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Cells for a Hecke Algebra Representation

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Abstract.

If Y is an affine symmetric variety for the reductive group G with Weyl group W, there exists by Lusztig and Vogan a representation of the Hecke algebra of W in a module which has a basis indexed by the set Λ of pairs (v, ξ), where v is an orbit in Y of a Borel group B and ξ is a B-equivariant rank one local system on v. We introduce cells in Λ and associate with a cell a two-sided cell in W.

Introduction.

Let G be a connected reductive group over an algebraically closed field of characteristic $\neq 2$. Let θ be an automorphism of G of order 2, with fixed point group K. In [LV] Lusztig and Vogan introduced a module $\mathcal{M}$ over the Hecke algebra of the Weyl group W of G, coming from the action of K on the flag manifold $\mathcal{F}$ of G. In the present note we introduce cells for that situation. Instead of K-orbits on $\mathcal{F}$ we prefer to work with orbits of a Borel group B on the affine symmetric variety $Y = G/K$. The module $\mathcal{M}$ then has a basis indexed by the set of pairs $\Lambda = (v, \xi)$, where v is a B-orbit on Y and ξ a B-equivariant rank one local system on v.

After the introductory Section 1 we define the cells of Λ in Section 2, in much the same way as the cells of W. In Section 3 we attach to a cell in Λ a representation of Lusztig's asymptotic ring $\mathcal{J}$. We also attach in 3.5 (ii) to a cell in Λ a two-sided cell in W. The final Section 4 discusses some complements and also two examples for $G = SL_3$.

The results of this note about cells in Λ are more or less well-known. In the case $k = \mathbb{C}$ they can probably be extracted from the literature on representations of real reductive groups.

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The definition of cells is also given in [G, 3.1] and [H, no. 4]. In [G, loc. cit.] it is also stated without proof that one can attach to a cell in Λ a two-sided cell in W (see also [H, 4.6]).
The examples at the end of Section 4 are counterparts of the examples of [V, 16.2, 16.3], which are discussed in the context of representation theory of real Lie groups (viz. $SL_3(\mathbb{R})$ and $SU(2,1)$). But cells do not occur there.

I am indebted to A. Henderson for a useful remark.

1. Notations and recollections.

1.1. Let B be a θ-stable Borel subgroup of G and T a θ-stable torus contained in B. The root system of (G,T) is R, the system of positive roots in R defined by B is R^+. The Weyl group of R is W and S is the set of simple reflections defined by B. The associated length function is l.

Denote by $\mathcal{H}$ the generic Hecke algebra defined by (W,S) (see [Cu, p. 16]). It is a free module over $\mathbb{Z}[t,t^{-1}]$, with basis $(e_x)_{x\in W}$. The multiplication is described in [loc. cit.]. In particular, $e_s^2 = (t^2-1)e_s + t^2$ $(s \in S)$.

1.2. Denote by V the set of B-orbits on Y. The results to be used about these orbits can be found in [RS1] and [S1].
For $v \in V$ denote by $\mathcal{L}_v$ the group of isomorphism classes of B-equivariant rank one local systems on v. Let Λ be the set of pairs $l = (v,\xi)$ with $v \in V$, $\xi \in \mathcal{L}_v$. Let $\mathcal{M}$ be the free module over $\mathbb{Z}[t,t^{-1}]$ with basis ϵ_l indexed by the elements $l \in \Lambda$. Then $\mathcal{M}$ has a left module structure over the Hecke algebra $\mathcal{H}$. The products $e_s\epsilon_l$ $(s \in S,\ l \in \Lambda)$ are described in [MS2, 4.3.1]. We shall not write down the formulas of [loc. cit.], as we shall not need them. (Notice that in the present case we have, with the notations of [loc. cit.], $\hat{\phi}_v\xi = 0$ since we are dealing with B-equivariant local systems. Moreover in the cases IIIb and IVb, $2a_v(\xi) = 0$, see [loc. cit., 6.7].)

The construction of $\mathcal{M}$ is sheaf-theoretic. One works over the algebraic closure of a sufficiently large finite field. The elements of $\mathcal{M}$ lie in a Grothendieck group built out of B-equivariant l-adic sheaves on Y with Frobenius action. In the general situation of [loc. cit.], $\mathcal{M}$ appears as a module over a large ring, which can in the present case be cut down to $\mathbb{Z}[t,t^{-1}]$.

