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On the middle dimension cohomology of A_l singularity

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Abstract.

Let (V, o) be a normal isolated singularity in a complex Euclidean space (C^N, o). Let M be the intersection of this singularity and the real hypersphere $S_\epsilon^{2N-1}(o)$, centered at the origin o with an ϵ radius. Then, naturally, this link M admits a CR structure, induced from V, and the deformation theory of this CR structures has been studied in [1], [2],[3]. Especially in [3], a particular subspace of the infinitesimal deformation space is found, and we propose to study the relation between this subspace and simultaneous deformation. We note that: if the canonical line bundle of the CR structure is trivial, then the infinitesimal space of the deformation of CR structures is a part of the middle dimension cohomology. And in this line, we conjecture that Z^1, introduced in [3], might be related to the simultaneous deformation of isolated singularity (V, o)(see also [2]). We discuss this problem for A_l singularities.

§1. Motivation and Z^1 - space

Let $(V^{(n)}, o)$ be an isolated singularity in a complex eucliean space (C^N, o). We consider the intersection

$$M = S_\epsilon^{2N-1}(o) \cap V.$$

Then M is a compact non-singular real $2n-1$ dimensional C^∞ manifold, and a CR structure $(M, {}^0T'')$ is induced from V, by ;

$${}^0T'' = C \otimes TM \cap T''(V - o).$$

Here $T''(V-o)$ means the space consisting of type $(1,0)$ vectors on $V-o$. This pair $(M, {}^0T'')$ is called a CR structure(or a CR manifold). For this CR structure, the deformation theory, related to the deformation theory

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of isolated singularities (V, o), is successfully developed by Kuranishi. After the great work of Kuranishi, we are interested in the mixed Hodge structure of CR manifolds. We take a supplement vector field ζ to ${}^0T'' + {}^0T'$, here ${}^0T' = \overline{{}^0T''}$. For this CR structure with the supplement vector field $\{(M, {}^0T''), \zeta\}$, we can introduce a mixed Hodge structure which should correspond to the mixed Hodge structure on a tubular neighborhood U of M in V. Here, we assume that there is a real vector field ζ satisfying:

$$(1) \qquad \zeta_p \notin {}^0T''_p + {}^0T'_p$$

$$(2) \qquad [\zeta, \Gamma(M, {}^0T'')] \subset \Gamma(M, {}^0T'').$$

While, during our studying deformation theory of CR structures, we learn that: for Calabi-Yau manifolds, the Kuranishi family is unobstructed. So, in order to obtain the analogy to isolated singularities, Z^1 space is found(see [3]).

$$(3) \qquad Z^1 = \{u : u \in F^{n-1,1}, d''u = 0, d'u = 0\}.$$

In the case complex manifolds, Z^1 might be translated as follows. For a tubular neighborhood U of M in V, we set

$$(4) \qquad \{u : u \in \Gamma(U, \wedge^{n-1}(T'U)^* \wedge (T''U)^*), \overline{\partial}u = 0, \partial u = 0\}.$$

If $X^{(n)}$ is a compact n-dimensional Kaehler manifold, then

$$(5) \quad \{u : u \in \Gamma(X^{(n)}, \wedge^{n-1}(T'X^{(n)})^* \wedge (T''X^{(n)})^*), \overline{\partial}u = 0, \partial u = 0\}.$$

includes the $\overline{\partial}$-harmonic space consisting of $(n-1, 1)$ forms. While, here, we are treating an open manifold U(tubular neighborhood of M). So even if the $(n-1, 1)$ Kohn-Rossi cohomology does not vanish(the existence of a non- trivial $\overline{\partial}$-harmonic space consisting of $(n-1, 1)$ forms), the above space might be 0. Here we give a program to obtain a non-trivial element of (4) from a non- trivial simultaneous deformation.

