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Invariant Measures for a Stochastic Porous Medium Equation

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Abstract.

We prove the existence of (infinitesimally) invariant measures for a stochastic version of the porous medium equation (of exponent $m = 3$) with Dirichlet Laplacian on an open set in $\mathbb{R}^d$.

§1. Introduction

The porous medium equation

$$\frac{\partial X}{\partial t} = \Delta(X^m), \quad m \in \mathbb{N}, \tag{1.1}$$

on a bounded open set $D \subset \mathbb{R}^d$ has been studied extensively. We refer to [1] for both the mathematical treatment and the physical background and also to [2, Section 4.3] for the general theory of equations of such type.

In this paper we are interested in a stochastic version of (1.1). Throughout this paper we assume

$$m = 3. \tag{$H1$}$$

We believe our approach can be extended for other odd values of m, but this would require a technically much more complicated proof. To avoid the latter and to explain the main idea we restrict to the above case.

We consider Dirichlet boundary conditions for the Laplacian Δ. So, the stochastic partial differential equation we would like to analyze for suitable initial conditions is the following:

$$dX(t) = \Delta(X^3(t))dt + \sqrt{C}\, dW(t), \quad t \geq 0. \tag{1.2}$$

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As in [3], where similar equations were studied (but with $x \to x^3$ replaced by some $\beta : \mathbb{R} \to \mathbb{R}$ of linear growth, satisfying, in particular, $\beta' \geq c > 0$), it turns out that the appropriate state space is $H^{-1}(D)$, i.e. the dual of the Sobolev space $H_0^1 := H_0^1(D)$. Below we shall use the standard $L^2(D)$ dualization $\langle \cdot, \cdot \rangle$ between $H_0^1(D)$ and $H = H^{-1}(D)$ induced by the embeddings

$$H_0^1(D) \subset L^2(D)' = L^2(D) \subset H^{-1}(D) = H$$

without further notice. Then for $x \in H$

$$|x|_H^2 = \int_D (-\Delta)^{-1} x(\xi)\, x(\xi) d\xi$$

and for the dual H' of H we have $H' = H_0^1$.

$(W_t)_{t \geq 0}$ is a cylindrical Brownian motion in H and C is a positive definite bounded operator on H of trace class. To be more concrete below we assume:

(H2) *There exists* λ_k, $k \in [0, +\infty)$, $k \in \mathbb{N}$, *such that for the eigenbasis* $\{e_k |\ k \in \mathbb{N}\}$ *of* Δ *(with Dirichlet boundary conditions) we have*

$$Ce_k = \sqrt{\lambda_k}\, e_k \text{ for all } k \in \mathbb{N}.$$

(H3) *For* $\alpha_k := \sup_{\xi \in D} |e_k(\xi)|^2$, $k \in \mathbb{N}$, *we have*

$$K := \sum_{k=1}^{\infty} \alpha_k \lambda_k < +\infty.$$

Our aim in this paper is to construct invariant measures for (1.2). Existence of solutions to (1.2) will be studied in another paper. To formulate what is meant by "invariant measure" without refering to a solution of (1.2) we need to consider the generator, also called Kolmogorov operator, corresponding to (1.2).

Applying Itô's formula (on a heuristic level) to (1.2) one finds what the corresponding Kolmogorov operator, let us call it N_0, should be, namely

$$N_0\varphi(x) = \frac{1}{2} \sum_{k=1}^{\infty} \lambda_k D^2\varphi(e_k, e_k) + D\varphi(x)(\Delta(x^3)), \quad x \in H, \tag{1.3}$$

where $D\varphi$, $D^2\varphi$ denote the first and second Fréchet derivatives of $\varphi : H \to \mathbb{R}$. So, we take $\varphi \in C_b^2(H)$.

In order to make sense of (1.3) one needs that $\Delta(x^3) \in H$ at least for "relevant" $x \in H$. Here one clearly sees the difficulties since x^3 is, of course, not defined for any Schwartz distribution in $H = H^{-1}$, not to mention that it will not be in $H_0^1(D)$. An invariant measure for (1.2) is now defined "infinitesimally" (cf.[4]), without having a solution to (1.2), as the solution to the equation

$$N_0^* \mu = 0 \tag{1.4}$$

with the property that μ is supported by those $x \in H$ for which x^3 makes sense and $\Delta(x^3) \in H$. (1.4) is a short form for

$$N_0\varphi \in L^1(H,\mu) \text{ and } \int_H N_0\varphi d\mu = 0 \text{ for all } \varphi \in C_b^2(H). \tag{1.5}$$

Any invariant measure for any solution of (1.2) in the classical sense will satisfy (1.4).

