Advanced Studies in Pure Mathematics 40, 2004
Representation Theory of Algebraic Groups and Quantum Groups
pp. 371–442

Tropical Robinson-Schensted-Knuth correspondence and birational Weyl group actions

Masatoshi Noumi and Yasuhiko Yamada

§ Introduction

This paper is an outcome of our attempt to understand internal connections among several appearances of the subtraction-free birational transformations.

There is a well-known procedure for passing from subtraction-free rational functions to piecewise linear functions. Roughly, this is the procedure of replacing the operations

$$a\,b \to a+b, \quad a/b \to a-b, \quad a+b \to \max\{a,b\} \quad (\text{or } \min\{a,b\}).$$

It can be applied consistently to an arbitrary rational function expressed as a ratio of two polynomials with positive real coefficients, in order to produce a combination of $+$, $-$ and max (or min), representing a piecewise linear function. In combinatorics, this procedure has been employed for the *algebraization* of combinatorial algorithms. A large class of combinatorial algorithms can be described as piecewise linear transformations among discrete variables which take integer values. For such a piecewise linear transformation, it is meaningful in many cases to find a good subtraction-free rational counterpart; algebraic computation of subtraction-free rational functions may possibly bring out unexpected solutions to combinatorial problems. For this *tropical approach* to combinatorics, we refer the reader to [1], [14] and the references therein.

In the context of discrete integrable systems, the same procedure is known as *ultra-discretization* [27]. A remarkable example is the ultra-discretization of discrete Toda equation which provides with soliton cellular automata, called the box-ball systems [28]. It is already recognized that the theory of box-ball systems is precisely the dynamics of

Received March 11, 2002.
Revised October 15, 2002.

crystal bases which arise as the $q \to 0$ limit of representations of quantum groups (see [6], for example). The ultra-discretization of certain q-Painlevé systems can also be understood as a non-autonomous deformation of box-ball systems [10, 11, 12]; the time evolution of such (ultra-)discrete systems arises from the translation lattice of affine Weyl groups.

Another important aspect is the connection with the theory of totally positive matrices. Totally positive matrices have been studied extensively from the viewpoint of geometric approach to canonical bases [19], [1], [4], [2]; they provide a basic tool for producing nice subtraction-free rational transformations.

The purpose of this paper is to develop a new, elementary approach to the application of subtraction-free birational transformations to combinatorial problems. Our method is based on the decomposition and exchange of matrices, and the path representation of minor determinants. We employ such techniques to construct both subtraction-free rational and piecewise linear transformations for typical combinatorial algorithms, such as the bumping procedure, the Schützenberger involution and the Robinson-Schensted-Knuth correspondence (RSK correspondence, for short). Our *matrix approach* can be regarded as an integration of the idea of totally positive matrices and the technique of discrete Toda equations. We also investigate certain birational and piecewise linear actions of (affine) Weyl groups on matrices and tableaux.

This work was motivated by the impressive paper [14] by A.N. Kirillov. It was a great surprise for the authors to find that many formulas in [14], arising from combinatorics, were essentially the same as what we had encountered with in the context of discrete Painlevé systems. The matrix approach, as we will develop below, was a natural consequence of our attempt to clarify the theoretical background of this remarkable coincidence.

In view of the elementary nature of our approach, we have tried to make this paper as self-contained as possible. Many of the explicit formulas discussed in this paper can already be found in the literature ([3], [14], [15]). Also, many of the statements on decomposition of matrices are essentially contained in a series of works [1], [4], [19] on totally positive matrices. We expect however that the results and techniques developed in this paper would be applicable to various problems both in combinatorics and in discrete integrable systems.

The authors would like to express their thanks to Professors S. Fomin, A.N. Kirillov, and A. Zelevinsky for valuable discussions.

Notes: The use of the phrase "tropical" comes originally from computer science; as in "tropical semirings", this word has been used in a restrictive way to refer to the semiring structure on various set of numbers with respect to the pair of operations $(\min, +)$. We thank Prof. Fomin for directing our attention to this point. In the combinatorial literature, the same phrase seems to be used in a broader sense, mostly in such a situation that subtraction-free rational functions and piecewise linear functions appear more or less exchangeably; it also depends on the author on which side emphasis is put. In this paper, following [14] we use the word "tropical" *tentatively* to refer to objects concerning subtraction-free rational functions (see Section 1.3). This may *not* be identical to the traditional usage, but we could not find a better alternative.

