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Cubic Schrödinger: The Petit Canonical Ensemble

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§1. Introduction

This report describes some aspects of the Gibbsian petit canonical ensemble for the cubic Schrödinger equation in the space of functions of period 1, say. §2–5 (defocussing case) represent joint work with K. Vaninsky[1)]. §6 is a brief report on the much more difficult focussing case. The original hope, that the petit ensemble might provide a picture of the typical solution, is far from being achieved.

1.1. Preliminaries[2)]

The mechanical state is a pair QP of nice functions of period 1, moving according to the defocussing flow:

$$\begin{aligned}\frac{\partial Q}{\partial t} &= -\frac{\partial^2 P}{\partial x^2} + (Q^2+P^2)\,P = \frac{\partial H_3}{\partial P}\\ \frac{\partial P}{\partial t} &= +\frac{\partial^2 Q}{\partial x^2} - (Q^2+P^2)\,Q = -\frac{\partial H_3}{\partial Q}\end{aligned}$$

This is a Hamiltonian system, relative to the classical bracket in function space, with Hamiltonian

$$H_3 = \frac{1}{2}\int_0^1 \left[(Q')^2 + (P')^2\right] + \frac{1}{4}\int_0^1 (Q^2+P^2)\,.$$

It is integrable in the full technical sense of the word, having an infinite series of (commuting) constants of motion $H_1 = \frac{1}{2}\int_0^1 (Q^2+P^2)$, $H_2 = \int_0^1 Q'P$, H_3, and so on. The flow is integrated with the help of the Dirac equation

$$M' = \left[\begin{pmatrix} Q & P \\ P & -Q\end{pmatrix} + \frac{\lambda}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}\right] M$$

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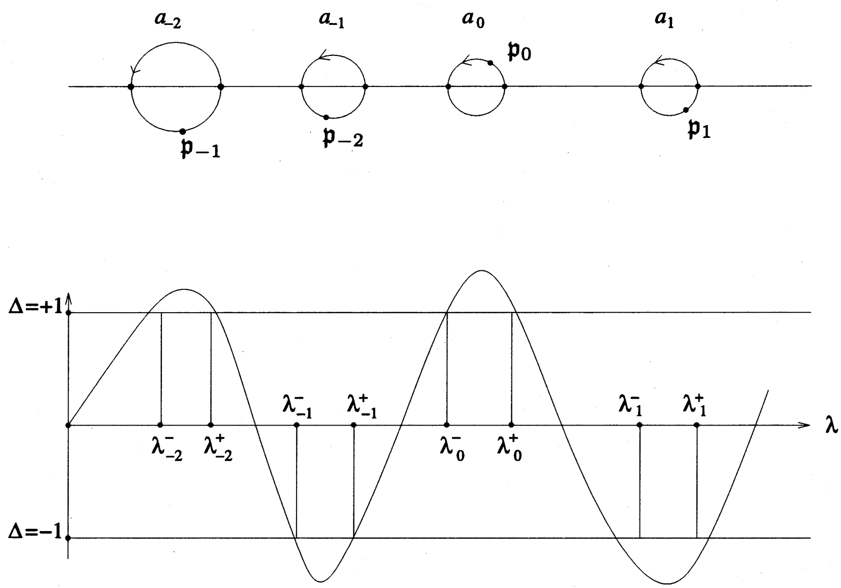
1) McKean-Vaninsky [1997]

2) Manakov et al. [1984] and/or McKean-Vaninsky [1997]

for the 2×2 monodromy matrix $M = [m_{ij} : 1 < i, j \leq 2]$ with $M(x = 0) = I$. Introduce the "discriminant" $\Delta(\lambda) = \frac{1}{2}\, sp\, M(x = 1)$ and the associated "Dirac curve" $\mathfrak{M}$ with points $\mathfrak{p} = \left[\lambda,\ \sqrt{\Delta^2(\lambda) - 1}\right]$. The latter is a double cover of the complex plane where λ lives, ramified over the roots