In §2 we construct a solution μ to (1.4) and prove the necessary support properties of μ, more precisely, that for all $M \in \mathbb{N}$, $M \geq 2$,

$$\mu\left(\{x \in L^2(D) |\, x^M \in H_0^1\}\right) = 1,$$

so that N_0 in (1.3) is μ–a.e. well defined for all $\varphi \in C_b^2(H)$. We rely on results in [3] which we apply to suitable approximations, i.e. the function $x \mapsto x^3$ is replaced by

$$\beta_\varepsilon(x) := \frac{x^3}{1+\varepsilon x^2} + \varepsilon x, \quad \varepsilon \in (0,1],$$

to which the results in [3] apply.

§2. Existence of an infinitesimal invariant measure

Throughout this section (H1)–(H3) are still in force. So, we first consider the following approximations for the Kolmogorov operator N_0. For $\varepsilon \in (0,1]$ we define for $\varphi \in C_b^2(H)$, $x \in L^2(D)$ such that $\beta_\varepsilon(x) \in H_0^1$

$$N_\varepsilon\varphi(x) := \frac{1}{2}\sum_{k=1}^{\infty} \lambda_k D^2\varphi(x)(e_k, e_k) + D\varphi(x)(\Delta\beta_\varepsilon(x)), \tag{2.1}$$

where

$$\beta_\varepsilon(r) := \frac{r^3}{1+\varepsilon r^2} + \varepsilon r, \quad r \in \mathbb{R}. \tag{2.2}$$

We note that β_ε is Lipschitz and recall the following result from [3] which is crucial for our further analysis, see [3, Theorems 3.1, 3.9, Remark 3.1].

Theorem 2.1. *Let $\varepsilon \in (0,1]$. Then there exists a probability measure μ_ε on H such that*

$$\mu_\varepsilon(H_0^1) = 1, \tag{2.3}$$

$$\int_H |x|_{H_0^1}^2 \mu_\varepsilon(dx) < +\infty, \tag{2.4}$$

$$\int_H |\beta_\varepsilon|_{H_0^1}^2 d\mu_\varepsilon = \int_H |\Delta\beta_\varepsilon|_H^2 d\mu_\varepsilon < +\infty \tag{2.5}$$

and

$$\int_H N_\varepsilon \varphi d\mu_\varepsilon = 0 \quad \text{for all } \varphi \in C_b^2(H). \tag{2.6}$$

Remark 2.2. (i). In [3] only

$$\mu_\varepsilon\left(\{x \in L^2(D) |\ \beta_\varepsilon(x) \in H_0^1\}\right) = 1$$

was proved. But since $\beta_\varepsilon(0) = 0$, $\beta_\varepsilon(\mathbb{R}) = \mathbb{R}$, and

$$\beta_\varepsilon'(r) = r^2 \frac{3 + \varepsilon r^2}{(1 + \varepsilon r^2)^2} + \varepsilon \geq \varepsilon \quad \text{for all } r \in \mathbb{R}, \tag{2.7}$$

it follows that the inverse β_ε^{-1} of β_ε is Lipschitz with $\beta_\varepsilon^{-1}(0) = 0$, so $\beta_\varepsilon(x) \in H_0^1$ is equivalent to $x \in H_0^1$ and (2.4) follows from (2.5), since

$$|\nabla x| = |\nabla \beta_\varepsilon^{-1}(\beta_\varepsilon(x))| \leq \varepsilon^{-1} |\nabla \beta_\varepsilon(x)|.$$

We thank V. Barbu for pointing this out to us.

(ii) By Theorem 2.1 we have that $N_\varepsilon\varphi(x)$ is well defined for μ_ε–a.e. $x \in H$.

For $N \in \mathbb{N}$ we define

$$P_N x = \sum_{k=1}^N \langle x, e_k \rangle_k e_k, \quad x \in H.$$

Note that, since $\{e_k |\ k \in \mathbb{N}\}$ is the eigenbasis of the Laplacian we have that the respective restriction P_N is also an orthogonal projection on $L^2(D)$ and H_0^1 and on both spaces $(P_N)_{N \in \mathbb{N}}$ also converges strongly to the identity.

The first new result on μ_ε, $\varepsilon \in (0,1]$, is the following:

Proposition 2.3. *$\{\mu_\varepsilon,\ \varepsilon\in(0,1]\}$ is tight on H. For any weak limit point μ*

$$\int_H |x|^2_{L^2(D)}\mu(dx)\le\int_D 1\,d\xi+\frac12\,\mathrm{Tr}\,C.$$

In particular, $\mu(L^2(D))=1$.