CONTENTS

§1. Preliminaries

In this section, we give some preliminary remarks on the matrix approach to nonintersecting paths. In the last part of this section, we also give a summary on a canonical procedure for passing from subtraction-free rational functions to piecewise linear functions. In what follows, we fix the ground field $\mathbb{K}$, and set $\mathbb{K}^* = \mathbb{K} \setminus \{0\}$. For a matrix $X = \left(x^i_j\right)_{i,j}$ given, we denote by

$$(1.1) \qquad X^{i_1,\dots,i_r}_{j_1,\dots,j_r} = \left(x^{i_a}_{j_b}\right)^r_{a,b=1}, \quad \det X^{i_1,\dots,i_r}_{j_1,\dots,j_r} = \det\left(x^{i_a}_{j_b}\right)^r_{a,b=1}$$

the $r \times r$ submatrix and the r-minor determinant of X with row indices $i_1, \dots, i_r$ and column indices $j_1, \dots, j_r$, respectively.

1.1. Path representation of minor determinants

For an n-vector $\boldsymbol{x} = (x_1, \dots, x_n) \in (\mathbb{K}^*)^n$ given, we introduce the following two matrices $E(\boldsymbol{x})$ and $H(\boldsymbol{x})$:

$$(1.2) \qquad E(\boldsymbol{x}) = \mathrm{diag}(\boldsymbol{x}) + \Lambda, \quad H(\boldsymbol{x}) = (\mathrm{diag}(\boldsymbol{x})^{-1} - \Lambda)^{-1},$$

where $\Lambda = \left(\delta_{j,i+1}\right)_{i,j=1}^{n}$ stands for the shift matrix. With the notation of matrix units $E_{ij} = \left(\delta_{a,i}\delta_{b,j}\right)_{a,b=1}^{n}$, these matrices can be written alternatively as

(1.3)
$$E(\boldsymbol{x}) = \sum_{i=1}^{n} x_i E_{ii} + \sum_{i=1}^{n-1} E_{i,i+1}, \quad H(\boldsymbol{x}) = \sum_{1\le i\le j\le n} x_i x_{i+1}\cdots x_j\, E_{ij}.$$

For a given sequence of n-vectors $\boldsymbol{x}^1,\dots,\boldsymbol{x}^m$, $\boldsymbol{x}^i = (x_1^i,\dots,x_n^i) \in (\mathbb{K}^*)^n$, we define

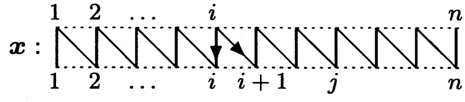
$$\begin{aligned} &E(\boldsymbol{x}^1,\dots,\boldsymbol{x}^m) = E(\boldsymbol{x}^1)E(\boldsymbol{x}^2)\cdots E(\boldsymbol{x}^m),\\ &H(\boldsymbol{x}^1,\dots,\boldsymbol{x}^m) = H(\boldsymbol{x}^1)H(\boldsymbol{x}^2)\cdots H(\boldsymbol{x}^m). \end{aligned} \tag{1.4}$$

Note that $H(\boldsymbol{x}) = DE(\overline{\boldsymbol{x}})^{-1}D^{-1}$, $D = \mathrm{diag}((-1)^{i-1})_{i=1}^{n}$, where $\overline{x} = (\overline{x}_1,\dots,\overline{x}_n)$, $\overline{x}_j = \frac{1}{x_j}$; we use the notation $\overline{x}$ for x^{-1} in order to avoid the conflict with that of upper indices. With this notation, the two matrices in (1.4) are related as

$$H(\boldsymbol{x}^1,\dots,\boldsymbol{x}^m) = DE(\overline{\boldsymbol{x}}^m,\dots,\overline{\boldsymbol{x}}^1)D^{-1}. \tag{1.5}$$

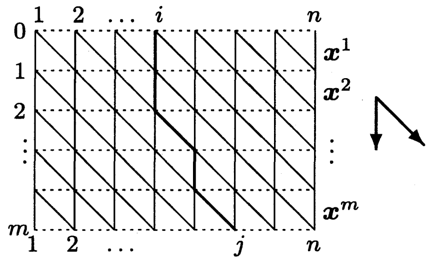
In the following, we propose graphical expressions for the minor determinants of $E(\boldsymbol{x}^1,\dots,\boldsymbol{x}^m)$ and $H(\boldsymbol{x}^1,\dots,\boldsymbol{x}^m)$, in terms of nonintersecting paths.

