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Quadratic Wiener Functionals, Kalman-Bucy Filters, and the KdV Equation

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Dedicated to Professor Kiyosi Itô on his 88th birthday

Abstract.

Soliton solutions and the tau function of the KdV equation are studied within the stochastic analytic framework. A key role is played by the Itô formula and the Cameron-Martin transformation.

§ Introduction

In this paper, we investigate the Korteweg-de Vries (KdV) equation within the framework of stochastic analysis. We shall study soliton solutions with the help of the Itô formula, whose original form was achieved in 1942 ([9]). The Cameron-Martin transformation, which was established in the early 1940's ([2, 3]), also plays a key role.

Let $x > 0$ and $\mathcal{W}^n$ be the space of $\mathbf{R}^n$-valued continuous functions on $[0, x]$ starting at the origin, and let P be the Wiener measure on $\mathcal{W}^n$. Following the idea of Cameron-Martin [3], we can show that

$$\begin{aligned}(1)\quad I(x,t) &:= \int_{\mathcal{W}^1} \exp\left[-\frac{a^2}{2}\int_0^x w(y)^2 dy - \frac{a}{2}\tanh(a^3 t) w(x)^2\right] P(dw) \\ &= \big(\cosh(a^3 t)\big)^{1/2}\big(\cosh(ax + a^3 t)\big)^{-1/2} \quad \text{for any } a > 0,\end{aligned}$$

where $w(y) \in \mathbf{R}$ denotes the position of $w \in \mathcal{W}^1$ at time y (see §4 and [8]). Then $u(x,t) = -4\partial_x^2 \log I(x,t)$, where $\partial_x = \partial/\partial x$, is a 1-soliton solution of the KdV equation

$$(2)\qquad \frac{\partial u}{\partial t} = \frac{3}{2} u \frac{\partial u}{\partial x} + \frac{1}{4}\frac{\partial^3 u}{\partial x^3}.$$

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A reflectionless potential with scattering data $\eta_j, m_j > 0$, $1 \le j \le n$, is by definition a function

$$q(x) = -2\frac{d^2}{dx^2}\log\det(I + A(x)), \tag{3}$$

where

$$A(x) = \left(\frac{\sqrt{m_i m_j}}{\eta_i + \eta_j} e^{-(\eta_i+\eta_j)x}\right)_{1\le i,j\le n}. \tag{4}$$

We denote by $\mathcal{Q}_n$ the totality of all reflectionless potentials with scattering data consisting of $2n$ positive numbers. Let Σ be the set of all pairs $\sigma = (\sigma_+, \sigma_-)$ of non-negative measures $\sigma_\pm$ on $(-\infty, 0]$ such that $\int_{(-\infty,0]} e^{\lambda\sqrt{-z}}\sigma_\pm(dz) < \infty$ for any $\lambda > 0$. For $\sigma \in \Sigma$, set

$$\begin{aligned} G(u, v; \sigma) = \frac{1}{4}\int_{-\infty}^{0} \frac{1}{\sqrt{-z}}\left(e^{\sqrt{-z}(u+v)} - e^{\sqrt{-z}|u-v|}\right)\sigma_+(dz) \\ + \frac{1}{4}\int_{-\infty}^{0} \frac{1}{\sqrt{-z}}\left(e^{-\sqrt{-z}|u-v|} - e^{-\sqrt{-z}(u+v)}\right)\sigma_-(dz). \end{aligned}$$

We consider a family $\mathcal{G}$ of all Gaussian processes X^σ with mean 0 and covariance function $G(u, v; \sigma)$, $\sigma \in \Sigma$. We also consider the totality $\mathcal{Q}$ of all functions q^σ, $\sigma \in \Sigma$, defined by

$$q^\sigma(x) = -4\frac{d^2}{dx^2}\log E\left[\exp\left(-\frac{1}{2}\int_0^x |X^\sigma(y)|^2 dy\right)\right], \quad X^\sigma \in \mathcal{G},$$

where E stands for the expectation with respect to the underlying probability measure. In [12], Kotani showed that $\mathcal{Q}$ includes all $\mathcal{Q}_n$, $n = 1, 2, \dots$, and any element of $\mathcal{Q}$ is obtained as a limit of reflectionless potentials in the topology of uniform convergence on compacts.

Furthermore, it is well known that $q(x,t)$ defined by (3) and (4) with $m_j(t) = m_j\exp[-2\eta_j^3 t]$ instead of m_j, $1 \le j \le n$, gives a rise of an n-soliton solution $u(x,t) = -q(x,t)$ of the KdV equation (2).

The facts mentioned above indicate that soliton solutions of the KdV equation may be represented in terms of Gaussian processes. In this paper, we shall establish such an expression of n-soliton solutions and the tau function, which plays a fundamental role in the study of the KdV hierarchy (see [14, 16, 17]), in the Wiener space.

If both components $\sigma_\pm$ of $\sigma \in \Sigma$ are discrete measures, then the corresponding Gaussian process belongs to $\bigcup_{n\in\mathbf{N}}\mathcal{G}_n$, where $\mathcal{G}_n$ is a set of Gaussian processes obtained as superpositions of n independent 1-dimensional Ornstein-Uhlenbeck processes (for the definition of $\mathcal{G}_n$, see

§1.2). In this case, the correspondence between $\mathcal{G}$ and $\mathcal{Q}$ is given concretely; for every $n \in \mathbf{N}$, we shall give a mapping from $\mathcal{G}_n$ to $\mathcal{Q}_n$. See §2. Moreover, not only reflectionless potentials but also n-soliton solutions and the tau function of the KdV equation can be represented in terms of Gaussian processes in $\bigcup_{n\in\mathbf{N}} \mathcal{G}_n$. See §4. These expressions show that "a superposition" to make an n-soliton solution out of 1-soliton ones can be realized in the Wiener space. Further, we can explicitly see how speed parameters of 1-solitons reflect on those of the n-solitons obtained as superpositions. See §4.

An exact expression of Wiener integrals of Wiener functionals of the form $\exp[-(a^2/2)\int_0^x X(y)^2dy+R(x)]$ of $X \in \bigcup_{n\in\mathbf{N}} \mathcal{G}_n$ plays a basic role in this paper, where $R(x)$ is a Wiener functional which varies according as we deal with reflectionless potentials, n-soliton solutions, and the tau function. Such exact expressions are achieved with the help of the Itô formula and the Cameron-Martin transformation. The Cameron-Martin transformation we deal with is determined by a second order ordinary differential equation. When a 1-dimensional Wiener process, which is in $\mathcal{G}_1$, is considered, the equation is the Sturm-Liouville one employed in [3]. The ordinary differential equations in this paper appear in different features from place to place, while they correspond to the same Wiener integral. Namely, we encounter several types of $n \times n$-matrix Riccati equations and second order $n \times n$-matrix and first order $2n \times n$-matrix linear ordinary differential equations. These different features are unified in terms of Grassmannians. See §1.1. It should be also mentioned that the above Riccati equations play an important role in the theory of the linear filtering problem by Kalman-Bucy. See §3.

Before closing this section, we note that the class $\mathcal{G}$ of Gaussian processes is closely related to the one studied by Hida-Streit [5] and Okabe [15].

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§1. Cameron-Martin transformation — Ornstein-Uhlenbeck process

1.1. Ordinary differential equations

We first recall several known facts about ordinary differential equations. For $n \in \mathbf{N}$, we set $\mathcal{A}_n = \mathcal{P}_n \times \mathcal{C}_n$, where

$$\mathcal{P}_n = \{\boldsymbol{p} = {}^t(p_1, \dots, p_n) \in \mathbf{R}^n : p_i \neq p_j \text{ for } i \neq j\},$$
$$\mathcal{C}_n = \{\boldsymbol{c} = {}^t(c_1, \dots, c_n) \in \mathbf{R}^n : c_i > 0 \text{ for } 1 \leq i \leq n\}.$$

For $(\boldsymbol{p}, \boldsymbol{c}) \in \mathcal{A}_n$ and $a > 0$, we define $n \times n$-matrices

$$D_{\boldsymbol{p}} = \operatorname{diag}[p_1, \dots, p_n] \quad \text{and} \quad E_{\boldsymbol{p},\boldsymbol{c}}(a) = D_{\boldsymbol{p}}^2 + a^2 \boldsymbol{c} \otimes \boldsymbol{c}.$$

We shall often write simply D and $E(a)$ for $D_{\boldsymbol{p}}$ and $E_{\boldsymbol{p},\boldsymbol{c}}(a)$, respectively.