Proof. For $n\in\mathbb{N}$ let $\chi_n\in C^\infty(\mathbb{R})$, $\chi_n(x)=x$ on $[-n,n]$, $\chi_n(x)=(n+1)\mathrm{sign}\,x$, for $x\in\mathbb{R}\backslash[-(n+2),n+2]$, $0\le\chi_n'\le1$ and $\sup_{n\in\mathbb{N}}|\chi_n''|<+\infty$. Define for $n,N\in\mathbb{N}$

$$\varphi_{N,n}(x):=\frac12\,\chi_n(|P_nx|^2_H).$$

Then $\varphi_{N,n}\in C_b^2(H)$ and for $x\in H$

$$\begin{aligned}N_\varepsilon\varphi_{N,n}(x)&=\frac12\sum_{k=1}^N\lambda_k\left[2\chi_n''(|P_nx|^2_H)\langle P_Nx,e_k\rangle^2_H+\chi_n'(|P_nx|^2_H)\right]\\&\quad+\chi_n'(|P_nx|^2_H)\langle P_Nx,\Delta\beta_\varepsilon(x)\rangle_H.\end{aligned}$$

Hence integrating with respect to μ_ε, by (2.6) we find

$$\begin{aligned}&\int_H\chi_n'(|P_nx|^2_H)\langle P_Nx,\beta_\varepsilon(x)\rangle_{L^2(D)}\mu_\varepsilon(dx)\\&=\frac12\sum_{k=1}^N\lambda_k\int_H\left[2\chi_n''(|P_nx|^2_H)\langle P_Nx,e_k\rangle^2_H+\chi_n'(|P_nx|^2_H)\right]\mu_\varepsilon(dx)\\&\le\frac12\sum_{k=1}^N\lambda_k+\sup_{k\in\mathbb{N}}\lambda_k\int_H|\chi_n''(|P_nx|^2_H)|\;|P_Nx|^2_H\mu_\varepsilon(dx).\end{aligned}$$

For all $n\in\mathbb{N}$ the integrand in the left hand side is bounded by

$$1_{\{|P_nx|^2_H\le n+2\}}\;|P_Nx|_H\;|\beta_\varepsilon(x)|_{H_0^1},$$

and similar bounds for the integrand in the right hand side hold. Therefore, (2.5) and Lebesgue's dominated convergence theorem allow us to take $N \to \infty$ and obtain

$$\int_H \chi_n'(|x|_H^2)\langle x, \beta_\varepsilon(x)\rangle_{L^2(D)}\mu_\varepsilon(dx)$$

$$\leq \frac{1}{2}\sum_{k=1}^\infty \lambda_k + \sup_{k\in\mathbb{N}} \lambda_k \int_H |\chi_n''(|x|_H^2)|\ |x|_H^2\ \mu_\varepsilon(dx).$$

$$\leq \frac{1}{2}\sum_{k=1}^\infty \lambda_k + \sup_{k\in\mathbb{N}} \lambda_k \int_{\{|x|_H^2\geq n\}} |x|_H^2\ \mu_\varepsilon(dx).$$

Hence taking $n \to \infty$ by (2.4) and using the definition (2.2) of β_ε we arrive at

$$\int_H\int_D \left(\frac{x^4(\xi)}{1+\varepsilon x^2(\xi)} + \varepsilon x^2(\xi)\right) d\xi\mu_\varepsilon(dx) \leq \frac{1}{2}\ \mathrm{Tr}\ C.$$

Since $\varepsilon \in (0,1]$, this implies

$$\text{(2.8)} \qquad \int_H |x|_{L^2(D)}^2\mu_\varepsilon(dx) \leq \int_D \left(1 + \frac{x^4(\xi)}{1+x^2(\xi)}\right) d\xi\mu_\varepsilon(dx) \leq \int_D 1\ d\xi + \frac{1}{2}\ \mathrm{Tr}\ C.$$

Since $L^2(D) \subset H$ is compact, this implies that $\{\mu_\varepsilon |\ \varepsilon \in (0,1]\}$ is tight on H. Since the map $x \to |x|_{L^2(D)}^2$ is lower semicontinuous and nonnegative on H all assertions follow. □

Later we need better support properties of μ. Therefore, our next aim is to prove the following:

Theorem 2.4. *Let $(H1) - (H3)$ hold. Then:*

(i) *For all $M \in \mathbb{N}$, $M \geq 2$, there exists a constant $C_M = C_M(D,K) > 0$ such that*

$$\sup_{\varepsilon\in(0,1]} \int_H\int_D x^{2(M-1)}(\xi)|\nabla x(\xi)|^2 d\xi\mu_\varepsilon(dx) \leq C_M.$$

(ii) *For all $M \in \mathbb{N}$, $M \geq 2$ and any limit point μ as in Proposition 2.3*

$$\int_H\int_D |\nabla(x^M)(\xi)|^2 d\xi\mu(dx) \leq C_M.$$

In particular, setting

$$H^1_{0,M} := \{x \in L^2(D) |\; x^M \in H^1_0\}$$

we have

$$\mu(H^1_{0,M}) = 1 \quad \text{for all } M \geq 2.$$

In order to prove Theorem 2.4 we need some preparation, i.e. more precise information about the μ_ε, $\varepsilon \in (0,1]$. This can be deduced from (2.6), i.e. from the fact that μ_ε is an infinitesimally invariant measure for N_ε. So, we fix $\varepsilon \in (0,1]$ and for the rest of this section we assume that $(H1) - (H3)$ hold.

