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Levi form of logarithmic distance to complex submanifolds and its application to developability

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§1. Introduction

Let M be a complex manifold of codimension q defined in an open subset U of $\mathbb{C}^n$ and let $\delta_M(P)$ be the Euclidean distance from $P \in U$ to M. Then it is well-known that the function $\varphi := -\log \delta_M$ is, near M, weakly q-convex i.e., the Levi form $L(\varphi)$ of φ has $n-q+1$ nonnegative eigenvalues. Moreover, $L(\varphi)$ is positive semi-definite in the tangential direction of dimension $n-q$ to M (cf. [M2]).

The purpose of the present article is to calculate the Levi form $L(\varphi)$ explicitly near M and to give a necessary and sufficient condition for defining functions of M that $L(\varphi)$ degenerates in the tangential direction (§2, Theorem 1). Such calculation was first done by Matsumoto-Ohsawa [M-O] to study Levi flat hypersurfaces in complex tori of dimension two. As its application, by combining it with the theorem of Fischer-Wu [F-W], developability of a complex submanifold M ($\subset \mathbb{C}^n$) is characterized by the Levi form of $-\log \delta_M$ if $\dim M = 1$, 2 or $n-1$ (§3, Theorem 2).

§2. Levi form of logarithmic distance

Let r, q and n be integers with $r+q=n$, $r \geq 1$ and $q \geq 1$, and let M be a complex submanifold of dimension r in $\mathbb{C}^n$ defined by

$$M = \{(t, f(t)) \mid t = (t_1, \ldots, t_r) \in V\}$$

for open $V \subset \mathbb{C}^r$ and holomorphic $f = (f_1, \ldots, f_q) : V \longrightarrow \mathbb{C}^q$. Let $(z, w) = (z_1, \ldots, z_r; w_1, \ldots, w_q)$ be a (given) coordinate system of $\mathbb{C}^n = \mathbb{C}^r \times \mathbb{C}^q$. By a translation and a unitary transformation of (z, w) if necessary we may assume that $0 = (0, \ldots, 0) \in V$ and

$$(1) \qquad f_\mu(0) = 0, \quad \frac{\partial f_\mu}{\partial t_i}(0) = 0$$

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for $1 \leq i \leq r$ and $1 \leq \mu \leq q$. We denote by $\delta_M(z,w)$ the Euclidean distance from $(z,w) \in \mathbb{C}^n$ to M and put $\varphi(z,w) := -\log \delta_M(z,w)$.

We define the (r,r)-matrices $\Phi(w)$ and $F_\mu(t)$, $1 \leq \mu \leq q$, by

$$\Phi(w) := \left(\frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j}(0,w) \right)_{1 \leq i,j \leq r}, \quad F_\mu(t) := \left(\frac{\partial^2 f_\mu}{\partial t_i \partial t_j}(t) \right)_{1 \leq i,j \leq r}$$

and put

$$\mathcal{F}(w) := \sum_{\mu=1}^{q} \overline{F_\mu(0)} w_\mu.$$

$F_\mu(t)$ and $\mathcal{F}(w)$ are symmetric and $\Phi(w)$ is Hermitian.

Then we obtain the following (see [M-O], Lemma for $q = r = 1$).

Theorem 1. *There exists $\varepsilon > 0$ such that*

$$\Phi(w) = \frac{1}{2||w||^2} \overline{\mathcal{F}(w)} \mathcal{F}(w) [E - \overline{\mathcal{F}(w)} \mathcal{F}(w)]^{-1}$$

for $0 < ||w|| < \varepsilon$, where $||w||^2 := \sum_{\mu=1}^{q} |w_\mu|^2$ and E denotes the identity matrix. In particular, two matrices $\Phi(w)$ and $\mathcal{F}(w)$ have the same rank for each w with $0 < ||w|| < \varepsilon$.