Let $\Phi_a(y)$ be a solution of a first order $2n \times n$-matrix ordinary differential equation

$$\Phi' + M_{\boldsymbol{p},\boldsymbol{c},a}\Phi = 0, \quad \text{where } M_{\boldsymbol{p},\boldsymbol{c},a} = \begin{pmatrix} D_{\boldsymbol{p}} & I \\ a^2 \boldsymbol{c} \otimes \boldsymbol{c} & -D_{\boldsymbol{p}} \end{pmatrix} \tag{5}$$

and f' stands for the derivative of f. For $n \times n$-matrices A and B, we often write $\Phi_a(y; A, B)$ to emphasize the initial condition $\Phi_a(0) = \begin{pmatrix} A \\ B \end{pmatrix}$. Denote by $\phi_a(y)$ and $\psi_a(y)$ the upper and the lower half $n \times n$-submatrices of $\Phi_a(y)$, respectively;

$$\Phi_a(y) = \begin{pmatrix} \phi_a(y) \\ \psi_a(y) \end{pmatrix}.$$

Then $\phi_a(y)$ obeys a second order ordinary differential equation

$$\phi'' - E(a)\phi = 0. \tag{6}$$

In the cases we deal with in this paper, $\phi_a(y)$ is always shown to be invertible for any $y \geq 0$. Moreover, if $\Phi_a(y) = \Phi_a(y; I, 0)$, which is the case investigated in §2 and §3, then $\psi_a(z)$ is also invertible for $z > 0$ (see a paragraph after Theorem 2.1). Hence, in the remainder of this subsection, we assume that $\phi_a(y)$ and $\psi_a(z)$ are both invertible for any $y \geq 0$ and $z > 0$. Then $\Phi_a(y)$ determines an n-frame in a $2n$-dimensional vector space $V(2n)$ over $\mathbf{R}$, and hence gives a rise of a dynamics on the Grassmannian $GM(n, V(2n))$ consisting of all n-dimensional vector subspaces of $V(2n)$. Moreover, $\Phi_a(y)$ is identified in $GM(n, V(2n))$ with

$$\begin{pmatrix} I \\ \psi_a(y)\phi_a^{-1}(y) \end{pmatrix} = \begin{pmatrix} I \\ \gamma_a(y) \end{pmatrix},$$

where $\phi_a^{-1}(y) = (\phi_a(y))^{-1}$ and $\gamma_a(y) = \psi_a(y)\phi_a^{-1}(y) = -\phi'(y)\phi_a^{-1}(y) - D$. Due to the Cole-Hopf transformation, γ_a obeys the $n \times n$-matrix Riccati equation

$$\gamma' - \gamma D - D\gamma - \gamma^2 + a^2 \boldsymbol{c} \otimes \boldsymbol{c} = 0$$

(see [18]). We next consider an n-frame obtained by reversing the time of $\Phi_a(\cdot)$; $\widetilde{\Phi}_a(y) = \Phi_a(x-y)$. Set $\mu(y) = \gamma_a(x-y)$ and $\nu(y) = \mu(y)^{-1}$, $y < x$. Then, for $y < x$, $\widetilde{\Phi}_a(y)$ determines a point in $GM(n, V(2n))$ identified with $\begin{pmatrix} I \\ \mu(y) \end{pmatrix}$ and $\begin{pmatrix} \nu(y) \\ I \end{pmatrix}$. Thus a dynamics of $\widetilde{\Phi}_a(y)$, $y < x$, in the Grassmannian is expressed in two different ways by Riccati equations

$$\begin{aligned} &\mu' + \mu D + D\mu + \mu^2 - a^2 \boldsymbol{c} \otimes \boldsymbol{c} = 0, \\ &\nu' - \nu D - D\nu + \nu(a^2 \boldsymbol{c} \otimes \boldsymbol{c})\nu - I = 0. \end{aligned}$$

The second equation is a Riccati equation which an error matrix appearing in the linear filtering theory obeys (see §3 and [1]).

Let $\alpha_{ij}(y)$ be the (i,j)-component of the n-frame $\Phi_a(y)$, $0 \le i \le 2n-1$, $0 \le j \le n-1$. The Plücker coordinate of $\Phi_a(y)$ is given by

$$\alpha_I(y) = \det\bigl[\bigl(\alpha_{i_k j}(y)\bigr)_{0 \le k,j \le n-1}\bigr], \qquad I = (i_0, \dots, i_{n-1}) \in \mathcal{I},$$

where $\mathcal{I}$ is the totality of $I = (i_0, \dots, i_{n-1}) \in \mathbf{Z}^n$ with $0 \le i_0 < \cdots < i_{n-1} \le 2n-1$. We set $F = \begin{pmatrix} -D & -I \\ -a^2 \boldsymbol{c} \otimes \boldsymbol{c} & D \end{pmatrix}$ and define a $\binom{2n}{n} \times \binom{2n}{n}$-matrix G by

$$G\alpha_I = \sum_{k=0}^{n-1} \sum_{j=0}^{2n-1} F_{i_k j} \alpha_{i_0, \dots, i_{k-1}, j, i_{k+1}, \dots, i_{n-1}}, \qquad I = (i_0, \dots, i_{n-1}) \in \mathcal{I},$$

where, for $0 \le k_0, \dots, k_{n-1} \le 2n-1$, $\alpha_{k_0, \dots, k_{n-1}}$ is defined in the same manner as α_I. We then have a dynamics on the Grassmannian in terms of the Plücker coordinate;

$$\frac{d}{dy}\bigl(\alpha_I(y)\bigr)_{I \in \mathcal{I}} = G\bigl(\alpha_I(y)\bigr)_{I \in \mathcal{I}}.$$

It should be mentioned that

$$\alpha_I(y) = \det \phi_a(y) \qquad \text{for } I = (0, 1, \dots, n-1).$$

For related results, see also [4].

1.2. Cameron-Martin transformation

For $\boldsymbol{p} \in \mathcal{P}_n$, let $\xi_{\boldsymbol{p}}(y) = {}^t(\xi_{\boldsymbol{p}}^1(y), \ldots, \xi_{\boldsymbol{p}}^n(y))$ be the unique solution of the $\mathbf{R}^n$-valued stochastic differential equation

$$(7) \qquad d\xi(y) = dw(y) + D\xi(y)dy, \quad \xi(0) = 0.$$

We set $\mathcal{OU}_n = \{\xi_{\boldsymbol{p}} : \boldsymbol{p} \in \mathcal{P}_n\}$. For $(\boldsymbol{p}, \boldsymbol{c}) \in \mathcal{A}_n$, we define a superposition of 1-dimensional Ornstein-Uhlenbeck processes $\xi_{\boldsymbol{p}}^1(y), \ldots, \xi_{\boldsymbol{p}}^n(y)$ by

$$X_{\boldsymbol{p},\boldsymbol{c}}(y) = \langle \boldsymbol{c}, \xi_{\boldsymbol{p}}(y) \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product on $\mathbf{R}^n$. $X_{\boldsymbol{p},\boldsymbol{c}}(y)$, $0 \le y \le x$, is a continuous Gaussian process with mean 0 and covariance function

$$(8) \qquad R(u, v) = \sum_{j=1}^{n} \frac{c_j^2}{2p_j} \left(e^{p_j(u+v)} - e^{p_j|u-v|} \right), \quad 0 \le u, v \le x.$$

We set

$$\mathcal{G}_n = \{X_{\boldsymbol{p},\boldsymbol{c}} : (\boldsymbol{p}, \boldsymbol{c}) \in \mathcal{A}_n\} = \{\langle \boldsymbol{c}, \xi_{\boldsymbol{p}} \rangle : (\boldsymbol{p}, \boldsymbol{c}) \in \mathcal{A}_n\}.$$

Obviously, $aX_{\boldsymbol{p},\boldsymbol{c}} = X_{\boldsymbol{p},a\boldsymbol{c}}$ for $a > 0$, and hence $\mathcal{G}_n$ is closed under the multiplication by positive numbers. Moreover, $X_{\boldsymbol{p},\boldsymbol{c}} \in \mathcal{G}_n$ is invariant under permutation in the sense that $X_{\boldsymbol{p}',\boldsymbol{c}'} = X_{\boldsymbol{p},\boldsymbol{c}}$ if $(\boldsymbol{p}, \boldsymbol{c}) \in \mathcal{A}_n$, σ is a permutation of $\{1, \ldots, n\}$, and $\boldsymbol{p}' = {}^t(p_{\sigma(1)}, \ldots, p_{\sigma(n)})$, $\boldsymbol{c}' = {}^t(c_{\sigma(1)}, \ldots, c_{\sigma(n)})$.