Let $l = (v, \xi) \in \Lambda$. The basis element ϵ_l of $\mathcal{M}$ is the element in the appropriate Grothendieck group defined by the sheaf on Y extending ξ by zero.
Denote by $A_{\xi,v}$ the irreducible perverse sheaf on Y supported by the closure $\bar{v}$ whose restriction to v is $\xi[\dim v]$ (the "perverse extension" of ξ). It defines an element γ_l of $\mathcal{M}$ (see [loc. cit., 3.1.2, p. 42]).
For $l = (v, \xi) \in \Lambda$ put $d(l) = \dim(v)$.

We next quote some results of Lusztig and Vogan, established in [LV] (see also [MS2, no. 7]).

1.3. Lemma. *There exists an additive duality map D of $\mathcal{M}$ such that for $\mu \in \mathcal{M}, s \in S, l \in \Lambda$*
(a) $D(t\mu) = t^{-1}D(\mu)$,
(b) $D(e_s\mu) = e_s^{-1}D(\mu)$,
(c) $D(\epsilon_l) = t^{-2d(l)}(\epsilon_l + \sum_{d(m)<d(l)} R_{m,l}(t^2)\epsilon_m)$, where $R_{m,l} \in \mathbb{Z}[T]$ has degree $\leq d(l) - d(m)$.
D is an algebraic reflection of Verdier duality.

1.4. Lemma. *γ_l is the unique element of $\mathcal{M}$ satisfying $D(\gamma_l) = \gamma_l$, of the form*

$$t^{-d(l)}(\epsilon_l + \sum_{d(m)<d(l)} P_{m,l}(t^2)\epsilon_m), \tag{1}$$

where $P_{m,l} \in \mathbb{Z}[T]$ has degree $\leq \frac{1}{2}(d(l) - d(m) - 1)$ and has positive coefficients.

If $l = (v, \xi)$, $m = (w, \eta)$ and $P_{m,l} \neq 0$ then w is contained in the closure of v.
For $d(m) < d(l)$ we denote by $\mu(m, l)$ the coefficient of $T^{\frac{1}{2}(d(l)-d(m)-1)}$ in $P_{m,l}$. If $d(l) < d(m)$ we put $\mu(m, l) = \mu(l, m)$.

Denote by b_x ($x \in W$) the Kazhdan-Lusztig elements of $\mathcal{H}$ (see [Cu, p. 30]). They can also be viewed as the elements $[A_{0,x}]$ of [MS2, 3.2]).

Let $l = (v, \xi)$. For $s \in S$ let $P_s = B \cup BsB$ be the parabolic subgroup defined by s. Denote by $\tau(l) \subset S$ the set of simple reflections s such that $\dim P_s v = \dim v$ and, moreover, ξ extends to a sheaf on $P_s v$. (In the notations of [MS2, 4.3.1] the $s \in \tau(l)$ are the simple reflections for which we have one of the cases I, IIb, IIIb or IVb with $a(\xi) = 0$.)

1.5. Proposition. *$b_s\gamma_l$ equals*

$$\sum_{s\in\tau(m)} \mu(m,l)\gamma_m \quad \text{if} \quad s \notin \tau(l), \tag{2}$$

$$(t+t^{-1})\gamma_l \quad \text{if} \quad s\in\tau(l). \tag{3}$$

Proof. (2) is proved in the same way as [LV, 5.3], using [loc. cit., 5.4]. For (3) see [loc. cit., 5.2].

1.6. Corollary. *Assume that $\gamma = \sum_{l\in\Gamma} f_l\gamma_l$, where the f_l are Laurent polynomials. If $b_s\gamma = (t+t^{-1})\gamma$ then $s\in\tau(l)$ if $f_l\neq 0$.*
Proof. Using (3) we see that it suffices to prove that if $f_l = 0$ for all l with $s\in\tau(l)$ then $f_l = 0$ for all l. This follows from (2).

1.7. Proposition. *Let $x\in W, l\in\Lambda$. Then*

$$b_x\gamma_l = \sum_{m\in\Lambda} g_{x,l,m}\gamma_m,$$

where the $g_{x,l,m}$ lie in $\mathbb{Z}[t,t^{-1}]$ and have non-negative coefficients. Moreover, they are invariant under the map $t\mapsto t^{-1}$.
Proof. The first part follows from the sheaf-theoretic construction of the product, using the decomposition theorem and the fact that the eigenvalues of Frobenius on the stalks of the cohomology sheaves of the perverse sheaves $A_{\xi,v}$ are powers of q (see [MS2, 7.1.2]). For a similar result see [MS1, 4.2.6]. The last point is a consequence of the relative hard Lefschetz theorem.

