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Quasi-convex decomposition in o-minimal structures. Application to the gradient conjecture

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Abstract.

We show that every subset of $\mathbb{R}^n$ definable in an o-minimal structure can be decomposed into a finite number of definable sets that are quasi-convex i.e. have comparable, up to a constant, the intrinsic distance and the distance induced from the embedding. We apply this result to study the limits of secants of the trajectories of gradient vector field ∇f of a C^1 definable function f defined in an open subset of $\mathbb{R}^n$. We show that if the o-minimal structure is polynomially bounded then the limit of such secants exists, that is an analog of the gradient conjecture of R. Thom holds. Moreover we prove that for $n = 2$ the result is true in any o-minimal structure.

§ 0. Introduction

Let f be a real analytic function on an open set $U \subset \mathbb{R}^n$ and let ∇f be its gradient in the Euclidean metric. Let $x(t)$ be a trajectory of ∇f. Then, after Łojasiewicz [16], if $x(t)$ has a limit point $x_0 \in U$, then the length of $x(t)$ is finite and $x(t) \to x_0$ as $t \to \infty$. Moreover, then the trajectory cannot spiral, that is the limit of secants

$$\lim_{t\to\infty} \frac{x(t) - x_0}{|x(t) - x_0|}$$

exists. The last result, known as the gradient conjecture of R. Thom, has been proven recently in [14]. The main purpose of this paper is to

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study this conjecture in the o-minimal set-up, that is for f that is C^1 and definable in an o-minimal structure.

Recall that the o-minimal structures are natural generalizations of the semi-algebraic or the subanalytic geometry satisfying important finiteness properties. The reader that is not familiar with this notion may refer to various introductory references as for instance [4], [5]. An o-minimal structure is called polynomially bounded if for each continuous function $\varphi : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$, $\varphi^{-1}(0) = 0$, definable in the structure, there is a constant $N > 0$ such that $|\varphi(r)| \geq |r|^N$. In o-minimal polynomially bounded structures the classical Łojasiewicz Inequalities (with exponents) hold. On the other hand the o-minimal structures that are not polynomially bounded contain the exponential function, cf. [17], and hence many flat functions. In what follows we suppose that we have fixed an o-minimal structure and the functions we consider are definable in this structure.

The trajectories of the gradient vector field of definable functions have been studied in [9], where an analog of Łojasiewicz's result of finiteness of length was proven. Thus we may place ourselves in the following set-up. We suppose that $f : U \to \mathbb{R}$ is a C^1 definable function defined in an open bounded definable $U \subset \mathbb{R}^n$. We consider a trajectory $x(s)$ of ∇f parameterized by the arc-length s. Since its length is finite the trajectory $x(s)$ has a unique limit point x_0, that is $x(s) \to x_0$ as $s \to s_0$, and either $x_0 \in U$, and then $\nabla f(x_0) = 0$, or $x_0 \in \overline{U} \setminus U$. In both cases we shall study the limits of secants

$$\lim_{s \to s_0} \frac{x(s) - x_0}{|x(s) - x_0|}.$$

Even in the subanalytic case this set-up is more general than the classical analytic one but of course the main difficulty to extend the gradient conjecture to this case is the presence of flat functions. In this paper, we were able to extend most of the properties of the trajectories of the gradient established in [14], but we came short of proving the conjecture in general. Our main results are the following

(1) *The length of the trajectory has the same asymptotic as the distance to the limit point*

$$\frac{|x(s) - x_0|}{|s - s_0|} \to 1 \text{ as } s \to s_0.$$

(2) *The gradient conjecture holds for $n = 2$. More precisely, in this case the trajectory is definable in an o-minimal structure, maybe bigger than the one that contains f.*

(3) *The gradient conjecture holds for polynomially bounded o-minimal structures.*

Moreover, similarly to [14], we were able to "capture" the trajectories arriving to a fixed limit point $x_0 \in \overline{U}$ into a finite number of sets. First of all only finitely many limiting values of f, $\lim_{s \to s_0} f(x(s))$, are allowed along the trajectories of ∇f that tend to x_0, see remark 6.2. Furthermore, if we suppose $|\nabla f| \geq 1$, that we can do by section 3, then we can describe the asymptotic behavior of f at the limit point more precisely. There exists a finite number of definable functions $\{\varphi(r)\}$ of one real variable r, where r stands for radius $r = |x - x_0|$, such that on each trajectory that tends to x_0, $f(x(s)) \sim \varphi(|x(s) - x_0|)$ for exactly one such function φ. We shall call these functions the characteristic functions associated to f at x_0. A more precise result on the asymptotic behavior of f along the trajectory is given in section 7.

Some parts of our argument are similar to that of [14]. Let us stress here the main differences. The characteristic exponents of [14] characterizing the possible asymptotic behavior of f along trajectories are replaced by characteristic functions. In order to show their existence we cannot use the argument of finitude of exponents as in [14] since it does not make sense in general. Instead we use a geometric argument on the structure of definable sets. Namely we show that each definable set can be decomposed into a finite union of quasi-convex cells, as explained in section 1 below. In a polynomially bounded case if $\varphi : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ is definable continuous then φ/r is locally integrable. This is not the case in general. We have to carefully distinguish those φ for which φ/r is integrable, we call them small, and the other ones, that we call unit-like. Many our arguments, in particular the proof of conjecture for $n = 2$, relies on the properties of small functions. We stopped short of carrying out the proof of the gradient conjecture for o-minimal structure because of the existence of small functions with unit-like square root.

The paper is organized as follows. In section 1 we show that each definable set can be decomposed in a finite union of quasi-convex cells, that is such cells in $\mathbb{R}^n$ for which the induced Euclidean distance is comparable, up to a constant, with the intrinsic one (i.e. along the cell). This part is quite technical. The reader interested mainly in the properties of the trajectories of gradient can go directly to section 2. In this section we study the germs of continuous definable functions $\varphi : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ and the question of integrability of φ/r. In section 3 we show analogs of Łojasiewicz and Bochnak-Łojasiewicz Inequalities for gradient in the o-minimal set-up. The characteristic functions are introduced in section 4. We show that there are finitely many such functions and that each trajectory $x(s)$ of ∇f with the origin as the limit point has to finally end-up in one of finitely many sets $U_\varphi = \{x | c\varphi(x) < |f(x)| < C\varphi(x)\}$, $C, c > 0$, φ being a characteristic function. This holds under the assumption $|\nabla f| \geq 1$, that can be always achieved by replacing f by $\Psi \circ f$ without affecting the trajectories of the gradient. Subsequently the function $F(x) = \frac{f(x)}{\varphi(r)}$ is used as a control function in the sense of Thom. The estimates along trajectories are carried out in sections 5 and 7. It is convenient, for each characteristic function, to make another change of target coordinate , that is to replace f by a function of the from $\Phi \circ f$, so that the corresponding characteristic function φ becomes equivalent to the distance to the origin r. This simplifies many formulae. After such a change we show not only that $F = \frac{f}{\varphi}$ is bounded from zero and infinity on the trajectory but also that it approaches a fixed value and only finitely many such values are allowed. These values, called asymptotic critical values, are studied in section 6. As application we show in section 8 the o-minimal gradient conjecture for $n = 2$ and in section 9 for the polynomially bounded structures.

The result of the first section has been obtained independently by W. Pawłucki [21]. During the redaction of this paper we also learned that some other of the results proven is this paper were obtained during a workshop at the Fields Institute (Toronto) by M. Aschenbrenner, S. Kuhlmann, C. Miller, D. Novikov, P.Speisseger, and S. Starchenko. In particular, we were informed that the gradient conjecture holds in the polynomially bounded o-minimal structures. The case $n = 2$ was stated as an open problem at this meeting.

Notation and convention.

We often write r instead of $|x|$ which is the Euclidean norm of x. We use the standard notation $\varphi = o(\psi)$ or $\varphi = O(\psi)$ to compare the asymptotic behavior of φ and ψ, usually when we approach the origin.

Sometimes we write $\varphi \ll \psi$ instead of $\varphi = o(\psi)$. We write $\varphi \sim \psi$ if $\varphi = O(\psi)$ and $\psi = O(\varphi)$, and $\varphi \simeq \psi$ if $\frac{\varphi}{\psi}$ tends to 1.

Erratum to [14]:

The formula on the line -6 on page 783 should be replaced by:

$$\frac{dF}{d\tilde{s}} = \frac{|\nabla' f|}{r^{l-1}} + \frac{|\partial_r f|}{r^{l-1}} \frac{|\partial_r f|}{|\nabla' f|} O(r^{2\omega}) = O(r^{\eta}) + \frac{|\partial_r f|}{|\nabla' f|} O(r^{2\omega}).$$

§1. L-regular cells

Consider $\mathbb{R}^n$ equipped with the canonical scalar product. We say that $A \subset \mathbb{R}^n$ verifies *the Whitney property with constant* $M > 0$, if any two points $x, y \in A$ can be joined in A by a piecewise smooth arc of length $\leq M|x - y|$. Following M. Gromov [6] one could also say that A is *quasi-convex*, or more precisely that A is M*-quasi-convex.* Any bounded semianalytic set can be covered by a finite number of quasi-convex (and semianalytic) sets as proven by the second named author [19] using the regular projections of T. Mostowski [18]. The construction proposed in [19] (extended in [20] to subanalytic sets) does not allow to estimate the constant M. Next the first named author [10] proved, by a different argument, that any bounded subanalytic subset can be decomposed (more precisely stratify) into a finite union of M-quasi-convex (and subanalytic) sets, with the constant M depending only on n - the dimension of the ambient space. This result was improved in [12], where it is shown that for any $M > 1$ such a finite decomposition into M-quasi-convex sets exists.

The construction from [10] can be adapted for o-minimal structures and actually can be done with parameters (which we need in the sequel). We shall explain it in this section.

We define, by induction on n, a class of subsets of $\mathbb{R}^n$. For any $x \in \mathbb{R}^n$ let us write $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$. We say that $A \subset \mathbb{R}^n$ is a *standard L-regular cell in* $\mathbb{R}^n$ *with constant* C, if $A = \{0\}$ for $n = 0$, and for $n > 0$ the set A is of one of the following forms:

(*graph*) $\qquad A = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R};\ x_n = h(x'),\ x' \in A'\}$

(we write often h instead of A), or
(band)
$$A = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R};\ f(x') < x_n < g(x'),\ x' \in A'\} = (f, g)$$

where A' is a standard L-regular cell in $\mathbb{R}^{n-1}$ with constant C, $f, g, h : A' \to \mathbb{R}$ are C^1 functions such that $f(x') < g(x')$ for $x' \in A'$. Moreover we require that

$$\|df(x')\| \le C,\ \|dg(x')\| \le C,\ \|dh(x')\| \le C \tag{1.1}$$

for all $x' \in A'$. We call A' *the base* of the cell A, and in the case of band the graphs of f and g *the horizontal part of the boundary* of A.

By induction, we obtain that A is a C^1 submanifold of $\mathbb{R}^n$ (not closed in general). So it make sense to define df, dg, dh and also their norms (with respect to the norm induced on tangent space to A' at x'). If in the above we drop the condition (1.1), but we still assume that the functions f, g, h are C^1 we say that the set A is a *standard C^1 cell* in $\mathbb{R}^n$. If the functions f, g, h are only continuous we shall say that A is a *standard cell* in $\mathbb{R}^n$.

Finally we say that $B \subset \mathbb{R}^n$ is an *L-regular cell in $\mathbb{R}^n$ with constant C*, if there exists an orthogonal change of variables $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ such that $\varphi(B)$ is a standard L-regular cell (with constant C) in $\mathbb{R}^n$. By convention the empty set is an L-regular set (with any constant), also it will be convenient not to distinguish between function and its graph.

It is easily seen by induction that

Lemma 1.1. *Any L-regular cell in $\mathbb{R}^n$ with constant C is M-quasi-convex, where $M = (C+1)^{n-1}$. Moreover $\overline{A}$ is also M-quasi-convex.*

As a piece of terminology we recall that by a *decomposition* we always understand a disjoint union. We say that a decomposition $\mathbb{R}^N = \bigcup_{i \in I} B^i$ is *compatible* with a collection $A^k \subset \mathbb{R}^N$, $k \in K$, if $B^i \cap A^k = \emptyset$ or $B^i \subset A^k$ for any $i \in I$, $k \in K$. We also say that a decomposition $\mathbb{R}^N = \bigcup_{i \in I} B^i$ is a *stratification* if each B^i is a C^1 submanifold and $\dim(\overline{B^i} \setminus B^i) < \dim B^i$, and moreover that this decomposition is compatible with the collection $\overline{B^i}, i \in I$.