Given an $n \times n$-matrix valued C^1 function κ on $[0, x]$ with $\det \kappa(y) \neq 0$ for each $y \in [0, x]$, we define two Cameron-Martin transformations K and L on $\mathcal{W}^n$ by

$$(9) \qquad K[w](y) = w(y) - \int_0^y \kappa'(u)\kappa^{-1}(u)w(u)du,$$

$$(10) \qquad L[w](y) = w(y) - \kappa(y) \int_0^y \left(\kappa^{-1}\right)'(u)w(u)du, \quad w \in \mathcal{W}^n.$$

By a change of variables formula on $[0, x]$, we see that

$$(11) \qquad K[L[w]] = L[K[w]] = w \quad \text{for any } w \in \mathcal{W}^n.$$

Set $\widetilde{\theta}(y) = \kappa'(y)\kappa^{-1}(y)$ and let $\widehat{\theta}$ be an $n \times n$-matrix valued continuous function on $[0, x]$. We then have

Lemma 1.1. *For any measurable $f : \mathcal{W}^n \to [0,\infty)$, it holds that*

$$\begin{aligned}(12)\quad &\int_{\mathcal{W}^n} f(\xi_{\boldsymbol{p}}) \exp\Big[-\frac{a^2}{2}\int_0^x X_{\boldsymbol{p},\boldsymbol{c}}(y)^2 dy \\ &\qquad + \frac{1}{2}\langle(\widetilde{\theta}(x) - D)\xi_{\boldsymbol{p}}(x), \xi_{\boldsymbol{p}}(x)\rangle - \frac{1}{2}\int_0^x |\widehat{\theta}(y)\xi_{\boldsymbol{p}}(y)|^2 dy\Big]\, dP \\ &= \int_{\mathcal{W}^n} f(w) \exp\Big[-\frac{1}{2}\int_0^x \langle E(a)w(y), w(y)\rangle dy \\ &\qquad + \frac{1}{2}\langle\widetilde{\theta}(x)w(x), w(x)\rangle - \frac{1}{2}\int_0^x |\widehat{\theta}(y)w(y)|^2 dy\Big]\, dP\, e^{-(x/2)\mathrm{tr}D}.\end{aligned}$$

Proof. By using the Maruyama-Girsanov transformation ([8, 13, 19]), we obtain that

$$\begin{aligned}&\text{the left hand side of (12)} \\ &= \int_{\mathcal{W}^n} f(w) \exp\Big[-\frac{a^2}{2}\int_0^x \langle \boldsymbol{c}, w(y)\rangle^2 dy \\ &\qquad + \frac{1}{2}\langle(\widetilde{\theta}(x) - D)w(x), w(x)\rangle - \frac{1}{2}\int_0^x |\widehat{\theta}(y)w(y)|^2 dy \\ &\qquad\qquad + \int_0^x \langle Dw(y), dw(y)\rangle - \frac{1}{2}\int_0^x |Dw(y)|^2 dy\Big]\, P(dw),\end{aligned}$$

where the identity may hold as $\infty = \infty$. Applying the Itô formula ([8, 13]) to $\langle Dw(x), w(x)\rangle$, it is easily seen that this implies (12). Q.E.D.

Suppose that a solution $\phi_a(y)$, $y \in [0,x]$, of the ordinary differential equation (6) satisfies the condition that $\det\phi_a(y) \neq 0$, $0 \le y \le x$, where the initial condition is not specified. We set

$$(13)\qquad \beta_{a,x}(y) = -(\phi_a'\phi_a^{-1})(x-y),$$

and denote by $\widetilde{\beta}_{a,x}(y)$ and $\widehat{\beta}_{a,x}(y)$ its symmetric and skew-symmetric parts, respectively. Let $\kappa_{a,x}(y)$ be the solution of the differential equation

$$\kappa'(y) = \widetilde{\beta}_{a,x}(y)\kappa(y), \quad \kappa(x) = I,$$

and define linear transformations $K_{a,x}, L_{a,x} : \mathcal{W}^n \to \mathcal{W}^n$ by (9) and (10) with $\kappa = \kappa_{a,x}$.

Proposition 1.1. *For any bounded measurable $f : \mathcal{W}^n \to [0,\infty)$, it holds that*

$$\begin{aligned}(14)\quad &\int_{\mathcal{W}^n} f(\xi_p)\exp\Big[-\frac{a^2}{2}\int_0^x X_{p,c}(y)^2dy \\ &\qquad +\frac{1}{2}\langle(\widetilde{\beta}_{a,x}(x)-D)\xi_p(x),\xi_p(x)\rangle-\frac{1}{2}\int_0^x |\widehat{\beta}_{a,x}(y)\xi_p(y)|^2dy\Big]dP \\ &\qquad = \big(\det\phi_a(0)\big)^{1/2}\big(e^{x\mathrm{tr}D}\det\phi_a(x)\big)^{-1/2}\int_{\mathcal{W}^n} f\circ L_{a,x}\,dP.\end{aligned}$$

Proof. For the sake of simplicity, we write β, $\widetilde{\beta}$, and $\widehat{\beta}$ for $\beta_{a,x}$, $\widetilde{\beta}_{a,x}$, and $\widehat{\beta}_{a,x}$, respectively. It follows from (6) that

$$(15)\qquad \beta' = E(a)-\beta^2 \quad\text{and}\quad \widetilde{\beta}' = E(a)-\widetilde{\beta}^2-\widehat{\beta}^2.$$

Since $\mathrm{tr}\widetilde{\beta}=\mathrm{tr}\beta$, by virtue of the Itô formula, we then have

$$\begin{aligned}&\frac{1}{2}\langle\widetilde{\beta}(x)w(x),w(x)\rangle \\ &=\frac{1}{2}\int_0^x\langle E(a)w(y),w(y)\rangle dy+\frac{1}{2}\int_0^x|\widehat{\beta}(y)w(y)|^2dy \\ &\qquad+\int_0^x\langle\widetilde{\beta}(y)w(y),dw(y)\rangle-\frac{1}{2}\int_0^x|\widetilde{\beta}(y)w(y)|^2dy+\frac{1}{2}\int_0^x\mathrm{tr}\beta(y)dy.\end{aligned}$$

Combining this with Lemma 1.1, we have

$$\begin{aligned}(16)\quad &\int_{\mathcal{W}^n} f(\xi_p)\exp\Big[-\frac{a^2}{2}\int_0^x X_{p,c}(y)^2dy \\ &\qquad +\frac{1}{2}\langle(\widetilde{\beta}(x)-D)\xi_p(x),\xi_p(x)\rangle-\frac{1}{2}\int_0^x |\widehat{\beta}(y)\xi_p(y)|^2dy\Big]dP \\ &=\int_{\mathcal{W}^n} f(w)\exp\Big[\int_0^x\langle\widetilde{\beta}(y)w(y),dw(y)\rangle-\frac{1}{2}\int_0^x|\widetilde{\beta}(y)w(y)|^2dy\Big]P(dw) \\ &\qquad\qquad \times\exp\Big[-\frac{x}{2}\mathrm{tr}D+\frac{1}{2}\int_0^x\mathrm{tr}\beta(y)dy\Big].\end{aligned}$$

Applying the Maruyama-Girsanov transformation to the equation

$$dz(y)=dw(y)+\widetilde{\beta}(y)z(y)dy,$$

and noting that $\widetilde{\beta} = \kappa'_{a,x}\kappa^{-1}_{a,x}$, we obtain that

$$(17)\quad \int_{\mathcal{W}^n} (g \circ K_{a,x})(w) \exp\left[\int_0^x \langle \widetilde{\beta}(y)w(y), dw(y)\rangle \right.$$
$$\left. - \frac{1}{2}\int_0^x |\widetilde{\beta}(y)w(y)|^2 dy\right] P(dw) = \int_{\mathcal{W}^n} g\, dP$$

for any bounded measurable $g : \mathcal{W}^n \to [0,\infty)$. By the definition of β, we have

$$(18)\qquad \exp\left[\int_0^x \mathrm{tr}\beta(y)dy\right] = \det\phi_a(0)\big(\det\phi_a(x)\big)^{-1}.$$

By (11), combining (17) and (18) with (16), we obtain (14). Q.E.D.