2. Cells.

2.1. We define a preorder relation $\leq$ on Λ as follows: $m\leq l$ if $g_{x,l,m}\neq 0$ for some $x\in W$, where $g_{x,l,m}$ is as in 1.7. An equivalent definition is: γ_m occurs with a non-zero coefficient in some element of $\mathcal{H}\gamma_l$.
Since the b_s $(s\in S)$ generate $\mathcal{H}$, it follows that the relation can also be defined to be the one generated by the elementary relations $\leq_s$ $(s\in S)$, where $m\leq_s l$ if $s\notin\tau(l)$ and γ_m occurs in $b_s\gamma_l$ with a non-zero coefficient. By (2) the latter condition is equivalent with: $s\in\tau(m)$ and $\mu(m,l)\neq 0$.
We define an equivalence relation $\sim$ on Λ by $l\sim m$ if $l\leq m$ and $m\leq l$. The equivalence classes are the *cells* of Λ. These definitions are similar to the well-known definition of cells in W, due to Kazhdan and Lusztig. For the results about such cells in W we refer to [Cu, Ch. II, III].

Let Γ be a cell in Λ. Write $m \leq \Gamma$ ($m < \Gamma$) if $m \leq l$ (respectively, $m \leq l$ and $m \not\sim l$) for some $l \in \Gamma$. The γ_l with $l \leq \Gamma$ (respectively, $l < \Gamma$) span a submodule $\mathcal{M}_\Gamma$ (respectively, $\mathcal{M}'_\Gamma$) of $\mathcal{M}$. Put

$$\mathcal{N} = \mathcal{N}_\Gamma = \mathcal{M}_\Gamma / \mathcal{M}'_\Gamma.$$

This is a free $\mathcal{H}$-module, with basis $\delta_l = \gamma_l + \mathcal{M}'_\Gamma$ ($l \in \Gamma$). We define an integer $a = a(\Gamma)$ by

$$a = \max_{x \in W; l,m \in \Gamma} (\deg g_{x,l,m}).$$

Clearly $a \geq 0$. For $x \in W$, $l, m \in \Lambda$ all Laurent polynomials $g_{x,l,m}$ have degree $\leq a$. Let $c_{x,l,m}$ be the coefficient of t^a in $g_{x,l,m}$. It is an integer ≥ 0.

In the proof of the next lemma the notations are as in [loc. cit., no. 6]: the $h_{x,y,z}$ are the structure constants of $\mathcal{H}$ for the Kazhdan-Lusztig basis (b_x), $a(z) = \max_{x,y} (\deg h_{x,y,z})$ is Lusztig's cell invariant and $\gamma_{x,y,z}$ is the coefficient of $t^{a(z)}$ in $h_{x,y,z}$.
We shall also use Lusztig's asymptotic ring, which we denote by $\mathcal{J}$, see [loc. cit., no. 9]. It is a free abelian group with basis j_z ($z \in W$), the $\gamma_{x,y,z}$ being the corresponding structure constants. By [loc. cit., 9.2] we may view $\mathcal{J}$ as a subring of $\mathbb{Q}(t) \otimes_{\mathbb{Z}[t]} \mathcal{H}$, such that for $x \in W$

$$b_x = \sum_{d \in \mathcal{D}, a(d) = a(z)} h_{x,d,z} j_z, \tag{4}$$

where $\mathcal{D} \subset W$ is the set of Duflo involutions in W (introduced in [loc. cit., 6.8 (ii)]).

2.2. Lemma. *If* $a(x) > a$ *then* $b_x \mathcal{N} = 0$.

Proof. If j_z occurs in the right-hand side of (4) with a non-zero coefficient then $z \leq_R x$, whence $a(z) \geq a(x)$ by [loc. cit., 6.9 (ii)]. Hence in order to prove the lemma it suffices to show that $j_x \mathcal{N} = 0$ for $a(x) > a$ ($\mathcal{N}$ being viewed as a subset of $\mathbb{Q}(t) \otimes \mathcal{N}$). Putting

$$b = \max\{a(x) \mid j_x \mathcal{N} \neq 0\},$$

this amounts to proving that $b \leq a$.
Let $j_x \mathcal{N} \neq 0$ and $a(x) = b$. Let d be the Duflo involution in the left cell of x (see [loc. cit., 6.11]). Then by [loc. cit., 9.5 (i)] we have $j_x = j_x j_d$,

whence $j_d\mathcal{N} \neq 0$, and $a(d) = a(x) = b$. So we may assume that $x = d$. Let tr be the trace function on $\mathbb{Q}(t) \otimes \mathcal{H}$, acting on $\mathbb{Q}(t) \otimes \mathcal{N}$. By (4)

$$\mathrm{tr}(b_d) = \sum_{e \in \mathcal{D}, a(e)=a(z)} h_{d,e,z}\mathrm{tr}(j_z).$$