Notation. For $B \subset \mathbb{R}^n \times \mathbb{R}^p$ and $t \in \mathbb{R}^p$ we write $B_t = \{x \in \mathbb{R}^n : (x, t) \in B\}$.

Now we state the main result on a decomposition of a definable set into a finite number of quasi-convex sets.

Theorem 1.2. *There exists $M = M(n) > 0$ such that any set $A \subset \mathbb{R}^n \times \mathbb{R}^p$ definable in an o-minimal structure can be decomposed into a finite (and disjoint) union $A = \bigcup_{i \in I} B^i$, such that for each $t \in \mathbb{R}^p$, every set B^i_t has the Whitney property with constant M (i.e. is M-quasi-convex). So, in particular, $A_t = \bigcup_{i \in I} B^i_t$ for each $t \in \mathbb{R}^p$.*

Corollary 1.3. *Let $A \subset \mathbb{R}^n \times \mathbb{R}^p$ be a family of definable sets such that each A_t, $t \in T \subset \mathbb{R}^p$, is connected. Then there is a constant $C > 0$ such that for every $t \in T$ and $x, x' \in A_t$ there is a definable continuous curve ξ joining x and x' in A_t such that*

$$\text{length}(\xi) \leq C \, \text{diam}(A_t),$$

where $\text{diam}(A_t)$ *stands for the diameter of A_t.*

What we actually prove below is more precise than theorem 1.2, namely we have:

Proposition 1.4. *Let $A^k \subset \mathbb{R}^n \times \mathbb{R}^p$, $k \in K$, be a finite collection of definable sets in an o-minimal structure. Then there exists finitely many disjoint definable sets $B^i \subset \mathbb{R}^n \times \mathbb{R}^p$, $i \in I$, and linear orthogonal mappings $\varphi^i : \mathbb{R}^n \to \mathbb{R}^n$, $i \in I$, such that:*

a) *for every $t \in \mathbb{R}^p$, each $\varphi^i(B^i_t)$ is a standard L-regular cell in $\mathbb{R}^n$ with constant C. The constant $C = C_n$ depends only on n.*
b) *For every $t \in \mathbb{R}^p$, the family $B^i_t \subset \mathbb{R}^n$, $i \in I$, is a stratification of $\mathbb{R}^n$.*
c) *For any $k \in K$ there exists $I_k \subset I$ such that $A^k_t = \bigcup_{i \in I_k} B^i_t$, for every $t \in \mathbb{R}^p$.*

Remark 1.5. Clearly, for a fixed $t \in \mathbb{R}^p$ some of B^i_t may be empty.

Proposition 1.4 will be proved at the end of this section. Before we give some preliminaries on the distances between linear subspaces and we recall some basic facts on cell decompositions in o-minimal structures. We establish also the definability of tangent mapping (with parameters).

1.1. Distances between linear subspaces

In this subsection by a line or a hypersurface in $\mathbb{R}^n$ we mean a linear subspace of dimension 1 and $n-1$ respectively. First we recall the definition of the angle (or the distance) between linear subspaces. If

P, S are (vector) lines we denote by $\delta(P,S)$ the sine of the angle between P and S, in other words $\delta(P,S) = \sqrt{1-\langle p,s\rangle^2}$, where $|p| = |s| = 1$, $p \in P$ and $s \in S$.

Let X be a linear subspace in $\mathbb{R}^n$, let P be a line in $\mathbb{R}^n$. We define the angle between P and X as

$$\delta(P,X) = \inf\{\delta(P,S); S \textit{ is a line in } X\}$$

Finally, if Y is a linear subspace in $\mathbb{R}^n$ we put

$$\delta(Y,X) = \sup\{\delta(P,X); P \textit{ is a line in } Y\}$$

Let us denote by $\mathbb{G}_{d,n}$ the grassmanian of all d-dimensional linear subspaces of $\mathbb{R}^n$ equipped with the natural structure of real algebraic variety. Then, it is easily seen by the Tarski-Seidenberg theorem that:

Lemma 1.6. *The function $\mathbb{G}_{d,n} \times \mathbb{G}_{e,n} \ni (Y,X) \longmapsto \delta(Y,X) \in \mathbb{R}$ is continuous and semialgebraic. Moreover, if $d = e$, then δ is a distance on $\mathbb{G}_{d,n}$, compatible with the standard topology on $\mathbb{G}_{d,n}$.*

Remark 1.7. Let X be a linear subspace and P a line in $\mathbb{R}^n$. Denote by $P^\perp$ the orthogonal complement of P and by π the orthogonal projection on $P^\perp$. Let $c > 0$. Assume that $\delta(P,X) > c$, then X is the graph of a linear mapping

$$\xi : P^\perp \cap \pi(X) \to P$$

satisfying $\|\xi\| \le C < +\infty$, where $C = \frac{\sqrt{1-c^2}}{c}$.

Given a finite system $X_1, ...X_r$ of hyperplanes of $\mathbb{R}^n$. Then we may find, in a uniform way, a line P transverse ot each X_i. More precisely we have the following fact of metric-combinatorial nature that will be crucial in the proof of proposition 1.4.

Lemma 1.8. *For any two positive integers r, n there exist constants $\tau = \tau(r,n) > 0$ and $c = c(r,n) > 0$ such that for given $X_1, ...X_r$ hyperplanes in $\mathbb{R}^n$, there exists a line P such that, if $Y_1, ..., Y_r$ are hyperplanes verifying $\delta(Y_i, X_i) < \tau$, $i = 1, ...r$, then*

$$\delta(P, Y_i) > c \quad \textit{for each } i = 1, ..., r. \tag{1.2}$$

Proof. We fix n, and consider the metric d on the sphere S^{n-1} induced by δ i.e. $d(p,q) = \delta(\mathbb{R}p, \mathbb{R}q)$ for $p, q \in S^{n-1}$. Let us denote

$X_i^\tau = \{p \in S^{n-1} : dist(p, X_i \cap S^{n-1}) < \tau\}$, where as usually $dist(p, Z) = \inf\{d(p,q) : q \in Z\}$. Note that $\delta(Y_i, X_i) < \tau$ means that $Y_i \cap S^{n-1} \subset X_i^\tau$ and $\delta(P, Y_i) > c$ means that $B(p,c) \cap Y_i \cap S^{n-1} = \emptyset$, where $p \in P \cap S^{n-1}$.

Claim 1.9. *For any $r \in \mathbb{N}$, there exists $\tau_r > 0$ and $c_r > 0$ such that the complement of $\bigcup_{i=1}^r X_i^\tau$ in S^{n-1} contains a ball of radius c_r (in the metric d).*

The claim implies lemma 1.8. Indeed, the line passing by the center of the ball has the property desired in (1.2). We show the claim by induction on r. The case $r = 1$ is obvious. Let us denote τ_r and c_r corresponding constants in the claim for r hyperplanes. Let $B(p, c_r)$ be a ball in S^{n-1} which is disjoint with each $X_i^{\tau_r}$, $i = 1, ...r$. Put $\tau_{r+1} = c_{r+1} = \min\{\tau_r, c_r\}/3$, then the set $B(p, c_r) \setminus X_{r+1}^{\tau_{r+1}}$ contains a ball of radius c_{r+1}. Q.E.D.

1.2. Cell decompositions in families

Recall that a finite decomposition $\mathbb{R}^N = \bigcup_{i \in I} B^i$ is called *a cell decomposition (resp. a C^1 cell decomposition)* if each B^i is a standard (resp. a C^1 standard) cell in $\mathbb{R}^N$, and the collection $\pi(B^i)$, $i \in I$, is a cell decomposition of $\mathbb{R}^{N-1}$, where $\pi : \mathbb{R}^N \to \mathbb{R}^{N-1}$ is the projection parallel to the x_N-axis. We say that a decomposition is definable if all its members are definable (in some fixed o-minimal structure). We have the following fundamental result in the theory of o-minimal structures due to Steinhorn, Pillay and Knight [22],[8] (see also [4], [3]):

Theorem 1.10 (Cell decomposition). *For any finite collection A^k, $k \in K$, of definable sets in $\mathbb{R}^N$ there exists a definable C^1 cell decomposition $\mathbb{R}^N = \bigcup_{i \in I} B^i$ compatible with the collection A^k, $k \in K$.*

Remark 1.11. The basic result (proved in [22],[8]) is the existence of a cell decomposition (without any smoothness assumption). The existence of C^1 decomposition is due to van den Dries and is valid in the C^k class for any finite k (cf. [4]). Moreover this decomposition can be refined to a stratification (loc.cit.).

What we need in the sequel is a decomposition with parameters (we rather say in a family). We say that that a definable set $A \subset \mathbb{R}^n \times \mathbb{R}^p$ is *a definable family of standard cells in $\mathbb{R}^n$* if: each A_t is either empty or is a standard cell in $\mathbb{R}^n$ and the type of A_t does not depend on $t \in \mathbb{R}^p$.

(We say that two cells $A_1, A_2 \subset \mathbb{R}^n$ *are of the same type* if they are both graphs or both bands over their bases which are of the same type.)

Clearly if $A \subset \mathbb{R}^{n+p}$ is a standard cell in $\mathbb{R}^{n+p} = \mathbb{R}^p \times \mathbb{R}^n$ then A is a definable family of standard cells in $\mathbb{R}^n$ (cf. eg. [4] chap 3). Hence all claims of existence of decomposition into a definable family of standard cells (or C^1 cells) in $\mathbb{R}^n$ follows from theorem 1.10.

We shall often use the following construction of cell decomposition in an o-minimal structure.

The CD (cell decomposition) construction:

Let $A^k \subset \mathbb{R}^{n+1} \times \mathbb{R}^p$, $k \in K$, be a finite collection of disjoint definable sets. Denote by $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ the projection which forgets the coordinate x_{n+1}. Suppose that for each $t \in \mathbb{R}^p$ every A_t^k (if nonempty) is a C^1 submanifold of $\mathbb{R}^{n+1}$ of dimension d and moreover that π restricted to A_t^k is an immersion. Each $\pi^{-1}(x) \cap A_t^k$ is discrete, so it must be finite, by o-minimality. Now, for every $r \in \mathbb{N}$ and $k \in K$ the set

$$\Sigma_r^k = \{(x,t) \in \mathbb{R}^n \times \mathbb{R}^p;\ \pi^{-1}(x) \cap A_t^k \text{ has } r \text{ elements}\}$$

is definable, moreover $\Sigma_r^k = \emptyset$ for r larger than some r_0. This is due to the fundamental property of o-minimal structures; if the fibers of a definable mapping have only isolated points, then there exists a uniform bound for the number of points in each fiber (cf. eg. [4],[3]).

Let $B^l \subset \mathbb{R}^n \times \mathbb{R}^p$, $l \in L$, be a finite collection of definable families of standard C^1 cells in $\mathbb{R}^n$ compatible with the family Σ_r^k; $r \le r_0$, $k \in K$, and such that, for each $t \in \mathbb{R}^p$, the collection B_t^l; $l \in L$, is a cell decomposition of $\mathbb{R}^n$. Fix $l \in L$ such that B_t^l is non-empty and hence is a C^1 submanifold of $\mathbb{R}^n$. We claim that all connected components of

$$\pi^{-1}(B_t^l) \cap A_t^k,\ \ k \in K$$

are the graphs of C^1 functions $f_t^j : B_t^l \to \mathbb{R}$, $1 \le j \le r$. Indeed, $\Gamma = \pi^{-1}(B_t^l) \cap A_t^k$ is a C^1 submanifold of $\mathbb{R}^{n+1}$ and the projection $\pi|_\Gamma : \Gamma \to B_t^l$ is a local diffeomorphism. Since $B_t^l \subset \Sigma_r^k$ for some $r \in \mathbb{N}$, the number of points in the fiber is constant, and it follows that $\pi|_\Gamma$ is a finite (r-sheeted) covering. Moreover, it is a diffeomorphism on each connected component of Γ, because B_t^l is simply connected (in fact homeomorphic to a ball). So the family

$$\pi^{-1}(B_t^l) = \bigcup_{1 \le j \le r} f_t^j \cup \bigcup_{0 \le j \le r} (f_t^j, f_t^{j+1}),$$

form a (standard) C^1 cell decomposition of $\pi^{-1}(B_t^l)$. We shall call this collection *subordinate to the collection* A^k, $k \in K$. (Recall that $(f_t^j, f_t^{j+1}) = \{(x, x_{n+1}) \in \mathbb{R}^n \times \mathbb{R};\ f_t^j(x) < x_{n+1} < f^{j+1}(x),\ x \in B_t^l\}$.) The functions are ordered in the way that $f_t^j < f_t^{j+1}$ and $f_t^0 \equiv -\infty$, $f_t^{r+1} \equiv +\infty$. Moreover each function (or rather its graph) f_t^j, $1 \le j \le r$, is contained in some A_t^k, where k may depend on t. Subdividing, if necessary, the set B^l, we may assume that $f_t^j \subset A_t^k$, where $k = k(j)$ does not depend on $t \in \mathbb{R}^p$. Of course for some t the set A_t^k may be empty and then by convention we set $f_t^j = \emptyset$, $1 \le j \le r$.