Proof. If we put

$$\alpha(z,w,t) := \sum_{i=1}^{r} |z_i - t_i|^2 + \sum_{\mu=1}^{q} |w_\mu - f_\mu(t)|^2 \tag{2}$$

for $(z,w) \in \mathbb{C}^r \times \mathbb{C}^q$ and $t \in V$, then

$$\frac{\partial \alpha}{\partial t_i} = \overline{t_i - z_i} + \sum_{\mu=1}^{q} \frac{\partial f_\mu}{\partial t_i} \{\overline{f_\mu(t) - w_\mu}\} \tag{3}$$

for $1 \leq i \leq r$. By the implicit function theorem we can find C^ω-functions $t_k = t_k(z,w)$, $1 \leq k \leq r$, defined near $(0,0) \in \mathbb{C}^r \times \mathbb{C}^q$ such that

$$\frac{\partial \alpha}{\partial t_i}(z,w,t(z,w)) = 0, \quad \frac{\partial \alpha}{\partial \bar{t}_i}(z,w,t(z,w)) = 0 \tag{4}$$

for $1 \leq i \leq r$ (cf. [M1]). Then by (1) we have $t_k(0,w) = 0$ for $1 \leq k \leq r$.

If we put $\beta(z,w) := \alpha(z,w,t(z,w))$ then $\beta(z,w) = \delta_M(z,w)^2$ near $(0,0) \in \mathbb{C}^r \times \mathbb{C}^q$. By applying (4) and (2) we have

$$\frac{\partial \beta}{\partial z_i} = \frac{\partial \alpha}{\partial z_i} = \overline{z_i - t_i}, \quad \frac{\partial^2 \beta}{\partial z_i \partial \bar{z}_j} = \delta_{ij} - \frac{\partial \bar{t}_i}{\partial \bar{z}_j} \tag{5}$$

for $1 \le i, j \le r$. By differentiating (4) we have

$$\text{(6)} \qquad \begin{cases} \dfrac{\partial^2 \alpha}{\partial t_i \partial z_j} + \displaystyle\sum_{k=1}^{r} \left(\dfrac{\partial^2 \alpha}{\partial t_i \partial t_k} \dfrac{\partial t_k}{\partial z_j} + \dfrac{\partial^2 \alpha}{\partial t_i \partial \bar{t}_k} \dfrac{\partial \bar{t}_k}{\partial z_j} \right) = 0 \\ \dfrac{\partial^2 \alpha}{\partial \bar{t}_i \partial z_j} + \displaystyle\sum_{k=1}^{r} \left(\dfrac{\partial^2 \alpha}{\partial \bar{t}_i \partial t_k} \dfrac{\partial t_k}{\partial z_j} + \dfrac{\partial^2 \alpha}{\partial \bar{t}_i \partial \bar{t}_k} \dfrac{\partial \bar{t}_k}{\partial z_j} \right) = 0 \end{cases}$$

and by differentiating (3) we have

$$\frac{\partial^2 \alpha}{\partial t_i \partial z_j} = 0, \quad \frac{\partial^2 \alpha}{\partial \bar{t}_i \partial z_j} = -\delta_{ij},$$

$$\frac{\partial^2 \alpha}{\partial t_i \partial t_j} = \sum_{\mu=1}^{q} \frac{\partial^2 f_\mu}{\partial t_i \partial t_j} \overline{\{f_\mu(t) - w_\mu\}}, \quad \frac{\partial^2 \alpha}{\partial t_i \partial \bar{t}_j} = \delta_{ij} + \sum_{\mu=1}^{q} \frac{\partial f_\mu}{\partial t_i} \frac{\partial \bar{f}_\mu}{\partial \bar{t}_j}.$$

Now if $(z, w) = (0, w)$ then $t(0, w) = 0$ and by (1) we have

$$\text{(7)} \qquad \frac{\partial^2 \alpha}{\partial t_i \partial t_j}(0, w, 0) = -\sum_{\mu=1}^{q} \frac{\partial^2 f_\mu}{\partial t_i \partial t_j}(0) \bar{w}_\mu, \quad \frac{\partial^2 \alpha}{\partial t_i \partial \bar{t}_j}(0, w, 0) = \delta_{ij}.$$

