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The Bergman kernel of Hartogs domains and transformation laws for Sobolev-Bergman kernels

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Introduction

If we consider the Bergman kernel of strictly pseudoconvex domains, we can discuss a scalar invariant theory associated with CR geometry of the boundaries. This is Fefferman's program proposed in [3] and then developed in [6], [10], [1], [11], [8] and others. What will happen if the Bergman kernel is replaced by reproducing kernels associated with spaces of holomorphic functions contained in L^2 Sobolev spaces? Let us restrict ourselves to the case where the Sobolev order is a half integer $s/2$ ($s \in \mathbb{Z}$). The case $s = 0$ corresponds to the Bergman kernel. The case $s = 1$ corresponds to the Szegö kernel, and the invariant theory is essentially the same as that of the Bergman kernel ([10], [11]). The situation changes with the signature of this s. More precisely, it is at first necessary that the inner product of the Hilbert space which admits the reproducing kernel must satisfy a transformation law under biholomorphic mappings. Existence of such an inner product is obvious when $s \leq 0$ ($s \in \mathbb{R}$), whereas it is unknown for $s > 0$ except for $s = 1$. Next, boundary invariants will be contained in the singularity of the reproducing kernel, and if the singularity is of the same type as that of the Bergman kernel ([3], [2]) then in particular $s \geq 0$ is necessary ([9]). This fact suggests that the type of the singularities of the reproducing kernels for $s < 0$ are different from that of the Bergman kernel. Is it possible to avoid considering such new singularities? In what follows, we shall give an almost affirmative answer by considering Hartogs domains and Hirachi's formulation in [8] of a biholomorphic transformation law for local defining functions of strictly pseudoconvex domains.

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§1. Hartogs domains and biholomorphic transformation laws

Points of $\mathbb{C}^{n+t} = \mathbb{C}^n \times \mathbb{C}^t$ $(n \geq 2)$ will be denoted by (z, z_0), (w, w_0), etc. Recall that a domain $\Omega \subset \mathbb{C}^n$ is said to have C^∞ boundary if there exists a real valued function $\rho \in C^\infty(\overline{\Omega}) = C^\infty(\mathbb{C}^n)|_{\overline{\Omega}}$ such that

$$\Omega = \{z \mid \rho(z) > 0\}, \qquad d\rho \neq 0 \text{ on } \partial\Omega;$$

we then write $\rho \in C^\infty_{\text{def}}(\overline{\Omega})$. Given such a *defining function* $\rho \in C^\infty_{\text{def}}(\overline{\Omega})$, the *Hartogs domain* $D = D^t_\rho \subset \mathbb{C}^{n+t}$ associated with it is defined by

$$D := \{(z, z_0) \mid \lambda(z, z_0) > 0\}, \qquad \lambda(z, z_0) := \rho(z) - |z_0|^2;$$

thus $\lambda \in C^\infty_{\text{def}}(\overline{D})$ which depends on $t \in \mathbb{N}$ and $\rho \in C^\infty_{\text{def}}(\overline{\Omega})$.

Remark 1. D is defined even when $\rho \notin C^\infty(\overline{\Omega})$, but if $\partial\Omega \in C^\infty$ then $\partial D \in C^\infty$ because $\partial\lambda = \partial\rho - \overline{z}_0 \cdot dz_0$. If in addition $\partial\Omega$ is strictly pseudoconvex, so is ∂D on $z_0 = 0$. If furthermore $-\rho$ is strictly plurisubharmonic, so is $-\lambda$ and thus ∂D is everywhere strictly pseudoconvex.

In what follows, we assume $\rho \in C^\infty_{\text{def}}(\overline{\Omega})$ and consider for simplicity only *strictly pseudoconvex domains* Ω. For subscripts $i = 1, 2$, we use the following notation:

$$\rho_i \in C^\infty_{\text{def}}(\overline{\Omega}_i), \quad \lambda_i = \rho_i - |z_0|^2, \quad D_i = D^t_{\rho_i} \subset \mathbb{C}^{n+t}.$$

By elementary operations on determinants, we have:

Fact 1. *The Levi determinants (i.e. the complex Monge-Ampère operators) on Ω and D satisfy*

$$J_\Omega[\rho] := (-1)^n \det \begin{pmatrix} \rho & \rho_{\overline{k}} \\ \rho_j & \rho_{j\overline{k}} \end{pmatrix} = J_D[\lambda],$$

where the subscripts j, $\overline{k}$ stand for differentiation with respect to z_j, $\overline{z}_k$.