The non-zero $h_{d,e,z}$ in the right-hand side are such that $a(z) \geq a(d) = b$. Our assumption implies that we can restrict the summation to the z with $a(z) = b$. Then $\deg(h_{d,e,z}) \leq b$. If equality holds we must have $\gamma_{d,e,z} \neq 0$, which can only be if $d = e = z$, by [loc. cit., 6.10 (i), 6.8 (ii)]. This implies that $\mathrm{tr}(b_d) - h_{d,d,d}\mathrm{tr}(j_d)$ is a Laurent polynomial of degree $< b$ (notice that all $\mathrm{tr}(j_z)$ are algebraic integers). Since $j_d^2 = j_d$ we have $\mathrm{tr}(j_d) > 0$. Hence $\mathrm{tr}(b_d)$ is a Laurent polynomial of degree b. Now

$$\mathrm{tr}(b_d) = \sum_{l \in \Gamma} g_{d,l,l},$$

from which we see that there is $l \in \Gamma$ with deg $(g_{d,l,l}) \geq b$. This implies that $b \leq a$, which we had to prove.

For $x \in W$ define $\tau(x) = \{s \in S \mid sx < x\}$.

2.3. Lemma. *Let $x \in W$, $l, m \in \Gamma$ and assume that $c_{x,m,l} \neq 0$.*
(i) $\tau(x) = \tau(l)$;
(ii) For any $l' \in \Gamma$ there exists $x' \in W$ such that $x' \leq_L x$ and $c_{x',m,l'} \neq 0$.

Proof. Assume that $l, l' \in \Gamma$ and $l' \leq_s l$ for some $s \in S$. Then $s \notin \tau(l)$. We have the associativity relation

$$(b_s b_x)\gamma_m = b_s(b_x\gamma_m).$$

If $sx < x$ we have $b_s b_x = (t + t^{-1})b_x$ by [Cu, 5.1], whence $b_s(b_x\delta_m) = (t+t^{-1})(b_x\delta_m)$. From 1.6 we infer that this is impossible, since δ_l occurs in $b_x\delta_m$ with a non-zero coefficient. It follows that $\tau(x) \subset \tau(l)$.
Now assume that $sx > x$. Writing out the associativity relation and comparing coefficients of l' on both sides we obtain, using [loc. cit.] and 1.5,

$$(5) \quad \sum_{sx'<x'} \mu(x', x)g_{x',m,l'} = (t + t^{-1})g_{x,m,l'} + \sum_{n,\ s\notin\tau(n)} g_{x,m,n}\mu(l', n).$$

In the left-hand side of (5), $\mu(x', x)$ is the usual Kazhdan-Lusztig coefficient.
If $c_{x,m,l} \neq 0$ and $\mu(l', l) \neq 0$, the right-hand side contains a non-zero

multiple of t^a. Since all Laurent polynomials occurring in (5) have coefficients ≥ 0, the left-hand side also contains a non-zero multiple of t^a. We conclude that there is $x' <_{L,s} x$ with $c_{x',m,l'} \neq 0$ (where $<_{L,s}$ is the elementary preorder relation on W defined by s, i.e. $sx' < x', sx > x$ and $\mu(x', x) \neq 0$, cf. [Cu, 5.2]). (ii) follows in the case that $l' \leq_s l$. The general case is a consequence.
Again, let $sx > x$ and consider (5) with l' arbitrary such that $s \notin \tau(l')$. Since the left-hand side has degree $\leq a$ we must have $c_{x,m,l'} = 0$. This implies that $\tau(l) \subset \tau(x)$ if $c_{x,m,l} \neq 0$ and (i) follows.

2.4. Lemma. *Let $l, m \in \Gamma$ and $x \in W$ be such that $c_{x,l,m} \neq 0$.*
(i) $a(x) = a$;
(ii) In 2.3 (ii) we have $x' \sim_L x$.

Proof. From the asssociativity relation $(b_x b_y)\gamma_n = b_x(b_y\gamma_n)$ $(x, y \in W, n \in \Gamma)$ we obtain for $m \in \Gamma$

$$\sum_{z \in W} h_{x,y,z} g_{z,n,m} = \sum_{p \in \Gamma} g_{x,p,m} g_{y,n,p}. \tag{6}$$

Let $c_{x,l,m} \neq 0$, then $g_{x,l,m}$ has degree a. By 2.3 (ii) there exists $y \leq_L x$ such that $\deg(g_{y,l,l}) = a$. Take $n = l$ in (6). The right-hand side has degree $2a$. If in the left-hand side of (6) we have $g_{z,l,m} \neq 0$ then $b_z\mathcal{N} \neq 0$ and $a(z) \leq a$, whence $\deg(h_{x,y,z}) \leq a$. Since the right-hand side has degree $2a$ there is $z \in W$ with $\deg(h_{x,y,z}) = a(z) = a$. Then $\gamma_{x,y,z} \neq 0$. By [Cu, 6.10] we have $x \sim_R z$ and $a(x) = a(z) = a$, proving (i).
(ii) is a consequence of (i) and [loc. cit.].