Remark 1.12. Note that by construction the horizontal parts of boundaries of cells are also cells.

1.3. Controlling tangents

First let us observe that each C^1 cell has a definable tubular neighborhood. More precisely

Lemma 1.13 (Definable tubular neighborhoods). *Let $A \subset \mathbb{R}^n \times \mathbb{R}^p$ be a definable family of standard C^1 cells of dimension d. Then there is a definable family of submersions*

$$\rho_t : \Omega_t \to A_t,\ t \in \mathbb{R}^p$$

such that $\Omega_t \subset \mathbb{R}^n$ is an open neighborhood of A_t and each ρ_t is the identity on A_t.

Proof. We sketch the construction only in the case without parameters. The reader may easily check that it works also with parameters. Let $A \subset \mathbb{R}^n$ be a standard C^1 cell of dimension $d < n$. We proceed by induction on n. Let $\rho' : \Omega' \to A'$ be a definable tubular neighborhood (in $\mathbb{R}^{n-1}$) of the base A' of cell A. In the case A is a band

$$A = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R};\ f(x') < x_n < g(x'),\ x' \in A'\}$$

we put $\rho(x', x_n) = (\rho'(x'), x_n)$ for $x' \in \Omega', x_n \in \mathbb{R}$, and $\Omega = \rho^{-1}(A)$.

In the case of graph

$$A = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R};\ x_n = h(x'),\ x' \in A'\}$$

we set $\rho(x', x_n) = (\rho'(x'), h(\rho'(x)))$ for $x' \in \Omega', x_n \in \mathbb{R}$, and $\Omega = \rho^{-1}(A) = \Omega' \times \mathbb{R}$. Q.E.D.

Lemma 1.14 (Definability of the tangent map). *Let $A \subset \mathbb{R}^n \times \mathbb{R}^p$ be a definable family of standard C^1 cells of dimension d. Then the mapping*

$$\sigma : A \ni (x,t) \mapsto T_x A_t \in \mathbb{G}_{d,n}$$

is definable, where $T_x A_t$ stands for the tangent space to A_t at x.

Proof. We proceed by induction on n. We may suppose that $n > 0$ and $d < n$. We construct (by induction) a definable family of mappings

$$\varphi_t : \Omega_t \to \mathbb{R}^{n-d}, \; t \in \mathbb{R}^p$$

such that Ω_t is an open neighborhood of A_t, $\varphi_t^{-1}(0) = A_t$ and φ_t is submersive on Ω_t.

The case of graph; each non empty A_t is the graph of a C^1 function $h_t : A'_t \to \mathbb{R}$, where $A' = \bigcup A'_t \subset \mathbb{R}^{n-1} \times \mathbb{R}^p$ is a definable family of C^1 cells (of dimension d) in $\mathbb{R}^{n-1}$. By lemma 1.13 each h_t can be extended to C^1 function in an open neighborhood Ω'_t of A'_t, moreover this can be done in a definable family. By induction we have family $\varphi'_t : \Omega'_t \to \mathbb{R}^{n-d-1}$, $t \in \mathbb{R}^p$, corresponding to A' . Clearly we may suppose that φ'_t and h_t are defined on the same Ω'_t. We put

$$\varphi_t(x', x_n) = (\varphi_t(x'), x_n - h_t(x'))$$

for $(x', x_n) \in \Omega'_t \times \mathbb{R} = \Omega_t$.

The case of band is similar and is left to the reader.

The derivative of φ_t i.e. the mapping

$$\varphi^{(1)} : (x,t) \mapsto d\varphi_t(x) \in L^*(\mathbb{R}^n, \mathbb{R}^{n-d})$$

is definable (cf. eg. [4] Chap 7.). (Here by $L^*(\mathbb{R}^n, \mathbb{R}^{n-d})$ we mean the space of linear epimorphisms from $\mathbb{R}^n$ to $\mathbb{R}^{n-d}$.) The mapping $L^*(\mathbb{R}^n, \mathbb{R}^{n-d}) \ni \phi \mapsto \ker \phi \in \mathbb{G}_{d,n}$ is semialgebraic, hence definable in any o-minimal structure. So our σ is definable as a composition of definable maps. Q.E.D.

Our next goal is to control the variation of tangent spaces to cells. Recall that we have the metric δ on the grasmannian $\mathbb{G}_{d,n}$. Let $\varepsilon > 0$, we say that Γ, a d-dimensional C^1 submanifold of $\mathbb{R}^n$, is *ε-flat* if for any $x, y \in \Gamma$ we have

$$\delta(T_x, T_y) \leq \varepsilon.$$

For each $\varepsilon > 0$ we fix a finite covering $\mathbb{G}_{d,n} = \bigcup \Theta_\nu^\varepsilon$, where each Θ_ν^ε is an open ball of diameter (with respect to δ) less then ε. Let A_t be a definable C^1 submanifold in $\mathbb{R}^n$, of dimension d and let $\sigma : A_t \to \mathbb{G}_{d,n}$ denote the tangent mapping. Then each nonempty $\sigma^{-1}(\Theta_\nu^\varepsilon)$ is ε-flat. Moreover, by lemma 1.14, it is a definable set. Indeed each Θ_ν^ε is semialgebraic (cf. lemma 1.6) and the inverse image of a definable set, by a definable map, is definable. Having this observation it is now routine to prove the following:

Proposition 1.15. *Given $\varepsilon > 0$, and let $A^k \subset \mathbb{R}^n \times \mathbb{R}^p$, $k \in K$, be a finite collection of definable sets. Then there exists finitely many disjoint definable sets $B^i \subset \mathbb{R}^n \times \mathbb{R}^p$, $i \in I$, such that:*

a) *for each $i \in I$, (B_t^i) is a definable family of ε-flat standard C^1 cells of dimension d. More precisely; for every $i \in I$ there exists ν_i such that*
$$T_x B_t^i \in \Theta_{\nu_i}^\varepsilon,\ (x,t) \in B^i;$$
b) *For every $t \in \mathbb{R}^p$ the collection $B_t^i \subset \mathbb{R}^n$, $i \in I'$, is a stratification of $\mathbb{R}^n$;*

c) *For any $k \in K$ there exists $I_k \subset I$ such that $A_t^k = \bigcup_{i \in I_k} B_t^i$, for every $t \in \mathbb{R}^p$.*

1.4. Proof of Proposition 1.4

We proceed by induction on n. The case $n = 0$ is trivial. Suppose that Proposition 1.4 holds for $n-1$. We argue now by induction on $d = \max\{\dim A_t^k\}$. For the sake of clarity we prove only the decomposition part, i.e. statements a) and c). The refinement to a stratification is routine (cf. [4]). At first we deal with the non-open cells, that is $d < n$, then we decompose the open ones.

Case of non-open cells.

Fix an $\varepsilon < 1/2$ and assume that we are given $A^k \subset \mathbb{R}^n \times \mathbb{R}^p$, $k \in K$, a finite collection of definable sets. Let B^i be one of the sets given by proposition 1.15, let $d < n$ be the dimension of nonempty B_t^i. We shall prove that:

Lemma 1.16. *There exists a finite collection of definable families of cells $D^l \subset \mathbb{R}^{n-1} \times \mathbb{R}^p$, $l \in \Lambda$, and linear orthogonal mappings $\varphi^l : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$, $l \in \Lambda$, such that:*

(1) *for every* $l \in \lambda$ *each (nonempty)* $\varphi^\lambda(D^l_t)$ *is a standard L-regular cell in* $\mathbb{R}^{n-1}$*;*

(2) *For every* $i \in I$ *there is* $\Lambda_i \subset \Lambda$ *such that* $B^i_t = \bigcup_{\lambda \in \Lambda_i} \pi^{-1}(D^\lambda_t)$*,*

for every $t \in \mathbb{R}^p$*.*

(Here $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ *denotes the projection on the first* $n-1$ *variables.* *)*

Proof. Since all tangent spaces to B^i_t are in a ball of diameter less than $\varepsilon < 1/2$ there exists a line P and $c = c(\varepsilon) > 0$ such that

$$\delta(P, T_{(x,t)}B^i_t) > c, \ (x,t) \in B^i.$$

Let $\varphi^i : \mathbb{R}^n \to \mathbb{R}^n$ be an orthogonal mapping which sends P to x_n-axis and $P^\perp$ to $\mathbb{R}^{n-1}$ (the first $n-1$ coordinates). According to remark 1.7, the set $\varphi^i(B^i)$ is locally the graph of a C^1 function defined on a submanifold in $\mathbb{R}^{n-1}$. Moreover, there exists $C < \infty$ (depending only on c) such that the norm of the differential of this function is bounded by C. Now it is enough to apply the induction hypothesis and the CD construction to obtain lemma 1.16. Q.E.D.

Case of open cells.

The main difficulty is to decompose an open cell into finitely many L-regular cells. This will be done in two steps: in the first one, using proposition 1.15, we construct a decomposition into C^1 cells such that the boundary of each open cell is contained in a union of at most $2n$ ε-flat submanifolds of dimension $n-1$. Then, in the second step, we apply to each such cell lemma 1.8. If $\varepsilon \le \tau(2n, n)$ then there exists a line P that makes angle with any tangent space to the boundary of the cell larger than some $c > 0$. After changing the coordinates in the way that P becomes the x_n-axis we apply the CD construction. This will give us C^1 cells with the horizontal parts of the boundary that are graphs of C^1 functions with differential of norm smaller than $C < \infty$ (cf. remark 1.7). Now by induction we may subdivide (in $\mathbb{R}^{n-1}$) the base of each above cell into L-regular cells. Hence the proof will be achieved. Now we explain the details.

Step 1. Let us fix $\varepsilon = \tau(2n, n)$ of lemma 1.8. Let $A^k \subset \mathbb{R}^n \times \mathbb{R}^p$, $k \in K$, be a finite collection of definable sets. By proposition 1.15, theorem 1.10, and the CD construction there exists a finite collection of disjoint definable families $B^l \subset \mathbb{R}^n \times \mathbb{R}^p$, $l \in \Lambda$, with properties we explain below.

Fix $l \in \Lambda$ such that each (nonempty) cell $B = B_t^l$ is open (we skip l, t for a moment to simplify the notation), that is of the form:

(1.3)
$$B = B_n = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R};\ f_n(x') < x_n < g_n(x'),\ x' \in B_{n-1}\}$$

and by induction:

$$B_i = \{(x_i', x_i) \in \mathbb{R}^{i-1} \times \mathbb{R};\ f_i(x_i') < x_i < g_i(x_i'),\ x_i' \in B_{i-1}\}, \tag{1.4}$$

$i = 1, \dots, n-1$. We may assume that each B_i, $i = 1, \dots, n-1$, is open in $\mathbb{R}^i$ and every f_i, g_i, $i = 1, \dots, n-1$, is a C^1 function such that its graph (in $\mathbb{R}^i$) is ε-flat. More precisely; independently of $t \in \mathbb{R}^p$, there exist $\tilde{\Theta}_i^f$, $\tilde{\Theta}_i^g$ two open balls, of diameter ε, in the grasmannian $\mathbb{G}_{i-1,i}$, such that the tangent spaces to the graph of f_i (resp. g_i) belong to $\tilde{\Theta}_i^f$ (resp. $\tilde{\Theta}_i^g$). Note that if $\tilde{X}, \tilde{Y} \in \mathbb{G}_{i-1,i}$ and $X = \tilde{X} \times \mathbb{R}^{n-i}$, $Y = \tilde{Y} \times \mathbb{R}^{n-i}$, then

$$\delta(X, Y) = \delta(\tilde{X}, \tilde{Y}), \tag{1.5}$$

since $\mathbb{R}^i \times 0$ and $0 \times \mathbb{R}^{n-i}$ are orthogonal. This implies that there exist Θ_i^f, Θ_i^g two open balls of diameter ε, in the grasmannian $\mathbb{G}_{n-1,n}$ such that independently of $t \in \mathbb{R}^p$ we have

(1.6)
$$\{X = \tilde{X} \times \mathbb{R}^{n-i};\ \tilde{X} \in \tilde{\Theta}_i^f\} \subset \Theta_i^f,\ \{X = \tilde{X} \times \mathbb{R}^{n-i};\ \tilde{X} \in \tilde{\Theta}_i^g\} \subset \Theta_i^g.$$

Denote by ∂B the boundary of B. Then clearly, by (1.3) and (1.4),

$$\partial B \subset \bigcup_{i=1}^{n} f_i \times \mathbb{R}^{n-i} \cup \bigcup_{i=1}^{n} g_i \times \mathbb{R}^{n-i} \tag{1.7}$$

Hence the tangent spaces to ∂B belong to the union of balls Θ_i^f, $i = 1, \dots, n$, and Θ_i^g, $i = 1, \dots, n$. Indeed we can take the decomposition (1.7) of the boundary of B.