$$\ldots \lambda_{-1}^{-} \leq \lambda_{-1}^{+} < \lambda_{-1}^{-} \leq \lambda_{-1}^{+} < \lambda_0^{-} \leq \lambda_0^{+} < \lambda_1^{-} \leq \lambda_1^{+} < \ldots, \lambda_n^{\pm} \simeq 2\pi n \ \textit{etc.}$$

of $\Delta(\lambda) = \pm 1$ indicated in the figure. These comprise the periodic/anti-periodic spectrum of the Dirac equation and may be interpreted as a



complete list of constants of motion, commuting among themselves and with the prior constants, H_1, H_2, H_3, etc. The cycles $a_n : n \in \mathbb{Z}$ seen in the upper part of the figure are the "real ovals" of $\mathfrak{M}$ covering the "gaps" $[\lambda_n^{-}, \lambda_n^{+}]$, these being all open for QP in general position, as is mostly assumed below. QP is encoded into a divisor $\mathfrak{P} = [\mathfrak{p}_n : n \in \mathbb{Z}]$ of $\mathfrak{M}$ having 1 point on each real oval: the numbers $\lambda(\mathfrak{p}_n) \equiv \mu_n \in [\lambda_n^{-}, \lambda_n^{+}]$ are the roots of $m_{12}(\mu) = 0$[3] and the radical $\sqrt{\Delta^2 - 1}\,(\mathfrak{p}_n)$ is declared to be $\frac{1}{2}\,(m_{11} - m_{12})\,(\mu_n)$[4]. The map $QP \to \mathfrak{P}$ is $1:1$ or to the product

3) $m_{12}(\lambda)$ looks much like $-\sin(\lambda/2)$.

4) $\det M(1) = 1$ so $m_{11}\, m_{12} = 1$ if $m_{12} = 0$ and $m_{11} + m_{12} = 2\Delta$ always, whence this possibility.

of all the ovals. The next actor in the play is the "Abel map" of the divisor into the (real) Jacobi variety Jac of $\mathfrak{M}$, determined as follows. DFK = the "differentials of the first kind" of $\mathfrak{M}$ are of the form $\omega = \phi_n(\lambda)\, d\lambda/\sqrt{\Delta^2(\lambda)-1}$ with certain entire functions ϕ, and a basis may be chosen so that $a_i(\omega_j) = 2\pi$ or 0 according as $i = j$ or not.[5] $\mathfrak{P}$ is now mapped to Jac via the "angles" $\theta_n = \sum_{k\in\mathbb{Z}} \int_{\mathfrak{o}_k}^{\mathfrak{p}_k} \omega_n$ construed *mod* 2π,[6] *i.e.*

$$\mathfrak{P} \to \Theta = [\theta_n : n \in \mathbb{Z}] \in (\mathbb{R}/2\pi\mathbb{Z})^\infty = \text{ Jac},$$

and this map likewise is 1 : 1 and onto. Now you have the composite map $QP \to$ divisor $\to$ Jac, the point of the whole exercise being that the (complicated) flow of QP is converted thereby into (simple) straight-line motion at constant speed in Jac which may be mapped back to the original (mechanical) variables with the help of a Riemann-like "theta" function. In this way, the flow is "integrated".

§2. Petit Ensemble at Levels 1 & 3

Level 1 is a warm-up for "level 3" to be described below. Introduce the "level 1 actions" $I_n = \frac{1}{4\pi}\, a_n\left(ch^{-1}\,\Delta,\, d\lambda\right)$[7] and note the trace formula $H_1 = \frac{1}{2}\int\left(Q^2+P^2\right) = \sum_{\mathbb{Z}} I_n$. The petit ensemble[8]

$$\begin{aligned} e^{-H_1} d^\infty Q\, d^\infty P &= \frac{e^{-\frac{1}{2}\int_0^1 Q^2}}{(2\pi/0_+)^{\infty/2}} d^\infty Q \times \frac{e^{-\frac{1}{2}\int_0^1 P^2}}{(2\pi/0_+)^{\infty/2}}\, d^\infty P \\ &= \prod_{\mathbb{Z}} e^{-I_n}\, dI_n \times \prod_{\mathbb{Z}} d\,\theta_n/2\pi \ : \end{aligned}$$

is descriptive of 2 independent copies of white noise; line 2 comes from the trace formula plus the formal identification of the volume elements $d^\infty Q\, d^\infty P$ & $d^\infty I\, d^\infty\, \theta/2\pi$ prompted by the fact that actions & angles are canonically conjugate and together form a full coordinate system in QP-space. Naturally, line 2 requires proof as does the invariance of the ensemble under the flow, for which see McKean-Vaninsky [1997].