Let $\tilde{V}$ be the resolution of the isolated singularity with complex dimension n in C^N, V, and π is the resolution map π; $\tilde{V} \to V$. And consider non-trivial deformations of isolated singularity (V, o) with this resolution. Namely, π_t is a resolution map of V_t in C^N , π_t; $\tilde{V}_t \to V_t$, $t \in T$, where V_t is a deformation of V, $\tilde{V}_t$ is a deformation of $\tilde{V}$, T is an analytic space with the origin, and at the origin, $\pi_o = \pi$, $\tilde{V}_o = \tilde{V}$, $V_o = V$. Furthermore, we assume that $V_t \subset C^N$. Now we take a C^∞ trivialization i_t : a tubular neighborhood of $M_o \to$ a tubular neighborhood of M_t, which satisfies $i_t(M_o) = M_t$. In this setting, our program is as follows.

- (<u>First Step</u>) By using the simultaneous resolution, we construct a non-trivial $(n, 0)$ form ω_t, which is not d exact on $\tilde{V}_t$ for a generic t, and depends on t complex analytically. In general, "to give an $(n, 0)$ form, satisfying a certain condition", might be easier than "to give an $(n-1, 1)$ form with the corresponding condition".
- (<u>Second Step</u>) By choosing a proper C^∞ trivialization of the simultaneous deformation, i_t,

$$i_t^* \omega_t = \omega_0 + \omega_1 t + \cdots, \quad \text{(expansion with respect to } t\text{)}.$$

- (<u>Third Step</u>) From $d\omega_t = 0$, it follows that: $d\omega_1 = 0$. By the definition, ω_1 is a form of type $(n, 0) + (n-1, 1)$ on $\tilde{V}_o - \pi^{-1}(o)$, we write it by;

$$\omega_1 = \omega_1^{(n,0)} + \omega_1^{(n-1,1)}.$$

As $d\omega_1 = 0$, this is equivalent to

$$\overline{\partial} \omega_1^{(n-1,1)} = 0,$$

$$\overline{\partial} \omega_1^{(n,0)} + \partial \omega_1^{(n-1,1)} = 0.$$

The $\overline{\partial}$-cohomology class, determined by $\omega_1^{(n-1,1)}$, is the induced one by the Kodaira-Spencer class of deformations. So, this must be non-trivial. In this setting, we would like to construct a non-trivial element of (4), associated with the given simultaneous deformation.

For the Third Step, we have to comment on a crucial point. The naive answer is that:

$$\partial \omega_1^{(n-1,1)} = 0 \ \ ?$$

This is too strong. There is an ambiguity to choose the C^∞ trivialization, i_t. By changing the C^∞ trivialization, ω_1(resp. $\omega_1^{(n-1,1)}$) is replaced by $\omega_1 - du$(resp. $\omega_1^{(n-1,1)} - \overline{\partial} u$), where u is an $(n-1, 1)$ form. Hence our problem(to obtain a non-trivial element of (4)) is reduced to that; is there any C^∞ $(n-1, 1)$ form u, satisfying: $\overline{\partial} \omega_1^{(n-1,1)} - \partial \overline{\partial} u = 0$? This is so called "$\partial\overline{\partial}$ lemma". For a compact Kaehler manifold, by taking the harmonic part, this is always solvable. However, for an open manifold, this is not an easy problem. One of our conjecture is that; if $\omega_1^{(n-1,1)}$ is induced by the simultaneous deformation, then this might be solvable. In the next section, we study this conjecture in A_l singularities.

§2. A_l singularities

Let

$$X = \{(z_1, \dots, z_{n+1}) : (z_1, \dots, z_{n+1}) \in C^{n+1}, z_1^2 + \cdots + z_{n+1}^{l+1} = 0\},$$

where l is a positive integer. We call this isolated singularity A_l singularity. Consider a family of deformations of X,

$$X_t = \{(z_1, \dots, z_{n+1}) : (z_1, \dots, z_{n+1}) \in C^{n+1}, z_1^2 + \cdots + z_{n+1}^{l+1} = t\}.$$