So we have proved the following:

Lemma 1.17. *For every $l \in \Lambda$ such that B_t^l is open there exist $2n$ balls of diameter ε in the grassmanian $\mathbb{G}_{n-1,n}$ such that for each $t \in \mathbb{R}^p$ any tangent space to the boundary of B_t^l belongs to one of these balls.*

Step 2. Recall $\varepsilon \le \tau(2n, n)$ of lemma 1.8 and we work with a fixed definable family B^l such that for each $t \in \mathbb{R}^p$ the set B_t^l is open (possibly empty) in $\mathbb{R}^n$, and B^l_t satisfies lemma 1.17. By lemma 1.8 there exist a line P and $c > 0$ such that if $Y \in \mathbb{G}_{n-1,n}$ is a tangent space to the boundary of B^l_y, then $\delta(P,Y) > c$. After a linear orthogonal change of variables in $\mathbb{R}^n$ we may assume that P is the x_n-axis. Applying the CD construction to ∂B^l_t, decomposed as in (1.7), we obtain finitely many disjoint definable families $D^s \subset \mathbb{R}^{n-1} \times \mathbb{R}^p$, $s \in S$, and such that B^l_t, $t \in \mathbb{R}^p$, is a union of the sets of the form

$$(f,g) = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R};\, f(x') < x_n < g(x'),\, x' \in D^s_t\}$$

and

$$h = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R};\, x_n = h(x'),\, x' \in D^s_t\},$$

with C^1 functions $f, g, h : D^s_t \to \mathbb{R}$. By remark 1.7 the norm of differential of each f, g, h is bounded by a constant C which depends only on n. On the other hand we may assume by induction, that after an orthogonal change of coordinates in $\mathbb{R}^{n-1}$ (independent of $t \in \mathbb{R}^p$), each D^s_t is an L-regular cell in $\mathbb{R}^{n-1}$, with constant C. So h and (f,g) are standard L-regular cells in $\mathbb{R}^n$, with constant C.

This ends the proof of proposition 1.4.

§2. Definable Functions of One Variable

First we shall recall some elementary properties of germs at 0 of definable functions. We denote $\mathbb{R}_{\geq 0} = \{r \in \mathbb{R}; r \geq 0\}$ and the variable in $\mathbb{R}_{\geq 0}$ will be usually denoted by r.

Lemma 2.1. *Let $\varphi(r)$ and $\psi(r)$ be two continuous definable functions $(\mathbb{R}_{\geq 0}, 0) \to (\mathbb{R}_{\geq 0}, 0)$, not identically equal to 0. Suppose $\psi \geq \varphi$. Fix $c > 1$. Then for r sufficiently small*

$$\psi'(r) \geq \varphi'(r) \tag{2.1}$$

$$c\frac{\varphi'(r)}{\varphi(r)} \geq \frac{\psi'(r)}{\psi(r)}. \tag{2.2}$$

Proof. Since $\psi - \varphi \geq 0$ and $(\psi - \varphi)(0) = 0$, $\psi - \varphi$ is increasing for small r and the first inequality follows. Similarly, $\rho(r) = \frac{(\varphi(r))^c}{\psi(r)}$ is non-negative and $\rho(r) \to 0$ as $r \to 0$. Hence ρ has to be increasing and

$$0 \leq \rho' = \frac{c\varphi^{c-1}\varphi'\psi - \varphi^c\psi'}{\psi^2} = \frac{\varphi^c}{\psi}\Big(\frac{c\varphi'}{\varphi} - \frac{\psi'}{\psi}\Big),$$

as claimed. Q.E.D.

Remark 2.2. If, moreover, $\varphi(r)/\psi(r) \to 1$ as $r \to 0$, then $\varphi(r)/\psi(r)$ is decreasing and

$$c\frac{\varphi'}{\varphi} \geq \frac{\psi'}{\psi} \geq \frac{\varphi'}{\varphi}.$$

Definition 1. Let $\varphi(r)$ be the germ at 0 of a continuous definable function $(\mathbb{R}_{\geq 0}, 0) \to (\mathbb{R}, 0)$. We shall say that φ is *small* if there is a continuous definable function $\psi : (\mathbb{R}_{\geq 0}, 0) \to (\mathbb{R}_{\geq 0}, 0)$, such that

$$\frac{|\varphi|}{r} \leq \psi'. \tag{2.3}$$

In particular, if φ is small then $\frac{\varphi}{r}$ is integrable and $\varphi(r) \to 0$ as $r \to 0$.

We shall say that φ is *unitlike* if there is a continuous function $\psi : (\mathbb{R}_{\geq 0}, 0) \to (\mathbb{R}_{\geq 0}, 0)$, C^1 for $r > 0$, such that

$$\varphi = \frac{r\psi'}{\psi}. \tag{2.4}$$

Clearly in a polynomially bounded o-minimal structure all continuous definable $\varphi : (\mathbb{R}_{\geq 0}, 0) \to (\mathbb{R}, 0)$ are small. This is not the case for the other o-minimal structures, see example 1 below.

Lemma 2.3. *Let $\psi : (\mathbb{R}_{\geq 0}, 0) \to (\mathbb{R}_{\geq 0}, 0)$ be continuous definable. Then $\frac{r\psi'}{\psi}$ is bigger than any small function.*

Let $\varphi_1(r)$ and $\varphi_2(r)$ be two continuous definable functions $(\mathbb{R}_{\geq 0}, 0) \to (\mathbb{R}, 0)$, not identically equal to 0. Suppose $\varphi_2(r) \geq r$. Then the function

$$\frac{\varphi_1'\varphi_2}{\varphi_1\varphi_2'} \tag{2.5}$$

is bigger than any small function.

Proof. Let $\psi : (\mathbb{R}_{\geq 0}, 0) \to (\mathbb{R}_{\geq 0}, 0)$ be definable. Then $(\log \psi)' = \frac{\psi'}{\psi}$ is not integrable and hence $\frac{r\psi'}{\psi}$ is bigger than any small function. By lemma 2.1, $\frac{\varphi_2'}{\varphi_2} \leq 2\frac{1}{r}$, and hence $\frac{\varphi_1'\varphi_2}{\varphi_1\varphi_2'}$ is bigger than any small function. Q.E.D.

We have a more precise result that, however, we do not use in this paper.

Proposition 2.4. *Each continuous definable* $\varphi : (\mathbb{R}_{\geq 0}, 0) \to (\mathbb{R}_{\geq 0}, 0)$ *is either small or unitlike. Moreover,* φ *is small iff* $\frac{\varphi}{r}$ *is integrable and then there is* $\psi : (\mathbb{R}_{\geq 0}, 0) \to (\mathbb{R}_{\geq 0}, 0)$ *such that*

$$\frac{\varphi}{r} = \psi'. \tag{2.6}$$

The functions ψ *of* (2.6) *and* (2.4) *belong to the Pfaffian closure of the o-minimal structure containing* φ.

Proof. Let $\varphi : (\mathbb{R}_{\geq 0}, 0) \to (\mathbb{R}_{\geq 0}, 0)$. Fix $a > 0$ small and consider

$$f(r) = \int_a^r \frac{\varphi(t)}{t}\, dt. \tag{2.7}$$

By [23], f is definable in the Pfaffian closure of the o-minimal structure containing φ. If $f(r)$ is bounded then φ is small and we may take in (2.3), $\psi(r) = f(r) - f(0)$.

Suppose $f(r)$ is not bounded that is $f(r) \to -\infty$ as $r \to 0$. Then the structure is not polynomially bounded and hence contains the exponential and the logarithmic functions, see [17]. Then we may take in (2.4), $\psi = e^f$. Q.E.D.

Example 1. Let $\alpha(r) = (-\ln r)^{-1}$ for $r > 0$ and $\alpha(0) = 0$. Then, $\alpha(r)$ satisfies

$$r\alpha'(r) = \alpha^2(r). \tag{2.8}$$

In particular, α^2 is small and $\alpha = \frac{r\alpha'}{\alpha}$ is unitlike.

§3. Łojasiewicz Inequalities in o-minimal Structures

We recall the main result of [9].

Theorem 3.1. *Let* $f : U \to \mathbb{R}$ *be a differentiable definable function defined in an open bounded* $U \subset \mathbb{R}^n$. *Then there exist* $c > 0, \rho > 0$, *and a continuous definable change of target coordinate* $\Psi : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ *such that*

$$|\nabla(\Psi \circ f)(x)| \geq c, \tag{3.1}$$

for $x \in U$ *and* $f(x) \in (-\rho, \rho)$.

Let us recall briefly after [9] the construction of Ψ. We suppose for simplicity that $f \geq 0$. Choose a definable curve $\gamma(t) : (\mathbb{R}_{\geq 0}, 0) \to \bar{U}$, such that $\gamma(t) \in U$ for $t > 0$, $f(\gamma(t)) = t$, and that

$$|\nabla f(\gamma(t))| \leq 2 \inf\{|\nabla f(x)|; f(x) = t\}, \tag{3.2}$$

in $\bar{U}$. Such a curve exists by the o-minimal version of the curve selection lemma and the fact that the right hand side of (3.2) is a definable function strictly bigger than 0 for $t > 0$ and sufficiently small, see [9]. Change the parameter by $\gamma(s) = \gamma(s(t))$ so that $|\frac{d\gamma}{ds}(0)| = 1$ and $\gamma(s)$ is definable of class C^1 (for instance we may use the distance to $\gamma(0)$ as the parameter). Then we define Ψ as the inverse function of $s \to f(\gamma(s))$ that is

$$\Psi(f(\gamma(s))) = s.$$

Hence for arbitrary $x \in U$, $t = f(x)$ close to 0, and $s = s(t)$,

(3.3)

$$|\nabla(\Psi \circ f)(x)| \geq \tfrac{1}{2}|\nabla(\Psi \circ f)(\gamma(t))| \geq 1/4\langle\nabla(\Psi \circ f)(\gamma(s)), \gamma'(s)\rangle = 1/4,$$

as required.

Corollary 3.2. *([9], Theorem 2) Let $f : U \to \mathbb{R}$ be a C^1-definable function defined in an open bounded $U \subset \mathbb{R}^n$. Then there exists a constant A such that all the trajectories of ∇f have length bounded by A. In particular, each trajectory $x(t)$ has a unique limit point $x_0 \in \overline{U}$, that is there is $t_0 \in \mathbb{R} \cup \{\infty\}$ such that*

$$\lim_{t \to t_0} x(t) = x_0$$

and $\nabla f(x_0) = 0$ if $x_0 \in U$.

We have as well an o-minimal version of Bochnak-Łojasiewicz Inequality [2].

Proposition 3.3. *Let $f : U \to \mathbb{R}$ be a differentiable definable function defined in an open $U \subset \mathbb{R}^n$. Suppose $0 \in \bar{U}$ and $f(x) \to 0$ as $x \to 0$. Then there exists a continuous definable change of target coordinate $\Phi : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ and constants $c_\Phi > 0, \rho > 0$, such that*

$$|x||\nabla(\Phi \circ f)| \geq c_\Phi |\Phi \circ f|, \tag{3.4}$$

for $x \in U$, close to the origin, and $f(x) \in (-\rho, \rho)$.

Proof. Again we suppose $f \geq 0$ leaving the general case to the reader. Define $\varphi_0(r) = \sup_{|x|=r} f(x)$. Let Φ be the inverse function of φ_0. Then

$$(\Phi \circ f)(x) \leq r. \tag{3.5}$$

Let γ be a definable curve going to the origin and parameterized by r. Then $|\gamma'(r)| \to 1$ as $r \to 0$. By the choice of parameterization and (3.5), $(\Phi \circ f)(\gamma(r)) \leq r$. Denote $\psi(r) = (\Phi \circ f)(\gamma(r))$. By Lemma 2.1, for any $c > 1$,

$$c\frac{\psi'(r)}{\psi(r)} \geq \frac{1}{r}. \tag{3.6}$$

On the other hand

$$\psi'(r) = \langle \nabla(\Phi \circ f), \gamma'(r) \rangle \leq 2|\nabla(\Phi \circ f)|. \tag{3.7}$$

The proposition follows from (3.6) and (3.7) by the curve selection lemma. Q.E.D.