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5) I should say differentials of the third kind as they have simple poles at the 2 points of $\mathfrak{M}$ covering ∞, but as they play the role of classical DFK, I keep the name. $\phi_n(\lambda)$ looks much like $m_{12}(\lambda)$ divided by $\lambda - \mu$, *i.e.* with 1 root left out.

6) $\mathfrak{o}_k = \left[\lambda_k^-, 0\right]$, some such choice being necessary for the convergence of the sum.

7) The name will be justified in §4.

8) Here and below, I will be free and easy with possibly infinite norming constants.

Level 3. The petit ensemble at "level 3":

$$e^{-H_3}\,d^\infty Q\,d^\infty P = \frac{e^{-\frac{1}{2}\int_0^1 (Q')^2}}{(2\pi 0_+)^{\infty/2}}\,d^\infty Q\,\frac{e^{-\frac{1}{2}\int_0^1 (P')^2}}{(2\pi 0_+)^{\infty/2}}\,d^\infty P \times e^{-\frac{1}{4}\int_0^1 (Q^2+P^2)^2}$$

is descriptive of 2 independent "circular" Brownian motions[9] coupled by the third factor; it is invariant under the flow as for level 1. To describe it in action/angle language requires a revision: DFK at level 3 is as before (level 1) but with a new basis $\omega'_n : n \in \mathbb{Z}$ normalized as in $a_i(\lambda^2\,\omega'_j) = 2\pi$ or 0 according as $i = j$ or not. The level 3 actions are $I'_n = \frac{1}{q\pi}\,a_n(\lambda^2\,ch^{-1}\,\Delta\,d\lambda)$ and you have the trace formula $H_3 = \sum_{\mathbb{Z}} I'_n$, whence

$$\begin{aligned} e^{-H_3} d^\infty Q\,d^\infty P &= \prod_{\mathbb{Z}} e^{-I'_n} \times \left[d^\infty Q d^\infty P = d^\infty I\ d^\infty\,\frac{d\theta}{2\pi}\ \text{at level 1}\right] \\ &= \prod_{\mathbb{Z}} e^{-I'_n}\,dI'_n \prod_{\mathbb{Z}} d\,\frac{\theta_n}{2\pi} \times \det\frac{\partial I}{\partial I'}\ , \end{aligned}$$

in which the third (Jacobian) factor is still to be understood. The level 3 actions are canonically paired to the level 3 angles[10] $\theta'_n = \sum_{r\in\mathbb{Z}} \int_{\mathfrak{o}_k}^{\mathfrak{p}_k} \omega'_n$, so

$$\begin{aligned} \det\frac{\partial I}{\partial I'} &= \det\frac{\partial \theta'}{\partial \theta} \\ &= \frac{\det\left[\omega'_i/d\lambda(\mathfrak{p}_j)\right]}{\det\left[\omega_i/d\lambda(\mathfrak{p}_j)\right]} \\ &\times \int \frac{\det\prod_{i>j}(\mu_i-\mu_j)}{\prod_{\mathbb{Z}}\sqrt{\Delta^2-1(\mathfrak{p}_n)}}\,d^\infty\,\mu \\ &\text{divided by} \\ &\int \prod_{\mathbb{Z}} \mu^2 \times \ \text{the same "volume element".} \end{aligned}$$

This rather fanciful expression comes from level 2 in case all but N gaps are closed and making $N \uparrow \infty$ with an (unpardonable) disregard of normalizing factors. Now the "volume element" seen in line 3 is

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[9] CBM is standard Brownian motion, conditioned to end where it began, with this common displacement distributed over $\mathbb{R}$ by flat Lebesgue measure. The coupling holds down the total mass so that normalization is possible.

