Advanced Studies in Pure Mathematics 41, 2004
Stochastic Analysis and Related Topics
pp. 333–350

Criticality of Generalized Schrödinger Operators and Differentiability of Spectral Functions

Masayoshi Takeda and Kaneharu Tsuchida

Abstract.

Let μ be a positive Radon measure in the Kato class. We consider the spectral bound $C(\lambda) = -\inf \sigma(\mathcal{H}^{\lambda\mu})$ $(\lambda \in \mathbb{R}^1)$ of a generalized Schrödinger operator $\mathcal{H}^{\lambda\mu} = -\frac{1}{2}\Delta - \lambda\mu$ on $\mathbb{R}^d$, and show that the spectral bound is differentiable if $d \leq 4$ and μ is Green-tight.

§1. Introduction

Let $(\mathbf{D}, H^1(\mathbb{R}^d))$ be the classical Dirichlet integral and μ a positive Radon measure in the Kato class. For a Schrödinger operator $\mathcal{H}^{\lambda\mu} = -\frac{1}{2}\Delta - \lambda\mu$, $\lambda \in \mathbb{R}^1$, define *the spectral function* $C(\lambda)$ by

$$\begin{aligned} C(\lambda) &= -\inf\{\theta : \theta \in \sigma(\mathcal{H}^{\lambda\mu})\} \\ &= -\inf\left\{\frac{1}{2}\mathbf{D}(u,u) - \lambda\int_{\mathbb{R}^d} \tilde{u}^2 d\mu : u \in H^1(\mathbb{R}^d),\ \int_{\mathbb{R}^d} u^2 dx = 1\right\}, \end{aligned}$$

where $\sigma(\mathcal{H}^{\lambda\mu})$ is the set of the spectrum of $\mathcal{H}^{\lambda\mu}$ and $\tilde{u}$ is a quasi-continuous version of u. In this paper, we study the differentiability of the function $C(\lambda)$.

When the potential μ is a function in a certain Kato class, Arendt and Batty [3] proved that the spectral function is differentiable at $\lambda = 0$ and its derivative equals to zero ([3, Corollary 2.10]). Using a large deviation principle for additive functionals of the Brownian motion, Wu [27] obtained a necessary and sufficient condition for the spectral function being differentiable at 0. In [24] one of the authors extended Wu's result to measures which may be singular with respect to the Lebesgue measure. Furthermore, one of the authors showed that if $d \leq 2$ and the measure μ is Green-tight (in notation, $\mu \in \mathcal{K}_d^\infty$), the spectral function is differentiable on $\mathbb{R}^1$. Here the class $\mathcal{K}_d^\infty$ was introduced in Zhao [29](see

Received February 12, 2003.

Definition 2.1 (II) below). A main objective of this paper is to extend the results in [24] as follows:

Theorem 1.1. *If* $d \le 4$ *and* $\mu \in \mathcal{K}_d^\infty$, *then the spectral function* $C(\lambda)$ *is differentiable for all* $\lambda \in \mathbb{R}^1$.

Define $\lambda^+ = \inf\{\lambda > 0 : C(\lambda) > 0\}$. We then see that $\lambda^+ = 0$ for $d \le 2$ and $\lambda^+ > 0$ for $d \ge 3$ and the proof of Theorem 1.1 is reduced to the proof of the differentiability of $C(\lambda)$ at $\lambda = \lambda^+$. In [24], the differentiablity at $\lambda = 0$ is derived from the fact that for $d \le 2$ the Brownian motion is a Harris recurrent process with infinite invariant measure, the Lebesgue measure. We will extend this method for $d = 3,\ 4$ by applying the criticality theory of Schrödinger operators.

We first extend the criticality theory to the generalized Schrödinger operator $\mathcal{H}^\mu$; we show in Corollary 3.5 below that if $d \ge 3$, then the operator $\mathcal{H}^{\lambda^+\mu}$ is *critical*, that is, $\mathcal{H}^{\lambda^+\mu}$ does not admit the minimal positive Green function but admits a positive continuous $\mathcal{H}^{\lambda^+\mu}$-harmonic function. This harmonic function is called a *ground state*, which is uniquely determined up to constant multiplication. Moreover, if $d = 3,\ 4$, $\mathcal{H}^{\lambda^+\mu}$ is *null critical*, that is, the ground state does not belong to L^2. In fact, denoting by h the ground state, we prove in section 5 that $h(x)$ is equivalent to the Green function $G(0, x)$ of the Brownian motion on a neighbourhood of the infinity; there exist positive constants c, C such that

$$\frac{c}{|x|^{d-2}} \le h(x) \le \frac{C}{|x|^{d-2}}, \quad |x| > 1. \tag{1}$$

The criticality and the null criticality are regarded as extended notions of recurrence and null recurrence respectively. Using these facts, we see that if $d = 3,\ 4$, the h-transformed process generated by the Markov semigroup

$$P_t^{\lambda^+\mu,h} f(x) = \frac{1}{h(x)} \exp(-t\mathcal{H}^{\lambda^+\mu})(hf)(x)$$

becomes a Harris recurrent Markov process with infinite invariant measure h^2dx. Furthermore, through the h-transformation a functional inequality for the *critical Schrödinger form* is derived (Theorem 4.4) ; the inequality is an extenstion of Oshima's inequality ([11]) which holds for the Dirichlet forms generated by symmetric Harris recurrent Markov processes. We now obtain Theorem 1.1 by applying the argument in [24] to the transformed process. This is a key idea of the proof of Theorem 1.1. The equation (1) tells us that if $d \ge 5$, $\mathcal{H}^{\lambda^+\mu}$ becomes *positive critical*, that is, the ground state belongs to L^2. Thus we can not use our method and have not known yet whether $C(\lambda)$ is differentable or not.

The criticality of Schrödinger operators is studied by many people (M. Murata, Y. Pinchover, R. Pinsky,...). In particular, the equation (1) was shown by Murata [10] for classical Schrödinger operators on $\mathbb{R}^d$ and extended by Pinchover [12] to second order elliptic operators in a domain of $\mathbb{R}^d$.

Our motivation lies in the proof of the large deviation principle for continuous additive functional A_t^μ in the Revuz correspondence with μ. The function $C(\lambda)$ is regarded as a *logarithmic moment generating function* of the additive functional A^μ (see [21]), and the differentiability of logarithmic moment generating functions play a crucial role in the Gärtner-Ellis Theorem (see [7]). In fact, using Theorem 1.1, we can show the large deviation principle for additive functional A_t^μ associated with $\mu \in \mathcal{K}_d^\infty$.