Recall by Fefferman [4] that if $\Phi : \Omega_1 \to \Omega_2$ is biholomorphic then $J_{\Omega_1}[u_1] = J_{\Omega_2}[u_2] \circ \Phi$ with $u_1 := |\det \Phi'|^{-2/(n+1)} (u_2 \circ \Phi)$ for functions u_2 in Ω_2, where Φ' denotes the holomorphic Jacobian matrix of Φ.

Lemma 1. *Given a biholomorphic map $\Phi : \Omega_1 \to \Omega_2$, let*

$$\Psi : (z, z_0) \mapsto (\Phi(z), m(z)\, z_0), \quad m(z) := [\det \Phi'(z)]^{1/(n+1)}.$$

Then $\Psi : D_1 \to D_2$ is a biholomorphic lift, provided

$$\rho_1 = |\det \Phi'|^{-2/(n+1)} (\rho_2 \circ \Phi). \tag{1.1}$$

Incidentally,

(1.2) $$\det \Psi'(z, z_0) = [\det \Phi'(z)]^{w(-t)/(n+1)}, \quad w(-t) := n + 1 + t.$$

Proof. It follows from (1.1) that $\lambda_2(\Psi(z, z_0)) = |m(z)|^2 \lambda_1(z, z_0)$ so that $\Psi(D_1) \subset D_2$, and similarly $\Psi^{-1}(D_2) \subset D_1$. Now (1.2) is easy.

Remark 2. The lift Ψ is motivated by that of Fefferman [4], [5]:

$$\Phi_{\#} : (z_{\mathrm{F}}, z) \mapsto \big(m(z)^{-1} z_{\mathrm{F}},\, \Phi(z)\big), \quad z_{\mathrm{F}} \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}.$$

The map $\Phi_{\#} : \mathbb{C}^* \times \Omega_1 \to \mathbb{C}^* \times \Omega_2$ is biholomorphic. The multiplicative factor of the variable z_{F} in $\Phi_{\#}$ is the inverse of that in Ψ. Thus it is natural to consider the Lorentz-Kähler potential $|z_{\mathrm{F}}|^2 \rho(z)$ upstairs (cf. [4], [5]), whereas we consider $\lambda(z, z_0) = \rho(z) - |z_0|^2$ in Lemma 1.

Following Hirachi [8], we lift the Levi determinant $J_\Omega[\,\cdot\,]$ on Ω to Fefferman's $\mathbb{C}^*$ bundle in [4], [5]. That is, we set, for functions $U = U(z_{\mathrm{F}}, z)$ in $\mathbb{C}^* \times \Omega$,

$$J_{\Omega,\#}[U] := (-1)^n \det \begin{pmatrix} U_{\mathrm{F}\overline{\mathrm{F}}} & U_{\mathrm{F}\overline{k}} \\ U_{j\overline{\mathrm{F}}} & U_{j\overline{k}} \end{pmatrix},$$

where the subscripts F, $\overline{\mathrm{F}}$ stand for differentiation with respect to z_{F}, $\overline{z}_{\mathrm{F}}$. Then, as in the proof of Fact 1, we have:

Fact 2. *Let $\Lambda(z_{\mathrm{F}}, z, z_0) := U(z_{\mathrm{F}}, z) - |z_{\mathrm{F}}|^2 |z_0|^2$ in $\mathbb{C}^* \times D$ for functions $U = U(z_{\mathrm{F}}, z)$ in $\mathbb{C}^* \times \Omega$. Then*

$$J_{D,\#}[\Lambda] := (-1)^{n+t} \det \begin{pmatrix} \Lambda_{\mathrm{F}\overline{\mathrm{F}}} & \Lambda_{\mathrm{F}\overline{k}} & \Lambda_{\mathrm{F}\overline{0}} \\ \Lambda_{j\overline{\mathrm{F}}} & \Lambda_{j\overline{k}} & \Lambda_{j\overline{0}} \\ \Lambda_{0\overline{\mathrm{F}}} & \Lambda_{0\overline{k}} & \Lambda_{0\overline{0}} \end{pmatrix} = |z_{\mathrm{F}}|^{2t} J_{\Omega,\#}[U].$$

