}^k)$.

DEFINITION. The *externally-supporting* (*internally-supporting*) *symmetry set* of the manifold M is the closure $\mathcal{C}_{ext}(M)$ ($\mathcal{C}_{int}(M)$, respectively) in $\mathbb{R}^k$ of the set of centers of externally-supporting (internally-supporting) hyperspheres that are tangent to M at two different points.

The supporting symmetry sets of the manifold M are subsets of its symmetry set. It is easy to see that $\mathcal{C}_{ext}(M) = \varrho(\Sigma_{ext}(M))$, $\mathcal{C}_{int}(M) = \varrho(\Sigma_{int}(M))$. Moreover, the projection ϱ defines a homeomorphism of the sets $\Sigma_{ext}(M)$ and $\Sigma_{int}(M)$ with the sets $\mathcal{C}_{ext}(M)$ and $\mathcal{C}_{int}(M)$, respectively.

PROPOSITION. *Let M be a smooth closed submanifold in $\mathbb{R}^k$. Then the set $\mathcal{C}_{ext}(M)$ is compact if an only if one of the connected components of M is a strictly convex hypersurface and all other components lie inside the compact domain in $\mathbb{R}^k$ bounded by this hypersurface.*

The similar statement is valid for the set $\mathcal{C}_{int}(M)$ as well. Moreover, if $M = M_0 \cup (M \setminus M_0)$ where M_0 is a smooth closed strictly convex hypersurface and $M \setminus M_0$ lies inside the compact domain in $\mathbb{R}^k$ bounded by the hypersurface M_0, then $\mathcal{C}_{int}(M) = \mathcal{C}_{int}(M_0)$.

PROOF. The set $\mathcal{C}_{ext}(M)$ (or $\mathcal{C}_{int}(M)$) is compact if and only if M has no singular supporting hyperplanes. The set of all supporting hyperplanes of such a manifold is diffeomorphic to S^{k-1}. The mapping $S^{k-1} \to \mathbb{R}^k$ that assigns to a supporting hyperplane the point of tangency with M is a smooth embedding ([8]). The image of this embedding is the desired strictly convex connected component of M. Proposition is proved.

Let M be a smooth closed submanifold in $\mathbb{R}^k$ where $k \leq 6$. Then the set of centers of singular externally-supporting (internally-supporting) $\mathcal{A}$-spheres of a generic M is a smooth submanifold of codimension $c(\mathcal{A}) - 1$ in $\mathbb{R}^k$ for any $\mathcal{A} \in \mathbb{A}_{odd}$. It is called the manifold of singularities of type $\mathcal{A}$ of the set $\Sigma = \mathcal{C}_{ext}(M)$ ($\Sigma = \mathcal{C}_{int}(M)$, respectively) and is denoted by $\mathcal{A}_\Sigma$. The numbers $c(\mathcal{A})$ and $d(\mathcal{A})$ are called the codimension and the degree of these singularities.

THEOREM 3. *Let M be a smooth closed submanifold in $\mathbb{R}^k$ where $k \leq 6$. Suppose that one of the connected components of M is a strictly convex hypersurface and all other components lies inside the compact domain in $\mathbb{R}^k$ bounded by this hypersurface. Then for a generic M, the statements* 1 – 5 *from Theorem* 2 *are valid in every of the following two cases:*

1) $\Sigma = \mathcal{C}_{ext}(M)$ *and* $\chi_0 = (-1)^k[1 - \chi(M)]$;
2) $\Sigma = \mathcal{C}_{int}(M)$ *and* $\chi_0 = 1$.

PROOF. Theorem 3 is a corollary of the main result of [13]. Indeed, the set $\mathcal{C}_{ext}(M)$ ($\mathcal{C}_{int}(M)$) is the Maxwell set of global minima (maxima, respectively) of the family of functions $F(x, \lambda)$ of $x \in M$ depending on the parameter $\lambda \in \mathbb{R}^k$, where $F(x, \lambda)$ is the square of the distance between λ and x (see [1]). The manifold $\mathcal{A}_\Sigma$ is the manifold of singularities of type $\mathcal{A}$ of the corresponding Maxwell set for any $\mathcal{A} \in \mathbb{A}_{odd}$.