2.5. Lemma. *For $x, y \in W$ and $m, n \in \Gamma$ we have $\sum_{z \in W} \gamma_{x,y,z} c_{z,n,m} = \sum_{l \in \Gamma} c_{x,l,m} c_{y,n,l}$.*

Proof. We use (6). From the proof of 2.4 we see that all structure constant occurring in (6) are Laurent polynomials of degree $\leq a$. The asserted identity then follows by comparing coefficients of t^{2a} in both sides of (6).

3. A $\mathcal{J}$-module.

3.1. Let $\mathcal{K} = \mathcal{K}_\Gamma$ the free abelian group with basis k_l indexed by the elements of Γ. For $x \in W$, $l \in \Lambda$ define

$$j_x k_l = \sum_{m \in \Gamma} c_{x,l,m} k_m,$$

and extend this to an additive map $\mathcal{J} \otimes_{\mathbf{Z}} \mathcal{K} \to \mathcal{K}$. By 2.5 we have

$$j_x(j_y k_l) = (j_x j_y) k_l.$$

This shows that we have defined a $\mathcal{J}$-module structure on $\mathcal{K}$. We have not yet established that $\mathcal{K}$ is a unital module, i.e that the identity element

$$\mathbf{1} = \sum_{d \in \mathcal{D}} j_d$$

of $\mathcal{J}$ acts as the identity on $\mathcal{K}$. We shall do this presently.

3.2. Proposition. *For $z \in \mathcal{J}$ the traces $tr(j_z, \mathcal{K})$ and $tr(j_z, \mathcal{N})$ are equal.*

Proof. It follows from (4) that for $z \in W$

$$j_z = \sum_{w \in W} \xi_{z,w} t^{a(z)} b_w,$$

where $(\xi_{z,w})$ is a matrix with entries in $\mathbb{Q}(t)$. Also, $\xi_{z,w}$ is defined at $t = 0$ and $\xi_{z,w}(0) = \delta_{z,w}$ (cf. [Cu, p. 54]). Hence

$$j_z \delta_l = \sum_{m \in \Gamma} \eta_{z,l,m} \delta_m,$$

with

$$\eta_{z,l,m} = \sum_{w} \xi_{z,w} t^{a(z)} g_{w,l,m}.$$

By 2.4 we may assume that $a(z) = a$. Since $g_{w,l,m}$ is invariant under the map $t \mapsto t^{-1}$ (see 1.7), it follows that $t^{a(z)} g_{w,l,m} \in \mathbb{Z}[t]$ and has value $c_{w,l,m}$ for $t = 0$. We conclude that $\eta_{z,m,l}$ is a rational function in t which is defined at $t = 0$ with value $c_{z,m,l}$.
We have

$$\mathrm{tr}(j_z, \mathcal{N}_{\mathbb{Q}(t)}) = \sum_{l \in \Gamma} \eta_{z,l,l},$$

a rational function of t which is defined at $t = 0$. Since $\mathrm{tr}(j_z)$ is an algebraic integer for all z, this rational function must be constant and its value is the value at 0, which is $\mathrm{tr}(j_z, \mathcal{K})$. The proposition follows.

3.3. Corollary. *$\mathcal{K}$ is unital.*

Proof. Put

$$\mathcal{K}_0 = \{k \in \mathcal{K} \mid \mathbf{1}.k = 0\},$$

this is a direct summand of $\mathcal{K}$. We have a structure of unital $\mathcal{J}$-module on $\mathcal{K}/\mathcal{K}_0$, whence $\mathrm{tr}(\mathbf{1},\mathcal{K}) = |\Gamma| - \mathrm{rank}(\mathcal{K}_0)$. The proposition shows that $\mathrm{tr}(\mathbf{1},\mathcal{K}) = |\Gamma|$ and it follows that $\mathcal{K}_0 = \{0\}$, i.e. that $\mathcal{K}$ is unital.

We write $\mathcal{H}_{\mathbb{Q}(t)} = \mathbb{Q}(t) \otimes_{\mathbb{Z}[t]} \mathcal{H}$, and similarly for other objects obtained by extending coefficients. We know that $\mathcal{H}_{\mathbb{Q}(t)} = \mathcal{J}_{\mathbb{Q}(t)}$ (recall that $\mathcal{J}$ is a subring of $\mathcal{H}_{\mathbb{Q}(t)}$). From 3.3 we see that $\mathcal{K}_{\mathbb{Q}(t)}$ is a $\mathcal{H}_{\mathbb{Q}(t)}$-module.