We first consider the case of $E(\boldsymbol{x}^1,\dots,\boldsymbol{x}^m)$. We represent the matrix $E(\boldsymbol{x})$ by the diagram

(1.6)


with weight x_j attached to the j-th *vertical edge* for each $j = 1,\dots,n$, and weight 1 to each slanted edge. The (i,j)-component of $E(\boldsymbol{x})$ can then be read off by the weight of paths from i at the top to j at the bottom. Piling up the diagrams for $E(\boldsymbol{x}^1)$, ... ,$E(\boldsymbol{x}^m)$ all together, we obtain the following diagram for $E(\boldsymbol{x}^1, \dots, \boldsymbol{x}^m)$.

(1.7)



Here we make all the edges oriented downward, to the south or to the southeast. The (i,j)-component of $E(\boldsymbol{x}^1, \dots, \boldsymbol{x}^m)$ is then given as the sum of weights over all paths from i at the top to j at the bottom. It can also be expressed as

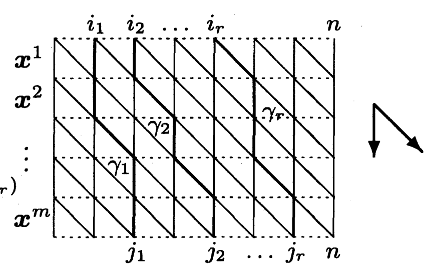
$$E(\boldsymbol{x}^1, \dots, \boldsymbol{x}^m)^i_j = \sum_{1 \le k_i < k_{i+1} < \dots < k_{j-1} \le m} \prod_{b=i}^{j} \prod_{a=k_{b-1}+1}^{k_b - 1} x^a_b, \tag{1.8}$$

where $k_{i-1} = 0$ and $k_j = m+1$. Furthermore, we have

Proposition 1.1. *For any choice of row indices $i_1 < \dots < i_r$ and column indices $j_1 < \dots < j_r$, the minor determinant $\det E(\boldsymbol{x}^1, \dots, \boldsymbol{x}^m)^{i_1, \dots, i_r}_{j_1, \dots, j_r}$ is expressed as the sum of weights over all r-tuples $(\gamma_1, \dots, \gamma_r)$ of nonintersecting paths γ_k from i_k at the top to j_k at the bottom $(k = 1, \dots, r)$.*

(1.9)

$$\det E(\boldsymbol{x}^1, \dots, \boldsymbol{x}^m)^{i_1, \dots, i_r}_{j_1, \dots, j_r} = \sum_{(\gamma_1, \dots, \gamma_r)}$$



This proposition is an immediate consequence of the theorem of Gessel-Viennot [5]. In our context, however, it is also meaningful to understand this passage to nonintersecting paths through the multiplicative properties of minor determinants. Proposition 1.1 is essentially reduced to the multiplicative formula

$$\det(XY)^{i_1,\dots,i_r}_{j_1,\dots,j_r} = \sum_{k_1<\cdots<k_r} \det X^{i_1,\dots,i_r}_{k_1,\dots,k_r} \det Y^{k_1,\dots,k_r}_{j_1,\dots,j_r} \tag{1.10}$$

for minor determinants of the product of matrices. A key step is the following simple lemma. Note that the matrix $E(\boldsymbol{x})$ is decomposed in the form

$$E(\boldsymbol{x}) = \operatorname{diag}(\boldsymbol{x})(1+\overline{x}_{n-1}E_{n-1,n})\cdots(1+\overline{x}_2E_{2,3})(1+\overline{x}_1E_{1,2}). \tag{1.11}$$