REMARK. If the sets $\mathcal{C}_{ext}(M)$ and $\mathcal{C}_{int}(M)$ are compact, then the Euler characteristic of the manifold of Morse global minima of the family

$F(x, \lambda)$ is equal to $\chi(M)$. The Euler characteristic of the manifold of Morse global maxima of the family $F(x, \lambda)$ is equal to $\chi(S^{k-1})$.

Now, take an arbitrary connected component U of the complement to a smooth closed generic submanifold M in $\mathbb{R}^k$. By M_U denote the union of the connected components of M lying strictly inside the closure $\overline{U}$ of the domain U.

Let $\Sigma_{sup}(U)$ be the set of singular supporting hyperspheres and hyperplanes of M lying in $\overline{U}$. The set $\Sigma_{sup}(U)$ is a connected component of the set $\Sigma_{sup}(M)$. Singularities of $\Sigma_{sup}(M)$ at points from $\Sigma_{sup}(U)$ are called singularities of $\Sigma_{sup}(U)$.

DEFINITION. The *middle points set* of the domain U is the set $\mathcal{C}(U)$ of points in U being the centers of singular externally-supporting hyperspheres of the manifold M. Sometimes, depending on M and U, this set is called *medial axe* or *conflict set*.

The set $\mathcal{C}(U)$ is a subset of the externally-supporting symmetry set of the manifold M. If the domain U is bounded, then its middle points set is a compact connected component $\mathcal{C}(U) = \varrho(\Sigma_{sup}(U))$ of the set $\mathcal{C}_{ext}(M)$. Singularities of the set $\mathcal{C}_{ext}(M)$ at points from $\mathcal{C}(U)$ are called singularities of $\mathcal{C}(U)$.

THEOREM 4. *Let M be a smooth closed submanifold in $\mathbb{R}^k$, where $k \leq 6$. Assume that the complement $\mathbb{R}^k \setminus M$ is disconnected and U is one of its connected components. Then for a generic M, the statements $1 - 5$ from Theorem 2 are valid for*

1) $\Sigma = \mathcal{C}(U)$ and $\chi_0 = \chi(\overline{U}) - (-1)^k \chi(M_U)$ if U is bounded;

2) $\Sigma = \Sigma_{sup}(U)$ and $\chi_0 = \chi(\overline{U}) - (-1)^k \chi(M_U) + (-1)^k$ if U is unbounded.

PROOF. Theorem 4 in the case of the bounded component U follows from [13] by analogy with Theorem 3. Namely, the set $\mathcal{C}(U)$ is the Maxwell set of global minima of the family of functions $F(x, \lambda)$ of $x \in M$ depending on the parameter $\lambda \in \overline{U} \setminus \partial\overline{U}$, where $F(x, \lambda)$ is the square of the distance between λ and x, and $\partial\overline{U}$ is the boundary of $\overline{U}$. The Euler characteristic of the manifold of Morse global minima of this family is equal to $\chi(\partial\overline{U}) + \chi(M_U)$.

The case of the unbounded component U is reduced to the previous one after the inversion of the space $\mathbb{R}^k$ with respect to a hypersphere of a small radius having the centre at a point from the complement $\mathbb{R}^k \setminus (\overline{U} \cup M)$.

EXAMPLE. Let M be a smooth closed connected curve in the plane $\mathbb{R}^2$. Suppose that it has no self-intersections and is generic. Take a connected component U of the complement $\mathbb{R}^2 \setminus M$. Let $\chi(A_3)$ be the number of the curvature circles of M lying in $\overline{U}$ and $\chi(3A_1)$ be the number of circles in $\overline{U}$ that are tangent to M at three points. Then Theorem 4 implies,

$$\chi(A_3) - \chi(3A_1) = 2. \tag{6}$$

REMARK. At the first time, the relation (6) was obtained in [2] for a convex curve M. In [7], it was proved in the non-convex case as well. Theorem 4 extends these results onto the case of non-connected curves and gives their many-dimensional generalizations.