1.3. Eigenvalues of $E(a)$

Let $(\boldsymbol{p},\boldsymbol{c}) \in \mathcal{A}_n$ and $a > 0$. We shall specify the eigenvalues of $E(a) = E_{\boldsymbol{p},\boldsymbol{c}}(a)$. As for $\boldsymbol{p} = {}^t(p_1,\ldots,p_n)$, rearranging if necessary, we may assume that there exist m and $1 \le j(1) < \cdots < j(m) \le n$ such that

(H)$_m$ $|p_j| \le |p_{j+1}|$ for $j = 1,2,\ldots,n-1$, $p_{j(\ell)} > 0$ and $p_{j(\ell)+1} = -p_{j(\ell)}$ for $\ell = 1,\ldots,m$, and $\#\{|p_1|,\ldots,|p_n|\} = n-m$.

When $m = 0$, this conditions means that $|p_1| < |p_2| < \cdots < |p_n|$. If $n = 1$, then (H)$_0$ holds. We define a Herglotz function $h_{\boldsymbol{p},\mathbf{c},a}$ on $\mathbf{C}^+ = \{z \in \mathbf{C} : \mathrm{Im} z > 0\}$ by

$$h_{\boldsymbol{p},\mathbf{c},a}(z) = \frac{1}{2}\int_0^\infty \frac{1}{u-z}\{\sigma_+ + \sigma_-\}(-du),$$

where

$$(19)\quad \sigma_+(du) = 2a^2 \sum_{i:p_i \ge 0} c_i^2 \delta_{-p_i^2}(du), \quad \sigma_-(du) = 2a^2 \sum_{i:p_i<0} c_i^2 \delta_{-p_i^2}(du)$$

(cf.[11]). Then $h_{\boldsymbol{p},\mathbf{c},a}(\lambda + t\sqrt{-1})$ converges to

$$h_{\boldsymbol{p},\mathbf{c},a}(\lambda + 0\sqrt{-1}) = a^2 \sum_{j=1}^n \frac{c_j^2}{p_j^2 - \lambda} \qquad \text{as } t \downarrow 0.$$

Under (H)$_m$, the equation $h_{\boldsymbol{p},\mathbf{c},a}(r + 0\sqrt{-1}) = -1$ possesses $n-m$ roots $0 < r_1 < \cdots < r_{n-m}$ such that $r_j^{1/2} \notin \{p_{j(1)},\ldots,p_{j(m)}\}$, $j = 1,\ldots,n-m$. Take $\eta_1,\ldots,\eta_n \in \mathbf{R}$ so that

$$\{|\eta_1|,\ldots,|\eta_n|\} = \{p_{j(1)},\ldots,p_{j(m)}, r_1^{1/2},\ldots,r_{n-m}^{1/2}\}.$$

Define an $n \times n$ matrix $U = (U_{ij})_{1 \le i,j \le n}$ by

$$
\text{(20)} \qquad U_{ij} = \begin{cases} \dfrac{c_i}{|(D^2 - r_k I)^{-1} \boldsymbol{c}|(p_i^2 - r_k)}, & \text{if } \eta_j^2 = r_k, \\[2ex] \dfrac{\delta_{i,j(\ell)+1} c_{j(\ell)} - \delta_{i,j(\ell)} c_{j(\ell)+1}}{(c_{j(\ell)}^2 + c_{j(\ell)+1}^2)^{1/2}}, & \text{if } \eta_j^2 = p_{j(\ell)}^2. \end{cases}
$$

Lemma 1.2. *$U \in O(n)$ and it holds that*

$$
E(a) = UR^2U^{-1}, \qquad \textit{where } R = \operatorname{diag}[\eta_1, \dots, \eta_n].
$$

Proof. It is easily checked that $p_{j(\ell)}^2$ is an eigenvalue of $E(a)$ with eigenvector

$$
u = {}^t(\underbrace{0, \dots, 0}_{j(\ell)-1}, -c_{j(\ell)+1}, c_{j(\ell)}, 0, \dots, 0).
$$

Noting that $D^2 - r_j I$ is invertible, we see that r_j is an eigenvalue with eigenvector $u = (D^2 - r_j I)^{-1} \boldsymbol{c}$. Thus we have obtained n distinct eigenvalues of $E(a)$ and the associated eigenvectors. In conjunction with the symmetry of $E(a)$, we obtain the desired assertion. Q.E.D.

§2. Reflectionless potential

For $(\boldsymbol{p}, \boldsymbol{c}) \in \mathcal{A}_n$ and $a > 0$, define $\sigma = (\sigma_+, \sigma_-) \in \Sigma$ by (19). Then $G(u, v; \sigma)$ coincides with $a^2 R(u, v)$, the covariance function described in (8). Hence we can identify X^σ with $aX_{\boldsymbol{p},\boldsymbol{c}}$, and $\mathcal{G}_n \subset \mathcal{G}$. We shall spell out a correspondence between $\mathcal{G}_n$ and $\mathcal{Q}_n$.

Assuming $(\mathrm{H})_m$, we define $0 < r_1 < \cdots < r_{n-m}$ as described before Lemma 1.2. Define $0 < \eta_1 < \cdots < \eta_n$ and $m_1, \dots, m_n > 0$ by

$$
\{\eta_1, \dots, \eta_n\} = \{p_{j(1)}, \dots, p_{j(m)}, r_1^{1/2}, \dots, r_{n-m}^{1/2}\}.
$$

$$
m_i = \begin{cases} 2\eta_i \dfrac{c_{j(\ell)+1}^2}{c_{j(\ell)}^2} \displaystyle\prod_{k \ne i} \frac{\eta_k + \eta_i}{\eta_k - \eta_i} \prod_{k \ne j(\ell), j(\ell)+1} \frac{p_k + \eta_i}{p_k - \eta_i}, & \text{if } i = j(\ell), \\[2ex] -2\eta_i \displaystyle\prod_{k \ne i} \frac{\eta_k + \eta_i}{\eta_k - \eta_i} \prod_{k=1}^{n} \frac{p_k + \eta_i}{p_k - \eta_i}, & \text{otherwise.} \end{cases}
$$

Mention that $\eta_{j(\ell)} = p_{j(\ell)}$, $\ell = 1, \dots, m$. We set

$$
\text{(21)} \qquad I_{\boldsymbol{p},\boldsymbol{c},a}(x) = \int_{\mathcal{W}^n} \exp\left[-\frac{a^2}{2} \int_0^x X_{\boldsymbol{p},\boldsymbol{c}}(y)^2 dy \right] dP.
$$

Theorem 2.1. *Define $A(x)$ by (4) with the above scattering data $\eta_i, m_i > 0$, $i = 1, \ldots, n$. Then it holds that*

$$\log(I_{\boldsymbol{p},\mathbf{c},a}(x)) = -\frac{1}{2}\log\det(I + A(x)) + \frac{1}{2}\log\det(I + A(0)) - \frac{x}{2}\sum_{i=1}^{n}(p_i + \eta_i). \tag{22}$$

In particular,

$$4\frac{d^2}{dx^2}\log(I_{\boldsymbol{p},\mathbf{c},a}(x)) = -2\frac{d^2}{dx^2}\log\det(I + A(x)).$$

Note that $I_{\boldsymbol{p},\mathbf{c},a}(x)$ is invariant under the permutation of parameters $(\boldsymbol{p}, \boldsymbol{c})$ in the sense as stated after (8). Thus, so is the reflectionless potential associated with $I_{\boldsymbol{p},\mathbf{c},a}(x)$.