[10] These must be construed, not $mod\,2\pi$, but relative to another, pretty complicated lattice of periods.

nothing but an un-normalized expression of the flat (level 1) volume element $d^\infty\theta/2\pi$ on Jac, written out in the language of the divisor; also $m_{12}(\lambda) = \frac{1}{2}(\mu_0 - \lambda)\prod_{\mathbb{Z}}(2\pi n)^{-1}(\mu_n - \lambda)$ precisely; and so it is an educated guess that, after proper normalization, the Jacobian det $\partial I/\partial I'$ ought to be the reciprocal of $N = \int_{Jac} m_{12}^2(0)\, d^\infty\,\theta/2\pi$.

This is correct as far as it goes[11], but what does N really look like? It is a function of actions alone, so the level 1 angles are still independent of them, with the same flat distribution as before. There are 10 integrals of products of 2 entries of $M(1)$, and I know 9 relations among them involving the constants of motion Δ and $\Delta^\bullet$, but the value of N is not revealed by these. Too bad! Crude estimates of N can be had but do not help to describe how the actions couple. I leave the subject in this unsatisfactory state.

§3. Some Tricks

I record here 3 amusing examples of averaging over Jac with respect to $d^\infty\theta/2\pi$, but first a general principle. Think of the (still to be normalized) expression

$$d^\infty\,\frac{\theta}{2\pi} = \prod_{i>j}(\mu_i - \mu_j)\, d^\infty\mu \text{ divided by } \prod_{\mathbb{Z}}\sqrt{\Delta^2 - 1}\,(\mathfrak{p}_n)$$

encountered in §3. The top, considered as a function of μ_n, say, is proportional to $m_{12}^\bullet(\mu_n)$, so you have the "splitting rule at $n \in \mathbb{Z}$":

$$d^\infty\frac{\theta}{2\pi} = \frac{m_{12}^\bullet(\mu_n)}{\sqrt{\Delta^2 - 1}\,(\mathfrak{p}_n)} \quad \text{on the oval } a_n$$

$\times$ a volume element on the product of all the other ovals.

This principle is now applied in 3 ways:

Example 1. $m_{12}(\lambda)$ looks like $-\sin(\lambda/2)$ and $\Delta(\lambda)$ like $\cos(\lambda/2)$, so you may expect $2\Delta^\bullet - m_{12}$ to be of "degree 1 lower" than m_{12} and that Lagrange interpolation would apply. This is correct:

$$2\Delta^\bullet(\lambda) - m_{12}(\lambda) =^{12)} \sum_{\mathbb{Z}} \frac{2\Delta^\bullet(\mu_n)}{m_{12}^\bullet(\mu_n)}\frac{m_{12}(\lambda)}{\lambda - \mu_n}\ .$$

[11] McKean-Vaninsky [1997].

[12] $m_{12}(\mu_n) = 0$ of course.

Now average over Jac, exchange sum and average, and split the volume at $n \in \mathbb{Z}$ to produce

$$2\Delta^\bullet(\lambda) - \int_{\mathrm{Jac}} m_{12}(\lambda)\,\frac{d^\infty\theta}{2\pi} = \sum_{\mathbb{Z}} 2 \int\limits_{\substack{\times a_k \\ k\neq n}} \frac{m_{12}(\lambda)}{\lambda-\mu_n} \int_{a_n} \frac{\Delta^\bullet d\mu_n}{\sqrt{\Delta^2-1}} = {}^{13)}0,$$

i.e. $2\Delta^\bullet =$ average m_{12}.