Table 1 (k is even)

$c = 2$	$2\chi(2A_1) = 6\chi(3A_1) + 2\chi(A_2 + A_1) + \chi(A_3)$ $- 40\chi(5A_1) - 18\chi(A_2 + 3A_1) - 8\chi(2A_2 + A_1)$ $- 11\chi(A_3 + 2A_1) - 5\chi(A_3 + A_2)$ $- 7\chi(A_4 + A_1) - 4\chi(A_5)$ $+ 672\chi(7A_1) + 320\chi(A_2 + 5A_1) + 152\chi(2A_2 + 3A_1)$ $+ 72\chi(3A_2 + A_1) + 204\chi(A_3 + 4A_1)$ $+ 97\chi(A_3 + A_2 + 2A_1) + 46\chi(A_3 + 2A_2)$ $+ 62\chi(2A_3 + A_1) + 129\chi(A_4 + 3A_1)$ $+ 61\chi(A_4 + A_2 + A_1) + 39\chi(A_4 + A_3)$ $+ 81\chi(A_5 + 2A_1) + 38\chi(A_5 + A_2)$ $+ 50\chi(A_6 + A_1) + 31\chi(A_7) + \ldots$
$c = 4$	$4\chi(4A_1) = 20\chi(5A_1) + 4\chi(A_2 + 3A_1) + 2\chi(A_3 + 2A_1)$ $- 280\chi(7A_1) - 100\chi(A_2 + 5A_1) - 32\chi(2A_2 + 3A_1)$ $- 8\chi(3A_2 + A_1) - 58\chi(A_3 + 4A_1)$ $- 18\chi(A_3 + A_2 + 2A_1) - 4\chi(A_3 + 2A_2)$ $- 10\chi(2A_3 + A_1) - 26\chi(A_4 + 3A_1)$ $- 6\chi(A_4 + A_2 + A_1) - 3\chi(A_4 + A_3) - 12\chi(A_5 + 2A_1)$ $- 2\chi(A_5 + A_2) - 4\chi(A_6 + A_1) - \chi(A_7) + \ldots$
	$2\chi(A_2 + 2A_1) = 6\chi(A_2 + 3A_1) + 4\chi(2A_2 + A_1)$ $+ 2\chi(A_3 + 2A_1) + \chi(A_3 + A_2) + 2\chi(A_4 + A_1)$ $- 40\chi(A_2 + 5A_1) - 36\chi(2A_2 + 3A_1) - 24\chi(3A_2 + A_1)$ $- 24\chi(A_3 + 4A_1) - 21\chi(A_3 + A_2 + 2A_1)$ $- 14\chi(A_3 + 2A_2) - 12\chi(2A_3 + A_1) - 24\chi(A_4 + 3A_1)$ $- 17\chi(A_4 + A_2 + A_1) - 9\chi(A_4 + A_3) - 16\chi(A_5 + 2A_1)$ $- 10\chi(A_5 + A_2) - 11\chi(A_6 + A_1) - 6\chi(A_7) + \ldots$
	$2\chi(2A_2) = 2\chi(2A_2 + A_1) + 2\chi(A_3 + A_2) + \chi(A_5)$ $- 4\chi(2A_2 + 3A_1) - 6\chi(3A_2 + A_1) - 4\chi(A_3 + A_2 + 2A_1)$ $- 5\chi(A_3 + 2A_2) - 4\chi(2A_3 + A_1) - 4\chi(A_4 + A_2 + A_1)$ $- 4\chi(A_4 + A_3) - 2\chi(A_5 + 2A_1) - 5\chi(A_5 + A_2)$ $- 3\chi(A_6 + A_1) - 4\chi(A_7) + \ldots$
	$\chi(A_3 + A_1) = 2\chi(A_3 + 2A_1) + \chi(A_3 + A_2)$ $+ \chi(A_4 + A_1) + \chi(A_5)$ $- 8\chi(A_3 + 4A_1) - 4\chi(A_3 + A_2 + 2A_1) - 2\chi(A_3 + 2A_2)$ $- 5\chi(2A_3 + A_1) - 6\chi(A_4 + 3A_1) - 3\chi(A_4 + A_2 + A_1)$ $- 4\chi(A_4 + A_3) - 6\chi(A_5 + 2A_1) - 3\chi(A_5 + A_2)$ $- 5\chi(A_6 + A_1) - 4\chi(A_7) + \ldots$
	$2\chi(A_4) = 2\chi(A_4 + A_1) + 2\chi(A_5)$ $- 4\chi(A_4 + 3A_1) - 2\chi(A_4 + A_2 + A_1) - \chi(A_4 + A_3)$ $- 4\chi(A_5 + 2A_1) - 2\chi(A_5 + A_2)$ $- 4\chi(A_6 + A_1) - 4\chi(A_7) + \ldots$