Also, the minor determinant

$$\det(1+aE_{k,k+1})^{i_1,\dots,i_r}_{j_1,\dots,j_r} \quad (i_1<\dots<i_r,\ j_1<\dots<j_r) \tag{1.12}$$

vanishes unless either the two index sets $I = \{i_1,\dots,i_r\}$ and $J = \{j_1,\dots,j_r\}$ are identical, or J is obtained from I by replacing $k \in I$ by $k+1$. From this remark, we have

Lemma 1.2. *For row indices $i_1 < \cdots < i_r$ and column indices $j_1 < \cdots < j_r$ given, the minor determinant $\det E(\boldsymbol{x})^{i_1,\dots,i_r}_{j_1,\dots,j_r}$ vanishes unless*

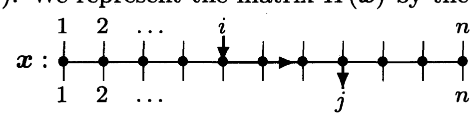
$$j_a = i_a \quad \text{or} \quad i_{a+1} \quad \text{for all} \quad a = 1,\dots,r. \tag{1.13}$$

If this is the case, $\det E(\boldsymbol{x})^{i_1,\dots,i_r}_{j_1,\dots,j_r}$ is the product of x_{j_a} over all a such that $j_a = i_a$.

Proposition 1.1 is then obtained from Lemma 1.2 by applying the multiplicative formula (1.10) to the decomposition $E(\boldsymbol{x}^1,\dots,\boldsymbol{x}^m) = E(\boldsymbol{x}^1)\cdots E(\boldsymbol{x}^m)$. Path representations as in (1.9) can also be translated into the language of tableaux; see for instance [21].

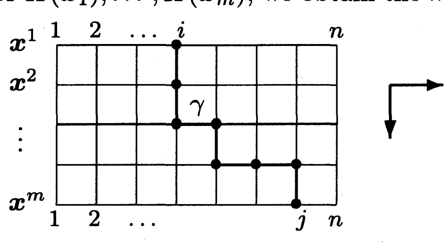
We now turn to the graphical representation of $H(\boldsymbol{x})$ and $H(\boldsymbol{x}^1,\dots,\boldsymbol{x}^m)$. We represent the matrix $H(\boldsymbol{x})$ by the diagram

(1.14)



with weight x_j attached to the j-th *vertex* ($j = 1,\dots,m$). Then piling up the diagrams for $H(\boldsymbol{x}_1),\dots,H(\boldsymbol{x}_m)$, we obtain the $m \times n$ rectangle.

(1.15)

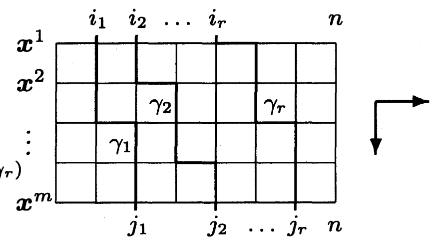


In this diagram for $H(\boldsymbol{x}_1, \dots, \boldsymbol{x}_m)$, for each $a = 1, \dots, m$ and $b = 1, \dots, n$, we attach the weight x_b^a to the vertex with coordinates (a, b). This time, the weight of a path γ is defined to be the product of all x_b^a's attached to the vertices on γ.

Proposition 1.3. *For any choice of row indices $i_1 < \cdots < i_r$ and column indices $j_1 < \dots < j_r$, the minor determinant $\det H(\boldsymbol{x}^1, \dots, \boldsymbol{x}^m)^{i_1, \dots, i_r}_{j_1, \dots, j_r}$ is expressed as the sum of weights over all r-tuples $(\gamma_1, \dots, \gamma_r)$ of nonintersecting paths $\gamma_k : (1, i_k) \to (m, j_k)$ $(k = 1, \dots, r)$.*

(1.16)

$$\det H(\boldsymbol{x}^1, \dots, \boldsymbol{x}^m)^{i_1, \dots, i_r}_{j_1, \dots, j_r} = \sum_{(\gamma_1, \dots, \gamma_r)}$$



The following corresponds to Lemma 1.2 for $E(\boldsymbol{x})$.