We need to apply (2.6) with φ replaced by $\varphi_M : L^2(D) \to [0,\infty]$, $M \in \mathbb{N}$, given by

$$\varphi_M(x) := \int_D x^{2M}(\xi)\, d\xi, \quad x \in L^2(D).$$

Clearly, such functions are not in $C^2_b(H)$ so we have to construct proper approximations. So, define for $\delta \in (0,1]$

$$f_{M,\delta}(r) := \frac{r^{2M}}{1+\delta r^2}, \quad r \in \mathbb{R}. \tag{2.9}$$

Then for $r \in \mathbb{R}$

$$f'_{M,\delta}(r) = (1+\delta r^2)^{-2}[2Mr^{2M-1} + 2\delta(M-1)r^{2M+1}] \tag{2.10}$$

and

(2.11)
$$f''_{M,\delta}(r) = 2(1+\delta r^2)^{-3}[M(2M-1)r^{2M-2} + \delta(4M^2-6M-1)r^{2M} + \delta^2(M-1)(2M-3)r^{2M+2}].$$

We have chosen this approximation since below (cf. Lemma 2.7) it will be crucial that $f''_{M,\delta}$ is nonnegative if $M \geq 2$. More precisely we have

$$\begin{aligned} &0 \leq f_{M,\delta}(r) \leq \tfrac{1}{\delta}\, r^{2M-2} \\ &0 \leq f'_{M,\delta}(r) \leq \tfrac{2M}{\delta}\, |r|^{2M-3} \\ &0 \leq f''_{M,\delta}(r) \leq 16M^2\, |r|^{2M-4} \inf\{r^2, 1/\delta\}. \end{aligned} \tag{2.12}$$

Remark 2.5. The following will be used below: if $x \in H_0^1$ is such that for $M \in \mathbb{N}$

$$\int_H x^{2(M-1)}(\xi)|\nabla x(\xi)|^2 d\xi < \infty, \tag{2.13}$$

then $x^M \in H_0^1$ and $x^{M-1}\nabla x = \frac{1}{M}\nabla x^M$, or using the notation introduced in Theorem 2.4–(ii) equivalently $x \in H^1_{0,M}$. The proof is standard by approximation. So, we omit it. We also note that by Poincaré's inequality, $H^1_{0,M} \subset L^{2M}(D)$. More precisely, there exists $C(D) \in (0,\infty)$ such that

$$\begin{aligned} C(D)\int_D x^{2M}(\xi)d\xi &\le \int_D |\nabla x^M(\xi)|^2 d\xi \\ &= M\int_D x^{2(M-1)}(\xi)|\nabla x^M(\xi)|^2 d\xi, \end{aligned} \tag{2.14}$$

for all x as above.

The following lemma is a consequence of (2.6) and crucial for our analysis of μ_ε, $\varepsilon \in (0,1]$ and their limit points.

Lemma 2.6. *Let $M \in \mathbb{N}$, $\delta \in (0,1]$. Assume that*

$$\int_H\int_D x^{2(M-2)}(\xi)|\nabla x(\xi)|^2 d\xi \mu_\varepsilon(dx) < \infty \quad \text{if } M \ge 3. \tag{2.15}$$

Then

$$\begin{aligned} &\frac{1}{2}\sum_{k=1}^\infty \lambda_k \int_H\int_D f''_{M,\delta}(x(\xi))e_k^2(\xi)d\xi\mu_\varepsilon(dx) \\ &= \int_H\int_D f''_{M,\delta}(x(\xi))\beta'_\varepsilon(x(\xi))|\nabla x(\xi)|^2 d\xi\mu_\varepsilon(dx). \end{aligned} \tag{2.16}$$

Proof. We first note that (2.15) holds for $M = 2$ by (2.3). For $\kappa \in (0,1]$ we define

$$f_{M,\delta,\kappa}(r) := f_{M,\delta}(r)e^{-\frac{1}{2}\kappa r^2}, \quad r \in \mathbb{R} \quad \text{if } M \ge 2$$

and $f_{1,\delta,\kappa} := f_{1,\delta}$. Then (2.11) implies that $f_{M,\delta,\kappa} \in C_b^2(\mathbb{R})$. Define

$$\varphi_{M,\delta,\kappa}(x) := \int_D f_{M,\delta,\kappa}(x(\xi))d\xi, \quad x \in L^2(D).$$

Then it is easy to check that $\varphi_{M,\delta,\kappa}$ is Gâteaux differentiable on $L^2(D)$ and that for all $y, z \in L^2(D)$

$$\varphi'_{M,\delta,\kappa}(x)(y) = \int_D f'_{M,\delta,\kappa}(x(\xi))y(\xi)d\xi, \tag{2.17}$$

$$\varphi''_{M,\delta,\kappa}(x)(y,z) = \int_D f''_{M,\delta,\kappa}(x(\xi))y(\xi)z(\xi)d\xi. \tag{2.18}$$