§2. Preliminaries

Let $\mathbb{W} = (P_x, B_t)$ be a Brownian motion on $\mathbb{R}^d$ $(d \geq 3)$. Let $p(t,x,y)$ be the transition density function of $\mathbb{W}$ and $G(x,y)$ its Green function, $G(x,y) = C(d)|x-y|^{2-d}$, where $C(d) = (2\pi)^{-1}\Gamma(\frac{d}{2}-1)$. For a measure μ, the 0-potential of μ is defined by $G\mu(x) = \int_{\mathbb{R}^d} G(x,y)\mu(dy)$. Let P_t be the semigroup of $\mathbb{W}$, $P_t f(x) = \int_{\mathbb{R}^d} p(t,x,y)f(y)dy = E_x[f(B_t)]$. The Dirichlet form of $\mathbb{W}$ is given by $(1/2\mathbf{D}, H^1(\mathbb{R}^d))$ where $\mathbf{D}$ denotes the classical Dirichlet integral and $H^1(\mathbb{R}^d)$ is the Sobolev space of order 1 ([8, Example 4.4.1]). Let $(1/2\mathbf{D}, H_e^1(\mathbb{R}^d))$ denote the extended Dirichlet form of $(1/2\mathbf{D}, H^1(\mathbb{R}^d))$ ([8, p.36]). Note that $H_e^1(\mathbb{R}^d)$ is a Hilbert space with inner product $\mathbf{D}$ because $\mathbb{W}$ is transient ([8, Theorem 1.5.3]). Let $G_\alpha(x,y)$ be the α-resolvent kernel of $\mathbb{W}$.

Throughout this paper, the Lebesgue measure is denoted by m and $m(dx)$ is abbriviated to dx. For $r > 0$, we denote by $B(r)$ an open ball with radius R centered at the origin. We use $c, C, ..., etc$ as positive constants which may be different at different occurrences. We now define classes of measures which play an important role in this paper.

Definition 2.1. (I) *A positive Radon measure μ on $\mathbb{R}^d$ is said to be in the Kato class ($\mu \in \mathcal{K}_d$ in notation), if*

$$\lim_{a\to 0} \sup_{x\in\mathbb{R}^d} \int_{|x-y|\leq a} G(x,y)\mu(dy) = 0. \tag{2}$$

(II) *A measure μ is in $\mathcal{K}_d^\infty$ if μ is in $\mathcal{K}_d$ and satisfies*

$$\lim_{R\to\infty}\sup_{x\in\mathbb{R}^d}\int_{|y|>R} G(x,y)\mu(dy)=0. \tag{3}$$

For $\mu\in\mathcal{K}_d$, define a symmetric bilinear form $\mathcal{E}^\mu$ by

$$\mathcal{E}^\mu(u,u)=\frac{1}{2}\mathbf{D}(u,u)-\int_{\mathbb{R}^d}\widetilde{u}^2 d\mu,\quad u\in H^1(\mathbb{R}^d), \tag{4}$$

where $\widetilde{u}$ is a quasi continuous version of u ([8, Theorem 2.1.3]). In the sequel, we always assume that every function $u\in H_e^1(\mathbb{R}^d)$ is represented by its quasi continuous version. Since $\mu\in\mathcal{K}_d$ charges no set of zero capacity by [2, Theorem 3.3], the form $\mathcal{E}^\mu$ is well defined. We see from [2, Theorem 4.1] that $(\mathcal{E}^\mu, H^1(\mathbb{R}^d))$ becomes a lower semi-bounded closed symmetric form. We call $(\mathcal{E}^\mu, H^1(\mathbb{R}^d))$ a *Schrödinger form.* Denote by $\mathcal{H}^\mu$ the self-adjoint operator generated by $(\mathcal{E}^\mu, H^1(\mathbb{R}^d))$: $\mathcal{E}^\mu(u,v)=(\mathcal{H}^\mu u,v)$. Let P_t^μ be the L^2-semigroup generated by $\mathcal{H}^\mu$: $P_t^\mu=\exp(-t\mathcal{H}^\mu)$. We see from [2, Theorem 6.3(iv)] that P_t^μ admits a symmetric integral kernel $p^\mu(t,x,y)$ which is jointly continuous on $(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$.

For $\mu\in\mathcal{K}_d$, A_t^μ denotes a positive continuous additive functional which is in the *Revuz correspondence* with μ: for any positive Borel function f and γ-excessive function h,

$$< h\mu, f >=\lim_{t\to 0}\frac{1}{t}E_{hm}\left[\int_0^t f(B_s)dA_s^\mu\right], \tag{5}$$

([8, p.188]). By the Feynman-Kac formula, the semigroup P_t^μ is written as

$$P_t^\mu f(x)=E_x[\exp(A_t^\mu)f(B_t)]. \tag{6}$$

§3. Criticality and ground state

Definition 3.1. *A real-valued function h is said to be harmonic on a domain D with respect to $\mathcal{H}^\mu$ if for any relatively compact open set $G\subset\bar{G}\subset D$,*

$$h(x)=E_x[\exp(A_{\tau_G}^\mu)h(B_{\tau_G})],\quad x\in G, \tag{7}$$

where τ_G is the first exit time from G, $\tau_G=\inf\{t>0: B_t\notin G\}$.

We formally write a $\mathcal{H}^\mu$-harmonic function h as $\mathcal{H}^\mu h=0$. An operator $\mathcal{H}^\mu$ is said to be *subcritical* if $\mathcal{H}^\mu$ possesses the minimal positive

Green function $G^{\mu}(x,y)$, that is,

$$G^{\mu}(x,y)=\int_0^{\infty}p^{\mu}(t,x,y)dt<\infty,\quad x\neq y.$$

The operator $\mathcal{H}^{\mu}$ is said to be *critical* if $G^{\mu}(x,y)=\infty$ and a positive continuous $\mathcal{H}^{\mu}$-harmonic function exists. If the operator $\mathcal{H}^{\mu}$ is neither subcritical nor critical, it is said to be *supercritical* (see [13, p.145]).

The *spectral function* $C(\lambda)$ is defined by the bottom of the spectrum of $\mathcal{H}^{\lambda\mu}$: for $\mu\in\mathcal{K}_d^{\infty}$,

$$C(\lambda)=-\inf\left\{\mathcal{E}^{\lambda\mu}(u,u)\ ;\ u\in H^1(\mathbb{R}^d),\int_{\mathbb{R}^d}u^2dx=1\right\}. \tag{8}$$

Define

$$\lambda^{+}\ =\ \inf\{\lambda>0:C(\lambda)>0\}.$$

We then see that $C(\lambda)=0$ for $\lambda\leq\lambda^{+}$ ([23]).