Lemma 1.4. *For row indices $i_1 < \cdots < i_r$ and column indices $j_1 < \cdots < j_r$ given, the minor determinant $\det H(\boldsymbol{x})^{i_1, \dots, i_r}_{j_1, \dots, j_r}$ vanishes unless*

$$i_1 \le j_1 < i_2 \le j_2 < \cdots < i_r \le j_r. \tag{1.17}$$

If this is the case, one has

$$\det H(\boldsymbol{x})^{i_1, \dots, i_r}_{j_1, \dots, j_r} = x_{i_1} \cdots x_{j_1} x_{i_2} \cdots x_{j_2} \cdots x_{i_r} \cdots x_{j_r}. \tag{1.18}$$

Note also

$$H(\boldsymbol{x}) = (1 + x_1 E_{1,2})(1 + x_2 E_{2,3}) \cdots (1 + x_{n-1} E_{n-1,n}) \operatorname{diag}(\boldsymbol{x}). \tag{1.19}$$

In the following, we apply the same idea to nonintersecting paths in triangles and trapezoids. For this purpose, we define

$$\Lambda_{\ge k} = \sum_{i=k}^{n-1} E_{i,i+1} \qquad (k = 1, \dots, n), \tag{1.20}$$

so that $\Lambda_{\geq 1} = \Lambda$ and $\Lambda_{\geq n} = 0$. With these *truncated shift matrices*, we introduce the following variations of $E(\boldsymbol{x})$ and $H(\boldsymbol{x})$:

$$E_k(\boldsymbol{x}) = \mathrm{diag}(\boldsymbol{x}) + \Lambda_{\geq k}, \quad H_k(\boldsymbol{x}) = (\mathrm{diag}(\overline{\boldsymbol{x}}) - \Lambda_{\geq k})^{-1} \tag{1.21}$$

for $k = 1, \dots, n$. When we use these notations, we will tacitly assume that $\boldsymbol{x} = (1, \dots, 1, x_k, \dots, x_n)$, i.e., $x_j = 1$ $(j < k)$, unless otherwise mentioned. Under this convention, $E_k(\boldsymbol{x})$ and $H_k(\boldsymbol{x})$ are expressed as

$$E_k(\boldsymbol{x}) = \begin{bmatrix} 1 & 0 \\ 0 & E(\boldsymbol{x}') \end{bmatrix}, \quad H_k(\boldsymbol{x}) = \begin{bmatrix} 1 & 0 \\ 0 & H(\boldsymbol{x}') \end{bmatrix}, \tag{1.22}$$

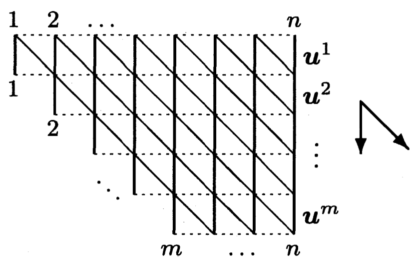
respectively, where $\boldsymbol{x}' = (x_k, x_{k+1}, \dots, x_n)$; we will often identify the $(n-k-1)$-vector $(x_k, \dots, x_n)$ with the n-vector $(1, \dots, 1, x_k, \dots, x_n)$, by putting 1's in front. Assuming that $m \leq n$, let us consider a sequence of n-vectors $\boldsymbol{u}^1, \dots, \boldsymbol{u}^m$, $\boldsymbol{u}^i = (u^i_i, \dots, u^i_n)$, and arrange u^i_j $(i \leq j)$ in the form

$$U = \left(u^i_j\right)_{i \leq j} = \begin{bmatrix} u^1_1 & u^1_2 & \dots & u^1_m & \dots & u^1_n \\ & u^2_2 & \dots & u^2_m & \dots & u^2_n \\ & & \ddots & \vdots & & \vdots \\ & & & u^m_m & \dots & u^m_n \end{bmatrix}. \tag{1.23}$$

For such a table U given, we define an $n \times n$ matrix E_U by

$$E_U = E_1(\boldsymbol{u}^1) E_2(\boldsymbol{u}^2) \cdots, E_m(\boldsymbol{u}^m). \tag{1.24}$$