Hence

$$\varphi_{M,\delta,\kappa} \circ P_N \in C_b^2(H)$$

and for all $x \in H_0^1$ (hence $\beta_\varepsilon(x) \in H_0^1$)

$$\begin{aligned} N_\varepsilon(\varphi_{M,\delta,\kappa} \circ P_N)(x) &= \frac{1}{2}\sum_{k=1}^{N} \lambda_k \int_D f''_{M,\delta,\kappa}(P_N x(\xi))e_k^2(\xi)d\xi \\ &\quad + \int_D f'_{M,\delta,\kappa}(P_N x(\xi))P_N(\Delta\beta_\varepsilon(x))(\xi)d\xi. \end{aligned}$$

Since $P_N\Delta = \Delta P_N$, integrating by parts we obtain

$$\begin{aligned} N_\varepsilon(\varphi_{M,\delta,\kappa} \circ P_N)(x) &= \frac{1}{2}\sum_{k=1}^{N} \lambda_k \int_D f''_{M,\delta,\kappa}(P_N x(\xi))e_k^2(\xi)d\xi \\ &\quad - \int_D f''_{M,\delta,\kappa}(P_N x(\xi))\langle\nabla(P_N x)(\xi), \nabla(P_N\beta_\varepsilon(x))(\xi)\rangle_{\mathbb{R}^d} d\xi. \end{aligned}$$

Since $(P_N)_{N\in\mathbb{N}}$ converges strongly to the identity in H_0^1, we conclude by (H3) that

$$\begin{aligned} \lim_{N\to\infty} N_\varepsilon(\varphi_{M,\delta,\kappa} \circ P_N)(x) &= \frac{1}{2}\sum_{k=1}^{\infty} \lambda_k \int_D f''_{M,\delta,\kappa}(x(\xi))e_k^2(\xi)d\xi \\ &\quad - \int_D f''_{M,\delta,\kappa}(x(\xi))\beta'_\varepsilon(x)(\xi)|\nabla x(\xi)|^2 d\xi. \end{aligned}$$

Since β_ε is Lipschitz, by (2.3)–(2.5) and (H3) this convergence also holds in $L^1(H, \mu_\varepsilon)$. Hence (2.6) implies that

$$\begin{aligned} &\frac{1}{2}\sum_{k=1}^{\infty} \lambda_k \int_H \int_D f''_{M,\delta,\kappa}(x(\xi))e_k^2(\xi)d\xi\mu_\varepsilon(dx) \\ &= \int_H \int_D f''_{M,\delta,\kappa}(x(\xi))\beta'_\varepsilon(x)(\xi)|\nabla x(\xi)|^2 d\xi\mu_\varepsilon(dx). \end{aligned} \tag{2.19}$$

So, for $M = 1$ the assertion is proved. If $M \geq 2$, an elementary calculation shows that by (2.12) there exists a constant $C(M, \delta) > 0$ (only depending on M and δ) such that

$$|f''_{M,\delta,\kappa}(r)| \leq C(M, \delta) r^{2(M-2)}, \quad r \in \mathbb{R}. \tag{2.20}$$

Hence by (H3), Remark 2.5 and assumption (2.15) we can apply Lebesgue's dominated convergence theorem to (2.19) and letting $\kappa \to 0$ we obtain the assertion. □

Lemma 2.7. *Let $M \in \mathbb{N}$ and assume that* (2.15) *holds if $M \geq 3$.*
(i) We have

$$\begin{aligned} &\frac{K}{2} \int_H \int_D x^{2(M-1)}(\xi)\, d\xi \mu_\varepsilon(dx) \\ &\geq \int_H \int_D x^{2(M-1)}(\xi) \left(\frac{x^2(\xi)}{1 + x^2(\xi)} + \varepsilon \right) |\nabla x(\xi)|^2 d\xi \mu_\varepsilon(dx). \end{aligned} \tag{2.21}$$

(ii) If $M \geq 2$, we have

$$\begin{aligned} &\frac{K}{2} \int_H \int_D \left(x^{2(M-1)}(\xi) + x^{2(M-2)}(\xi) \right) d\xi \mu_\varepsilon(dx) \\ &\geq \int_H \int_D x^{2(M-1)}(\xi)\, |\nabla x(\xi)|^2 d\xi \mu_\varepsilon(dx) \\ &= \frac{1}{M^2} \int_H \int_D |\nabla x^M(\xi)|^2 d\xi \mu_\varepsilon(dx). \end{aligned} \tag{2.22}$$

(iii)

$$\int_H \int_D |\nabla x(\xi)|^2 d\xi \mu_\varepsilon(dx) \leq \frac{K}{2\varepsilon}.$$