Remark 3. Roughly speaking, there does not exist any natural family, in the context of local biholomorphic invariant theory, of C^∞ (local) defining functions which satisfy the transform law (1.1) (cf. Theorem 2 of [9] for a precise statement). That is, (1.1) necessarily contains an error (cf. [5], [6], [7], [1], [11]). According to Hirachi's theory in [8], this difficulty in Fefferman's program for the invariant theory of the Bergman kernel can be avoided by considering asymptotic solutions of the complex Monge-Ampère equation upstairs

$$J_{\Omega,\#}[U] = |z_{\mathrm{F}}|^{2n} \;\&\; U > 0 \text{ in } \mathbb{C}^* \times \Omega, \quad U = 0 \text{ on } \mathbb{C}^* \times \partial\Omega.$$

More precisely, asymptotic solutions are of the form

$$U = \rho_\# + \rho_\# \sum_{k=1}^{\infty} \eta_{k,\Omega} \left(\rho^{n+1} \log \rho_\#\right)^k, \qquad \eta_{k,\Omega} \in C^\infty(\overline{\Omega}),$$

where $\rho_\#$ takes the form $\rho_\#(z_{\mathrm{F}}, z) = |z_{\mathrm{F}}|^2 \, \rho(z)$ with special $\rho \in C^\infty_{\mathrm{def}}(\overline{\Omega})$. (This ρ involves an *ambiguity parameter* but transforms by (1.1), because the class of these ρ's are so chosen and an action is defined on the ambiguity parameter. See [8] for the detail.) On the other hand, $\Lambda := U - |z_{\mathrm{F}}|^2 \, |z_0|^2$ in Fact 2 formally satisfies

$$J_{D,\#}[\Lambda] = |z_{\mathrm{F}}|^{2n+2t} \ \ \& \ \ \Lambda > 0 \ \text{ in } \ \mathbb{C}^* \times D, \qquad \Lambda = 0 \ \text{ on } \ \mathbb{C}^* \times \partial D.$$

It might be interesting to study the role of Λ in the framework of Hirachi's theory [8].

§2. Sobolev-Bergman kernels of Ω in terms of the Bergman kernel of D

We denote the Bergman kernel of a Hartogs domain $D = D^t_\rho \subset \mathbb{C}^{n+t}$ by

$$K^{\mathrm{B}}_D((z, z_0), (w, w_0)) \qquad \big((z, z_0), (w, w_0) \in D\big),$$

and the restriction to the diagonal by $K^{\mathrm{B}}_D((z, z_0)) = K^{\mathrm{B}}_D((z, z_0), (z, z_0))$.

Lemma 2. *If* $\Phi : \Omega_1 \to \Omega_2$ *is biholomorphic, then under the condition* (1.1) *in Lemma 1,*

$$K^{\mathrm{B}}_{D_1}((z, 0)) = K^{\mathrm{B}}_{D_2}((\Phi(z), 0)) \, |\det \Phi'(z)|^{2w(-t)/(n+1)}. \tag{2.1}$$

More precisely, for the lift $\Psi : D_1 \to D_2$ *in Lemma 1,*

$$K^{\mathrm{B}}_{D_1}((z, z_0)) = K^{\mathrm{B}}_{D_2}(\Psi(z, z_0)) \, |\det \Phi'(z)|^{2w(-t)/(n+1)}. \tag{2.2}$$

Proof. It follows from the transformation law in general for the Bergman kernel that if $\Psi : D_1 \to D_2$ is biholomorphic then

$$K^{\mathrm{B}}_{D_1}((z, z_0)) = K^{\mathrm{B}}_{D_2}(\Psi(z, z_0)) \, |\det \Psi'(z, z_0)|^2.$$

Thus (2.2) follows from (1.2). Setting $z_0 = 0$ in (2.2), we get (2.1).

This lemma makes sense when it is combined with the following elementary observation by Ligocka in [12]. Recall by definition that the Bergman kernel of D is the reproducing kernel associated with the