The actual constants in both (3.1) and (3.4) can be made arbitrarily small. For instance for (3.4) it suffices to replace $\Phi \circ f$ by its power $(\Phi \circ f)^\alpha$.

Remark 3.4. Unlike in the analytic case, in general, it is not possible to find a definable change of target coordinate which gives both Łojasiewicz type inequalities. We may take as example $f : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ given in polar coordinates by

$$f(r, \theta) = \alpha(r) \sin \theta,$$

where α is the function of example 1. Indeed, suppose $\Phi : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ is the change of target coordinate such that $\Phi \circ f$ satisfies both inequalities. In what follows we suppose Φ increasing and restrict ourselves to the set $(\Phi \circ f) \geq 0$. Then

$$\begin{aligned} |\nabla(\Phi \circ f)| &= (\Phi' \circ f)|(\alpha'(r) \sin \theta, r^{-1}\alpha(r) \cos \theta)| \\ &= (\Phi' \circ f) r^{-1} |(\alpha^2(r) \sin \theta, \alpha(r) \cos \theta)|. \end{aligned}$$

For $\sin \theta = 1, \cos \theta = 0$, the Bochnak-Łojasiewicz Inequality gives

$$r|\nabla(\Phi \circ f)| = (\Phi' \circ f)\alpha^2 = r(\Phi \circ \alpha)'(r) \geq \tilde{c}(\Phi \circ \alpha)(r)$$

that is

$$\frac{(\Phi \circ \alpha)'}{\Phi \circ \alpha} \geq \frac{\tilde{c}}{r}.$$

and by integration (or Lemma 2.1)

$$r^c \geq (\Phi \circ \alpha)(r), \tag{3.8}$$

for $c < \tilde{c}$. Since $\tilde{c} > 0$ may choose $c > 0$ as well.

On the other hand, consider the set $\{\theta < r\}$. Then $\sin\theta \to 0$ and $\cos\theta \to 1$ as $r \to 0$. By Łojasiewicz Inequality (3.1)

$$r^{-1}\alpha(r)\Phi'(\alpha(r)\sin\theta) \geq c. \tag{3.9}$$

Define $\gamma(r,\theta)$ by $\alpha(r)\sin\theta = \alpha(\gamma(r,\theta))$. Then (3.9) is equivalent to

$$\alpha(r)(\Phi \circ \alpha)'(\gamma(r,\theta)) \geq c_1 r\alpha'(\gamma(r,\theta)),$$

that gives by (3.8)

$$c\alpha(r)(\gamma(r,\theta))^{c-1} \geq c_1 r\alpha'(\gamma(r,\theta)).$$

Equivalently

$$c\alpha(r)/r \geq c_1\alpha'(\gamma(r,\theta))/(\gamma(r,\theta))^{c-1}$$

that is impossible since the right hand side $\alpha'(\gamma(r,\theta))/(\gamma(r,\theta))^{c-1} = \alpha^2(\gamma(r,\theta))/(\gamma(r,\theta))^c$ tends to ∞ as $\theta \to 0$ and r is fixed and the left hand side does not depend on θ.

Remark 3.5. Suppose that there is a positive exponent a and a constant $c > 0$ such that $r^a \geq |f(x)| \geq cr$. Then f itself, without any change of target coordinate , satisfies both inequalities. Indeed, by construction, it suffices to check these inequalities on a definable curve and in this case they are obvious.

§4. Characteristic Functions

In this section we suppose that $f : U \to \mathbb{R}$ is a differentiable definable function defined in an open $U \subset \mathbb{R}^n$, $0 \in \bar{U}$. We shall assume f bounded. The gradient ∇f of f splits into its radial component $\frac{\partial f}{\partial r}\frac{\partial}{\partial r}$ and the spherical one $\nabla' f = \nabla f - \frac{\partial f}{\partial r}\frac{\partial}{\partial r}$. Fix $\varepsilon > 0$ and consider

$$V^\varepsilon = \{0 \leq |x| \leq r_0; f(x) \neq 0, |f(x)| \geq \varepsilon r|\nabla' f(x)|\},$$

where $r_0 > 0$ is small. By the local conical structure of definable sets we may suppose that, for $r_0 > 0$ sufficiently small, $V^\varepsilon \ni x \to |x| \in (0, r_0]$ is a topologically trivial fibration. In particular for $0 < r \le r_0$, the inclusion $S(r) \cap V^\varepsilon \subset V^\varepsilon$, where $S(r) = \{x; |x| = r\}$, is a homotopy equivalence. Let V be a connected component of V^ε. Denote

$$\varphi(r) = \varphi_V(r) = \inf\{|f(x)|; x \in V \cap S(r)\}. \tag{4.1}$$

Proposition 4.1. *There exists $C > 0$ such that*

$$\varphi(|x|) \le |f(x)| \le C\varphi(|x|), \qquad \text{for } x \in V. \tag{4.2}$$

In particular $\varphi(r) > 0$ for $r > 0$.

Proof. By corollary 1.3 there exists a constant $M > 0$ such that for every $x, x' \in V$, that satisfy $|x| = |x'| = r$, there is a continuous definable curve $\xi(t)$ joining x and x' in $V \cap S(r)$ and of length $\le Mr$. Then, by the definition of V^ε,

$$|\frac{d}{dt} f(\xi(t))| = |\langle \nabla' f, \xi'(t)\rangle| \le |\nabla' f||\xi'(t)| \le \varepsilon^{-1}\frac{|f|}{r}|\xi'(t)|.$$

Hence

$$|\frac{d}{dt}\ln|f(\xi(t))|| \le \frac{\varepsilon^{-1}}{r}|\xi'(t)|.$$

Finally, by integration of both sides along curve $\xi(t)$, $|\ln|f(x)| - \ln|f(x')||$ $\le M' = M\varepsilon^{-1}$, which gives

$$|\frac{f(x)}{f(x')}| \le e^{M'}.$$

The proposition follows by the curve selection lemma. Q.E.D.

We shall call the (finite) set of functions φ_V defined by (4.1), where V goes over the connected components of V^ε, *the characteristic functions of f*. They depend on the choice of ε though it may be shown that the number of connected components of V^ε at the origin stabilizes as $\varepsilon \to 0$. Each of these connected components give rise to a family of characteristic functions $\varphi_{\varepsilon,V}$. It can be shown that they can be compared as follows: if $\varepsilon' < \varepsilon$ then there exists $C = C(\varepsilon', \varepsilon)$ such that $\varphi_{\varepsilon',V} \le \varphi_{\varepsilon,V} \le C(\varepsilon', \varepsilon)\varphi_{\varepsilon',V}$. In what follows shall consider ε fixed and small and we will be interested mostly in those connected components V of V^ε such that $\varphi_V(r) \to 0$ as $r \to 0$. Let V be such a component and let $\gamma(t), t \geq 0$ be a definable curve such that $\gamma(t) \to 0$ as $t \to 0$ and $\gamma(t) \in V$ for $t \neq 0$. In order to simplify the notation we reparametrize γ by the distance to the origin, that is to say $|\gamma(t(r))| = r$. Write in spherical coordinates $\gamma(r) = r\theta(r)$, $|\theta(r)| \equiv 1$. Then $r|\theta'(r)| \to 0$ as $r \to 0$. Moreover, $r|\theta'(r)|$ is small in sense of definition 1. Denote $\psi(r) = |f(\gamma(r))|$. Then

$$\frac{df(\gamma(r))}{dr} = \frac{|f|}{r}\frac{r\psi'}{\psi} \geq \varepsilon|\nabla' f|\frac{r\psi'}{\psi} \gg r|\theta'(r)||\nabla' f|,$$

since, by lemma 2.3, $\frac{r\psi'}{\psi}$ is much bigger than $r|\theta'(r)|$. In particular,

$$\frac{df(\gamma(r))}{dr} = \partial_r f + \langle \nabla' f, r\theta'(r)\rangle \simeq \partial_r f. \tag{4.3}$$

We shall consider as well

$$W^\varepsilon = \{x; f(x) \neq 0, |\partial_r f| \geq \varepsilon|\nabla' f|\}.$$

Unlike V^ε, the sets W^ε do not change if we replace f by $\Psi \circ f$, for any definable change of target coordinate Ψ at $0 \in \mathbb{R}$.

Proposition 4.2.

$$W^\varepsilon \cap \{x;\ |f(x)| \geq |x|\} \subset V^{\varepsilon'} \cap \{x;\ |f(x)| \geq |x|\} \quad \textit{if} \quad \varepsilon' < \varepsilon. \tag{4.4}$$

Let W be a connected component of $W^\varepsilon \cap \{x, |f(x)| \geq |x|\}$ and define $\varphi(r) = \inf\{|f(x)|; x \in W \cap S(r)\}$. There exists $C > 0$ such that

$$\varphi(|x|) \leq |f(x)| \leq C\varphi(|x|), \qquad \textit{for } x \in W. \tag{4.5}$$

Proof. It suffices to check (4.4) on definable curves. Fix a definable curve $\gamma(r)$ in $W^\varepsilon \cap \{x||f(x) \geq |x|\}$ parameterized by the distance to the origin. Denote $\psi(r) = |f(\gamma(r))|$. Suppose first $\psi(r) \to 0$ as $r \to 0$. Then $\psi \geq r$ and hence by lemma 2.1, $\psi \geq cr\psi'$, where we may take $1 > c > \frac{\varepsilon'}{\varepsilon}$. Then, by (4.3),

$$\psi \geq cr\psi' \geq \varepsilon' r|\nabla' f|,$$

as claimed. The proof for the curves on which $|f(\gamma(r))| \to c_0 > 0$ is similar since (4.3) holds for the curves in W^ε.

The last claim of the proposition follows from (4.4) and proposition 4.1. Q.E.D.

§5. Estimates on a trajectory. I

Let $f : U \to \mathbb{R}$ be a C^1 definable function defined in an open and bounded $U \subset \mathbb{R}^n$ and let $x(t)$ be a trajectory of ∇f with limit point $x_0 \in \overline{U}$, cf. corollary 3.2. We shall suppose, for simplicity of notation, that $x_0 = 0$ and usually we parameterize $x(t)$ by its arc-length s, starting from point $p_0 = x(0)$. Then

$$\dot{x} = \frac{dx}{ds} = \frac{\nabla f}{|\nabla f|}.$$

By corollary 3.2 the length of $x(s)$ is finite. Denote it by s_0. Then

$$x(s) \to 0 \quad \text{as } s \to s_0.$$

Our purpose is to study the geometric behavior of $x(s)$ as it approaches its limit point. We shall also assume that

$$f(x(s)) \to 0 \quad \text{as } s \to s_0.$$

Note that it means in particular, as being increasing, that f has negative along the trajectory.

By theorem 3.1 we may assume that $|\nabla f| \geq 1$ that we shall do. Then

$$|f(x(s))| \geq \text{length}\{x(s'); s \leq s' < s_0\} \geq |x(s)|. \tag{5.1}$$

Fix a definable $\varphi(r) : (\mathbb{R}_{\geq}, 0) \to (\mathbb{R}_{\geq}, 0)$ and consider $F = \frac{f}{\varphi(r)}$. Then

$$\begin{aligned} \frac{dF(x(s))}{ds} &= \langle \frac{\nabla f}{|\nabla f|}, \frac{\nabla' f}{\varphi} + (\frac{\partial_r f}{\varphi} - \frac{\varphi' f}{\varphi^2})\partial_r \rangle \\ &= \frac{1}{|\nabla f|\varphi}\Big(|\nabla' f|^2 + |\partial_r f|^2 - \partial_r f \frac{\varphi' f}{\varphi}\Big) \\ &= \frac{1}{|\nabla f|\varphi}\Big(|\nabla' f|^2 + |\partial_r f|^2 (1 - \frac{\varphi' f}{\varphi \partial_r f})\Big) \end{aligned} \tag{5.2}$$

Lemma 5.1. *Let $\varphi(r) \geq r$ and let $F = \frac{f}{\varphi(r)}$. Suppose $\varepsilon < 1$. Then in the complement of $V^\varepsilon = \{x; |f| \geq \varepsilon r |\nabla' f|\}$*

$$\frac{dF(x(s))}{ds} \geq \frac{1}{2} \frac{|\nabla f|}{\varphi}. \tag{5.3}$$

Proof. By (5.2), it is sufficient to show that

$$|\nabla f|^2 \geq 2f\partial_r f\frac{\varphi'}{\varphi}$$

on the complement of V^ε. Since $\varphi \geq r$ we have by lemma 2.1

$$\varepsilon\frac{\varphi'}{\varphi} \leq \frac{1}{r}. \tag{5.4}$$

Consequently, since we are away of V^ε,

$$|\nabla f|^2 \geq 2|\nabla' f||\partial_r f| \geq 2\varepsilon^{-1}\frac{|f|}{r}|\partial_r f| \geq 2f\partial_r f\frac{\varphi'}{\varphi}$$

as required. Q.E.D.