Lemma 3.1. *For $\mu\in\mathcal{K}_d^{\infty}$, there exists a positive continuous function such that $\mathcal{H}^{\lambda^{+}\mu}h=0$.*

Proof. Let λ_n be the bottom of spectrum of $\mathcal{H}^{\lambda^{+}\mu}$ for the Dirichlet problem on $B(n)$. Since $0=-C(\lambda^{+})<\lambda_{n+1}<\lambda_n$, $\mathcal{H}^{\lambda^{+}\mu}$ is subcritical on $B(n)$. Let G^n denotes the Green operator of $\mathcal{H}^{\lambda^{+}\mu}$ on $B(n)$. We define a function h_n by $h_n(x)=c_nG^{n+1}I_{A_n}(x)$, where I_{A_n} is the indicator function of $A_n(=B(n+1)\setminus B(n))$ and c_n is the normalized constant, $c_n=(G^{n+1}I_{A_n}(0))^{-1}$. Then h_n is a harmonic function on $B(m)$, $m<n$. Indeed, for $x\in B(m)$

$$\begin{aligned}&E_x[\exp(\lambda^{+}A^{\mu}_{\tau_m})h_n(B_{\tau_m})]=c_nE_x[\exp(\lambda^{+}A^{\mu}_{\tau_m})G^{n+1}I_{A_n}(B_{\tau_m})]\\&=c_nE_x\left[\exp(\lambda^{+}A^{\mu}_{\tau_m})E_{B_{\tau_m}}\left[\int_0^{\tau_{n+1}}\exp(\lambda^{+}A^{\mu}_t)I_{A_n}(B_t)dt\right]\right],\end{aligned}$$

where $\tau_m=\inf\{t>0:B_t\notin B(m)\}$. By the strong Markov property, the right hand side is equal to

$$\begin{aligned}&c_nE_x\left[\int_0^{\tau_{n+1}\circ\theta_{\tau_m}}\exp(\lambda^{+}(A^{\mu}_{\tau_m}+A^{\mu}_t\circ\theta_{\tau_m})I_{A_n}(B_{t+\tau_m})dt\right]\\&=\ c_nE_x\left[\int_{\tau_m}^{\tau_{n+1}\circ\theta_{\tau_m}+\tau_m}\exp(\lambda^{+}A^{\mu}_t)I_{A_n}(B_t)dt\right].\end{aligned}$$

Noting that $\tau_{n+1}\circ\theta_{\tau_m}+\tau_m=\tau_{n+1}$ and $\int_0^{\tau_m}\exp(\lambda^{+}A^{\mu}_t)I_{A_n}(B_t)dt=0$, we see that the last term is equal to $h_n(x)$. Therefore h_n satisfies (7) for $G=B(m)$.

Now by [4, Corollary 7.8], $\{h_n\}$ is uniformly bounded and equicontinuous on $B(1)$, so we can choose a subsequence of $\{h_n\}$ which converges uniformly on $B(1)$. We denote the subsequence by $\{h_n^{(1)}\}$. Next take a subsequence $\{h_n^{(2)}\}$ of $\{h_n^{(1)}\}$ so that it converges uniformly on $B(2)$. By the same procedure, we take a subsequence $\{h_n^{(m+1)}\}$ of $\{h_n^{(m)}\}$ so that it converges uniformly on $B(m+1)$. Then the function, $h(x) = \lim_{n\to\infty} h_n^{(n)}(x)$, is a desired one. Q.E.D.

Lemma 3.2. *The following statements are equivalent:*

(i) $\inf\left\{\frac{1}{2}\mathbf{D}(u,u) : u \in H^1(\mathbb{R}^d),\ \int_{\mathbb{R}^d} u^2 d\mu = 1\right\} < 1;$

(ii) $\inf\left\{\mathcal{E}^\mu(u,u) : u \in H^1(\mathbb{R}^d),\ \int_{\mathbb{R}^d} u^2 dx = 1\right\} < 0.$

Proof. Assume (i). Then there exists a $\varphi_0 \in C_0^\infty(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} \varphi_0^2 d\mu = 1$ and $1/2\mathbf{D}(\varphi_0,\varphi_0) < 1$. Letting $u_0 = \varphi_0/\sqrt{\int_{\mathbb{R}^d}\varphi_0^2 dx}$, we have $\mathcal{E}^\mu(u_0,u_0) < 0$.
(ii) $\Longrightarrow$ (i) follows similarly. Q.E.D.

Remark 3.3. We see from [25, Lemma 3.5] that if

$$\inf\left\{\frac{1}{2}\mathbf{D}(u,u) : \int_{\mathbb{R}^d} u^2 d\mu = 1\right\} \leq 1,$$

then

$$\inf\left\{\mathcal{E}^\mu(u,u) : \int_{\mathbb{R}^d} u^2 dx = 1\right\} \leq 0.$$

However, the converse does not hold in general. Indeed, let $\mu = \sigma_R$, the surface measure of the sphere $\partial B(R)$. Then if $R < \frac{d-2}{2}$, the first infimum is greater than 1, while the second infimum is equal to 0 ([25]).

Lemma 3.4. *Let $\mu \in \mathcal{K}_d^\infty$. Then the number λ^+ is characterized as a unique positive number such that*

$$\inf\left\{\frac{1}{2}\mathbf{D}(u,u) : \lambda^+ \int_{\mathbb{R}^d} u^2 d\mu = 1\right\} = 1. \tag{9}$$

Proof. Define

$$F(\lambda) = \inf\left\{\frac{1}{2}\mathbf{D}(u,u) : \lambda \int_{\mathbb{R}^d} u^2(x)\mu(dx) = 1\right\},$$

Note that $F(\lambda) = F(1)/\lambda$. Then $F(1)$ is nothing but the bottom of spectrum of the time changed process by the additive functional A_t^μ ([22, Lemma 3.1]). We see by [23, Lemma 3.1] that 1-resolvent R_1^μ of the time changed process satisfies $R_1^\mu 1 \in C_\infty(\mathbb{R}^d)$. Hence it follows from [17, Corollary 3.2] and [23, Corollary 2.2] that $F(1) > 0$. Consequently we see that $\lambda^0 = F(1)$ is a unique positive constant such that $F(\lambda^0) = 1$. Lemma 3.2 leads us that $\lambda^0 = \lambda^+$. Q.E.D.

Corollary 3.5. *For $\mu \in \mathcal{K}_d^\infty$, the operator $\mathcal{H}^{\lambda^+\mu}$ is critical.*

Proof. Let $F(\lambda)$ be the function in the proof of Lemma 3.4. Then it is known in [25, Theorem 3.9] that the operator $\mathcal{H}^{\lambda\mu}$ is subcritical if and only if $F(\lambda) > 1$. Hence by Lemma 3.1 and Lemma 3.4, $\mathcal{H}^{\lambda^+\mu}$ is critical. Q.E.D.