Hilbert space $H^{\mathrm{B}}(D) = L^2(D) \cap \mathcal{O}(D)$, where $\mathcal{O}(D)$ denotes the totality of holomorphic functions in D. Let us set, for $k \in \mathbb{N}_0$,

$$(2.3) \qquad g_k(z) = g_k[\rho](z) := c_k(t)\, \rho(z)^{t+k}, \qquad c_k(t) := \frac{1}{t+k} \frac{\pi^t}{\Gamma(t)}$$

and consider the Hilbert space $H^{\mathrm{B}}(\Omega, g_k) := L^2(\Omega, g_k) \cap \mathcal{O}(\Omega)$ with respect to the measure having each $g_k(z)$ as the weight function. Denoting the reproducing kernel by $K^{\mathrm{B}}_{g_k}(z,w)$, we set $K^{\mathrm{B}}_{g_k}(z) = K^{\mathrm{B}}_{g_k}(z,z)$. It will be sometimes clearer if we factor out the positive constant $c_k(t)$ and consider the following (then $\ell = t + k$ is not necessary):

$$H^{\mathrm{B}}(\Omega, \rho^\ell) = L^2(\Omega, \rho^\ell) \cap \mathcal{O}(\Omega), \quad K^{\mathrm{B}}_{\rho^\ell}(z) = K^{\mathrm{B}}_{\rho^\ell}(z,z) \quad (\ell \in \mathbb{N}_0).$$

$K^{\mathrm{B}}_{\rho^\ell}$ is called the *Sobolev-Bergman kernel* of Sobolev order $-\ell/2$ in [9]. Then, it is shown in Ligocka [12] that

$$(2.4) \qquad K^{\mathrm{B}}_D((z,z_0),(w,w_0)) = \sum_{k \in \mathbb{N}_0} K^{\mathrm{B}}_{g_k}(z,w) \sum_{|\alpha|=k} z_0^\alpha\, \overline{w}_0^\alpha.$$

Theorem. *Given a biholomorphic map $\Phi : \Omega_1 \to \Omega_2$, consider the Hartogs domains $D_i = D^t_{\rho_i} \subset \mathbb{C}^{n+t}$ $(i = 1, 2)$ defined by $\rho_i \in C^\infty_{\mathrm{def}}(\overline{\Omega}_i)$ satisfying the condition* (1.1) *in Lemma 1. Then the reproducing kernel $K^{\mathrm{B}}_{g_k[\rho]}(z)$ associated with the Hilbert space $H^{\mathrm{B}}(\Omega, g_k[\rho])$ defined via the function $g_k = g_k[\rho]$ in* (2.3) *satisfies the following transformation law*

$$(2.5) \qquad K^{\mathrm{B}}_{g_k[\rho_1]}(z) = K^{\mathrm{B}}_{g_k[\rho_2]}(\Phi(z))\, |\det \Phi'(z)|^{2w(-t-k)/(n+1)}.$$

That is, $K^{\mathrm{B}}_{\rho_1^{t+k}}(z) = K^{\mathrm{B}}_{\rho_2^{t+k}}(\Phi(z))\, |\det \Phi'(z)|^{2w(-t-k)/(n+1)}$.

Proof. If we set $z_0 = 0$ or $w_0 = 0$ in (2.4), then all terms in the right vanish except for $\alpha = 0$ (i.e. $k = 0$), so that $K^{\mathrm{B}}_D((z,0)) = K^{\mathrm{B}}_{g_0}(z)$. Thus (2.5) for $k = 0$ follows from (2.1) in Lemma 2. The result (2.5) for general $k \in \mathbb{N}_0$ also follows similarly by using (2.2) and Lemma 1.

Remark 4. Taking $k = 0$, we may write $K^{\mathrm{B}}_{\rho^t}(z) = c_0(t)\, K^{\mathrm{B}}_D((z,0))$ with $c_0(t) = \pi^t/\Gamma(t+1)$. Varying the dimension t, we get Sobolev-Bergman kernels of any negative half-integral order $-t/2$ $(t \in \mathbb{N})$. On the other hand, if we take $t = 1$, then we have $g_k(z) = c_k(1)\, \rho(z)^{k+1}$ with $c_k(1) = \pi/(k+1)$ and

$$K^{\mathrm{B}}_D((z,z_0)) = \sum_{k=0}^{\infty} K^{\mathrm{B}}_{g_k}(z)\, |z_0|^{2k} \qquad ((z,z_0) \in D \subset \mathbb{C}^{n+1}).$$

Varying this time the power k of $|z_0|^2$, we again get Sobolev-Bergman kernels of any negative half-integral order $-(k+1)/2$ ($k \in \mathbb{N}_0$).

Remark 5. The singularities of these Sobolev-Bergman kernels of Ω are computable from that of the Bergman kernel K_D^{B} of the Hartogs domain $D = D_\rho^t$, but there remains a problem of localizing the singularity of K_D^{B}. The author expects that the argument here will be used rather as a heuristics of formulating a local or microlocal version.

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