Corollary 5.2. *The trajectory $x(s)$ passes through V^ε in any neighborhood of the origin.*

Proof. Let $q > 0$ and consider $\varphi(r) = r^{1+q}$. Then $r\frac{\varphi'}{\varphi} = 1+q$ (5.4) is satisfied for $\varepsilon < (1+q)^{-1}$. Consequently the statement of lemma 5.1 holds for $F = \frac{f}{\varphi}$. Suppose, contrary to our claim, that $x(s)$ stays away of V^ε. Then, by lemma 5.1, $F = \frac{f}{r^{1+q}}$ is increasing on the trajectory. Hence it is bounded (recall $f(x(s))$ is negative). That is there exists a constant $C > 0$ such that

$$|f(x(s))| \leq C|x(s)|^{1+q},$$

which contradicts (5.1). Q.E.D.

Fix $\varepsilon < 1$. By Proposition 4.1 there is a finite family of functions of one variable $\{\varphi(r)\}$ such that

$$V^\varepsilon = \bigcup V^\varepsilon_\varphi, \tag{5.5}$$

so that $V^\varepsilon_\varphi \subset U_\varphi = \{x | c\varphi < |f| < C\varphi\}$. We regroup together the φ's with the same asymptotic behavior at 0, that is in the same equivalence classe of relation $\varphi_1(r) \sim \varphi_2(r)$. Thus we may actually assume that the U_φ's are mutually disjoint and so is the union in (5.5).

Fix one of such φ satisfying $\varphi(r) \geq r$ and consider $F = \frac{f}{\varphi}$. Recall that F is negative on the trajectory. Define

$$\partial^- U_\varphi = \{x; F(x) = -C\}, \qquad \partial^+ U_\varphi = \{x; F(x) = -c\}.$$

Then, by lemma 5.1, $F(x(s))$ is strictly increasing on $\partial^- U_\varphi \cup \partial^+ U_\varphi$. That is to say, the trajectory may enter U_φ only through $\partial^- U_\varphi$ and leave it only through $\partial^+ U_\varphi$. If the latter happens then the trajectory leaves U_φ definitely and never enters it again. Hence, by corollary 5.2,

Corollary 5.3. *The trajectory $x(s)$ has to end up in one of $U_\varphi = \{x | c\varphi < |f| < C\varphi\}$.*

Note that $\varphi(r) \geq r$ by (5.1). We shall fix such φ. Now we have the following strenghtened versions of lemma 5.1 and corollary 5.2.

Lemma 5.4. *Let $F = \frac{f}{\varphi}$. Then for any $\varepsilon > 0$ there is $c' > 0$ such that in the complement of W^ε in U_φ*

$$\frac{dF(x(s))}{ds} \geq c' \frac{|\nabla f|}{\varphi} \geq \frac{c'}{\varphi}. \tag{5.6}$$

Proof. Fix $\varepsilon > 0$. By (5.2), it is sufficient to show that there is $c > 1$ such that

$$|\nabla f|^2 \geq c f \partial_r f \frac{\varphi'}{\varphi} \tag{5.7}$$

on $U_\varphi \setminus W^\varepsilon$. This we show on an arbitrary definable curve $\gamma(r)$ in $U_\varphi \setminus W^\varepsilon$. Again we denote $\psi(r) = |f(\gamma(r))|$ and write in the spherical coordinates $\gamma(r) = r\theta(r)$. Then,

$$\psi'(r) = |\partial_r f + \langle \nabla' f, r\theta'(r) \rangle| \leq |\partial_r f| + |\nabla' f||r\theta'(r)|, \tag{5.8}$$

where $r\theta'(r) \to 0$.

Suppose first that $|\partial_r f| \gg |\nabla' f||r\theta'(r)|$ as $r \to 0$. Then, since we are away of W^ε,

$$|\nabla f|^2 \geq (1 + \varepsilon^{-2})|\partial_r f|^2 \geq (1 + \tilde{\varepsilon}^{-2})|\partial_r f|\psi', \tag{5.9}$$

for any $\tilde{\varepsilon} > \varepsilon$. An even stronger bound holds if $|\partial_r f| \gg |\nabla' f||r\theta'(r)|$ fails. Indeed, then $|\partial_r f| \ll |\nabla' f|$ and $|\nabla f|^2 \simeq |\nabla' f|^2 \gg |\partial_r f|\psi'$.

On the other hand, by lemma 2.1, for any $c > 1$ and on U_φ

$$c\psi' \geq \psi \frac{\varphi'}{\varphi}.$$

This and (5.9) show (5.7). The proof is complete. Q.E.D.

Corollary 5.5. *The trajectory $x(s)$ passes through W^ε in any neighborhood of the origin.*

Proof. Suppose, contrary to our claim, that $x(s)$ stays away of W^ε. Let $\varphi(r)$ be such that $x(s)$ stays in U_φ for s close to s_0. Then, there exists a constant $\tilde{c} > 0$, such that

$$\frac{dF(x(s))}{ds} \geq c' \frac{|\nabla f|}{\varphi} \geq \tilde{c}\frac{df}{ds}|f|^{-1} = \tilde{c}\frac{d(-\ln|f|)}{ds}.$$

But this is impossible since F is bounded and $-\ln|f|$ is not on $x(s)$. Q.E.D.

Let W be the union of those connected components of W^ε such that the trajectory $x(s)$ passes through them in any neighborhood of the origin. W is non-empty by corollary 5.5. Denote

$$\varphi(r) = \inf_{W \cap S(r)} |f(x)|.$$

Let Φ be the inverse function of φ and consider

$$\tilde{f}(x) = -\Phi(-f(x)). \tag{5.10}$$

Then, by the definition of φ,

$$|\tilde{f}(x)| \geq |x|, \qquad \text{for } x \in W. \tag{5.11}$$

Proposition 5.6. *There is a $C > 0$ such that on the trajectory $x(s)$ and for $x(s)$ sufficiently close to the origin*

$$-Cr \leq \tilde{f}(x(s)) \leq -r. \tag{5.12}$$

Proof. By definition of φ, $F = \frac{f}{\varphi} \leq -1$ on W and by lemma 5.4, $F(x(s))$ is strictly increasing in the complement of W. If $F(x(s)) > -1$ for one s then it rests bigger than -1 which contradicts the fact that the trajectory crosses W in any neighborhood of the origin. Thus, on the trajectory,

$$|f(x(s))| \geq \varphi(|x(s)|).$$

This implies $r \leq |\tilde{f}(x(s))|$.

By proposition 4.2 applied to W and $\tilde{f}$, $|\tilde{f}| \leq Cr$ on W. Now the second inequality of (5.12) follows from (5.11) and the fact that $F(x(s)) = \frac{f}{\varphi}$ is increasing in the complement of W. Q.E.D.

§6. Asymptotic critical values

Let F be a C^1 definable function F defined on an open definable set U such that $0 \in \overline{U}$. We say that $a \in \mathbb{R}$ is *an asymptotic critical value of F at the origin* if there exists a sequence $x \to 0$, $x \in U$, such that

(a) $|x||\nabla F(x)| \to 0$,
(b) $F(x) \to a$.

Proposition 6.1. *(see also* [1]*)*
The set of asymptotic critical values is finite.

Proof. Let $X = \{(x,t); F(x)-t = 0\}$ be the graph of F. Consider X and $T = \{0\} \times \mathbb{R}$ as a pair of strata in $\mathbb{R}^n \times \mathbb{R}$. Then the (w)-condition of Kuo-Verdier at $(0,a) \in T$ reads

$$1 = |\partial/\partial t(F(x) - t)| \leq C|x||\partial/\partial x(F(x) - t)| = Cr|\nabla F|.$$

In particular, $a \in \mathbb{R}$ is an asymptotic critical value if and only if the condition (w) fails at $(0,a)$. The set of such a's is finite by the genericity of (w) condition, see [15] or [1]. Q.E.D.

Remark 6.2. Suppose $x(s)$ is a trajectory of ∇f and let $a = \lim_{s\to s_0} f(x(s))$. Then a is an asymptotic critical value of f. Indeed, suppose contrary to our claim that $r|\nabla f(x)| \geq c > 0$ for $f(x)$ close to a and we may assume $a = 0$. By corollary 5.5, $x(s)$ passes through W^ε in any neighborhood of the origin. Let $\gamma(r)$ be a definable curve in W^ε such that $f(\gamma(r)) \to 0$ as $r \to 0$. Denote, as before, $\psi(r) = f(\gamma(r))$. Then, by (4.3) and since we are in W^ε,

$$r|\psi'(r)| \simeq r|\partial_r f| \geq \varepsilon' r|\nabla f| \geq \varepsilon' c > 0$$

that is impossible since the left-hand side is small.

In particular, only finitely many values of f are allowed as limits along the trajectories of the gradient.

One may ask whether we have an analogue of Łojasiewicz Inequality (3.1) for asymptotic critical values. More precisely, whether for an asymptotic critical value a there exists a continuous definable change of target coordinate $\Psi : (\mathbb{R}, a) \to (\mathbb{R}, 0)$ such that

$$r|\nabla(\Psi \circ F)| \geq c > 0, \tag{6.1}$$

at least if $F(x)$ is close to a. This is not the case in general, but it holds if we approach the singularity "sufficiently slowly".

Proposition 6.3. *Let F be as above and let $a \in \mathbb{R}$. Let $\eta(r)$ be small in sense of definition 1. Then there exists a continuous definable change of parameter $\Psi : (\mathbb{R}, a) \to (\mathbb{R}, 0)$ and a constant $c_a > 0$ such that (6.1) holds on $\{x \in U; |\partial_r F| \leq \eta(r)|\nabla' F|, |F(x) - a| \leq c_a\}$.*

Proof. The proof follows the main ideas of the proof of Łojasiewicz Inequality (3.1). We may assume that a is an asymptotic critical value of F. Choose first $c_a > 0$ so that there is no other asymptotic critical value in $\{t \in \mathbb{R};\ |t - a| \leq c_a\}$. For simplicity of notation we suppose also $a = 0$, $c_a = c_0$. We may also suppose $F \geq 0$, otherwise we replace F by F^2.

Denote $U_0 = \{x \in U; |\partial_r F| \leq \eta(r)|\nabla' F|, |F(x) - a| \leq c_a\}$. Choose a definable curve $\gamma(t) \not\equiv 0$ such that $F(\gamma(t)) = t$, and

$$r|\nabla F(\gamma(t))| \leq 2 \min\{r|\nabla F(x)|; F(x) = t\},$$

in U_0. Such a curve exists by the o-minimal version of curve selection lemma.

Let $x_0 = \lim_{t\to 0} \gamma(t)$. Suppose first that $x_0 \neq 0$. By [9] there exists Ψ such that $\nabla(\Psi \circ F) \geq 1$. Therefore, by the choice of γ,

$$r|\nabla(\Psi \circ F)(x)| \geq \tfrac{1}{2} r|\nabla(\Psi \circ F)(\gamma(F(x))| \geq c|x_0| > 0. \tag{6.2}$$

So suppose $x_0 = 0$. In this case we may use r as the parameter on γ, $\gamma(r) = \gamma(t(r))$, and write as before $\gamma(r) = r\theta(r)$ in spherical coordinates. Define $\psi(r) = F(\gamma(r))$. Then

$$|\psi'(r)| = |\partial_r F + \langle \nabla' F, r\theta' \rangle| \leq \tilde\eta(r)|\nabla' F|, \tag{6.3}$$

where $\tilde\eta = \eta + r|\theta'|$ is small. In particular there exists a germ of continuous definable function $h : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ such that

$$\tilde\eta(r) \leq r|h'(r)|. \tag{6.4}$$

Then $\Psi := h \circ \psi^{-1}$ satisfies the statement. Indeed, by (6.3) and (6.4),

$$r|\nabla(\Psi \circ F)(\gamma(r)| = r\frac{h'(r)}{\psi'(r)}|\nabla F(\gamma(r))| \geq 1.$$

Hence for x close to the origin, $t = f(x)$,

$$r|\nabla(\Psi \circ F)(x)| \geq \tfrac{1}{2} r|\nabla(\Psi \circ F)(\gamma(t)| \geq \tfrac{1}{2},$$

as required. Q.E.D.