Proof. (i) By (H3) the left hand side of (2.16) is dominated by

$$\frac{K}{2} \int_H \int_D f''_{M,\delta}(x(\xi)) d\xi \mu_\varepsilon(dx).$$

If $M \geq 2$, by assumption (2.15) and Remark 2.5 we know that

$$\int_H \int_D x^{2(M-1)}(\xi) d\xi \mu_\varepsilon(dx) < \infty$$

which trivially also holds for $M = 1$. So, by (2.11), (2.12) and Lebesgue's dominated convergence theorem we obtain that for $M \geq 2$

$$\frac{K}{2} \int_H \int_D 2M(2M-1)x^{2(M-1)}(\xi)d\xi\mu_\varepsilon(dx)$$

$$\geq \liminf_{\delta \to 0} \int_H \int_D f''_{M,\delta}(x(\xi))\beta'_\varepsilon(x(\xi))|\nabla x(\xi)|^2 d\xi\mu_\varepsilon(dx).$$

Since $f''_{M,\delta} \geq 0$ for $M \geq 2$ and

$$\beta'_\varepsilon(r) \geq \frac{r^2}{1+r^2} + \varepsilon \geq 0 \quad \text{for all } r \in \mathbb{R},$$

we can apply Fatou's lemma to prove the assertion. If $M = 1$ we conclude in the same way by (2.3) and Lebesgue's dominated convergence theorem which applies since β'_ε is bounded and $|f''_{1,\delta}| \leq 6$ for all $\delta \in (0,1]$.

(ii) Since (2.15) holds for $M = 2$, by Hölder's inequality (2.15) holds with $M-1$ replacing M, since by assumption it holds for M. So, the inequality in (i) also holds with $M-2$ replacing $M-1$. Estimating ε on the right hand sides from below by 0 and adding both resulting inequalities we obtain the inequality in (2.22). The equality in (2.22) follows by Remark 2.5.

(iii) The assertion follows from (2.21) setting $M = 1$. □

By an induction argument we shall now prove that the integrals in (2.22) are all finite and at the same time prove the bounds claimed in Theorem 2.4.

Proof of Theorem 2.4. (i). If $M = 2$, then the left hand side of (2.22) is finite by (2.8) and moreover (2.22) applies, so that by (2.8) we have

(2.23)
$$\int_H \int_D x^2(\xi)|\nabla(x(\xi))|^2 d\xi\mu_\varepsilon(dx) \leq \frac{K}{2}\left(\frac{1}{2}\operatorname{Tr} C + 2\int_D 1 d\xi\right) < \infty.$$

Suppose the left hand side of (2.22) is finite for $M \in \mathbb{N}, M \geq 2$, and (2.15) holds. Then (2.22) holds and by Remark 2.5

$$\infty > \int_H \int_D x^{2(M-1)}(\xi)|\nabla(x(\xi))|^2 d\xi\mu_\varepsilon(dx)$$

$$= \frac{1}{M^2}\int_H \int_D |\nabla(x^M(\xi))|^2 d\xi\mu_\varepsilon(dx) \tag{2.24}$$

$$\geq \frac{C(D)^2}{M^2}\int_H \int_D x^{2M}(\xi)d\xi\mu_\varepsilon(dx).$$

Hence (2.15) holds with $M-1$ replacing $M-2$ and the left hand side of (2.22) is finite for $M+1$ replacing M, hence by induction for all $M \in \mathbb{N}$. Furthermore, for all M first applying (2.22) and then applying (2.24) first with $M-1$ replacing M and then with $M-2$ replacing M respectively we obtain

$$\begin{aligned}
&\int_H \int_D x^{2(M-1)}(\xi)|\nabla(x(\xi))|^2 d\xi \mu_\varepsilon(dx) \\
&\leq \frac{K}{2}\left[\left(\frac{M-1}{C(D)}\right)^2 \int_H \int_D x^{2(M-2)}(\xi)|\nabla(x(\xi))|^2 d\xi \mu_\varepsilon(dx) \right. \qquad (2.25)\\
&\left. + \int_H \int_D x^{2(M-2)}(\xi) d\xi \mu_\varepsilon(dx)\right]
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{K}{2C(D)^2}\left[(M-1)^2 \int_H \int_D x^{2(M-2)}(\xi)|\nabla(x(\xi))|^2 d\xi \mu_\varepsilon(dx)\right. \\
&\left. +(M-2)^2 \int_H \int_D x^{2(M-3)}(\xi)|\nabla(x(\xi))|^2 d\xi \mu_\varepsilon(dx)\right]. \qquad (2.26)
\end{aligned}$$

If $M=3$ we cannot use (2.26) since for the second summand we have no bound which is independent of ε, but from (2.25) we obtain by (2.23) and (2.8) that

$$\begin{aligned}
&\int_H \int_D x^4(\xi)|\nabla(x(\xi))|^2 d\xi \mu_\varepsilon(dx) \\
&\leq \frac{K}{2}\left[\left(\frac{2}{C(D)}\right)^2 \frac{K}{2}\left(\frac{1}{2}\operatorname{Tr} C + 2\int_D 1 d\xi\right) + \frac{1}{2}\operatorname{Tr} C + \int_D 1 d\xi\right].
\end{aligned}$$

Now assertion (i) follows from (2.26) by induction.