$c=6$	$2\chi(6A_1) = 14\chi(7A_1) + 2\chi(A_2+5A_1) + \chi(A_3+4A_1) + \dots$
	$2\chi(A_2+4A_1) = 10\chi(A_2+5A_1)$ $+ 4\chi(2A_2+3A_1) + 2\chi(A_3+4A_1)$ $+ \chi(A_3+A_2+2A_1) + 2\chi(A_4+3A_1) + \dots$
	$2\chi(2A_2+2A_1) = 6\chi(2A_2+3A_1) + 6\chi(3A_2+A_1)$ $+ 2\chi(A_3+A_2+2A_1) + \chi(A_3+2A_2)$ $+ 2\chi(A_4+A_2+A_1) + \chi(A_5+2A_1) + \dots$
	$2\chi(3A_2) = 2\chi(3A_2+A_1)$ $+ 2\chi(A_3+2A_2) + \chi(A_5+A_2) + \dots$
	$\chi(A_3+3A_1) = 4\chi(A_3+4A_1) + \chi(A_3+A_2+2A_1)$ $+ \chi(2A_3+A_1) + \chi(A_4+3A_1) + \chi(A_5+2A_1) + \dots$
	$\chi(A_3+A_2+A_1) = 2\chi(A_3+A_2+2A_1) + 2\chi(A_3+2A_2)$ $+ 2\chi(2A_3+A_1) + \chi(A_4+A_2+A_1)$ $+ \chi(A_4+A_3) + \chi(A_5+A_2) + \chi(A_6+A_1) + \dots$
	$2\chi(2A_3) = 2\chi(2A_3+A_1) + 2\chi(A_4+A_3) + \chi(A_7) + \dots$
	$2\chi(A_4+2A_1) = 6\chi(A_4+3A_1) + 2\chi(A_4+A_2+A_1)$ $+ \chi(A_4+A_3) + 2\chi(A_5+2A_1) + 2\chi(A_6+A_1) + \dots$
	$\chi(A_4+A_2) = \chi(A_4+A_2+A_1) + \chi(A_4+A_3)$ $+ \chi(A_5+A_2) + \chi(A_7) + \dots$
	$\chi(A_5+A_1) = 2\chi(A_5+2A_1) + \chi(A_5+A_2)$ $+ \chi(A_6+A_1) + \chi(A_7) + \dots$
	$\chi(A_6) = \chi(A_6+A_1) + \chi(A_7) + \dots$

The dots in formulas of Table 1 denote universal linear combinations of the Euler characteristics of manifolds of singularities of odd codimensions starting from the codimension 9.