The proof of the theorem is divided into two steps, each step being a lemma. In the sequel, we write R for $\mathrm{diag}[\eta_1, \ldots, \eta_n]$ and let $\phi_a(y)$ be the upper half of $\Phi_a(y; I, 0)$. Then

$$\phi_a(y) = U\{\cosh(yR) - \sinh(yR)R^{-1}U^{-1}DU\}U^{-1}, \tag{23}$$

where $U \in O(n)$ is defined by (20), and, for $n \times n$-matrix A,

$$\cosh(A) = (e^A + e^{-A})/2 \quad \text{and} \quad \sinh(A) = (e^A - e^{-A})/2.$$

By computing the product matrix $\phi_a\psi_a$, we see that $\phi_a(y)$ and $\psi_a(z)$ are both invertible for $y \geq 0$, $z > 0$. Then we can define $\beta_{a,x}$ by (13) with this ϕ_a. Obviously $\beta_{a,x}(x) = D$. Since $\beta_{a,x}$ obeys the Riccati equation (15), it turns out to be symmetric. Applying Proposition 1.1, we obtain

$$I_{\boldsymbol{p},\mathbf{c},a}(x) = \big(\det\phi_a(0)\big)^{1/2}\big(\det\phi_a(x)\big)^{-1/2}e^{-(x/2)\mathrm{tr}D}. \tag{24}$$

In this expression, for the latter use, we left $\det\phi_a(0)$ while it is equal to one. If $(\mathrm{H})_0$ holds, then we set

$$\begin{gathered}
X_{ij} = \big(p_j + r_i^{1/2}\big)^{-1}, \quad Y_{ij} = \big(p_j - r_i^{1/2}\big)^{-1}, \quad 1 \leq i, j \leq n, \\
X = \big(X_{ij}\big)_{1\leq i,j\leq n}, \quad Y = \big(Y_{ij}\big)_{1\leq i,j\leq n}, \\
V(\boldsymbol{c}) = \mathrm{diag}\big[|(D^2 - r_1 I)^{-1}\boldsymbol{c}|^{-1}, \ldots, |(D^2 - r_n I)^{-1}\boldsymbol{c}|^{-1}\big], \\
\sigma(i) = \mathrm{sgn}\Bigg[\prod_{\beta=1}^{n}(p_\beta - \eta_i)\Bigg], \quad b(i) = \sigma(i)\Bigg\{-2\eta_i\frac{\prod_{\alpha\neq i}(\eta_\alpha^2 - \eta_i^2)}{\prod_{\beta=1}^{n}(p_\beta^2 - \eta_i^2)}\Bigg\}^{1/2},
\end{gathered}$$

and $B = \mathrm{diag}[b(1), \ldots, b(n)]$.

Lemma 2.1. *Suppose that* $(\mathrm{H})_0$ *holds. Then it holds that*

$$\phi_a(y) = -\frac{1}{2}UV(\boldsymbol{c})R^{-1}B\big(I + A(y)\big)e^{yR}B^{-1}XC(\boldsymbol{c}), \tag{25}$$

where $C(\boldsymbol{c}) = \mathrm{diag}[c_1, \ldots, c_n]$. *Moreover, the identity* (22) *holds.*

Proof. Due to (23), we have

$$\phi_a(y) = \frac{1}{2}UR^{-1}\big\{e^{yR}(RU^{-1} - U^{-1}D) + e^{-yR}(RU^{-1} + U^{-1}D)\big\}.$$

Set $Z = (Z_{ij})_{1\le i,j\le n}$, where

$$Z_{ij} = (p_i^2 - r_j)^{-1}, \quad 1 \le i, j \le n.$$

Then it holds that

$$U = C(\boldsymbol{c})ZV(\boldsymbol{c}), \quad U^{-1} = V(\boldsymbol{c})\,{}^tZC(\boldsymbol{c}).$$

Since $R, D, C(\boldsymbol{c})$, and $V(\boldsymbol{c})$ are all diagonal matrices, we have that

$$RU^{-1} - U^{-1}D = -V(\boldsymbol{c})XC(\boldsymbol{c}), \quad RU^{-1} + U^{-1}D = V(\boldsymbol{c})YC(\boldsymbol{c}).$$

Hence we obtain

$$\phi_a(y) = -\frac{1}{2}UR^{-1}V(\boldsymbol{c})\big\{I - e^{-yR}YX^{-1}e^{-yR}\big\}e^{yR}XC(\boldsymbol{c}). \tag{26}$$

We now compute YX^{-1}. Applying Cauchy's identity (cf. [14])

$$\det\left(\left(\frac{1}{\alpha_i + \beta_j}\right)_{1\le i,j\le n}\right) = \frac{\prod_{1\le i<j\le n}(\alpha_i - \alpha_j)(\beta_i - \beta_j)}{\prod_{i,j=1}^n(\alpha_i + \beta_j)} \quad \text{for } \alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \in \mathbf{C}, \tag{27}$$

to the cofactor matrix of X, we obtain

$$\left(X^{-1}\right)_{k\ell} = \frac{\prod_{\alpha\ne\ell}(p_k + \eta_\alpha)\prod_{\beta=1}^n(p_\beta + \eta_\ell)}{\prod_{\beta\ne k}(p_\beta - p_k)\prod_{\alpha\ne\ell}(\eta_\alpha - \eta_\ell)}, \quad 1 \le k, \ell \le n.$$

Using Lagrange's interpolation formula

$$\sum_{k=1}^n \frac{\prod_{j=1}^{n-1}(a_k + b_j)\prod_{\beta\ne k}(a_\alpha - z)}{\prod_{\beta\ne k}(a_\alpha - a_k)} = \prod_{j=1}^{n-1}(z + b_j), \quad z \in \mathbf{C},$$

we have

$$(YX^{-1})_{ij} = \frac{\prod_{\alpha\ne i}(\eta_\alpha + \eta_i)}{\prod_{\beta=1}^n(p_\beta - \eta_i)}\,\frac{2\eta_i}{\eta_i + \eta_j}\,\frac{\prod_{\beta=1}^n(p_\beta + \eta_j)}{\prod_{\alpha\ne j}(\eta_\alpha - \eta_j)}, \quad 1 \le i, j \le n.$$

Since $\operatorname{sgn}\big[\big(\prod_{\beta=1}^n(p_\beta+\eta_i)\big)/\big(\prod_{\alpha\neq i}(\eta_\alpha-\eta_i)\big)\big]=-\sigma(i)$, it holds that

$$b(i)\sqrt{m_i}=2\eta_i\frac{\prod_{\alpha\neq i}(\eta_\alpha+\eta_i)}{\prod_{\beta=1}^n(p_\beta-\eta_i)},\quad \frac{\sqrt{m_i}}{b(i)}=-\frac{\prod_{\beta=1}^n(p_\beta+\eta_i)}{\prod_{\alpha\neq i}(\eta_\alpha-\eta_i)},\quad 1\le i\le n.$$

Hence

$$(YX^{-1})_{ij}=-\frac{b(i)\sqrt{m_i}\sqrt{m_j}(b(j))^{-1}}{\eta_i+\eta_j},\quad 1\le i,j\le n.$$

Combining this with (26), we obtain (25).

The identity (25) implies

$$\det\phi_a(0)\big(\det\phi_a(x)\big)^{-1}=\det(I+A(0))\big(e^{x\operatorname{tr}R}\det(I+A(x))\big)^{-1}.$$

Thus the second assertion follows from this and (24). Q.E.D.

Lemma 2.2. *Let $m\ge 1$ and suppose that* $(\mathrm{H})_m$ *is satisfied. Then* (22) *holds.*

Proof. For $\varepsilon>0$, set

$$p_i^\varepsilon=\begin{cases}p_i-\varepsilon, & \text{if } i=j(\ell)+1 \text{ for some } \ell,\\ p_i, & \text{otherwise,}\end{cases}\qquad 1\le i\le n.$$

Choosing a sufficiently small $\varepsilon>0$, we may assume that

$$|p_1^\varepsilon|<|p_2^\varepsilon|<\cdots<|p_n^\varepsilon|.$$

Let $0<r_1^\varepsilon<\cdots<r_n^\varepsilon$ be roots of $a^2\sum_{i=1}^n c_i^2/\{(p_i^\varepsilon)^2-r\}=-1$. Then it holds that

$$(p_i^\varepsilon)^2<r_i^\varepsilon<(p_{i+1}^\varepsilon)^2<r_{i+1}^\varepsilon,\quad i=1,2,\dots,n-1. \tag{28}$$