To prove (ii) we start with the following
Claim: For all $M \in \mathbb{N}$

$$\Theta_M(x) := 1_{H^1_{0,M}}(x) \int_D |\nabla x^M(\xi)|^2 d\xi + \infty \cdot 1_{H \setminus H^1_{0,M}}(x), \quad x \in H \qquad (2.27)$$

is a lower semicontinuous function on H.

Since μ is a weak limit point of $\{\mu_\varepsilon |\ \varepsilon \in (0,1]\}$ and $\Theta_M \geq 0$, the claim immediately implies assertion (ii).

To prove the claim let $\alpha > 0$ and $x_n \in \{\Theta_M \leq \alpha\}$, $n \in \mathbb{N}$, such that $x_n \to x$ in H as $n \to \infty$. By Poincaré's inequality $\{x_n |\ n \in \mathbb{N}\}$ is a bounded set in $L^{2M}(D)$. So $x_n \to x$ as $n \to \infty$ also weakly in $L^2(D)$, in particular $x \in L^2(D)$. Since $\{x_n^M |\ n \in \mathbb{N}\}$ is bounded in H_0^1, there exists a subsequence $(x_{n_k}^M)_{k\in\mathbb{N}}$ and $y \in H_0^1$ such that $x_{n_k}^M \to y$ as $k \to \infty$ weakly in H_0^1 and

$$\int_D |\nabla y(\xi)|^2 d\xi \leq \alpha.$$

Since the embedding $H_0^1 \subset L^2(D)$ is compact, $x_{n_k}^M \to y$ as $k \to \infty$ in $L^2(D)$. Selecting another subsequence if necessary, this convergence is $d\xi$–a.e., hence

$$x_{n_k} \to y^{\frac{1}{M}} \quad \text{as } k \to \infty,\ d\xi\text{–a.e.}$$

Since (selecting another subsequence if necessary) we also know that the Cesaro mean of $(x_{n_k})_{k\in\mathbb{N}}$ has x as an accumulation point in the topology of $d\xi$–a.e. convergence, we must have $x^M = y$, so $x \in \{\Theta_M \leq \alpha\}$. □

As a consequence of the previous proof we obtain:

Corollary 2.8. *Let $M \in \mathbb{N}$. Then Θ_M has compact level sets in H.*

Proof. We already know from the previous proof that Θ_M is lower semicontinuous. The relative compactness of their level sets is, however, clear by Poincaré's inequality since $L^{2M}(D) \subset H$ is compact. □

Since for $M \in \mathbb{N}$ and $x \in H_{0,M}^1$

$$|\Delta x^M|_H = \int_D |\nabla x^M(\xi)|^2 d\xi, \tag{2.28}$$

so $\Delta x^M \in H$, we can define the Kolmogorov operator in (1.3) rigorously for $x \in H_{0,3}^1$. So, for $\varphi \in C_b^2(H)$

$$N_0\varphi(x) := \frac{1}{2}\sum_{k=1}^{\infty} \lambda_k D^2\varphi(x)(e_k, e_k) + D\varphi(x)(\Delta x^3). \tag{2.29}$$

We note that by Theorem 2.4–(ii) and (2.28), $N_0\varphi \in L^2(H,\mu)$ for any weak limit point μ of $\{\mu_\varepsilon |\ \varepsilon \in (0,1]\}$ on H. Now we can prove our main result, namely that any such μ is an infinitesimally invariant measure for N_0 in the sense of [4], i.e. satisfies (1.4).

Theorem 2.9. *Assume that* $(H1)$–$(H3)$ *hold. Let* μ *be as in Proposition* 2.3. *Then*

$$\int_H N_0\varphi d\mu = 0 \quad \text{for all } \varphi \in C_b^2(H).$$

Proof. Let $\varphi \in C_b^2(H)$. For $N \in \mathbb{N}$ define $\varphi_N := \varphi \circ P_N$. Then for $x \in H_{0,3}^1$

$$\begin{aligned} N_0\varphi_N(x) &= \frac{1}{2}\sum_{k=1}^{\infty}\lambda_k D^2\varphi(P_Nx)(P_Ne_k, P_Ne_k) + D\varphi_N(x)\ (\Delta x^3) \\ &= \frac{1}{2}\sum_{k=1}^{N}\lambda_k D^2\varphi(P_Nx)(e_k, e_k) + D\varphi(P_Nx)(P_N(\Delta x^3)). \end{aligned}$$