Example 2. The numerator ϕ_n of $\omega_n \in$ DFK at level 1 looks like m_{12} with 1 root factored out, so it, too, should be capable of interpolation:

$$\phi_n(\lambda) = \sum_{i\in\mathbb{Z}} \frac{\phi(\mu_i)}{m_{12}^\bullet(\mu_i)}\,\frac{m_{12}^\bullet(\lambda)}{\lambda-\mu_i}\ .$$

But this object has nothing to do with angles, so an average over Jac does it no harm, and proceeding as in ex. 1, you find

$$\begin{aligned}
\phi_n(\lambda) &= \sum_{i\in\mathbb{Z}} \int\limits_{\substack{\times a_j \\ j\neq i}} \frac{m_{12}(\lambda)}{\lambda-\mu_i} \int\limits_{a_i} \frac{\phi_n(\mu_i)\,d\mu_i}{\sqrt{\Delta^2-1}\,(\mathfrak{p}_i)} \\
&= \int\limits_{\substack{\times a_j \\ j\neq n}} \frac{m_{12}(\lambda)}{\lambda-\mu_n} \times 2\pi \\
&= \int\limits_{\times a_j = \,\mathrm{Jac.}} \frac{m_{12}(\lambda)}{m_{12}^\bullet(\mu_n)(\lambda-\mu_n)}\, d^\infty\,\frac{\theta}{2\pi}
\end{aligned}$$

divided by

$$\frac{1}{2\pi}\int\limits_{a_n} \frac{d\mu}{\sqrt{\Delta^2-1}}\,,$$

i.e.

$$\omega_n = \quad \text{average} \quad \frac{m_{12}(\lambda)}{m_{12}^\bullet(\mu_n)(\lambda-\mu_n)} \quad \text{normalized to have mass } 2\pi \text{ on } a_n\ .$$

This seems to be a new way of writing DFK.

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13) $\Delta^\bullet/\sqrt{\Delta^2-1} = d\,ch^{-1}\,\Delta$.

Example 3 identifies $I_n = \frac{1}{4\pi}\, a_n(ch^{-1}\,\Delta\, d\lambda)$ with a true mechanical action, as promised at start of §3.[14] The physical actions are $A_n = (2\pi)^{-1} a_n(PdQ) : n \in \mathbb{Z}$. To implement their evaluation, take the flow $e^{t\mathbb{X}}$ with Hamiltonian I_n which carries $\mathfrak{p}_n$ once about its private cycle a_n in time 2π, leaving the rest of the divisor fixed, and equate A_n with

$$\frac{1}{2\pi}\int_0^{2\pi} e^{t\mathbb{X}}\left[\int_0^1 P(x)\,\mathbb{X}\,Q(x)\,dx\right] dt.$$

Now A_n has nothing to do with angles, so you can average over Jac, exchange this average with the time-average, and use the invariance of the flat volume under the present flow and the flow of translation produced by $H_2 = \int_0^1 Q'P$ to reduce the previous display to $\int_{\mathrm{Jac}} P(0)\,\mathbb{X}\,Q(0)\, d^\infty\theta/2\pi$. Here,

$$\mathbb{X}\,Q(0) = \frac{1}{4\pi}\int_{a_n} (1/2)\,(m_{12} + m_{21})\,\frac{d\lambda}{\sqrt{\Delta^2 - 1}}\,,$$

$m_{12} - m_{21}$ is invariant under the "phase flow" $Q^\bullet = P$ and $P^\bullet = -Q$ produced by $H_1 = \frac{1}{2}\int_0^1 \left(Q^2 + P^2\right)$, and the average of $P(0)$ under *this* flow is 0, permitting a further reduction to

$$A_n = \frac{1}{4\pi}\int_{a_n} \frac{d\lambda}{\sqrt{\Delta^2 - 1}}\int_{\mathrm{Jac}} P(0)\,[m_{12}(\lambda) - 2\Delta^\bullet(\lambda)]\, d^\infty\,\frac{\theta}{2\pi}\,.$$

The trace formula $P(0) =$ [15] $\frac{1}{2}\sum(\mu_i - \lambda_i)$ and the interpolation of $2\Delta^\bullet - m_{12}$ from example 1 are now inserted under the average, sums and avarage are exchanged, and the volume element $d^\infty\theta/2\pi$ is split at

[14] The level 3 actions have also a mechanical interpretation, but I do not go into it here.

[15] $\lambda_n^\bullet : n \in \mathbb{Z}$ are the roots of $\Delta^\bullet(\lambda) = 0$.