Define scattering data $\eta_i^\varepsilon, m_i^\varepsilon>0$, $i=1,\dots,n$, with these p_i^ε's, r_i^ε's and c_i's as described before Theorem 2.1. By Lemma 2.1, we have

$$\log\big(I_{\boldsymbol{p}^\varepsilon,\boldsymbol{c},a}(x)\big)=-\frac{1}{2}\log\det(I+A^\varepsilon(x))+\frac{1}{2}\log\det(I+A^\varepsilon(0))-\frac{x}{2}\sum_{i=1}^n(p_i^\varepsilon+\eta_i^\varepsilon)$$

where $\boldsymbol{p}^\varepsilon={}^t(p_1^\varepsilon,\dots,p_n^\varepsilon)$ and $A^\varepsilon(x)$ is defined by (4) with $\eta_i^\varepsilon, m_i^\varepsilon$, $i=1,\dots,n$. Since $p_i^\varepsilon\to p_i$ as $\varepsilon\downarrow 0$, $i=1,\dots,n$, we have that

$$\log\big(I_{\boldsymbol{p}^\varepsilon,\boldsymbol{c},a}(x)\big)\to\log\big(I_{\boldsymbol{p},\boldsymbol{c},a}(x)\big)\quad\text{as } \varepsilon\downarrow 0.$$

Moreover, recalling that $\eta_i^\varepsilon = (r_i^\varepsilon)^{1/2}$, $i = 1, \ldots, n$, it follows from (28) that $\eta_i^\varepsilon \to \eta_i$ as $\varepsilon \downarrow 0$, $i = 1, \ldots, n$. Hence

$$\sum_{i=1}^{n}(p_i^\varepsilon + \eta^\varepsilon) \to \sum_{i=1}^{n}(p_i + \eta_i), \quad \text{as } \varepsilon \downarrow 0.$$

Thus the proof completes once we have shown the convergence of $A^\varepsilon(y)$ to $A(y)$ as $\varepsilon \downarrow 0$.

To see the convergence of $A^\varepsilon(y)$, it suffices to show that m_i^ε tends to m_i as $\varepsilon \downarrow 0$ for every $i = 1, \ldots, n$. If $i \neq j(\ell)$ for any ℓ, then it is easily seen that $m_i^\varepsilon \to m_i$ as $\varepsilon \downarrow 0$. We now consider the case that $i = j(\ell)$. Since $\eta_{j(\ell)}^\varepsilon \to \eta_{j(\ell)} = p_{j(\ell)}$ as $\varepsilon \downarrow 0$,

$$\frac{p_{j(\ell)}^\varepsilon + \eta_{j(\ell)}^\varepsilon}{p_{j(\ell)+1}^\varepsilon - \eta_{j(\ell)}^\varepsilon} \longrightarrow -1, \quad \text{as } \varepsilon \downarrow 0. \tag{29}$$

It follows from (28) and the identity $a^2 \sum_{j=1}^n c_j^2/\{(p_j^\varepsilon)^2 - r_{j(\ell)}^\varepsilon\} = -1$ that

$$c_{j(\ell)}^2\{(p_{j(\ell)} + \varepsilon)^2 - r_{j(\ell)}^\varepsilon\} + c_{j(\ell)+1}^2\{p_{j(\ell)}^2 - r_{j(\ell)}^\varepsilon\} = O(\varepsilon^2), \quad \text{as } \varepsilon \downarrow 0.$$

This yields that

$$(p_{j(\ell)}^\varepsilon)^2 - (\eta_{j(\ell)}^\varepsilon)^2 = \frac{-2p_{j(\ell)}c_{j(\ell)}^2}{c_{j(\ell)}^2 + c_{j(\ell)+1}^2}\,\varepsilon + O(\varepsilon^2), \tag{30}$$

$$(p_{j(\ell)+1}^\varepsilon)^2 - (\eta_{j(\ell)}^\varepsilon)^2 = \frac{2p_{j(\ell)}c_{j(\ell)+1}^2}{c_{j(\ell)}^2 + c_{j(\ell)+1}^2}\,\varepsilon + O(\varepsilon^2), \quad \text{as } \varepsilon \downarrow 0. \tag{31}$$

Hence

$$\frac{p_{j(\ell)+1}^\varepsilon + \eta_{j(\ell)}^\varepsilon}{p_{j(\ell)}^\varepsilon - \eta_{j(\ell)}^\varepsilon} \longrightarrow c_{j(\ell)+1}^2 c_{j(\ell)}^{-2}, \quad \text{as } \varepsilon \downarrow 0.$$

Combining this with (29) and the definition of $m_{j(\ell)}^\varepsilon$, we see that $m_{j(\ell)}^\varepsilon \to m_{j(\ell)}$ as $\varepsilon \downarrow 0$. Q.E.D.

Remark 2.1. (i) The exponent of the integrand in the right hand side of (21) is a sum of a quadratic Wiener functional and a constant. Hence the right hand side of (24) can be expressed in terms of the Carleman-Fredholm determinant of the symmetric Hilbert-Schmidt operator determining the quadratic Wiener functional. Moreover, ϕ_a is a solution of the Jacobi equation associated with the Lagrangian related to the Wiener functional (cf. [6, 7]).

(ii) In Theorem 2.1, for each $n \in \mathbf{N}$, a mapping from $\mathcal{A}_n$ to the space

$\{(\eta_1, \ldots, \eta_n, m_1, \ldots, m_n) : 0 < \eta_1 < \cdots < \eta_n, m_1, \ldots, m_n > 0\}$ of scattering data was established. If $n = 2$, the mapping is invertible.

§3. Filtering theory

In this section, we shall see that the change of variables formula (24) relates to the filtering theory. On the $(n+1)$-dimensional Wiener space $\mathcal{W}^{n+1}$, consider the following filtering problem.

$$\begin{aligned} d\xi_{\boldsymbol{p}}(y) &= dw(y) + D\xi_{\boldsymbol{p}}(y)dy, & \xi(0) &= 0, & &\text{(system)}, \\ dY(y) &= db(y) + a\langle \boldsymbol{c}, \xi_{\boldsymbol{p}}(y)\rangle dy, & Y(0) &= 0, & &\text{(observation)}, \end{aligned}$$

where $(\boldsymbol{p}, \boldsymbol{c}) \in \mathcal{A}_n$, $a > 0$ and $(w, b) \in \mathcal{W}^n \times \mathcal{W}^1 = \mathcal{W}^{n+1}$. Let $\mathcal{F}_y^Y$ be the σ-field generated by $Y(u)$, $u \le y$. A solution $\widehat{\xi}_{\boldsymbol{p}}(y)$ to the filtering problem with respect to $Y(u)$, $u \le y$, is realized as a function whose error matrix is minimal in the space of error matrices of $\mathbf{R}^n$-valued $\mathcal{F}_y^Y$-measurable functions, where the order is the one inherited from the non-negative definiteness ([1, Theorem 4.1]). In our case, $\widehat{\xi}_{\boldsymbol{p}}(y)$ coincides with the conditional expectation $E[\xi_{\boldsymbol{p}}(y)|\mathcal{F}_y^Y]$ of $\xi_{\boldsymbol{p}}(y)$ given $\mathcal{F}_y^Y$, which is called the Kalman-Bucy filter. The corresponding error matrix

$$P_a(y) = \int_{\mathcal{W}^{n+1}} (\xi_{\boldsymbol{p}}(y) - \widehat{\xi}_{\boldsymbol{p}}(y))^t (\xi_{\boldsymbol{p}}(y) - \widehat{\xi}_{\boldsymbol{p}}(y))\, dP$$

obeys the $n \times n$-matrix Riccati equation

$$P' = DP + PD - P(a^2\boldsymbol{c} \otimes \boldsymbol{c})P + I, \quad P(0) = 0,$$

which we have already seen as an equation for $\nu(y)$ in §1.1. Let $\rho_{\boldsymbol{p},\mathbf{c},a}(y)$ be the error variance of $X_{\boldsymbol{p},\mathbf{c}}(y)$;

$$\rho_{\boldsymbol{p},\mathbf{c},a}(y) = \int_{\mathcal{W}^{n+1}} \left| X_{\boldsymbol{p},\mathbf{c}}(y) - E[X_{\boldsymbol{p},\mathbf{c}}(y)|\mathcal{F}_y^Y] \right|^2 dP.$$

It then holds that

$$\rho_{\boldsymbol{p},\mathbf{c},a}(y) = \operatorname{tr}\left[(\boldsymbol{c} \otimes \boldsymbol{c})P_a(y)\right].$$