3.4. Proposition. *The $\mathcal{H}_{\mathbb{Q}(t)}$-modules $\mathcal{N}_{\mathbb{Q}(t)}$ and $\mathcal{K}_{\mathbb{Q}(t)}$ are isomorphic.*

Proof. The algebra $\mathcal{H}_{\mathbb{Q}(t)}$ is split semi-simple (see [Cu, 8.3]). Using the orthogonality relations for its irreducible representations (cf. [MS1, 11.1.4]) it follows from 3.2 that the multiplicities of an irreducible representation of $\mathcal{H}_{\mathbb{Q}(t)}$ in $\mathcal{N}_{\mathbb{Q}(t)}$ and $\mathcal{K}_{\mathbb{Q}(t)}$ are the same. This proves 3.4.

3.5. Proposition. *(i) For every $l \in \Gamma$ there is a unique Duflo involution d with $j_d k_l = k_l$;*
(ii) The involutions of (i) lie in a unique two-sided cell of W.

Proof. By 3.3 we have

$$\sum_{d \in \mathcal{D}} j_d k_l = k_l.$$

Now any product $j_d k_l$ is a positive integral linear combination of k_m's. (i) follows from the observation that the left-hand side of the formula can contain only one non-zero term.
Let $d, e \in \mathcal{D}$ and $l, m \in \Gamma$ be such that $j_d k_l = k_l$ and $j_e k_m = k_m$. It follows from 2.3 (ii) and 2.4 (ii) that there is $x \sim_L d$ such that $j_x k_l$ contains k_m with a non-zero coefficient. Then $j_e j_x k_l \neq 0$, in particular $j_e j_x \neq 0$. This implies that $e \sim_L x^{-1}$ (see [Cu, 9.5 (ii)]). Then $e \sim_R x \sim_L d$, whence $d \sim_{LR} e$, proving the proposition.

4. Complements and examples.

4.1. A bilinear form. Let $v \in V$. For $s \in S$ denote by $P_s \supset B$ the parabolic subgroup of semi-simple rank 1 associated to s. Let $m(s)v$ be the unique open orbit of B in $P_s v$. We recall the notion of a reduced decomposition $\mathbf{v} = ((v_0, ..., v_l), \mathbf{s} = (s_1, ..., s_l))$ of v: the v_i lie in V and the s_j in S, v_0 is a closed orbit, $v_l = v$ and $v_i = m(s_i)v_{i-1} \neq v_{i-1}$ (see [RS1, 5.7, no. 7]). Then $d(v_i) = d(v_{i-1}) + 1$ $(1 \leq i \leq l)$. Let $\lambda_c(v)$ $(\lambda_i(v))$ be the number of i such that s_i is complex (respectively, imaginary) for v_{i-1}. See [loc.cit., 4.3], the cases correspond

to the cases IIa (respectively, IIIa or IVa) of [MS2, 4.1.4]. It follows from the definitions, using [RS1, 3.7], that these numbers depend only on v, and not on the choice of the reduced decomposition $(\mathbf{v}, \mathbf{s})$.
Denote by U the unipotent part of B, so $B = TU$. It is known that $B \cap K$ $(T \cap K)$ is a Borel subgroup (respectively, a maximal torus) of K. Let $v_0 \in V$ be the closed orbit BK/K. It is isomorphic to $B/B \cap K$. In fact, this is true for any closed orbit $v_0 \in V$, as follows from [S1, 6.6]. If $v \in V$ we put

$$d_c(v) = \lambda_c(v) + \dim U/U \cap K,\ d_i(v) = \lambda_i(v) + \dim T/T \cap K.$$

Then

$$d(v) = d_c(v) + d_i(v).$$

If $(x, \xi) \in \Lambda$ we put $d_i(l) = d_i(v),\ d_c(l) = d_l(v)$.
Denote by N be the normalizer of T. Let $v \in V$. There exists $x \in G$ with $xK \in v$ such that $x(\theta x)^{-1} \in N$ (see [loc.cit., 4.2]). Denote isotropy subgroups of x by a suffix x.

4.2. Lemma. *(i) v is isomorphic as a variety to B/B_x;*
(ii) $B_x = T_x U_x$.
(iii) $\dim T/T_x = d_i(v)$, $\dim U/U_x = d_c(v)$.

Proof. It is clear that there is a bijective morphism of homogeneous spaces $B/B_x \rightarrow v$. It is separable (see [MS2, 6.3]), hence is an isomorphism. This proves (i). (ii) and (iii) follow from [S1, 4.7].