Table 2 (k is odd)

$c=3$	$2\chi(3A_1) = 8\chi(4A_1) + 2\chi(A_2+2A_1) + \chi(A_3+A_1) - 80\chi(6A_1) - 32\chi(A_2+4A_1) - 12\chi(2A_2+2A_1) - 4\chi(3A_2) - 19\chi(A_3+3A_1) - 7\chi(A_3+A_2+A_1) - 4\chi(2A_3) - 10\chi(A_4+2A_1) - 3\chi(A_4+A_2) - 5\chi(A_5+A_1) - 2\chi(A_6) + \ldots$
	$2\chi(A_2+A_1) = 4\chi(A_2+2A_1) + 4\chi(2A_2) + 2\chi(A_3+A_1) + 2\chi(A_4) - 16\chi(A_2+4A_1) - 16\chi(2A_2+2A_1) - 12\chi(3A_2) - 12\chi(A_3+3A_1) - 11\chi(A_3+A_2+A_1) - 8\chi(2A_3) - 12\chi(A_4+2A_1) - 10\chi(A_4+A_2) - 10\chi(A_5+A_1) - 8\chi(A_6) + \ldots$
	$\chi(A_3) = \chi(A_3+A_1) + \chi(A_4) - 2\chi(A_3+3A_1) - \chi(A_3+A_2+A_1) - \chi(2A_3) - 2\chi(A_4+2A_1) - \chi(A_4+A_2) - 2\chi(A_5+A_1) - 2\chi(A_6) + \ldots$
$c=5$	$2\chi(5A_1) = 12\chi(6A_1) + 2\chi(A_2+4A_1) + \chi(A_3+3A_1) + \ldots$
	$2\chi(A_2+3A_1) = 8\chi(A_2+4A_1) + 4\chi(2A_2+2A_1) + 2\chi(A_3+3A_1) + \chi(A_3+A_2+A_1) + 2\chi(A_4+2A_1) + \ldots$
	$2\chi(2A_2+A_1) = 4\chi(2A_2+2A_1) + 6\chi(3A_2) + 2\chi(A_3+A_2+A_1) + 2\chi(A_4+A_2) + \chi(A_5+A_1) + \ldots$
	$\chi(A_3+2A_1) = 3\chi(A_3+3A_1) + \chi(A_3+A_2+A_1) + \chi(2A_3) + \chi(A_4+2A_1) + \chi(A_5+A_1) + \ldots$
	$\chi(A_3+A_2) = \chi(A_3+A_2+A_1) + 2\chi(2A_3) + \chi(A_4+A_2) + \chi(A_6) + \ldots$
	$\chi(A_4+A_1) = 2\chi(A_4+2A_1) + \chi(A_4+A_2) + \chi(A_5+A_1) + \chi(A_6) + \ldots$
	$\chi(A_5) = \chi(A_5+A_1) + \chi(A_6) + \ldots$

The dots in formulas of Table 2 denote universal linear combinations of the Euler characteristics of manifolds of singularities of even codimensions starting from the codimension 8.

Table 3 (k is even)