Let $M = X \cap \{(z_1, \dots, z_{n+1}) : | z_1 |^2 + \cdots + | z_{n+1} |^2 = 1\}$. And consider a C^∞ trivialization of this deformation over a neighborhood of M in X. Let $i_t : (z_1, \dots, z_{n+1}) \to (z_1(t), \dots, z_{n+1}(t))$, where

$$\begin{aligned}
z_1(t) &= z_1 + \frac{1}{2k(z,\overline{z})}\overline{z}_1(1+ | z_{n+1} |^2 + \cdots + | z_{n+1} |^{2l})t, \\
&\cdots \\
z_n(t) &= z_n + \frac{1}{2k(z,\overline{z})}\overline{z}_n(1+ | z_{n+1} |^2 + \cdots + | z_{n+1} |^{2l})t \\
z_{n+1}(t) &= z_{n+1} + \frac{1}{(l+1)k(z,\overline{z})}\overline{z}_{n+1}^l t
\end{aligned}$$

Here

$$k(z,\overline{z}) = (1+| z_{n+1} |^2 + \cdots + | z_{n+1} |^{2(l-1)})(| z_1 |^2 + \cdots + | z_n |^2) + | z_{n+1} |^2 \; l.$$

So, on M, because of $| z_1 |^2 + \cdots + | z_n |^2 = 1 - | z_{n+1} |^2$, $k(z,\overline{z}) = 1$ holds. And,

$$\begin{aligned}
&z_1(t)^2 + \cdots + z_n(t)^2 + z_{n+1}(t)^{l+1} \\
&\quad = z_1^2 + \cdots + z_n^2 + z_{n+1}^{l+1} \\
&\qquad + \frac{1}{k(z,\overline{z})}\{(1+| z_{n+1} |^2 + \cdots + | z_{n+1} |^{2(l-1)})(| z_1 |^2 + \cdots + | z_n |^2) \\
&\qquad + | z_{n+1} |^{2l}\}t + \text{higher order term of } t \\
&\quad \equiv t \mod t^2
\end{aligned}$$

By adjusting higher order term, we have a C^∞ trivialization $i_t : X \to X_t$ over a neighborhood of M. However, in this paper, we discuss only differential forms of type $(n-1, 1)$. So the above map is enough.

§3. An approach to the First Step

In this section, we give a non-trivial holomorphic $(n, 0)$ form on $X_t \cap$ (a neighborhood of M in C^{n+1}), which depends on t, complex analytically. Let $f = z_1^2 + \cdots + z_n^2 + z_{n+1}^{l+1}$. Like in [2], we, first, set a type $(1,0)$ vector field Z_f, defined on a neighborhood of M in the C^{n+1}, as follows. Let Ω be the standard symplectic form.

$$\Omega = \sum_{i=1}^{n+1} \sqrt{-1} dz_i \wedge d\overline{z}_i.$$

By using this metric, we define a $(1,0)$ vector field Z_f on a neighborhood of M by;

$$df(X) = \Omega(X, \overline{Z}_f), \quad \text{for all } (1,0) \text{ vector field } X.$$

This Z_f is easily written down as follows.

$$\begin{aligned} Z_f &= \sqrt{-1} \sum_{i=1}^{n+1} \overline{(\frac{\partial f}{\partial z_i})} \frac{\partial}{\partial z_i} \\ &= \sqrt{-1} \{ \sum_{i=1}^{n} 2\overline{z}_i \frac{\partial}{\partial z_i} + (l+1) \overline{z}_{n+1}^l \frac{\partial}{\partial z_{n+1}} \}. \end{aligned}$$

So,

$$\begin{aligned} Z_f(f) &= \sqrt{-1}(2^2 \sum_{i=1}^{n} \mid z_i \mid^2 + (l+1)^2 \mid z_{n+1} \mid^{2l}) \\ &\neq 0 \quad \text{on a neighborhood of } M. \end{aligned}$$

Let $\omega = dz_1 \wedge \cdots \wedge dz_{n+1}$. For X_t, we set a holomorphic $(n,0)$ form $\omega'(t)$, which depends on t, complex analytically by ;

$$\omega'(t) = Z_f \rfloor \omega \quad \text{on} \quad X_t \text{ (inner product with vector field } Z_f).$$

And set

$$\omega_t' = \frac{1}{\sum_{i=1}^n 2^2 \mid z_i \mid^2 + (l+1)^2 \mid z_{n+1} \mid^{2l}} \omega'(t).$$

By the type of ω, our ω_t' is of type $(n,0)$ on X_t. We must show that our ω_t' is holomorphic on X_t. For this, we recall the following lemma.