We introduce the $\mathbf{Z}[t, t^{-1}]$-bilinear form β on $\mathcal{M}$ with

$$\beta(\epsilon_l, \epsilon_m) = \delta_{l,m}(t^2 - 1)^{d_i(l)} t^{2d_c(l)}.$$

Clearly, it is symmetric and nondegenerate.

4.3. Proposition. *For $x \in W$, $\mu, \nu \in \mathcal{M}$ we have*

$$\beta(e_x \mu, \nu) = \beta(\mu, e_{x^{-1}} \nu).$$

Proof. It suffices to prove this in the case that x is a simple reflection s and $\mu = \epsilon_l, \nu = \epsilon_m$ $(l, m \in \Lambda)$. Using the explicit formulas of [MS2, 4.3.1] for the products $e_s \epsilon_l$, the verification of the asserted formula is straightforward. It is left to the reader. (The explicit formulas in our special case are also described, somewhat differently, in [RS2, 7.3]).

4.4. Corollary. *Let $l, m \in \Lambda$.*
(i) $\beta(\gamma_l, \gamma_m) - \delta_{l,m} \in t^{-1}\mathbb{Z}[t^{-1}]$;
(ii) For $x \in W$, $l, m \in \Lambda$ we have $\deg g_{x,m,l} = \deg g_{x^{-1},l,m}$;
(iii) Let Γ be a cell in Λ. For $x \in W$, $l, m \in \Gamma$ we have $c_{x,m,l} = c_{x^{-1},l,m}$.

Proof. Inserting the expressions of (1) for γ_l and γ_m and using the degree estimates for the polynomials $P_{m,l}$ of (1), (i) readily follows.
Let M be the matrix $(\beta(\epsilon_l, \epsilon_m))_{l,m\in\Lambda}$. For $x \in W$, multiplication by b_x in $\mathcal{M}$ is given (relative to the basis (ϵ_l)) by the matrix $M_x = (h_{x,m,l})$. By the proposition, the matrix of $b_{x^{-1}}$ is given by the transpose of M_x relative to β, which is $M^{-1}({}^tM_x)M$. Using (i) we see that $M^{-1} - I$ is a matrix with entries in $t^{-1}\mathbb{Z}[[t^{-1}]]$. (ii) then follows from (i) and this observation. (iii) also follows.

4.5. Corollary. *Let $x \in W$, $l, m \in \Gamma$ and assume that $c_{x,m,l} \neq 0$.*
(i) $\tau(x^{-1}) = \tau(m)$;
(ii) For any $m' \in \Gamma$ there exists $x' \in W$ such that $x' \leq_R x$ and $c_{x',m',l} \neq 0$.

Proof. This follows from 4.4 (iii) and 2.3.

4.6. Examples. We briefly discuss two examples with $G = SL_3$. We take B and T to be the subgroups of upper triangular, respectively diagonal, matrices. The Weyl group is S_3.
The simple roots are the characters α_1, α_2 of T sending $(a_1, a_2, a_3) \in T$ to $a_1a_2^{-1}$, respectively $a_2a_3^{-1}$. The corresponding simple reflections are the transpositions (12) and (23). The corresponding generators of the Hecke algebra $\mathcal{H}$ are denoted by e_1 and e_2.

(a) $\theta(g) = a({}^tg)^{-1}a^{-1}$ where a is such that θ stabilizes B and T. Then $K \simeq SO_3$.
The set V of B-orbits in G/K has 4 elements v_0, v_1, v_1', v_2, of respective dimensions $3, 4, 4, 5$, as follows from [RS1, p. 432-433]. One checks that the group $\mathcal{L}_v$ of B-equivariant local systems on the orbit v is trivial except if $v \neq v_2$, in which case it is the character group of the subgroup of T of elements of order ≤ 2.
We abbreviate $\epsilon_{v_0,0}$ to ϵ_0. Similarly, we have ϵ_1 and ϵ_1'. We have 4 basis elements $\epsilon_{v_2,\xi}$ denoted by $\epsilon_{20}, \epsilon_{21}, \epsilon_{22}, \epsilon_{23}$, where ϵ_{20} corresponds to the constant sheaf on v_2. We use similar notations for the Kazhdan-Lusztig elements γ_l.
The action of e_1 and e_2 on the basis elements is described in [RS2, p. 141] (in the first formula of line 6 of that page f_1 should be replaced by

f_1').