Let $\Phi_a(y; I, 0) = \begin{pmatrix} \phi_a(y) \\ \psi_a(y) \end{pmatrix}$. As was seen before (24), $\det \phi_a(y) \neq 0$, $y \ge 0$. We set $\gamma_a(y) = \psi_a(y)\phi_a^{-1}(y)$ and $\gamma_{a,x}(y) = \gamma_a(x - y)$. Note that

$$\gamma'_{a,x} = -D\gamma_{a,x} - \gamma_{a,x}D - \gamma_{a,x}^2 + a^2(\boldsymbol{c} \otimes \boldsymbol{c}) \quad \text{on } [0, x], \quad \gamma_{a,x}(x) = 0.$$

Then it holds that

$$\{\det(I-\gamma_{a,x}P_a)\}' = \left(-\mathrm{tr}\left[a^2(\boldsymbol{c}\otimes\boldsymbol{c})P_a+\gamma_{a,x}\right]\right)\det(I-\gamma_{a,x}P_a).$$

Since $P(0)=\gamma_{a,x}(x)=0$,

$$1=\det(I-\gamma_{a,x}(x)P_a(x))=\exp\left[-\int_0^x \mathrm{tr}\left[a^2(\boldsymbol{c}\otimes\boldsymbol{c})P_a(y)+\gamma_{a,t}(y)\right]dy\right].$$

Combined with (18), this implies

$$\exp\left[-\int_0^x \mathrm{tr}\left[a^2(\boldsymbol{c}\otimes\boldsymbol{c})P_a(y)\right]dy\right]=\frac{e^{-x\mathrm{tr}D}}{\det\phi_a(x)}.$$

By (24), we obtain

$$\int_{\mathcal{W}^n}\exp\left[-\frac{a^2}{2}\int_0^x X_{\boldsymbol{p},\boldsymbol{c}}(y)^2dy\right]dP=\exp\left[-\frac{a^2}{2}\int_0^x \rho_{\boldsymbol{p},\boldsymbol{c},a}(y)dy\right],$$

which can be also shown by applying the result due to M.L. Kleptsyna and A. Le Breton [10].

Let $\delta(y,u)$ be the unique solution of the integral equation

$$(32)\qquad \delta(y,u)=a^2R(y,u)-\int_0^y\delta(y,v)\delta(u,v)dv,\quad 0\le y,u\le x.$$

In [10], they also have shown that $a^2\rho_{\boldsymbol{p},\boldsymbol{c},a}(y)$ coincides with $\delta(y,y)$.

§4. KdV equation

Throughout this section, we assume $(\mathrm{H})_m$. Let scattering data $0<\eta_1<\cdots<\eta_n$, $m_1,\dots,m_n>0$ and $n\times n$-matrices U and R be the ones stated before and after Theorem 2.1. Set

$$\mathbf{T}=\{\boldsymbol{t}=(t_1,t_2,\dots,):t_j\in\mathbf{R},\#\{j:t_j\neq 0\}<\infty\}.$$

For $x\in\mathbf{R},\boldsymbol{t}\in\mathbf{T}$, define

$$\zeta_j(x,\boldsymbol{t})=x\eta_j+\sum_{\alpha=1}^{\infty}t_\alpha\eta_j^{2\alpha+1},\quad \zeta(x,\boldsymbol{t})=\mathrm{diag}[\zeta_1(x,\boldsymbol{t}),\dots,\zeta_n(x,\boldsymbol{t})].$$

The tau function $\tau(x,\boldsymbol{t})$ of the KdV equation is of the form

$$(33)\quad \tau(x,\boldsymbol{t})=1+\sum_{p=1}^{n}\sum_{1\le i_1<\cdots<i_p\le n}\prod_{j=1}^{p}\frac{m_{i_j}}{2\eta_{i_j}}\prod_{1\le j<k\le p}\left(\frac{\eta_{i_j}-\eta_{i_k}}{\eta_{i_j}+\eta_{i_k}}\right)^2 \times\exp\left[-2\sum_{j=1}^{p}\zeta_{i_j}(x,\boldsymbol{t})\right].$$

For details, see [14, 16]. If we set

$$A(x,\boldsymbol{t}) = \left(\frac{\sqrt{m_i m_j}}{\eta_i + \eta_j} e^{-\{\zeta_i(x,\boldsymbol{t})+\zeta_j(x,\boldsymbol{t})\}} \right)_{1\le i,j\le n},$$

then, with the help of Cauchy's identity (27), we can show that

$$\tau(x,\boldsymbol{t}) = \det(I + A(x,\boldsymbol{t})), \quad x \in \mathbf{R},\ \boldsymbol{t} \in \mathbf{T}. \tag{34}$$

We shall show that $\tau(x,\boldsymbol{t})$ can be expressed in terms of Wiener integral. To do this, let

$$\phi_a(x,\boldsymbol{t}) = U\big\{\cosh(\zeta(x,\boldsymbol{t})) - \sinh(\zeta(x,\boldsymbol{t}))R^{-1}U^{-1}DU\big\}U^{-1}. \tag{35}$$

Then, for each $\boldsymbol{t} \in \mathbf{T}$, $\phi_a(\cdot,\boldsymbol{t})$ obeys the differential equation (6) with initial condition

$$\begin{aligned}
\phi_a(0) &= U\big\{\cosh(\zeta(0,\boldsymbol{t})) - \sinh(\zeta(0,\boldsymbol{t}))R^{-1}U^{-1}DU\big\}U^{-1},\\
\phi_a'(0) &= U\big\{R\sinh(\zeta(0,\boldsymbol{t})) - \cosh(\zeta(0,\boldsymbol{t}))U^{-1}DU\big\}U^{-1}.
\end{aligned}$$

As we shall see in Lemma 4.1 below, $\det\phi_a(x,\boldsymbol{t}) \neq 0$, and then we can define

$$\beta_{a,x,\boldsymbol{t}}(y) = -((\partial_x\phi_a)\phi_a^{-1})(x-y,\boldsymbol{t}).$$

We shall show that $\beta_{a,x,\boldsymbol{t}}(y)$ is symmetric. To do this, write $\beta(y,\boldsymbol{t})$ for $\beta_{a,x,\boldsymbol{t}}(y)$. For each $k = 1,2,\dots$, the partial derivative $\partial_{t_k}\phi_a$ of ϕ_a with respect to t_k satisfies that

$$\partial_{t_k}\phi_a(x,\boldsymbol{t}) = U\big\{R^{2k+1}\sinh(\zeta(x,\boldsymbol{t})) - R^{2k}\cosh(\zeta(x,\boldsymbol{t}))U^{-1}DU\big\}U^{-1}.$$

This implies that

$$\partial_{t_k}\phi_a(x,\boldsymbol{t}) = E(a)^k\partial_x\phi_a(x,\boldsymbol{t}) \ \text{ and } \ \partial_{t_k}^2\phi_a(x,\boldsymbol{t}) = E(a)^{2k+1}\phi_a(x,\boldsymbol{t})$$

for any $k = 1,2,\dots$ We then obtain that

$$\partial_{t_k}\beta(y,\boldsymbol{t}) = -E(a)^{k+1} + \beta(y,\boldsymbol{t})E(a)^k\beta(y,\boldsymbol{t}), \quad k = 1,2,\dots \tag{36}$$

Since $E(a)$ is symmetric, the transpose ${}^t\beta(y,\boldsymbol{t})$ of $\beta(y,\boldsymbol{t})$ satisfies the same identities in (36). Hence $\beta(y,\boldsymbol{t})$ is symmetric if and only if so is $\beta(y,\boldsymbol{t}[k])$ for some $k = 1,2,\dots$, where $\boldsymbol{t}[k]$ is obtained from $\boldsymbol{t}$ by replacing t_k by 0. As was seen in the paragraph before (24), $\beta(y,\mathbf{0})$ is symmetric, where $\mathbf{0} = (0,0,\dots) \in \mathbf{T}$. In conjunction with the above observation, this implies that $\beta(y,\boldsymbol{t})$ is symmetric for $\boldsymbol{t} = (t_1,0,0,\dots)$, $t_1 \in \mathbf{R}$. Apply the above observation again, it follows that $\beta(y,\boldsymbol{t})$ is symmetric for $\boldsymbol{t} = (t_1, t_2, 0, 0, \dots)$, $t_1, t_2 \in \mathbf{R}$. Repeating this argument successively, we can conclude that $\beta(y, \boldsymbol{t})$ is symmetric for any $(y, \boldsymbol{t}) \in \mathbf{R} \times \mathbf{T}$.