Consider F of the form $F = \frac{f}{\varphi(r)}$. Then

$$\partial_r F = \frac{\partial_r f}{\varphi(r)}\Big(1 - \frac{f\varphi'}{\varphi \partial_r f}\Big), \quad \nabla' F = \frac{\nabla' f}{\varphi(r)}. \tag{6.5}$$

Proposition 6.4. *Suppose that there is an exponent $N > 0$ such that $r^N \leq \varphi(r) \leq r^{1/N}$. Then $a \neq 0$ is an asymptotic critical value of $F = \frac{f}{\varphi(r)}$ if and only if there exists a sequence $x \to 0$, $x \neq 0$, such that*

(a') $\frac{|\nabla' f(x)|}{|\partial_r f(x)|} \to 0$,

(b') $F(x) \to a$

Proof. The proof is similar to that of Proposition 5.3 of [14] and is left to the reader. Q.E.D.

§7. Estimates on a trajectory. II

Let $x(s)$ be a trajectory and let W the union of connected components of W^ε (for any fixed $\varepsilon > 0$) such that $x(s)$ passes through them in any neighborhood of the origin. Restricting ourselves to a smaller neighborhood of the origin, if necessary, we may suppose that the trajectory stays away of $W^\varepsilon \setminus W$. Recall after proposition 5.6 that we may assume that

$$x(s) \in U_C = \{x; -C\varphi(r) \leq \tilde{f}(x) \leq -c\varphi(r)\}, \tag{7.1}$$

$0 < c < C < \infty$ and $\varphi(r) \sim r$, and $\tilde{f}(x)$ is given by (5.10). (Actually by propostion 5.6 we may assume $c = 1$ and $\varphi(r) = r$ but we do not need it.) In particular $\tilde{f}$ on U_C satisfies both Łojasiewicz and Bochnak-Łojasiewicz Inequalities, see remark 3.5. In order to simplify the notation we shall write f for $\tilde{f}$. Define

$$F(x) = \frac{f(x)}{\varphi(r)}, \tag{7.2}$$

Then

$$\frac{dF(x(s))}{ds} = \frac{1}{\varphi(r)|\nabla f|}\Big(|\nabla' f|^2 + |\partial_r f|^2 (1 - \frac{f\varphi'}{\varphi(r)\partial_r f})\Big)$$

By lemma 5.4

$$\frac{dF(x(s))}{ds} \geq c' \frac{|\nabla f|}{\varphi(r)} \geq c'' \frac{1}{r} \quad \text{on } U_C \setminus W. \tag{7.3}$$

Lemma 7.1. *There exists a continuous definable function $\tilde{\omega}$, $\tilde{\omega}(r) \to 0$ as $r \to 0$, such that*

$$|1 - \frac{f\varphi'}{\varphi\partial_r f}| \leq \tfrac{1}{2}\tilde{\omega}(r) \quad \textit{on } W. \tag{7.4}$$

Moreover, $\tilde{\omega}$ may be chosen small in sense of definition 1.

Proof. Let $\gamma(r)$ be a definable curve such that $|(1 - \frac{f\varphi'}{\varphi\partial_r f})(\gamma(r)| \geq \frac{1}{2}\sup_{W \cap S(r)} |(1 - \frac{f\varphi'}{\varphi\partial_r f})|$. Denote $\psi(r) = f(\gamma(r))$. Then, by (4.3)

$$\psi'(r) = \partial_r f + < \nabla' f, r\theta'(r) >, \tag{7.5}$$

and $r|\theta'(r)|$ is small. Consequently, since recall $|\nabla' f| \leq \varepsilon^{-1}|\partial_r f|$ on W,

$$(1 - \frac{f\varphi'}{\varphi\partial_r f}) = \frac{\varphi\partial_r f - \varphi' f}{\varphi\partial_r f} = \frac{\varphi\psi' - \varphi'\psi}{\varphi\psi'} + \tau(r), \tag{7.6}$$

and

$$|\tau(r)| \leq 2\frac{|\nabla' f|}{|\partial_r f|} r|\theta'| \leq 2\varepsilon^{-1} r|\theta'| \tag{7.7}$$

is small. Note that $\psi'(r) \sim 1$. Hence

$$\frac{\varphi\psi' - \varphi'\psi}{\varphi\psi'} = (\frac{\psi}{\varphi})' \frac{\varphi}{\psi'} \tag{7.8}$$

is small. This ends the proof of lemma. Q.E.D.

We list below some other properties of f on W which follows from (4.3). By (4.3), we get $\partial_r f \simeq \varphi' \sim 1$ on any definable curve in W. Thus, by the curve selection lemma, for any constant $c_1 < 1$, and some positive constants C', c'

$$-C' \leq -c_1^{-1}\varphi' \leq \partial_r f \leq -c_1\varphi' \leq -c' < 0 \quad \text{on W.} \tag{7.9}$$

In particular, $\partial_r f$ is negative on W.

We shall show in the proposition below that $F(x(s))$ has a limit as $s \to 0$. For this we use an auxiliary function $g = F - \alpha(r)$ where $\alpha : (\mathbb{R}_{\geq 0}, 0) \to (\mathbb{R}_{\geq 0}, 0)$ satisfies $\tilde{\omega} \leq C'^{-1}\varphi\alpha'$. Such an α exists since $\tilde{\omega}$ is small.

Proposition 7.2. *Let $\alpha : (\mathbb{R}_{\geq 0}, 0) \to (\mathbb{R}_{\geq 0}, 0)$ be a continuous definable function and suppose $\tilde{\omega} \leq C'^{-1}\varphi\alpha'$. Then the function $g(x) = F(x) - \alpha(r)$ is strictly increasing on the trajectory $x(s)$. In particular $F(x(s))$ has a nonzero limit*

$$(7.10) \qquad F(x(s)) \to a_0 < 0, \qquad \textit{as } s \to s_0.$$

Furthermore, a_0 has to be an asymptotic critical value of F at the origin.

Proof. First we show that $g(x(s))$ is increasing for $x(s) \in U_C \setminus W$. Recall that on $U_C \setminus W$, $|\partial_r f| < \varepsilon |\nabla' f|$ and (7.3) holds. On the other hand

$$(7.11) \qquad |\frac{d\alpha}{ds}| = |\alpha'(r)\frac{\partial_r f}{|\nabla f|}| \leq \varepsilon|\alpha'(r)| \ll r^{-1}.$$

Consequently, in this case,

$$\frac{dg}{ds}(x(s)) \geq c' r^{-1}.$$

This shows that g is increasing on $U_C \setminus W$ as claimed.

In general we have

$$(7.12) \quad \frac{dg}{ds}(x(s)) = \frac{1}{\varphi|\nabla f|}\Big(|\nabla' f|^2 + |\partial_r f|^2(1 - \frac{f\varphi'}{\varphi\partial_r f})\Big) - \alpha'(r)\frac{\partial_r f}{|\nabla f|}.$$

Now we consider $x(s) \in W$. By (7.9) and by the choice of α

$$\alpha'(r)\frac{|\partial_r f|}{|\nabla f|} \geq C'\varphi^{-1}\tilde{\omega}\frac{|\partial_r f|}{|\nabla f|} \geq \frac{1}{\varphi|\nabla f|}\Big(|\partial_r f|^2(1 - \frac{f\varphi'}{\varphi\partial_r f})\Big),$$

and hence the right-hand side expression in (7.12) is positive (recall $\partial_r f$ is negative on W).

Thus, since $g(x(s))$ is increasing, negative, and bounded from zero on U_C, it has a limit $a_0 < 0$. We shall show that a_0 is an asymptotic critical value of F.

Suppose, by contradiction, that $F(x(s)) \to a$ and a is not an asymptotic critical value of F at the origin. Then, by Proposition 6.4, there is $\tilde{c} > 0$ such that

$$|\nabla' f(x(s))| \geq \tilde{c} |\partial_r f(x(s))|,$$

for s close to s_0. Hence on W

$$\frac{dF}{ds} = \frac{|\nabla' f|^2}{\varphi |\nabla f|} + \frac{|\partial_r f|^2}{\varphi |\nabla f|}\left(1 - \frac{f\varphi'}{\varphi \partial_r f}\right) \geq c' \frac{1}{r}. \tag{7.13}$$

A similar bound holds on $U_C \setminus W$ by (7.3).

But (7.13) is not possible since $|\frac{dr}{ds}| \leq 1$. Indeed, (7.13) implies $\frac{dF}{dr} \geq c\frac{1}{r}$ with the right-hand side not integrable which contradicts the fact that F is bounded on the trajectory. This ends the proof. Q.E.D.

Corollary 7.3. *Let $\sigma(s)$ denote the length of the trajectory between $x(s)$ and the origin. Then*

$$\frac{\sigma(s)}{|x(s)|} \to 1 \text{ as } s \to s_0.$$

Proof. The proof follows from Proposition 7.2 and is similar to the one of Corollary 6.5 of [14]. Q.E.D.

§8. Gradient Conjecture on the Plane

In this section we show the following finiteness result.

Theorem 8.1. *Let $f : U \to \mathbb{R}$ be a differentiable definable function, where $U \subset \mathbb{R}^2$ is open definable and $0 \in \bar{U}$. Let $x(t)$ be a trajectory of ∇f such that $x(t) \to 0, f(x(t)) \to 0$ as $t \to 0^-$. Given a definable curve $\Gamma \subset U$. Then, there is $\varepsilon > 0$ such that the set $\{x(t); -\varepsilon < t < 0\}$ either lies entirely in Γ or does not intersect Γ at all.*

Proof. By a standard argument, see the proof of Proposition 2.1 of [14], it suffices to show that the trajectory cannot spiral, that is the statement of theorem holds for at least one curve Y, $0 \in \overline{Y}$. Indeed, consider an arbitrary definable curve $\Gamma \subset U$ parameterized in polar coordinates (r, θ) by $\gamma(r) = r\theta(r)$. Write

$$f(r, \theta) = f(r\cos\theta, r\sin\theta).$$

Denote $\partial_\theta f = \partial f/\partial\theta$. Then $|\partial_\theta f| = r|\nabla' f|$ and $\partial_\theta f$ is positive if and only if $\nabla' f$ is directed anti-clockwise. If Γ is not a trajectory itself, that is if ∇f is not tangent to Γ, then, near the origin, the trajectories of ∇f cross Γ only in one direction. Fix a point $x_0 = \gamma(r) = r\theta(r)$ and the orthonormal basis of $\mathbb{R}^2$ with the first vector being $\frac{x_o}{\|x_o\|} = \theta(r)$. Comparing in this basis the tangent vector $(1, r\theta'(r)$ to the curve Γ and the gradient $(\partial_r f, r^{-1}\partial_\theta f)$ of f we see that the trajectories cross Γ anti-clockwise if and only if

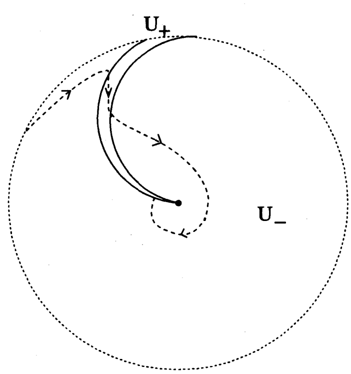
$$\partial_\theta f > r^2\theta'(r)\partial_r f(r). \tag{8.1}$$

Thus if the trajectory does not spiral and is not contained in Γ then, in a small neighborhood of the origin, it may cross Γ only once. In particular if U does not contain a punctured disc of the form $\{0 < r < r_0\}$ then any trajectory going to the origin cannot spiral otherwise it would hit the boundary of U. Thus we may suppose that U contains a punctured disc centered at the origin.

Divide U into two pieces

$$U_+ = \{\partial_\theta f \geq 0\}, \quad U_- = \{\partial_\theta f \leq 0\}.$$

Both of them are non-empty as germs at the origin since $f(r,\theta)$ is periodic for r fixed. On U_- the trajectory moves clockwise and on U_+ anti-clockwise. It is clear that the trajectory cannot spiral if each $U_\pm$ contains a non-empty sector of the form $\{\theta_1 < \theta < \theta_2\}$. This is the case for f analytic, see [14]. But for f definable in an o-minimal structure or even for f subanalytic it may happen that one of $U_\pm$ does not contain a sector, see the picture below.



(One may construct such example easily by choosing two definable curves $r = \gamma_1(\theta)$, $r = \gamma_2(\theta)$ and definable $f(r,\theta)$, periodic in θ, and such that $\partial_\theta f(r,\theta) \geq 0$ exactly on $\gamma_1(\theta) \leq r \leq \gamma_2(\theta)$.)

In what follows we shall assume that U_- contains a non-empty sector but U_+ not necessarily. If we show that U_+ contains a definable curve which $x(t)$ crosses anti-clockwise then we are done.