We now deal with the duality operator D. It follows from 1.3 (c) that $D(\epsilon_0) = t^{-6}\epsilon_0$. By [loc. cit.] we have $e_1\epsilon_0 = \epsilon_1'$, $e_2\epsilon_0 = \epsilon_1$. Then $D(\epsilon_1), D(\epsilon_1')$ can be determined from 1.3 (b). One checks that

$$\gamma_0 = t^{-3}\epsilon_0,\ \gamma_1 = t^{-4}(\epsilon_1 + \epsilon_0),\ \gamma_1' = t^{-4}(\epsilon_1' + \epsilon_0)$$

have the properties of 1.4 and thus are the correct Kazhdan-Lusztig elements. Next, since G/K is smooth its intersection cohomology complex is the shifted constant sheaf $E[5]$, from which it follows that

$$\gamma_{20} = t^{-5}(\epsilon_{20} + \epsilon_1 + \epsilon_1' + \epsilon_0).$$

Since γ_{20} is D-invariant, this formula determines $D(\epsilon_{20})$.

We have

$$e_2\epsilon_1' = \epsilon_{20} + \epsilon_{22} + \epsilon_1'.$$

By 1.3 (b) one knows how D acts on the right-hand side. Using what is already known one finds $D(\epsilon_{22})$, and similarly $D(\epsilon_{21})$. Then

$$\gamma_{21} = t^{-5}(\epsilon_{21} + \epsilon_1),\ \gamma_{22} = t^{-5}(\epsilon_{22} + \epsilon_1')$$

satisfy the requirements of 1.4.

Finally, we claim that

$$\gamma_{23} = t^{-10}\epsilon_{23}.$$

To see this it suffices to show that $D(\epsilon_{23}) = t^{-5}\epsilon_{23}$. Now by the formulas of [loc. cit.], ϵ_{23} is annihilated by $e_1 + 1$ and $e_2 + 1$. By 1.3 (b) the same must be true of $\mu = D(\epsilon_{23})$. Then μ must be orthogonal, with respect to the bilinear form β of 4.3, to $(e_1+1)\mathcal{M}$ and $(e_2+1)\mathcal{M}$. The formulas of [loc. cit.] show that this can only be if μ is a multiple of ϵ_{23}. By 1.3 (c) we then must have $\mu = t^{-10}\epsilon_{23}$.

Let $c_i = c_{s_i}$ $(i = 1, 2)$. The products $c_i\gamma_l$ which are not 0 or $(t+t^{-1})\gamma_l$ are the following: $c_1\gamma_0 = \gamma_1'$, $c_2\gamma_0 = \gamma_1$, $c_1\gamma_1 = \gamma_{20} + \gamma_{21}$, $c_2\gamma_1' = \gamma_{20} + \gamma_{22}$, $c_1\gamma_{22} = \gamma_1'$, $c_2\gamma_{21} = \gamma_1$. Using these formulas we see that the cells are: $\Gamma_0 = \{v_0\}$, $\Gamma_1 = \{v_1,\ \gamma_{21}\}$, $\Gamma_1' = \{v_1', v_{22}\}$, $\Gamma_3 = \{v_{20}\}$, $\Gamma_0' = \{v_{23}\}$. The index denote the a-value on the cell.

The two-sided cells in S_3 are $\Delta_0 = \{1\}$, $\Delta_1 = \{(12), (23), (123), (132)\}$, $\Delta_3 = \{(13)\}$. The two-sided cell in S_3 attached to a cell in Λ is the one with the same suffix.

(b) $\theta(g) = aga^{-1}$, where $a = \text{diag}(-\zeta, \zeta, \zeta)$ with $\zeta^3 = -1$. Now $K \simeq GL_2$.

This case is discussed (more generally, for SL_n) in [RS1, 10.5]. We have three closed orbits v_1, v_2, v_3 of dimension 2, two orbits v_{12}, v_{23} of

dimension 3 and the open orbit v_{13} of dimension 4. The numbering is such that $v_i \leq v_{jk}$ if and only if $i = j$ or $i = k$.
One checks that all groups $\mathcal{L}_v$ are trivial. Using [loc. cit.] and the formulas of [MS2, 4.3.1] or [RS2, 7.3] it is straightforward to determine the products $e_i \epsilon_v$. Proceeding as in the previous example one determines the various $D(\epsilon_v)$ and the Kazhdan-Lusztig elements γ_v.
It turns out that for all $v \in V$

$$\gamma_v = t^{-\dim v} \sum_{w \leq v} \epsilon_w$$

(which means that all orbit closures $\bar{v}$ are rationally smooth). We can then determine the products $c_i \gamma_v$. The upshot is that the cells are $\Gamma_0 = \{v_2\}$, $\Gamma_1 = \{v_1, v_{12}\}$, $\Gamma_1' = \{v_2, v_{23}\}$, $\Gamma_3 = \{v_{13}\}$. Again, the suffixes denote the a-values.

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