We set

$$I_{\boldsymbol{p},\boldsymbol{c},a}(x, \boldsymbol{t}) = \int_{\mathcal{W}^n} \exp\left[-\frac{a^2}{2}\int_0^x X_{\boldsymbol{p},\boldsymbol{c}}(y)^2 dy + \frac{1}{2}\langle(\beta_{a,x,\boldsymbol{t}}(x) - D)\xi_{\boldsymbol{p}}(x), \xi_{\boldsymbol{p}}(x)\rangle\right] dP.$$

To state our result, we introduce a set $J = \{(j(\ell)+1, j(\ell)) : \ell = 1, \dots, m\}$ and a quantity

$$Z_m(\boldsymbol{p}, \boldsymbol{c}) = (-1)^m \prod_{i=1}^n \frac{c_i}{2\eta_i} \prod_{1 \le i < j \le n} (p_i - p_j)(\eta_i - \eta_j) \left\{ \prod_{(i,j) \notin J} (p_i + \eta_j) \times \prod_{k=1}^{n-m} |(D_{\boldsymbol{p}}^2 - r_k I)^{-1} \boldsymbol{c}| \prod_{\ell=1}^m \frac{c_{j(\ell)+1}\left(c_{j(\ell)}^2 + c_{j(\ell)+1}^2\right)^{1/2}}{2p_{j(\ell)} c_{j(\ell)}} \right\}^{-1},$$

where, if $m = 0$, then $J = \emptyset$, "$(i,j) \notin J$" means "$1 \le i, j \le n$", and $\prod_{\ell=1}^0 (\cdots) = 1$.

Theorem 4.1. (i) *It holds that*

$$\det \phi_a(x, \boldsymbol{t}) = \tau(x, \boldsymbol{t}) e^{\operatorname{tr}\zeta(x,\boldsymbol{t})} Z_m(\boldsymbol{p}, \boldsymbol{c}), \tag{37}$$

$$\log\bigl(I_{\boldsymbol{p},\boldsymbol{c},a}(x, \boldsymbol{t})\bigr) = -\frac{1}{2}\log\tau(x, \boldsymbol{t}) + \frac{1}{2}\log\tau(0, \boldsymbol{t}) - \frac{x}{2}\sum_{i=1}^n (p_i + \eta_i). \tag{38}$$

(ii) *Let* $\boldsymbol{t} = (t, 0, \dots)$. *We write* t *for* $\boldsymbol{t}$.
(a) *Set*

$$q_{\boldsymbol{p},\boldsymbol{c},a}(x, t) = -4\partial_x^2 \log\bigl(I_{\boldsymbol{p},\boldsymbol{c},a}(x, t)\bigr).$$

Then $q_{\boldsymbol{p},\boldsymbol{c},a}(x, t)$ *solves the KdV equation* (2).
(b) *Both* $\det \phi_a(x, t)$ *and* $\bigl(I_{\boldsymbol{p},\boldsymbol{c},a}(x, t)\bigr)^{-2}$ *solve the Hirota equation:*

$$(4D_t D_x - D_x^4) u \cdot u = 0, \tag{39}$$

where (D_x, D_t) *denotes the Hirota derivatives with respect to the variables* (x, t) ([14]).

While $q_{\boldsymbol{p},\boldsymbol{c},a}(x, t)$ is defined only on $[0, \infty) \times \mathbf{R}$ in our framework, by virtue of (37), it extends to $\mathbf{R} \times \mathbf{R}$ so that the extension also solves the KdV equation (2).

To prove Theorem 4.1, we first show a relation between $\phi_a(x, \boldsymbol{t})$ and $A(x, \boldsymbol{t})$.

Lemma 4.1. *It holds that*

$$\det \phi_a(y, \boldsymbol{t}) = \det(I + A(y, \boldsymbol{t}))e^{\mathrm{tr}\zeta(y,\boldsymbol{t})} Z_m(\boldsymbol{p}, \boldsymbol{c}). \tag{40}$$

In particular, (37) *holds and* $\det \phi_a(y,t) \neq 0$, $y \geq 0$.

Proof. For $\varepsilon > 0$, define $\boldsymbol{p}^\varepsilon = {}^t(p_1^\varepsilon, \dots, p_n^\varepsilon) \in \mathcal{P}_n$ as in the proof of Lemma 2.2. For sufficiently small ε, we see that $\boldsymbol{p}^\varepsilon$ satisfies the condition $(\mathrm{H})_0$. In the sequel, as in the proof of Lemma 2.2, we use the superscript ε to indicate the dependence on $\boldsymbol{p}^\varepsilon$; given a quantity f defined with $\boldsymbol{p}$, we write f^ε for the same quantity defined with $\boldsymbol{p}^\varepsilon$ instead of $\boldsymbol{p}$. Then, in repetition of the argument employed to prove Lemma 2.1, we can show that

$$\phi_a^\varepsilon(y,t) = -\frac{1}{2} U^\varepsilon (R^\varepsilon)^{-1} V^\varepsilon(\boldsymbol{c}) B^\varepsilon \{I + A^\varepsilon(y,t)\}(B^\varepsilon)^{-1} e^{\zeta^\varepsilon(y,\boldsymbol{t})} X^\varepsilon C(\boldsymbol{c}).$$

Applying Cauchy's identity (27) to computing $\det X^\varepsilon$ and $\det U^\varepsilon$, we have

$$\det \phi_a^\varepsilon(y,t) = \det\big(I + A^\varepsilon(y,\boldsymbol{t})\big) e^{\mathrm{tr}\zeta^\varepsilon(y,\boldsymbol{t})} Z_0(\boldsymbol{p}^\varepsilon, \boldsymbol{c}). \tag{41}$$

As we have already seen in the proof of Lemma 2.2, as $\varepsilon \downarrow 0$, $\eta_i^\varepsilon \to \eta_i$ and $m_i^\varepsilon \to m_i$ for $i = 1, \dots, n$. Moreover, taking the advantage of (30) and (31), we can show that, as $\varepsilon \downarrow 0$, $U^\varepsilon \to U$ and $Z_0(\boldsymbol{p}^\varepsilon, \boldsymbol{c}) \to Z_m(\boldsymbol{p}, \boldsymbol{c})$. Hence, letting $\varepsilon \downarrow 0$ in (41), we obtain (40).

(37) is an immediate consequence of (34) and (40). The non-singularity of $\phi_a(y, \boldsymbol{t})$ follows from that of $I + A(y, \boldsymbol{t})$ and (40). Q.E.D.

Proof of Theorem 4.1. (i) We have already seen (37) in Lemma 4.1. By the same lemma, $\phi_a(y, \boldsymbol{t}) \neq 0$, $y \geq 0$. Applying Proposition 1.1 with $\phi_a(y) = \phi_a(y, \boldsymbol{t})$, we have

$$I_{\boldsymbol{p},\boldsymbol{c},a}(x, \boldsymbol{t}) = \big(\det \phi_a(0, \boldsymbol{t})\big)^{1/2} \big(e^{x \mathrm{tr} D} \det \phi_a(x, \boldsymbol{t})\big)^{-1/2}.$$

By (37), it holds that

$$\det \phi_a(0, \boldsymbol{t}) \big(\det \phi_a(x, \boldsymbol{t})\big)^{-1} = \tau(0, \boldsymbol{t}) \tau(x, \boldsymbol{t})^{-1} \exp\big[\mathrm{tr}\zeta(0, \boldsymbol{t}) - \mathrm{tr}\zeta(x, \boldsymbol{t})\big].$$

Since $\zeta(0, \boldsymbol{t}) - \zeta(x, \boldsymbol{t}) = -xR$, we obtain (38).

(ii)(a) The assertion follows form (34) and (38).

(b) It is well known ([14]) that $\tau(x,t)$ solves the Hirota equation (39). Since $\det \phi_a(x,t)$ and $\big(I_{\boldsymbol{p},\boldsymbol{c},a}(x,t)\big)^{-2}$ are both of the form $k(t)e^{cx}\tau(x,t)$ with a constant c and a function $k : \mathbf{R} \to \mathbf{R}$, they also obey the same Hirota equation (39) that $\tau(x,t)$ does. Q.E.D.

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