Lemma 3.1. *$\omega = -\sqrt{-1} df \wedge \omega_t'$ on a neighborhood of M.*

We sketch the proof of this lemma. For a point p of a neighborhood of M in C^{n+1}, $T_p' C^{n+1}$ is spanned by Z_f and $\{X_i(p)\}_{1 \leq i \leq n}$, which satisfy $X_i(p) f = 0$. So, with these vector fields, just by a direct computation, we have our lemma.

By this lemma, on X_t,

$$d\omega'_t = 0.$$

We have to see that our ω'_o is not a d-exactn on $X_o = X$. But if we restric ω_t to

$$\{(z_1, \cdots, z_n, z_{n+1}) : z_1^2 + \cdots + z_n^2 + z_{n+1}^{l+1} = 0, z_{n+1} = 0\}$$

a complex $n-1$ dimensional A_1 singularity, then it gives a non-trivial $n-1$ dimensional cohomology(by the definition of our ω'_t, it coincides with nontrivial element, constructed in [2]). So, we have a non trivial form.

§4. An approach to the Third Step

By the C^∞ trivialization of the simultaneous deformations, i_t, constructed in Section 2, on a tubular neighborhood of M,

$$i_t^*\omega_t = \omega_0 + \omega_1 t + \cdots, \quad \text{(expansion with respect to } t\text{)}.$$

We explain a difficulty about this part. For example, we take A_1 singularity (in our notations, $l = 1$). Then, in the C^∞ isomorphism map, i_t, as a denominator, $k(z, \overline{z})$ appears. Only on the boundary case(CR case)

$$k(z, \overline{z}) = 1 \text{ on the boundary.}$$

But we are treating the tubular neighborhood case. So, it is not so valid that there is no extra non-trivial $(n, 0)$ term of ω_1 (we write it by $\omega_1^{(n,0)}$). Fortunately, for the case $l = 1$ (the case of an ordinary double point), $(n, 0)$ term doesn't appear(this means that it is not necessary to change the C^∞ trivialization i_t, constructed in Section 2). So, in this case, $d\omega_1 = 0$ means that; $\partial\omega_1 = 0$ and $\overline{\partial}\omega_1 = 0$. For the other l, we have to control the difficulty which arises from the term $k(z, \overline{})$. In another paper, we discuss the other case.

For the case $l = 1$, the C^∞ isomorphism map is as follows.

$$z_i(t) = z_i + \frac{1}{2\sum_{i=1}^{n+1} | z_i |^2}\overline{z}_i t, \quad i = 1, \ldots, n+1.$$

And

$$Z_f = 2(\sum_{i=1}^{n+1} \overline{z}_i \frac{\partial}{\partial z_i}).$$

In order to simplify the sketch, we assume $n = 2$. Then,

$$Z_f = 2(\overline{z}_1 \frac{\partial}{\partial z_1} + \overline{z}_2 \frac{\partial}{\partial z_2} + \overline{z}_3 \frac{\partial}{\partial z_3})$$

And so,

$$Z_f \rfloor \omega = 2(\overline{z}_1 dz_2 \wedge dz_3 - \overline{z}_2 dz_1 \wedge dz_3 + \overline{z}_3 dz_1 \wedge dz_2),$$

$$\begin{aligned} Z_f(f) &= 4(|z_1|^2 + |z_2|^2 + |z_3|^2) \\ &= 4r^2. \end{aligned}$$

Here $r^2 = |z_1|^2 + |z_2|^2 + |z_3|^2$. And

$$\begin{aligned} z_1(t) &= z_1 + \frac{1}{2}\frac{1}{r^2}\overline{z}_1 t, \\ z_2(t) &= z_2 + \frac{1}{2}\frac{1}{r^2}\overline{z}_2 t, \\ z_3(t) &= z_3 + \frac{1}{2}\frac{1}{r^2}\overline{z}_3 t. \end{aligned}$$