$j \in \mathbb{Z}$, with the result that

$$A_n = \frac{1}{4\pi}\int_{a_n} \frac{d\lambda}{\sqrt{\Delta^2-1}} \sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} \int_{\times a_k, k\neq j} \frac{m_{12}}{\lambda-\mu_j} \int_{a_j} (\lambda_i^\bullet - \mu_i)\, d\, ch^{-1}\,\Delta(\mathfrak{p}_j)$$

$$= \frac{1}{4\pi}\int_{a_n} \frac{d\lambda}{\sqrt{\Delta^2-1}} \sum_{i\in\mathbb{Z}} \int_{\times a_k, k\neq i} \frac{m_{12}}{\lambda-\mu_i} \int_{a_n} ch^{-1}\,\Delta\, d\mu_i$$

$$= \frac{1}{2\pi}\int_{a_n} \frac{d\lambda}{\sqrt{\Delta^2-1}} \sum_{i\in\mathbb{Z}} \left\{ \begin{array}{l} \displaystyle\int_{\times a_k = \text{Jac.}} \frac{m_{12}}{m_{12}^\bullet(\mu_i)(\lambda-\mu_i)}\, d^\infty \frac{\theta}{2\pi} \\ \text{divided by } \displaystyle\frac{1}{2\pi}\int_{a_i} \frac{d\mu_i}{\sqrt{\Delta^2-1}} \\ \text{multiplied by } \displaystyle\frac{1}{4\pi}\int_{a_i} ch^{-1}\,\Delta\, d\mu_i \end{array}\right.$$

$$= \frac{1}{2\pi}\int_{a_n} \frac{d\lambda}{\sqrt{\Delta^2-1}} \sum_{i\in\mathbb{Z}} \phi_i\, I_i$$

$$= I_n,$$

as advertised.

§4. Thermodynamic Limit

Now let Q & P have period L and take the large volume limit $L \uparrow \infty$. What happens to the petit ensemble $e^{-H_3}\, d^\infty Q\, d^\infty P$? The answer is nice and simple. Let ψ be the ground state of $-\frac{1}{2}\,\Delta + \frac{1}{4}\, r^4$ in $\mathbb{R}^2$. Then the mechanical variables $[Q(x), P(x)] : x \in \mathbb{R}$ tend (in law) to the stationary diffusion with infinitessimal operator $\frac{1}{2}\,\Delta + (\text{grad}\, \ell n\psi) \bullet \text{grad}$. This is even easy to prove.

§5. Focussing Case

This is much harder. The Hamiltonian is changed to $\frac{1}{2}\int\left[(Q')^2+(P')^2\right]$ *minus* $\frac{1}{4}\int\left(Q^2+P^2\right)$ and the associated petit ensemble has total mass $+\infty$. This prompted Lebowitz-Rose-Speer [1989] to introduce the microcanonical ensemble obtained by conditioning upon the value N of the constant of motion $H_1 = \frac{1}{2}\int (Q^2 + P^2)$.[16] Their interest was in the thermodynamic limit: with fixed "density" D, "particle number" $N = DL$, and $L \uparrow \infty$, they found by numerical simulation, that the temperature dependent ensemble $e^{-H_3/T}\, d^\infty Q\, d^\infty P$ favors "solitons" / "radiation" at low/high temperatures, *i.e.* some kind of phase change takes place. Chorin [private communication] used a more sophisticated simulation of the Brownian motion and found the opposite: no phase change. This made me curious and, subsequently[17], I claimed to prove that the thermodynamical limit does not exist, explaining (as I thought) the discrepancy just described. But alas, all the big boys were wrong: in fact, my student B. Rider[18] proved that, at any values of temperature and density, the whole ensemble collapses onto $Q \equiv 0$ & $P \equiv 0$. A pity.

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[16] The total mass is now finite, so the ensemble can be normalized. For a probabilistic proof of the existence of the flow and for the invariance of the micro-canonical ensemble under it, see McKean [1995].

[17] McKean [1995]

[18] Rider [2002]