Lemma 3.6. *A positive $\mathcal{H}^{\lambda^+\mu}$-harmonic function h satisfies $P_t^{\lambda^+\mu}h(x) \le h(x)$.*

Proof. Let $x \in B(m)$. By Definition 3.1, h satisfies

$$h(x) = E_x[\exp(\lambda^+ A^\mu_{\tau_n})h(B_{\tau_n})]$$

for any $n > m$. Here τ_n is the first exit time from $B(n)$. It follows from the Markov property that

$$\begin{aligned} &E_x[\exp(\lambda^+ A_t^\mu)h(B_t); t < \tau_m] \\ &= E_x[\exp(\lambda^+ A_t^\mu)\exp(\lambda^+ A^\mu_{\tau_n}\circ\theta_t)h(B_{\tau_n}\circ\theta_t); t < \tau_m] \\ &= E_x[\exp(\lambda^+ A^\mu_{\tau_n})h(B_{\tau_n}); t < \tau_m] \le h(x). \end{aligned}$$

Hence we have

$$P_t^{\lambda^+\mu}h(x) = \lim_{m\to\infty} E_x[\exp(\lambda^+ A_t^\mu)h(B_t); t < \tau_m] \le h(x).$$

Q.E.D.

Let P_t be a positive semigroup with integral kernel $p(t,x,y)$. A positive function h is called *P_t-excessive* if h satisfies $P_t h(x) \le h(x)$. For a P_t-excessive function $h(x)$, the h-transformed semigroup P_t^h is defined by

$$(10) \qquad P_t^h f(x) = \int_{\mathbb{R}^d} \frac{1}{h(x)} p(t,x,y)h(y)f(y)dy, \quad t > 0,\ x,y \in \mathbb{R}^d.$$

Then P_t^h becomes a Markovian semigroup.

Let h be the function defined in Lemma 3.1. We see from Lemma 3.6 that the h-transformed semigroup $P_t^{\lambda^+\mu,h}$ generates a h^2m-symmetric Markov process $\mathbb{W}^{\lambda^+\mu,h} = (P_x^{\lambda^+\mu,h}, X_t)$. Note that $\mathbb{W}^{\lambda^+\mu,h}$ is recurrent because of the criticality of $\mathcal{H}^{\lambda^+\mu}$.

Lemma 3.7. *Finely continuous $P_t^{\lambda^+\mu}$-excessive functions are unique up to constant multiplication.*

Proof. We follow the argument in [13, Theorem 4.3.4]. Let h, h' be finely continuous $P_t^{\lambda^+\mu}$-excessive functions. Since

$$E_x\left[\exp(\lambda^+ A_t^\mu)h(B_t)\left(\frac{h'}{h}\right)(B_t)\right] \le h \cdot \frac{h'}{h}(x),$$

we have

$$E_x^{\lambda^+\mu,h}\left[\frac{h'}{h}(X_t)\right] \le \frac{h'}{h}(x),$$

where $E_x^{\lambda^+\mu,h}$ is the expectation of h-transformed process $\mathbb{W}^{\lambda^+\mu,h}$. For $y \in \mathbb{R}^d$ and $\epsilon > 0$, we put $U_\epsilon(y) = \{z : |h(z) - h(y)| < \epsilon\}$. Since $U_\epsilon(y)$ is finely open, $\sigma_{U_\epsilon(y)} < \infty$, $P_x^{\lambda^+\mu,h}$-a.s [8, Problem 4.6.3]. Replacing t by σ_ϵ, we have

$$E_x^{\lambda^+\mu,h}\left[\frac{h'}{h}(X_{\sigma_\epsilon})\right] \le \frac{h'}{h}(x). \tag{11}$$

Note that the left hand side of (11) converges to $\frac{h'}{h}(y)$ as $\epsilon \to 0$. We then have

$$\begin{aligned}\frac{h'}{h}(y) &= E_x^{\lambda^+\mu,h}\left[\liminf_{\epsilon\to 0}\frac{h'}{h}(X_{\sigma_\epsilon})\right] \le \liminf_{\epsilon\to 0} E_x^{\lambda^+\mu,h}\left[\frac{h'}{h}(X_{\sigma_\epsilon})\right] \\ &\le \frac{h'}{h}(x).\end{aligned}$$

Since x and y are arbitrary, h'/h is a constant function. Q.E.D.

Now we give known facts on the Kato class.

Theorem 3.8 ([20]). *Let $\mu \in \mathcal{K}_d$. Then for any $u \in H^1(\mathbb{R}^d)$*

$$\int_{\mathbb{R}^d} u^2(x)\mu(dx) \le \|G_\alpha\mu\|_\infty\left(\mathbf{D}(u,u) + \alpha\int_{\mathbb{R}^d} u^2(x)dx\right). \tag{12}$$

It is known from [1] (also see [28]) that $\mu \in \mathcal{K}_d$ if and only if

$$\lim_{\alpha\to\infty}\|G_\alpha\mu\|_\infty = 0. \tag{13}$$

Therefore we see that for any ϵ there exists a constant $M(\epsilon)$ such that for any $u \in H^1(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} u^2(x)\mu(dx) \le \epsilon\mathbf{D}(u,u) + M(\epsilon)\int_{\mathbb{R}^d} u^2(x)dx. \tag{14}$$

For a measure μ, let $\mu_R(\cdot) = \mu(\cdot \cap B(R))$ and $\mu_{R^c} = \mu(\cdot \cap B(R)^c)$.

Lemma 3.9. *If $\mu \in \mathcal{K}_d^\infty$, then the embedding of $H_e^1(\mathbb{R}^d)$ to $L^2(\mu)$ is compact.*

Proof. Let $\{u_n\}$ be a sequence in $H_e^1(\mathbb{R}^d)$ such that

$$u_n \to u_0 \in H_e^1(\mathbb{R}^d), \ \mathbf{D}\text{-weakly}.$$

Rellich's theorem says that for any compact set $K \subset \mathbb{R}^d$

$$u_n I_K \to u_0 I_K \quad L^2(m)\text{-strongly}. \tag{15}$$

Now, for $\varphi \in C_0^\infty(\mathbb{R}^d)$ with $\varphi = 1$ on $B(R)$

$$\int_{\mathbb{R}^d} |u_n - u_0|^2 \mu_R(dx) = \int_{\mathbb{R}^d} |u_n\varphi - u_0\varphi|^2 \mu_R(dx)$$
$$\leq \epsilon \mathbf{D}(u_n\varphi - u_0\varphi, u_n\varphi - u_0\varphi) + M(\epsilon) \int_{\mathbb{R}^d} |u_n\varphi - u_0\varphi|^2 dx$$

by (14), and the second term converges to 0 as $n \to \infty$ by (15). Since

$$\sup_n \mathbf{D}(u_n\varphi - u_0\varphi, u_n\varphi - u_0\varphi) < \infty$$

by the principle of uniform boundedness and ϵ is arbitrary, u_n converges to u_0 in $L^2(\mu_R)$. Moreover, since by Theorem 3.8,

$$\int_{\mathbb{R}^d} |u_n - u_0|^2 \mu(dx) = \int_{\mathbb{R}^d} |u_n - u_0|^2 \mu_R(dx) + \int_{\mathbb{R}^d} |u_n - u_0|^2 \mu_{R^c}(dx)$$
$$\leq \int_{\mathbb{R}^d} |u_n - u_0|^2 \mu_R(dx) + \|G\mu_{R^c}\|_\infty \mathbf{D}(u_n - u_0, u_n - u_0),$$

$$\limsup_{n\to\infty} \int_{\mathbb{R}^d} |u_n - u_0|^2 \mu(dx) \leq \|G\mu_{R^c}\|_\infty \sup_n \mathbf{D}(u_n - u_0, u_n - u_0).$$

Hence according to the definition of $\mathcal{K}_d^\infty$ the right hand side converges to 0 by letting R to ∞. Therefore $\{u_n\}$ is an $L^2(\mu)$-convergent sequence. Q.E.D.