$c=2$	$2\chi(2A_1) = 3\chi(3A_1) + \chi(A_3)$ $- 5\chi(5A_1) - \chi(A_3+2A_1) - \chi(A_5)$ $+ 21\chi(7A_1) + 5\chi(A_3+4A_1) + \chi(2A_3+A_1)$ $+ 3\chi(A_5+2A_1) + \chi(A_7)$ $- 153\chi(9A_1) - 41\chi(A_3+6A_1) - 11\chi(2A_3+3A_1)$ $- 3\chi(3A_3) - 19\chi(A_5+4A_1) - 5\chi(A_5+A_3+A_1)$ $- 7\chi(A_7+2A_1) - 3\chi(A_9) + \dots$
$c=4$	$4\chi(4A_1) = 10\chi(5A_1) + 2\chi(A_3+2A_1)$ $- 35\chi(7A_1) - 9\chi(A_3+4A_1) - 3\chi(2A_3+A_1)$ $- 4\chi(A_5+2A_1) - \chi(A_7)$ $+ 252\chi(9A_1) + 70\chi(A_3+6A_1) + 20\chi(2A_3+3A_1)$ $+ 6\chi(3A_3) + 31\chi(A_5+4A_1) + 9\chi(A_5+A_3+A_1)$ $+ 12\chi(A_7+2A_1) + 5\chi(A_9) + \dots$
	$\chi(A_3+A_1) = \chi(A_3+2A_1) + \chi(A_5)$ $- \chi(A_3+4A_1) - \chi(A_5+2A_1) - \chi(A_7)$ $+ 3\chi(A_3+6A_1) + \chi(2A_3+3A_1) + 3\chi(A_5+4A_1)$ $+ \chi(A_5+A_3+A_1) + 2\chi(A_7+2A_1) + \chi(A_9) + \dots$
$c=6$	$4\chi(6A_1) = 14\chi(7A_1) + 2\chi(A_3+4A_1)$ $- 84\chi(9A_1) - 20\chi(A_3+6A_1) - 5\chi(2A_3+3A_1)$ $- \chi(3A_3) - 6\chi(A_5+4A_1) - \chi(A_5+A_3+A_1)$ $- \chi(A_7+2A_1) + \dots$
	$2\chi(A_3+3A_1) = 4\chi(A_3+4A_1)$ $+ 2\chi(2A_3+A_1) + 2\chi(A_5+2A_1)$ $- 10\chi(A_3+6A_1) - 5\chi(2A_3+3A_1) - 3\chi(3A_3)$ $- 8\chi(A_5+4A_1) - 4\chi(A_5+A_3+A_1)$ $- 5\chi(A_7+2A_1) - 2\chi(A_9) + \dots$
	$4\chi(2A_3) = 2\chi(2A_3+A_1) + 2\chi(A_7)$ $- \chi(2A_3+3A_1) + 3\chi(3A_3) - \chi(A_5+A_3+A_1)$ $- \chi(A_7+2A_1) - 2\chi(A_9) + \dots$
	$\chi(A_5+A_1) = \chi(A_5+2A_1) + \chi(A_7)$ $- \chi(A_5+4A_1) - \chi(A_7+2A_1) - \chi(A_9) + \dots$
$c=8$	$2\chi(8A_1) = 9\chi(9A_1) + \chi(A_3+6A_1) + \dots$
	$\chi(A_3+5A_1) = 3\chi(A_3+6A_1)$ $+ \chi(2A_3+3A_1) + \chi(A_5+4A_1) + \dots$
	$2\chi(2A_3+2A_1) = 3\chi(2A_3+3A_1) + 3\chi(3A_3) +$ $+ 2\chi(A_5+A_3+A_1) + \chi(A_7+2A_1) + \dots$
	$2\chi(A_5+3A_1) = 4\chi(A_5+4A_1)$ $+ \chi(A_5+A_3+A_1) + 2\chi(A_7+2A_1) + \dots$
	$2\chi(A_5+A_3) = \chi(A_5+A_3+A_1) + 2\chi(A_9) + \dots$
	$\chi(A_7+A_1) = \chi(A_7+2A_1) + \chi(A_9) + \dots$

The dots in formulas of Table 3 denote universal linear combinations of the Euler characteristics of manifolds of singularities of odd codimensions starting from the codimension 11.

Table 4 (k is odd)