If we put

$$\text{(8)} \qquad \mathcal{F}(w)_{ij} := \sum_{\mu=1}^{q} \frac{\partial^2 \bar{f}_\mu}{\partial \bar{t}_i \partial \bar{t}_j}(0) w_\mu$$

then $\mathcal{F}(w)_{ij}$ is the (i, j)-component of the symmetric matrix $\mathcal{F}(w)$. By substituting (7) and (8) for (6) we have

$$\text{(9)} \qquad \begin{cases} \dfrac{\partial \bar{t}_i}{\partial z_j}(0, w) = \displaystyle\sum_{k=1}^{r} \overline{\mathcal{F}(w)}_{ik} \dfrac{\partial t_k}{\partial z_j}(0, w) \\ \dfrac{\partial t_i}{\partial z_j}(0, w) - \delta_{ij} = \displaystyle\sum_{k=1}^{r} \mathcal{F}(w)_{ik} \dfrac{\partial \bar{t}_k}{\partial z_j}(0, w) \end{cases}$$

and hence

$$\frac{\partial t_i}{\partial z_j}(0, w) - \delta_{ij} = \sum_{k=1}^{r} \mathcal{F}(w)_{ik} \sum_{l=1}^{r} \overline{\mathcal{F}(w)}_{kl} \frac{\partial t_l}{\partial z_j}(0, w).$$

Since $\mathcal{F}(0)$ is the zero matrix, we thus obtain

$$(\partial t_i / \partial z_j(0, w))_{1 \le i, j \le r} = [E - \mathcal{F}(w)\overline{\mathcal{F}(w)}]^{-1}$$

for sufficiently small w and therefore by (5) we have

$$\begin{aligned}(\partial^2\beta/\partial z_i\partial\bar{z}_j(0,w))_{1\le i,j\le r} &= E-[E-\overline{\mathcal{F}(w)}\mathcal{F}(w)]^{-1}\\ &= -\overline{\mathcal{F}(w)}\mathcal{F}(w)[E-\overline{\mathcal{F}(w)}\mathcal{F}(w)]^{-1}.\end{aligned}$$

On the other hand, $\beta = {\delta_M}^2$ and

$$\frac{\partial^2(-\log\delta_M)}{\partial z_i\partial\bar{z}_j} = \frac{1}{2}\left(-\frac{1}{\beta}\frac{\partial^2\beta}{\partial z_i\partial\bar{z}_j}+\frac{1}{\beta^2}\frac{\partial\beta}{\partial z_i}\frac{\partial\beta}{\partial\bar{z}_j}\right).$$

Moreover by (2) and (5) we have $\beta(0,w)=||w||^2$ and $\partial\beta/\partial z_i(0,w)=0$ for $1\le i\le r$. This proves the theorem. Q.E.D.

Remark. The complex Hessian matrix of $\varphi(z,w) := -\log\delta_M(z,w)$ at $(z,w)=(0,w)$, $0<||w||<\varepsilon$, is written as

$$\begin{pmatrix}(\partial^2\varphi/\partial z_i\partial\bar{z}_j) & (\partial^2\varphi/\partial z_i\partial\bar{w}_\nu)\\ (\partial^2\varphi/\partial w_\mu\partial\bar{z}_j) & (\partial^2\varphi/\partial w_\mu\partial\bar{w}_\nu)\end{pmatrix}(0,w) = \begin{pmatrix}\Phi(w) & O\\ O & \Psi(w)\end{pmatrix},$$

where $\Phi(w)$ is the (r,r)-matrix defined as above and $\Psi(w)$ is the (q,q)-matrix defined by $\Psi(w) := (\partial^2(-\log||w||)/\partial w_\mu\partial\bar{w}_\nu)_{1\le\mu,\nu\le q}$.