Assume that $\mathcal{H}^\mu$ is subcritical or critical. Let h be a positive $\mathcal{H}^\mu$-harmonic function. We denote by $\mathcal{D}_e(\mathcal{E}^\mu)$ the family of m-measurable function u on $\mathbb{R}^d$ such that $|u| < \infty$ m-a.e. and there exists an $\mathcal{E}^\mu$-Cauchy sequence $\{u_n\}$ of functions in $H^1(\mathbb{R}^d)$ such that $\lim_{n\to\infty} u_n = u$ m-a.e. We call $\{u_n\}$ as above an *approximating sequence* for $u \in \mathcal{D}_e(\mathcal{E}^\mu)$.

Note that the Dirichlet form $(\mathcal{E}^{\mu,h}, \mathcal{D}(\mathcal{E}^{\mu,h}))$ associated with the Markov semigroup $P_t^{\mu,h}$ is given by

$$\begin{aligned}\mathcal{E}^{\mu,h}(u,v) &= \mathcal{E}^{\mu}(hu,hv)\\ \mathcal{D}(\mathcal{E}^{\mu,h}) &= \left\{u \in L^2(\mathbb{R}^d; h^2dx) : \; hu \in \mathcal{D}(\mathcal{E}^{\mu})\right\}.\end{aligned}$$

Then we see that $u \in \mathcal{D}_e(\mathcal{E}^{\mu})$ if and only if $u/h \in \mathcal{D}_e(\mathcal{E}^{\mu,h})$, where $\mathcal{D}_e(\mathcal{E}^{\mu,h})$ is the entended Dirichlet space of $(\mathcal{E}^{\mu,h}, \mathcal{D}(\mathcal{E}^{\mu,h}))$. Consequently, the Schrödinger form $\mathcal{E}^{\mu}$ can be well extended to $\mathcal{D}_e(\mathcal{E}^{\mu})$ as a symmetric form: for $u \in \mathcal{D}_e(\mathcal{E}^{\mu})$ and its approximating sequence $\{u_n\}$

$$\mathcal{E}^{\mu}(u,u) = \lim_{n\to\infty} \mathcal{E}^{\mu}(u_n,u_n), \qquad u \in \mathcal{D}_e(\mathcal{E}^{\mu}) \tag{16}$$

(see [8, p.35]). We call $(\mathcal{E}^{\mu}, \mathcal{D}_e(\mathcal{E}^{\mu}))$ the *extended Schrödinger form.* We see from [18, Definition 1.6] that a function u belongs to $\mathcal{D}_e(\mathcal{E}^{\mu})$ if there exists a sequence $\{u_n\}$ of functions in $H^1(\mathbb{R}^d)$ such that $\lim_{n\to\infty} u_n = u$ m-a.e. and

$$\sup_n \mathcal{E}^{\mu}(u_n,u_n) < \infty.$$

If $(\mathcal{E}^{\mu}, H^1(\mathbb{R}^d))$ is a subcritical Schrödinger form, that is, the associated operator $\mathcal{H}^{\mu}$ be subcritical, then $(\mathcal{E}^{\mu}, \mathcal{D}_e(\mathcal{E}^{\mu}))$ becomes a Hilbert space by [8, Lemma 1.5.5]. In particular, a positive $\mathcal{H}^{\mu}$-harmonic function h does not belong to $\mathcal{D}_e(\mathcal{E}^{\mu})$. If $(\mathcal{E}^{\mu}, H^1(\mathbb{R}^d))$ is a critical Schrödinger form, that is, the associated operator $\mathcal{H}^{\mu}$ be critical, its ground state h belongs to $\mathcal{D}_e(\mathcal{E}^{\mu})$ on account of [8, Theotem 1.6.3]. Noting that for $\mu \in \mathcal{K}_d^{\infty}$

$$\mathcal{E}^{\mu}(u,u) \le (1/2 + \|G\mu\|_{\infty})\mathbf{D}(u,u)$$

by Theorem 3.8, we see that $\mathcal{D}_e(\mathcal{E}^{\mu})$ includes $H_e^1(\mathbb{R}^d)$.

For $w \ge 0 \in C_0(\mathbb{R}^d)$ define $\nu = \lambda^+\mu - w\cdot m$. We then see that $\mathcal{H}^{\nu}$ is subcritical. Let $G^{\nu}(x,y)$ be the Green function of $\mathcal{H}^{\nu}$ and G^{ν} the Green operator,

$$G^{\nu}f(x) = \int_{\mathbb{R}^d} G^{\nu}(x,y)f(y)dy. \tag{17}$$

By [26, Theorem 3.1], the Green function $G^{\nu}(x,y)$ is equivalent to $G(x,y)$: there exist positive constants c, C such that

$$cG(x,y) \le G^{\nu}(x,y) \le CG(x,y) \quad \text{for } x \ne y. \tag{18}$$

Lemma 3.10. *For a positive function $\varphi \in C_0(\mathbb{R}^d)$, $G^{\nu}\varphi$ belongs to $\mathcal{D}_e(\mathcal{E}^{\nu})$*

Proof. Let G^{ν}_{β} be the β-resolvent associated with $\mathcal{H}^{\nu}$. Then $G^{\nu}_{\beta}\varphi$ belongs to $H^1(\mathbb{R}^d)$ and $G^{\nu}_{\beta}\varphi \to G^{\nu}\varphi$ as $\beta \to 0$. Moreover,

$$\mathcal{E}^{\nu}(G^{\nu}_{\beta}\varphi, G^{\nu}_{\beta}\varphi) \leq \mathcal{E}^{\nu}_{\beta}(G^{\nu}_{\beta}\varphi, G^{\nu}_{\beta}\varphi) = (\varphi, G^{\nu}_{\beta}\varphi) \leq (\varphi, G^{\nu}\varphi)$$

and the right hand side is not greater than $C(\varphi, G\varphi) < \infty$ by (18). Q.E.D.

The next theorem is first obtained by Murata [10, Theorem 2.2] when the potential μ is absolutely continuous with respect to the Lebesgue measure.