$c = 3$	$4\chi(3A_1) = 8\chi(4A_1) + 2\chi(A_3 + A_1)$ $- 20\chi(6A_1) - 5\chi(A_3 + 3A_1) - 2\chi(2A_3) - 3\chi(A_5 + A_1)$ $+ 112\chi(8A_1) + 30\chi(A_3 + 5A_1) + 8\chi(2A_3 + 2A_1)$ $+ 15\chi(A_5 + 3A_1) + 4\chi(A_5 + A_3) + 6\chi(A_7 + A_1) + \dots$
	$4\chi(A_3) = 2\chi(A_3 + A_1) - \chi(A_3 + 3A_1)$ $+ 2\chi(2A_3) - \chi(A_5 + A_1)$ $+ 2\chi(A_3 + 5A_1) + 2\chi(A_5 + 3A_1)$ $+ \chi(A_5 + A_3) + \chi(A_7 + A_1) + \dots$
$c = 5$	$4\chi(5A_1) = 12\chi(6A_1) + 2\chi(A_3 + 3A_1)$ $- 56\chi(8A_1) - 14\chi(A_3 + 5A_1) - 4\chi(2A_3 + 2A_1)$ $- 5\chi(A_5 + 3A_1) - \chi(A_5 + A_3) - \chi(A_7 + A_1) + \dots$
	$4\chi(A_3 + 2A_1) = 6\chi(A_3 + 3A_1) + 4\chi(2A_3) + 4\chi(A_5 + A_1)$ $- 10\chi(A_3 + 5A_1) - 4\chi(2A_3 + 2A_1) - 9\chi(A_5 + 3A_1)$ $- 5\chi(A_5 + A_3) - 7\chi(A_7 + A_1) + \dots$
	$4\chi(A_5) = 2\chi(A_5 + A_1) - \chi(A_5 + 3A_1)$ $+ \chi(A_5 + A_3) - \chi(A_7 + A_1) + \dots$
$c = 7$	$2\chi(7A_1) = 8\chi(8A_1) + \chi(A_3 + 5A_1) + \dots$
	$2\chi(A_3 + 4A_1) = 5\chi(A_3 + 5A_1)$ $+ 2\chi(2A_3 + 2A_1) + 2\chi(A_5 + 3A_1) + \dots$
	$2\chi(2A_3 + A_1) = 2\chi(2A_3 + 2A_1)$ $+ 2\chi(A_5 + A_3) + \chi(A_7 + A_1) + \dots$
	$2\chi(A_5 + 2A_1) = 3\chi(A_5 + 3A_1)$ $+ \chi(A_5 + A_3) + 2\chi(A_7 + A_1) + \dots$
	$2\chi(A_7) = \chi(A_7 + A_1) + \dots$

The dots in formulas of Table 4 denote universal linear combinations of the Euler characteristics of manifolds of singularities of odd codimensions starting from the codimension 10.

Table 5

$k = 2$	$\chi(A_3) - \chi(3A_1) = 2\chi_0$
$k = 4$	$2\chi(A_5) - \chi(A_3 + 2A_1) + \chi(5A_1) = 6\chi_0$
$k = 6$	$5\chi(A_7) - 2\chi(A_5 + 2A_1) - \chi(2A_3 + A_1)$ $+ \chi(A_3 + 4A_1) - \chi(7A_1) = 20\chi_0$
$k = 8$	$14\chi(A_9) - 5\chi(A_7 + 2A_1) - 2\chi(A_5 + A_3 + A_1)$ $+ 2\chi(A_5 + 4A_1) - \chi(3A_3) + \chi(2A_3 + 3A_1)$ $- \chi(A_3 + 6A_1) + \chi(9A_1) = 70\chi_0$
$k = 10$	$42\chi(A_{11}) - 14\chi(A_9 + 2A_1) - 5\chi(A_7 + A_3 + A_1)$ $+ 5\chi(A_7 + 4A_1) - 4\chi(2A_5 + A_1) - 2\chi(A_5 + 2A_3)$ $+ 2\chi(A_5 + A_3 + 3A_1) - 2\chi(A_5 + 6A_1)$ $+ \chi(3A_3 + 2A_1) - \chi(2A_3 + 5A_1)$ $+ \chi(A_3 + 8A_1) - \chi(11A_1) = 252\chi_0$
$k = 12$	$132\chi(A_{13}) - 42\chi(A_{11} + 2A_1) - 14\chi(A_9 + A_3 + A_1)$ $+14\chi(A_9 + 4A_1) - 10\chi(A_7 + A_5 + A_1) - 5\chi(A_7 + 2A_3)$ $+5\chi(A_7 + A_3 + 3A_1) - 5\chi(A_7 + 6A_1) - 4\chi(2A_5 + A_3)$ $+ 4\chi(2A_5 + 3A_1) + 2\chi(A_5 + 2A_3 + 2A_1)$ $- 2\chi(A_5 + A_3 + 5A_1) + 2\chi(A_5 + 8A_1)$ $+ \chi(4A_3 + A_1) - \chi(3A_3 + 4A_1) + \chi(2A_3 + 7A_1)$ $- \chi(2A_3 + 10A_1) + \chi(13A_1) = 924\chi_0$