Now we compute ω_1.

$$\begin{aligned} i_t^*(\frac{1}{4r^2} Z_f \rfloor \omega) &= \frac{1}{2} i_t^*(\frac{1}{r^2}(\overline{z}_1 dz_2 \wedge dz_3 - \overline{z}_2 dz_1 \wedge dz_3 + \overline{z}_3 dz_1 \wedge dz_2)) \\ &= \frac{1}{2}(\frac{\overline{z}_1(t) dz_2(t) \wedge dz_3(t) - \overline{z}_2(t) dz_1(t) \wedge dz_3(t) + \overline{z}_3(t) dz_1(t) \wedge dz_2(t))}{z_1(t)\overline{z}_1(t) + z_2(t)\overline{z}_2(t) + z_3(t)\overline{z}_3(t)}) \\ &\equiv \frac{1}{2}(\frac{\overline{z}_1 dz_2(t) \wedge dz_3(t) - \overline{z}_2 dz_1(t) \wedge dz_3(t) + \overline{z}_3 dz_1(t) \wedge dz_2(t))}{z_1(t)\overline{z}_1 + z_2(t)\overline{z}_2 + z_3(t)\overline{z}_3}) \mod(t^2, \overline{t}) \\ &= \frac{1}{2}(\frac{\overline{z}_1 dz_2(t) \wedge dz_3(t) - \overline{z}_2 dz_1(t) \wedge dz_3(t) + \overline{z}_3 dz_1(t) \wedge dz_2(t))}{z_1\overline{z}_1 + z_2\overline{z}_2 + z_3\overline{z}_3}) \\ &\quad \text{because of } z_1^2 + z_2^2 + z_3^2 = 0. \end{aligned}$$

While

$$\begin{aligned} \overline{z}_1 dz_2(t) \wedge dz_3(t) &= \overline{z}_1(dz_2 + \frac{1}{2}(d(\frac{1}{r^2}))\overline{z}_2 t + \frac{1}{2}\frac{1}{r^2} d\overline{z}_2 t) \wedge (dz_3 + \frac{1}{2}(d(\frac{1}{r^2}))\overline{z}_3 t + \frac{1}{2}\frac{1}{r^2} d\overline{z}_3 t) \\ &\equiv \overline{z}_1 dz_2 \wedge dz_3 + \{\overline{z}_1 \frac{1}{2}(d(\frac{1}{r^2}))\overline{z}_2 \wedge dz_3 + \overline{z}_1 \frac{1}{2}\frac{1}{r^2} d\overline{z}_2 \wedge dz_3 \\ &\quad + \overline{z}_1 dz_2 \wedge \frac{1}{2}(d(\frac{1}{r^2}))\overline{z}_3 + \overline{z}_1 dz_2 \frac{1}{2}\frac{1}{r^2} d\overline{z}_3\} t \mod t^2. \end{aligned}$$

Therefore from this term, $(2,0)$ part is

$$\frac{1}{2}\overline{z}_1\overline{z}_2 \partial(\frac{1}{r^2}) \wedge dz_3 + \frac{1}{2}\overline{z}_1\overline{z}_3 dz_2 \wedge \partial(\frac{1}{r^2}).$$

By the same way, from $-\overline{z}_2 dz_1(t) \wedge dz_3(t)$, as a $(2,0)$ part,

$$-\frac{1}{2}\overline{z}_1\overline{z}_2\partial(\frac{1}{r^2}) \wedge dz_3 - \frac{1}{2}\overline{z}_2\overline{z}_3 dz_1 \wedge \partial(\frac{1}{r^2}).$$

And from $\overline{z}_3 dz_1(t) \wedge dz_2(t)$, $(2,0)$ part is

$$\frac{1}{2}\overline{z}_1\overline{z}_3\partial(\frac{1}{r^2}) \wedge dz_2 + \frac{1}{2}\overline{z}_2\overline{z}_3 dz_1 \wedge \partial(\frac{1}{r^2}).$$

So summing up these three terms, in this case, we see that $(2,0)$ part does not appear.

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