Lemma 8.2. *Let $\Gamma \in U_+$ be a germ at the origin of a definable curve parameterized by $\gamma(r)$. If*

$$r \to \lambda_\gamma(r) = \frac{|\nabla' f(\gamma(r))|}{|\partial_r f(\gamma(r))|}$$

is not small then the trajectories of ∇f cross Γ anti-clockwise.

Proof. Let $\gamma(r) = r\theta(r)$. It suffices to show (8.1). By lemma 2.3 $\lambda_\gamma(r) \gg r\theta'(r)$ and hence we have

$$\partial_\theta f = \lambda_\gamma(r) r |\partial_r f| \gg r^2\theta'(r)\partial_r f,$$

as required. Q.E.D.

Thus in what follows it suffices to suppose that

$$\lambda(r) = \sup_{x\in S(r)\cap U_+} \frac{|\nabla' f|}{|\partial_r f|} \tag{8.2}$$

is small. Then, in particular, $\lambda(r) \to 0$ as $r \to 0$. Thus $U_+ \subset W^\varepsilon$ for any $\varepsilon > 0$.

Suppose, contrary to our claim, that there exists a trajectory $x(t)$ of ∇f which spirals. By the previous sections we may suppose that

$$F(x(t)) = \frac{f(x(t))}{|x(t)|}$$

goes to -1 as $t \to 0$. The trajectory $x(t)$, since it spirals, has to cross infinitely many times any component of W^ε. Thus on W^ε, and hence on $U_+ \subset W^\varepsilon$, $f \simeq r$ and, by (7.9), $\partial_r f \simeq -1$.

Denote

$$\psi(r) = \min_{x\in S(r)} f(x) = \min_{x\in S(r)\cap U_+} f(x) \tag{8.3}$$

$$\varphi(r) = \max_{x\in S(r)} f(x) = \max_{x\in S(r)\cap U_+} f(x). \tag{8.4}$$

Lemma 8.3. *Under the above assumptions* $\frac{\varphi(r)-\psi(r)}{r}$ *is small.*

Proof. $\partial_r f \simeq -1$ on U_+. Hence $|\partial_\theta f| = r|\nabla' f| \simeq r\lambda(r)$. By integration in U_+,

$$|f(r,\theta_1) - f(r,\theta_2)| = |\int_{\theta_1}^{\theta_2} \partial_\theta f \, d\theta| \le C(\theta_2 - \theta_1) r\lambda(r).$$

Therefore

$$\frac{\varphi(r) - \psi(r)}{r} \le \tilde{C}\lambda(r),$$

and the right-hand side is small by assumption. Q.E.D.

Lemma 8.4. *Suppose* $\frac{\varphi(r)-\psi(r)}{r}$ *small and assume that* U_- *contains a non-empty sector* $\{\theta_1 \le \theta \le \theta_2\}$. *Then the set*

$$\{x \in U_-; |\partial_\theta f| \le \frac{3(\varphi(r) - \psi(r))}{\theta_2 - \theta_1}\}$$

contains a non-empty sector.

Proof. Otherwise

$$|f(r,\theta_1) - f(r,\theta_2)| = -\int_{\theta_1}^{\theta_2} \partial_\theta f \, d\theta \ge 2(\varphi(r) - \psi(r))$$

that contradicts the definition of φ and ψ. Q.E.D.

Let U_0 be a sector satisfying the statement of lemma 8.4. On this sector $|\nabla' f|$ is bounded by $\frac{3(\varphi(r)-\psi(r))}{r(\theta_2-\theta_1)}$ that is small and hence $|\nabla' f| \to 0$ as $r \to 0$. Therefore, by remark 3.5, $|\nabla f| \simeq |\partial_r f| \sim 1$. This means that U_0 is contained in W^ε for any $\varepsilon > 0$. Consider the part of the trajectory that is in U_0. Since the trajectory spirals we may find such a part in any neighborhood of the origin. Since $U_0 \subset W^\varepsilon$, $\partial_r f < 0$ and r is strictly decreasing on the trajectory. Parameterizing the trajectory by r

$$\frac{d\theta}{dr} = \frac{|\nabla' f|}{r|\partial_r f|} \simeq \frac{|\partial_\theta f|}{r^2} \le C \frac{1}{r} \frac{\varphi(r) - \psi(r)}{r}$$

and the right-hand side is integrable by lemma 8.3. This means that the trajectory cannot cross U_0 if it remains in a small neighborhood of the origin $\{0 < r < r_0\}$. Indeed, by integrability, on the part of the trajectory that is in $U_0 \cap \{0 < r < r_0\}$ the difference of the maximum and the minimum of θ goes to 0 as $r_0 \to 0$.

This ends the proof. Q.E.D.

Corollary 8.5. *If f is defined in an o-minimal structure $\tilde{\mathbb{R}}$ then the trajectory $x(t)$ is definable in the pfaffian closure of $\tilde{\mathbb{R}}$.*

Proof. This follows directly from [23]. Indeed, it suffices to show that the image L of $x(t)$ is a Rolle leaf. For this we fix U a definable "horn" neighborhood of that contains $L \setminus 0$, that is divided by $L \setminus 0$ into two connected components, and f is C^1 on U. The existence of such U follows from theorem 8.1. Clearly, $L \setminus 0$ is a Rolle leaf of

$$\omega = \frac{\partial f}{\partial y} dx - \frac{\partial f}{\partial x} dy$$

by the Rolle-Khovanskii Lemma [7]. Q.E.D.

§9. Gradient Conjecture for Polynomially Bounded o-minimal Structures

In this section we place ourselves in the situation described in section 7. We may suppose that $\varphi \equiv r$ which we shall do just for simplicity of notation. We shall also make the following additional assumption:

Assumption. There exists a continuous definable function ω, small in sense of definition 1, such that

$$|1 - \frac{f}{r\partial_r f}| \leq \tfrac{1}{2}\omega^2(r) \quad \text{on } W. \tag{9.1}$$

By lemma 7.1 such ω exists for any polynomially bounded o-minimal structure. Indeed, in this case it suffices to take $\omega = \sqrt{\tilde{\omega}}$. On the other hand example 1 shows that, in an o-minimal structure which is not polynomially bounded, $\tilde{\omega}$ small does not imply necessarily that $\sqrt{\tilde{\omega}}$ is small.

Theorem 9.1. *Let $x(s)$ be a trajectory of $\frac{\nabla f}{|\nabla f|}$, $x(s) \to 0$ as $s \to s_0$. Denote by $\tilde{x}(s)$ the projection of $x(s)$ onto the unit sphere, $\tilde{x}(s) = \frac{x(s)}{|x(s)|}$. Then $\tilde{x}(s)$ is of finite length.*

Proof. Let $F = \frac{f}{r}$ be given by section 7. Then we may suppose that the trajectory is contained in $U_C = \{x; -C \leq F(x) \leq -1\}$ and, by proposition 7.2, that (7.10) holds, and that $\lim_{s\to s_0} F(x(s)) \to a_0 \leq -1$.

We use the arc-length parameterization $\tilde{s}$ of $\tilde{x}(s)$ given by

$$\frac{ds}{d\tilde{s}} = \frac{r|\nabla f|}{|\nabla' f|}, \tag{9.2}$$

Reparametrize $x(s)$ using $\tilde{s}$ as parameter. Then

$$\frac{dF}{d\tilde{s}} = \frac{1}{|\nabla' f|}\Big(|\nabla' f|^2 + |\partial_r f|^2\big(1 - \frac{f}{r\partial_r f}\big)\Big) = r|\nabla' F| + r\partial_r F \frac{\partial_r f}{|\nabla' f|}, \tag{9.3}$$

where $\nabla' F = \frac{\nabla' f}{r}$.

Lemma 9.2. *There exists a continuous definable change of parameter $\Psi : (\mathbb{R}, a_0) \to (\mathbb{R}, 0)$ and a constant $c' > 0$ such that*

$$\frac{d\Psi(F(x(s)) - a_0)}{d\tilde{s}} \geq c' \tag{9.4}$$

holds on $\{x \in U_C;\ \omega \leq \frac{|\nabla' f|}{|\partial_r f|}\}$.

Proof. On the set $\{x \in U_C; \omega \leq \frac{|\nabla' f|}{|\partial_r f|}\}$, by formulae (6.5),

$$|\partial_r F| = |\frac{\partial_r f}{r}(1 - \frac{f}{r\partial_r f})| \leq \frac{\omega^2 |\partial_r f|}{r} \leq \omega |\nabla' F|.$$

By assumption ω is small and we may use Proposition 6.3. Thus there is Ψ such that (6.1) holds. We shall show that Ψ satisfies the statement of lemma.

First we suppose that we are also in W that is in the set $\{x \in W; \omega \leq \frac{|\nabla' f|}{|\partial_r f|}\}$. Then, by (9.1)

$$\frac{|\partial_r f|^2}{|\nabla' f|}(1 - \frac{f}{r\partial_r f}) \leq \tfrac{1}{2}\omega^2 \frac{|\partial_r f|^2}{|\nabla' f|} \leq \tfrac{1}{2}|\nabla' f|.$$

Consequently

$$\frac{dF}{d\tilde{s}} \geq |\nabla' f| + \frac{|\partial_r f|^2}{|\nabla' f|}(1 - \frac{f}{r\partial_r f}) \geq \tfrac{1}{2} r|\nabla' F|,$$

and the lemma follows from (6.1).

A similar argument works on $U_C \setminus W$ since, by (7.3),

$$\frac{dF}{d\tilde{s}} \geq c' r|\nabla' F| = c'|\nabla' f| \geq \mathit{const} > 0.$$

This ends the proof. Q.E.D.

Given $\alpha : (\mathbb{R}_{\geq 0}, 0) \to (\mathbb{R}_{\geq 0}, 0)$ such that

$$\omega \leq \tilde{c}^{-1} r\alpha', \tag{9.5}$$

where $\tilde{c}$ will be specified later. Define $\tilde{\alpha} = \Psi \circ \alpha$. We consider $g = \Psi(F - a_0) - \tilde{\alpha}(r)$ as a control function. Then

$$\frac{dg}{d\tilde{s}}(x(s)) = \Psi'|\nabla' f| + \frac{\partial_r f}{|\nabla' f|}\Psi'\Big(\partial_r f\big(1 - \frac{f}{r\partial_r f}\big) - r\tilde{\alpha}'(r)\Big). \tag{9.6}$$

We may also suppose that $\Psi' \geq 1$.

Lemma 9.3. *There is a constant $c' > 0$ such that*

$$\frac{dg}{d\tilde{s}} \geq c' \tag{9.7}$$

holds on $\{x \in U_C;\ \omega \leq \frac{|\nabla' f|}{|\partial_r f|}\}$.

Proof. On W, $\partial_r f$ is negative and hence

$$-\frac{d\tilde{\alpha}}{d\tilde{s}} = -r\tilde{\alpha}'(r)\frac{\partial_r f}{|\nabla' f|} \geq 0. \tag{9.8}$$

Thus the statement follows from lemma 9.2.

On $U_C \setminus W$

$$|\frac{d\tilde{\alpha}}{d\tilde{s}}| = |r\tilde{\alpha}'(r)\frac{\partial_r f}{|\nabla' f|}| \leq \varepsilon |r\tilde{\alpha}'(r)| = o(1)$$

and the lemma follows again from lemma 9.2. Q.E.D.

It remains to show that (9.7) holds on $U_C \setminus \{x \in U_C;\ \omega \leq \frac{|\nabla' f|}{|\partial_r f|}\}$ that is contained in W. We denote it by $W(\omega)$ that is $W(\omega) = \{x \in W; \omega > \frac{|\nabla' f|}{|\partial_r f|}\}$. Firstly we note that on $W(\omega)$

$$-\frac{d\tilde{\alpha}}{d\tilde{s}} = r\tilde{\alpha}'(r)\frac{|\partial_r f|}{|\nabla' f|}| \geq \Psi' \frac{r\alpha'(r)}{\omega} \geq \tilde{c}\Psi' \geq \tilde{c}. \tag{9.9}$$

On the other hand

$$r\alpha' \geq \tilde{c}\omega \gg \tfrac{1}{2}\omega^2 \partial_r f\big(1 - \frac{f}{r\partial_r f}\big) \tag{9.10}$$

This shows that $-\frac{d\tilde{\alpha}}{d\tilde{s}} = -\frac{\partial_r f}{|\nabla' f|}\Psi' r\tilde{\alpha}'(r)$ dominates in the second term of the right hand side of (9.6). Since the first part cannot be negative we get, by (9.9), (9.7) as required.

This ends the proof of the theorem. Q.E.D.

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