Theorem 3.11. *For $w \in C_0(\mathbb{R}^d)$ with $w \geq 0$, $w \not\equiv 0$, let $\nu = \lambda^+\mu - w\cdot m$. The positive continuous $\mathcal{H}^{\lambda^+\mu}$-harmonic function h satisfies*

$$h(x) = \int_{\mathbb{R}^d} G^{\nu}(x,y)h(y)w(y)dy. \tag{19}$$

Proof. Note that by Lemma 3.9 there exists a function $u_0 \in H^1_e(\mathbb{R}^d)$ such that u_0 attains the infimum:

$$\inf\left\{\frac{1}{2}\mathbf{D}(u,u) : u \in H^1_e(\mathbb{R}^d),\ \lambda^+\int_{\mathbb{R}^d} u^2 d\mu = 1\right\} = 1.$$

The function u_0 then satisfies the following equation:

$$\frac{1}{2}\mathbf{D}(u_0, f) = \lambda^+\int_{\mathbb{R}^d} u_0 f d\mu \quad \text{for all } f \in H^1_e(\mathbb{R}^d),$$

and thus by the definition of ν

$$\mathcal{E}^{\nu}(u_0, f) = \int_{\mathbb{R}^d} u_0 f w dx \quad \text{for all } f \in H^1_e(\mathbb{R}^d).$$

On account of the definition of the extended Schrödinger form, we see that the equation above is extended to any $f \in \mathcal{D}_e(\mathcal{E}^{\nu})$. Since $G^{\nu}\varphi \in \mathcal{D}_e(\mathcal{E}^{\nu})$ for any $\varphi \in C_0(\mathbb{R}^d)$ by Lemma 3.10, we obtain, by substituting $G^{\nu}\varphi$ for f

$$\int_{\mathbb{R}^d} u_0(x)\varphi(x)dx = \int_{\mathbb{R}^d} u_0(x)w(x)G^{\nu}\varphi(x)dx = \int_{\mathbb{R}^d} G^{\nu}(u_0 w)(x)\varphi(x)dx,$$

thus

$$u_0(x) = \int_{\mathbb{R}^d} G^{\nu}(x,y)u_0(y)w(y)dy, \quad m\text{-a.e.}$$

Let

$$v(x) = E_x\left[\int_0^\infty \exp(A_t^\nu)u_0(B_t)w(B_t)dt\right],$$

Then the function $v(x)$ equals to $u_0(x)$ m-a.e. and satisfies

$$v(x) = \int_{\mathbb{R}^d} G^\nu(x,y)v(y)w(y)dy, \ m\text{-a.e.}$$

Moreover, $v(x)$ is a finely continuous $P_t^{\lambda^+\mu}$-excessive function. Indeed,

$$\begin{aligned}
v(B_s) &= E_{B_s}\left[\int_0^\infty \exp(A_t^\nu)u_0(B_t)w(B_t)dt\right] \\
&= E_x\left[\int_0^\infty \exp(A_t^\nu \circ \theta_s)u_0(B_{t+s})w(B_{t+s})dt\Big|\mathcal{F}_s\right] \\
&= \exp(-A_s^\nu)E_x\left[\int_0^\infty \exp(A_t^\nu)u_0(B_t)w(B_t)dt\Big|\mathcal{F}_s\right] \\
&\quad - \exp(-A_s^\nu)\int_0^s \exp(A_t^\nu)u_0(B_t)w(B_t)dt.
\end{aligned} \tag{20}$$

and the first term of the last equality is right continuous because of the right continuity of $\mathcal{F}_s$. Hence v is finely continuous ([10,Theorem A.2.7]), and thus $v(x) = u_0(x)$ q.e. Consequently

$$v(x) = E_x\left[\int_0^\infty \exp(A_t^\nu)v(B_t)w(B_t)dt\right] \text{ for any } x. \tag{21}$$

Let $M_t = E_x[\int_0^\infty \exp(A_t^\nu)v(B_t)w(B_t)dt|\mathcal{F}_s]$. Then according to (20) and (21)

$$\begin{aligned}
&\exp(A_t^{\lambda^+\mu})v(B_t) = \exp(\int_0^t w(B_u)du)(\exp(A_t^\nu)v(B_t)) \\
= \ & v(B_0) + \int_0^t \exp(\int_0^s w(B_u)du)dM_s - \int_0^t \exp(A_s^{\lambda^+\mu})v(B_s)w(B_s)ds \\
&+ \int_0^t \exp(A_s^\nu)v(B_s)\exp(\int_0^s w(B_u)du)w(B_s)ds \\
= \ & v(B_0) + \int_0^t \exp(\int_0^s w(B_u)du)dM_s,
\end{aligned}$$

which implies that

$$E_x[\exp(A_t^{\lambda^+\mu})v(B_t)] \leq v(x).$$

Hence $h(x) = cv(x)$ by Lemma 3.7, and thus for all x

$$h(x) = \int_{\mathbb{R}^d} G^{\nu}(x,y)h(y)w(y)dy. \tag{22}$$

Q.E.D.

§4. An extension of Oshima's inequality

In this section, we extend Oshima's inequality in [11] to critical Schrödinger forms. The inequality plays a crucial role for the proof of the differentiability of $C(\lambda)$.

Lemma 4.1. *Let h be a positive continuous $\mathcal{H}^{\lambda^+\mu}$-harmonic function. Then the h-transformed semigroup $P_t^{\lambda^+\mu,h}$ of $P_t^{\lambda^+\mu}$ has the strong Feller property.*

Proof. Following the argument in [6, Corollary 5.2.7], we can prove this lemma. Q.E.D.

Proposition 4.2. *For the ground state h, the h-trasformed process $\mathbb{W}^{\lambda^+\mu,h} = (P_x^{\lambda^+\mu,h}, X_t)$ is Harris recurrent, that is, for a non-negative function f,*

$$\int_0^{\infty} f(X_t)dt = \infty, \quad P_x^{\lambda^+\mu,h}\text{-a.s.} \tag{23}$$

whenever $m(\{x : f(x) > 0\}) > 0$.

Proof. Since $P_t^{\lambda^+\mu,h}$ generates the h^2m-symmetric recurrent Markov process, we see from [8, Theorem 4.6.6] that

$$P_x[\sigma_A \circ \theta_n < \infty, \ \forall n \geq 0] = 1 \quad \text{for q.e. } x \in \mathbb{R}^d, \tag{24}$$

where $A = \{x : f(x) > 0\}$. Moreover, since the Markov process $\mathbb{W}^{\lambda^+\mu,h}$ has transition density with respect to h^2m, (24) holds for all $x \in \mathbb{R}^d$ by [8, Problem 4.6.3]. Hence according to [16, Chapter X, Proposition (3.11)], we have the equation (23). Q.E.D.