§3. Developability of complex submanifolds

Let $M=\{(t,f(t))\mid t\in V\}$ $(\subset\mathbb{C}^n)$ be as in §2. If we put $J(t) := (F_1(t),\dots,F_q(t))$ then ${}^tJ(t)$ is the Jacobian matrix of the Gauss map

$$t\longmapsto\left(\frac{\partial f_1}{\partial t_1},\dots,\frac{\partial f_1}{\partial t_r},\dots,\frac{\partial f_q}{\partial t_1},\dots,\frac{\partial f_q}{\partial t_r}\right).$$

By Fischer-Wu [F-W] (cf. [F-P]), the complex submanifold M of dimension r is developable almost everywhere (i.e., at each point $(t,f(t))$ where $\operatorname{rank}J(t)$ is maximal) if and only if $\operatorname{rank}J(t)<r$ for all t.

As an application of Theorem 1, we can obtain the following.

Theorem 2. *In the case* $\dim M = 1$, 2 *or* $n-1$, *M is developable almost everywhere if and only if the Levi form of* $-\log\delta_M$ *degenerates in the tangential direction at each point near M.*

For the proof we use the following.

Lemma. *Let $A_1,\dots,A_q$ be complex symmetric matrices of degree r and let $w=(w_1,\dots,w_q)\in\mathbb{C}^q$. Then*

(i) $\max_{w\in\mathbb{C}^q}\operatorname{rank}\sum_{\mu=1}^q A_\mu w_\mu \le \operatorname{rank}(A_1,\dots,A_q)$.

(ii) *The equality holds if* $r=1,2$ *or if* $q=1$.

(iii) *The equality does not hold in general if* $r\ge 3$ *and* $q\ge 2$.

Proof. (i) is trivial and (ii) is also trivial if $r = 1$ or $q = 1$. (In these cases the matrices $A_1, \dots, A_q$ need not be symmetric.)

If $(2,2)$-matrices $A_1, \dots, A_q$ are symmetric and $\det(\sum_{\mu=1}^{q} A_\mu w_\mu) \equiv 0$ then $\det(A_{\mu_1} w_{\mu_1} + A_{\mu_2} w_{\mu_2}) \equiv 0$ for any pair (μ_1, μ_2) with $1 \leq \mu_1 < \mu_2 \leq q$, and the coefficients of the polynomial of degree 2 with respect to (w_{μ_1}, w_{μ_2}) are all zero. From this it is easy to see that $\operatorname{rank}(A_{\mu_1}, A_{\mu_2}) \leq 1$ for all (μ_1, μ_2) and hence $\operatorname{rank}(A_1, \dots, A_q) \leq 1$, which proves (ii).

(iii) follows from the next example. Q.E.D.

Example. Consider the real symmetric matrices

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Then $\operatorname{rank}(A_1, A_2) = 3$, although $\det(A_1 w_1 + A_2 w_2) \equiv 0$. Therefore, if $M \subset \mathbb{C}^5 = \mathbb{C}^3 \times \mathbb{C}^2$ is the complex submanifold defined by

$$M = \{(z, w) \in \mathbb{C}^5 \mid w_1 = z_1 z_2, w_2 = z_1 z_2 + z_1 z_3\}$$

then $-\log \delta_M$ degenerates in the tangential direction at $(0, w)$ for all w near $0 \in \mathbb{C}^2$, but M is not developable at the origin $(0,0) \in M$.

References

[F-P] G. Fischer and J. Piontkowski, "Ruled Varieties", Vieweg, Braunschweig, 2001.

[F-W] G. Fischer and H. Wu, Developable complex analytic submanifolds, Internat. J. Math., **6** (1995), 229–272.

[M1] K. Matsumoto, A note on the differentiability of the distance function to regular submanifolds of Riemannian manifolds, Nihonkai Math. J., **3** (1992), 81–85.

[M2] K. Matsumoto, Boundary distance functions and q-convexity of pseudoconvex domains of general order in Kähler manifolds, J. Math. Soc. Japan, **48** (1996), 85–107.

[M-O] K. Matsumoto and T. Ohsawa, On the real analytic Levi flat hypersurfaces in complex tori of dimension two, Ann. Inst. Fourier (Grenoble), to appear.

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