References

[1] V. I. Arnold, V. A. Vassiliev, V. V. Goryunov and O. V. Lyashko, Singularities 2, Itogi Nauki i Tekhn., Sovr. Probl. Mat., Fund. Napravl., **39**, Moscow, VINITI, 1989; English translation Encycl. of Math. Sci., **39**, Springer.

[2] R. C. Bose, On the number of circles of curvature perfectly enclosing or perfectly enclosed by a closed convex oval, Math. Zeitsch, **35** (1932), 16–24.

[3] J. W. Bruce, P. J. Giblin and C. G. Gibson, Symmetry sets, Proc. Roy. Soc. Edinburgh Sect. A, **101** (1985), 163–186.

[4] E. A. Feldman, On parabolic and umbilic points of immeresed hypersurfaces, Trans. Amer. Math. Soc., **127** (1967), 1–28.

[5] P. J. Giblin, Symmetry sets and medial axes in two and three dimensions, The Mathematics of Surfaces IX, (eds. Roberto Cipolla and Ralph Martin), Springer-Verlag, 2000, 306–321.

[6] P. J. Giblin and B. B. Kimia, On the local form and Transitions of Symmetry Sets, Medial Axes, and Shocks, Int. J. Computer Vision, **54** (2003), 143–157.

[7] O. Haupt, Verallgemeinerung eines Satzes von R. C. Bose über die Anzahl der Schmiegkreise eines Ovals, die vom Oval umschlossen werden

oder das Oval umschließen, J. Reine Angew. Math., **239/240** (1969), 339–352.

[8] V. D. Sedykh, Invariants of strictly convex manifolds, Functsional. Anal. i Prilozhen, **27** (1993), no. 3, 67–75; English transl. in Functional Anal. Appl., **27** (1993), 205–210.

[9] V. D. Sedykh, A relationship between Lagrange and Legendre singularities in stereographic projection, Mat. Sb. (N.S.), **185** (1994), no. 12, 123–130; English transl. in Russian Acad. Sci. Sb. Math., **83** (1995), 533–540.

[10] V. D. Sedykh, Invariants of nonflat manifolds, Funktsional. Anal. i Prilozhen, **29** (1995), no. 3, 41–50; English transl. in Functional Anal. Appl., **29** (1995), 180–187.

[11] V. D. Sedykh, Relations between Euler numbers of manifolds of corank 1 singularities of a generic front, Dokl. Akad. Nauk, **383** (2002), no. 6, 735–739; English transl. in Russian Acad. Sci. Dokl. Math., **65** (2002), 276–279.

[12] V. D. Sedykh, Resolution of corank 1 singularities of a generic front, Funktsional. Anal. i Prilozhen, **37** (2003), no. 2, 52–64; English transl. in Functional Anal. Appl., **37** (2003), 123–133.

[13] V. D. Sedykh, On the topology of singularities of Maxwell sets, Moscow Mathematical Journal, **3** (2003), 1097–1112.

[14] V. D. Sedykh, On topology of stable corank 1 singularities on the boundary of a connected component of the complement to a front, Mat. Sb., **195** (2004), no. 8, 91–130; English transl. in Sb. Math., **195** (2004), 1165–1203.

[15] D. Siersma, Properties of conflict sets in the plane, Banach Center Publications, **50** (1999), 267–276.

[16] V. M. Zakalyukin, Singularities of convex hulls of smooth manifolds, Functsional. Anal. i Prilozhen, **11** (1977), no. 3, 76–77.

Department of Higher Mathematics
Russian State Gubkin University of Oil and Gas
Leninsky prosp. 65, Moscow 119991
Russia
e-mail: sedykh@mccme.ru