Theorem 4.3. *For the form $\mathcal{E}^{\lambda^+\mu}$ and its ground state h, there exist a positive function $g \in L^1(h^2m)$ and a function $\psi \in C_0(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} \psi h^2 dx = 1$ such that for $u \in \mathcal{D}(\mathcal{E}^{\lambda^+\mu,h})$*

$$\int_{\mathbb{R}^d} |u(x) - h(x)L(\frac{u}{h})|g(x)h(x)dx \leq C\mathcal{E}^{\lambda^+\mu}(u,u)^{1/2}, \tag{25}$$

where C is a positive constant and

$$L(u) = \int_{\mathbb{R}^d} u\psi h^2 dx.$$

Proof. We can apply Oshima's inequality to the Dirichlet form $(\mathcal{E}^{\lambda^+\mu,h}, \mathcal{D}(\mathcal{E}^{\lambda^+\mu,h}))$ satisfying the Harris recurrence condition: there exist a positive function $g \in L^1(h^2m)$ and a function $\psi \in C_0(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} \psi h^2 dx = 1$ such that for any $u \in \mathcal{D}(\mathcal{E}^{\lambda^+\mu,h})$

$$\int_{\mathbb{R}^d} |u(x) - L(u)| g(x) h^2(x) dx \leq C\mathcal{E}^{\lambda^+\mu,h}(u,u)^{1/2}, \tag{26}$$

where

$$L(u) = \int_{\mathbb{R}^d} u\psi h^2 dx.$$

Therefore substituting v/h for u in (26) and noting the relation

$$\mathcal{E}^{\lambda^+\mu,h}(v,v) = \mathcal{E}^{\lambda^+\mu}(hv,hv),$$

we obtain the equality (25). Q.E.D.

§5. Differentiability of spectral function

Lemma 5.1 ([24, Lemma 4.3]). *Let $\mu \in \mathcal{K}_d^\infty$. Then for any $\lambda > \lambda^+$, the negative spectrum of $\sigma(\mathcal{E}^{\lambda\mu})$ consists of isolated eigenvalues with finite multiplicities.*

Let $\mathcal{H}^\mu$ be critical and h its groung state. Then we call $\mathcal{H}^\mu$ *null critical* if the function h does not belong to $L^2(m)$,

Theorem 5.2. *Let $\mu \in \mathcal{K}_d^\infty$. If $\mathcal{H}^{\lambda^+\mu}$ is null critical, then its spectral function $C(\lambda)$ is diffentiable.*

Proof. Note that by Lemma 5.1, for $\lambda > \lambda^+$, $-C(\lambda)$ is the principal eigenvalue of Schrödinger operator $\mathcal{H}^{\lambda\mu} = -\frac{1}{2}\Delta - \lambda\mu$. By analytic perturbation theory [9, Chapter VII], we can see that $C(\lambda)$ is differentiable on $\lambda > \lambda^+$. Hence we only need to prove the differentiability of $C(\lambda)$ at $\lambda = \lambda^+$. Since $C(\lambda)$ is convex, it is enough to prove that there exists a sequence $\{\lambda_n\}$ such that $\lambda_n \downarrow \lambda^+$ and $dC(\lambda_n)/d\lambda \downarrow 0$. By the perturbation theory [9, p.405, Chapter VII (4.44)], we see

$$\frac{dC(\lambda)}{d\lambda} = \int_{\mathbb{R}^d} u_\lambda^2 d\mu, \tag{27}$$

where u_λ is the L^2-normalized eigenfunction corresponding to $-C(\lambda)$, that is,

$$C(\lambda) = \lambda \int_{\mathbb{R}^d} u_\lambda^2 d\mu - \frac{1}{2}\mathbf{D}(u_\lambda, u_\lambda). \tag{28}$$

Using (14) and taking $\epsilon > 0$ so small that $\lambda_n \epsilon < 1/2$, we have

$$\mathbf{D}(u_{\lambda_n}, u_{\lambda_n}) \le \frac{-C(\lambda_n) + \lambda_n M(\epsilon)}{1/2 - \lambda_n \epsilon}.$$

Noting that $C(\lambda_n) \to 0$ as $n \to \infty$, we see

$$\sup_n \mathbf{D}(u_{\lambda_n}, u_{\lambda_n}) < \infty. \tag{29}$$

Since

$$\mathcal{E}^{\lambda^+\mu}(u_{\lambda_n}, u_{\lambda_n}) - \mathcal{E}^{\lambda_n\mu}(u_{\lambda_n}, u_{\lambda_n}) \le (\lambda_n - \lambda^+)\|G\mu\|_\infty \mathbf{D}(u_{\lambda_n}, u_{\lambda_n})$$

the right hand side converges to 0 as $n \to \infty$ by (29). Thus we obtain

$$\lim_{n\to\infty} \mathcal{E}^{\lambda^+\mu}(u_{\lambda_n}, u_{\lambda_n}) = 0. \tag{30}$$

For the ground state h of $\mathcal{H}^{\lambda^+\mu}$ let $\mathcal{H}^{\lambda^+\mu,h}$ be the h-transformed operator. For $\psi \in C_0(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} \psi h^2 dx = 1$, let $L(u) = \int_{\mathbb{R}^d} u(x)\psi(x)h^2(x)dx$. Then we have

$$\left|L(\frac{u_{\lambda_n}}{h})\right| \le \sqrt{\int_{\mathbb{R}^d} u_{\lambda_n}^2 dx}\sqrt{\int_{\mathbb{R}^d} \psi^2(x)h^2(x)dx} < \infty.$$

Hence we can choose a sequence $\{\lambda_n\}$ tending to λ^+ such that $L(u_{\lambda_n}/h)$ converges to a certain constant C. Noting by Thorem 4.3,

$$\begin{aligned} & \int_{\mathbb{R}^d} |u_{\lambda_n} - Ch| g h dx \\ \le\ & \int_{\mathbb{R}^d} |u_{\lambda_n} - hL(\frac{u_{\lambda_n}}{h})| g h dx + \int_{\mathbb{R}^d} |hL(\frac{u_{\lambda_n}}{h}) - Ch| g h dx \\ \le\ & C\mathcal{E}^{\lambda^+\mu}(u_{\lambda_n}, u_{\lambda_n})^{1/2} + \int_{\mathbb{R}^d} |L(\frac{u_{\lambda_n}}{h}) - C| g h^2 dx \to 0, \end{aligned}$$

we see $u_{\lambda_n} \to Ch$ a.e. by choosing a subsequence if necessary. Since

$$1 = \liminf_{n\to\infty} \int_{\mathbb{R}^d} u_{\lambda_n}^2 dx \ge \int_{\mathbb{R}^d} \liminf_{n\to\infty} u_{\lambda_n}^2 dx = C^2 \int_{\mathbb{R}^d} h^2 dx,$$

the constant C must be equal to 0 on account of the null criticality. Since $C(\lambda_n)$ is an eigenvalue for $-\mathcal{H}^{\lambda_n\mu}$, $u_{\lambda_n} = e^{-C(\lambda_n)t}P_t^{\lambda_n\mu}u_{\lambda_n}$. Thus we have by [2, Theorem 6.1 (iii)]

$$\|u_{\lambda_n}\|_\infty \le e^{-C(\lambda_n)t}\|P_t^{\lambda_n\mu}\|_{2,\infty} \le \|P_t^{\lambda_1\mu}\|_{2,\infty} < \infty.$$

Also we can assume that $u_{\lambda_n} \to 0$ $q.e.$ as $k \to \infty$ by choosing a subsequence. Therefore we have

$$\begin{aligned} &\limsup_{n\to\infty} \int_{\mathbb{R}^d} u_{\lambda_n}^2 d\mu \\ \le\ & \limsup_{n\to\infty} \int_{\mathbb{R}^d} u_{\lambda_n}^2 d\mu_R + \limsup_{n\to\infty} \|G\mu_{R^c}\|_\infty \mathbf{D}(u_{\lambda_n}, u_{\lambda_n}) \\ \le\ & \|G\mu_{R^c}\|_\infty M. \end{aligned}$$

By letting R to ∞, we complete the proof. Q.E.D.