If we can prove that

$$\int_H N_0\varphi_N d\mu = 0 \quad \text{for all } N \in \mathbb{N}, \tag{2.30}$$

the same is true for φ by Lebesgue's dominated convergence theorem. So, fix $N \in \mathbb{N}$. Then by (2.6)

$$\begin{aligned} \int_H N_0\varphi_N d\mu &= \lim_{\varepsilon\to 0}\int_H \frac{1}{2}\sum_{k=1}^{\infty}\lambda_k D^2\varphi_N(x)(e_k, e_k)\mu_\varepsilon(dx) \\ &\quad + \int_H D\varphi_N(x)(\Delta x^3)\mu(dx) \\ &= -\lim_{\varepsilon\to 0}\int_H D\varphi_N(x)(\Delta\beta_\varepsilon(x))\mu_\varepsilon(dx) \\ &\quad + \int_H D\varphi_N(x)(\Delta x^3)\mu(dx) \\ &= \lim_{\varepsilon\to 0}\sum_{i=1}^{N}\int_H \Big[D\varphi(P_Nx)(e_i)\langle e_i, \Delta x^3\rangle_H\mu(dx) \\ &\quad - D\varphi(P_Nx)(e_i)\langle e_i, \Delta\beta_\varepsilon(x)\rangle_H\mu_\varepsilon(dx)\Big]. \end{aligned} \tag{2.31}$$

For $i \in \{1, ..., N\}$ fixed we have

$$
\begin{aligned}
&\Big| \int_H D\varphi(P_N x)(e_i)\langle e_i, \Delta x^3\rangle_H \mu(dx) \\
&\quad - \int_H D\varphi(P_N x)(e_i)\langle e_i, \Delta\beta_\varepsilon(x)\rangle_H \mu_\varepsilon(dx)\Big| \\
&\le \left| \int_H D\varphi(P_N x)(e_i)\langle e_i, \Delta x^3\rangle_H (\mu - \mu_\varepsilon)(dx)\right| \\
&\quad + \left| \int_H D\varphi(P_N x)(e_i)\langle e_i, \Delta(x^3 - \beta_\varepsilon(x))\rangle_H \mu_\varepsilon(dx)\right|.
\end{aligned} \tag{2.32}
$$

The right hand side's second summand is bounded by
(2.33)

$$
|e_i|_{L^2(D)} \sup_{x\in H} |D\varphi(x)|_{H_0^1} \int_H \left(\int_D |x^3(\xi) - \beta_\varepsilon(x(\xi))|^2 \, d\xi \right)^{1/2} \mu_\varepsilon(dx).
$$

We have

$$
|r^3 - \beta_\varepsilon(r)| = \left| \frac{\varepsilon r^5}{1 + \varepsilon r^2} - \varepsilon r \right| \le \varepsilon(|r|^5 + |r|), \quad r \in \mathbb{R}.
$$

So, the term in (2.33) is dominated by

$$
\varepsilon |e_i|_{L^2(D)} \sup_{x\in H} |D\varphi(x)|_{H_0^1} \int_H \left(||x|^5|_{L^2(D)} + |x|_{L^2(D)} \right) \mu_\varepsilon(dx)
$$

which by Theorem 2.4–(i), Remark 2.5 and Poincaré's inequality converges to 0 as $\varepsilon \to 0$.

Now we estimate the first summand in the right hand side of (2.32). So, we define

$$
f(x) := D\varphi(P_N x)(e_i)\langle e_i, \Delta x^3\rangle_H.
$$

Then since $\langle e_i, \Delta(x^3)\rangle_H = \langle e_i, x^3\rangle_{L^2(D)}$, it follows by the proof of the lower semicontinuity of Θ_3 that f is continuous on the level sets of Θ_3 (with Θ_3 defined as in (2.27)). Furthermore, since

$$
|f(x)| \le \sup_{x\in H} |D\varphi(x)|_{H_0^1} \, |x^3|_{L^2(D)},
$$

it follows that

$$
\lim_{R\to\infty} \sup_{\{\Theta_3 \ge R\}} \frac{|f(x)|}{1 + \Theta_3(x)} = 0.
$$

Furthermore, by Corollary 2.8 the function $1 + \Theta_3$ has compact level sets. Hence by [8, Theorem 5.1 (ii)], there exist $f_n \in C_b(H)$, $n \in \mathbb{N}$, such that

$$\lim_{n\to\infty} \sup_{x\in H} \frac{|f(x) - f_n(x)|}{1 + \Theta_3(x)} = 0. \tag{2.34}$$

But

$$\left|\int_H D\varphi(P_N x)(e_i)\langle e_i, \Delta x^3\rangle_H (\mu - \mu_\varepsilon)(dx)\right|$$

$$\leq \int_H |f(x) - f_n(x)|(\mu + \mu_\varepsilon)(dx) + \left|\int_H f_n(x)(\mu - \mu_\varepsilon)(dx)\right|.$$

For fixed n the second summand tends to 0 as $\varepsilon \to 0$ and the first is dominated by

$$\sup_{x\in H} \frac{|f(x) - f_n(x)|}{1 + \Theta_3(x)} \sup_{\varepsilon>0} \int_H (1 + \Theta_3) d(\mu + \mu_\varepsilon),$$

which in turn by Theorem 2.4 and (2.34) tends to zero as $n \to \infty$. So, also the first summand in (2.32) tends to zero as $\varepsilon \to 0$. Hence the right hand side of (2.31) is zero and (2.30) follows which completes the proof. □

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