Finally we consider the situation in Theorem 5.2. By Theorem 3.11 we have

$$c\int_K G^\nu(x,y)w(y)dy \le h(x) \le C\int_K G^\nu(x,y)w(y)dy,$$

where K is the support of w. Let $B(R) \supset K$. Applying the Harnack inequality to $G^\nu(x,\cdot)$, $x \in B(R)^c$, we see that

$$cG^\nu(x,0) \le h(x) \le CG^\nu(x,0) \quad \text{on } x \in B(R)^c.$$

We see from the equation (18) that the ground state h satisfies

$$cG(x,0) \le h(x) \le CG(x,0), \text{ on } x \in B(R)^c. \tag{31}$$

Hence we see that if $d \le 4$, h is not in L^2, that is, $\mathcal{H}^{\lambda^+\mu}$ is null critical. Therefore combining [24, Theorem 4.3] and Theorem 5.2, we obtain Theorem 1.1.

References

[1] Aizenman, M., Simon, B.: Brownian motion and Harnack inequality for Schrödinger operators, Comm. Pure Appl. Math. **35**, 209-273, (1982).

[2] Albeverio, S., Blanchard, P., Ma, Z.M.: Feynman-Kac semigroups in terms of signed smooth measures, in "Random Partial Differential Equations" ed. U. Hornung et al., Birkhäuser, 1-31, (1991).

[3] Arendt, W., Batty, C.J.K.: The spectral function and principal eigenvalues for Schrödinger operators, Potential Anal. **7**, 415-436, (1997).

[4] Boukricha, A., Hansen, W., Hueber, H.: Continuous solutions of the generalized Schrödinger equation and perturbation of harmonic spaces, Expo. Math. **5**, 97-135, (1987).

[5] Brasche, J.F., Exner, P., Kuperin, Y., Seba, P.: Schrödinger operators with singular interactions, J. Math. Anal. Appl. **184**, 112-139, (1994).

[6] Davies, E.B., Heat Kernals and Spectral Theory, Cambridge Univ. Press, Cambridge, U.K., (1989).

[7] Dembo, A., Zeitouni, O.: Large deviation techniques and applications, Second edition, Applications of Mathematics **38**, Springer-Verlag, New York, (1998).

[8] Fukushima, M., Oshima, Y., Takeda, M.: Dirichlet Forms and Symmetric Markov Processes, Walter de Gruyter, Berlin, (1994).

[9] Kato, T.: Perturbation Theory for Linear Operators, Springer-Verlag, Berlin-Heidelberg-New York, (1984).

[10] Murata, M.: Structure of positive solutions to $(-\Delta + V)u = 0$ in R^n, Duke Math. J. **53**, 869–943, (1986).

[11] Oshima, Y.: Potential of recurrent symmetric Markov processes and its associated Dirichlet spaces, in Funtional Analysis in Markov Processes, ed. M. Fukushima, Lecture Notes in Math. **923**, Springer-Verlag, Berlin-Heidelberg-New York, 260-275, (1982).

[12] Pinchover, Y.: Criticality and ground states for scond-order elliptic equations, J. Differential Equations, **80**, 237-250, (1989).

[13] Pinsky, R.G.: Positive Harmonic Functions and Diffusion, Cambridge Studies in Advanced Mathematics **45**, Cambridge University Press, (1995).

[14] Reed, M., Simon, B.: Methods of Modern Mathematical Physics, Volume I, Functional Analysis, Academic Press, (1972).

[15] Reed, M., Simon, B.: Methods of Modern Mathematical Physics, Volume IV, Analysis of Operators, Academic Press, (1979).

[16] Revuz, D., Yor, M.: Continuous Martingales and Brownian Motion, 3rd edition, Springer, New York, (1998).

[17] Sato, S.: An inequality for the spectral radius of Markov processes, Kodai Math. J. **8**, 5-13, (1985).

[18] Silverstein, M. L.: Symmetric Markov processes, Lecture Notes in Math ematics, **426**, Springer-Verlag, Berlin-New York, (1974).

[19] Simon, B.: Schrödinger semigroups, Bull. Amer. Math. Soc. **7**, 447-536, (1982).

[20] Stollmann, P., Voigt, J.: Perturbation of Dirichlet forms by measures, Potential Anal. **5**, 109-138, (1996).

[21] Takeda, M.: Asymptotic properties of generalized Feynman-Kac functionals, Potential Analysis **9**, 261-291, (1998).

[22] Takeda, M.: Exponential decay of lifetimes and a theorem of Kac on total occupation times, Potential Analysis **11**, 235-247, (1999).

[23] Takeda, M.: L^p-independence of the spectral radius of symmetric Markov semigroups, Stochastic Processes, Physics and Geometry: New Interplays. **II**: A Volume in Honor of Sergio Albeverio, Edited by Fritz Gesztesy, et.al, (2000).

[24] Takeda, M.: Large deviation principle for additive functionals of Brownian motion corresponding to Kato measures, To appear in Potential Analysis.

[25] Takeda, M.: Conditional gaugeability and subcriticality of generalized Schrödinger operators, J. Funct. Anal. **191**, 343-376, (2002).

[26] Takeda, M., Uemura, T.: Subcriticality and Gaugeability for Symmetric α-Stable Processes, To appear in Forum Math.

[27] Wu,L.: Exponential convergence in probability for emprical means of Brownian motion and random walks, J. Theoretical Probab. **12**, 661-673, (1999).

[28] Zhao, Z.: A probabilistic principle and generalized Schrödinger perturbation. J. Funct. Anal. **101**, 162-176, (1991).

[29] Zhao, Z.: Subcriticality and gaugeability of the Schrödinger operator, Trans. Amer. Math. Soc. **334**, 75-96, (1992).

Masayoshi Takeda
Mathematical Institute
Tohoku University
Aoba, Sendai, 980-8578
Japan
Kaneharu Tsuchida
Mathematical Institute
Tohoku University
Aoba, Sendai, 980-8578
Japan