The entries of this matrix can be represented by the diagram



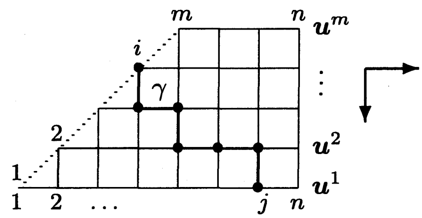
(1.25)

with the weights u^i_j attached to the vertical edges; for each (i, j) with $1 \leq i \leq j \leq n$, $(E_U)^i_j$ is the sum of weights over all paths γ from i at the top to j along the lower rim. The minor determinants of E_U are also represented by nonintersecting paths in diagram (1.25). We also introduce

$$H_U = H_m(\boldsymbol{u}^m) \cdots H_2(\boldsymbol{u}^2) H_1(\boldsymbol{u}^1), \tag{1.26}$$

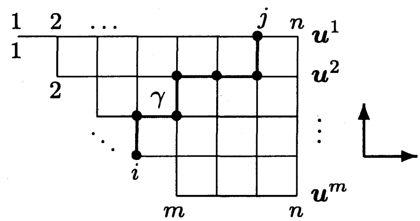
so that $H_U = DE_{\overline{U}}^{-1}D^{-1}$. The diagram for H_U is given by

(1.27)



or alternatively by

(1.28)



with edges oriented upward or rightward. The minor determinants of H_U are expressed in terms of nonintersecting paths in diagram (1.27) or (1.28).

1.2. Minor determinants of triangular matrices

The matrices E_U discussed above can be thought of as *canonical forms* of generic upper triangular matrices M of the form

(1.29)
$$M = \begin{array}{c} \begin{array}{ccccc} 1 & & m & & n \end{array} \\ \left[\begin{array}{ccccc} * & \cdots & * & 1 & 0 \\ & \ddots & & \ddots & 1 \\ & & a^i_j & & * \\ 0 & & & \ddots & \vdots \\ & & & & * \end{array}\right] \end{array}.$$

Let $M = \left(a^i_j\right)_{i,j=1}^n$ be an $n \times n$ upper triangular matrix satisfying the condition

(1.30)
$$a^i_j = 0 \quad (j < i \ \text{ or } \ j > i + m), \qquad a^i_j = 1 \quad (j = i + m);$$

when $m = n$, this simply means that M is upper triangular. For each (i, j) with $1 \le i \le j \le n$, introduce the notation of minor determinants

(1.31)
$$Q_{i,j} = Q_{i,j}(M) = \det M^{1,\dots,j-i+1}_{i,i+1,\dots,j};$$

when $j-i+1=0$, we set $Q_{i,j}=1$. We remark that the condition (1.30) for an upper triangular matrix M is equivalent to the condition

$$Q_{m+1,j}=1, \qquad Q_{i,j}=0 \quad (m+1<i\leq n), \tag{1.32}$$

for minor determinants. The following proposition is due to A. Berenstein, S. Fomin and A. Zelevinsky [1].

Proposition 1.5. *Let $M=\left(a_j^i\right)_{i,j=1}^n$ be an $n\times n$ upper triangular matrix satisfying the condition* (1.30) *for some m $(1\leq m\leq n)$. Suppose that $Q_{i,j}\neq 0$ for any (i,j) with $i\leq j$ and $i\leq m$. Then M can be decomposed uniquely in the form*

$$M=E_1(\boldsymbol{v}^1)E_2(\boldsymbol{v}^2)\cdots E_m(\boldsymbol{v}^m), \tag{1.33}$$

where $\boldsymbol{v}^i=(1,\ldots,1,v_i^i,\ldots,v_n^i)$, $v_j^i\neq 0$, for $i=1,\ldots,m$. Furthermore, v_j^i are determined by

$$v_i^i=Q_{i,i}, \quad v_j^i=\frac{Q_{i,j}\,Q_{i+1,j-1}}{Q_{i+1,j}\,Q_{i,j-1}} \qquad (i<j,\ i\leq m). \tag{1.34}$$

Proof. Assume first that M is decomposed as in (1.33). Then the minor determinants of M are expressed in terms of nonintersecting paths in diagram (1.25) for $V=\left(v_j^i\right)_{i\leq j}$. In particular we have

$$Q_{i,j}=\det M_{i,\ldots,j}^{1,\ldots,j-i+1}=\prod_{(a,b):\,a\geq i,b\leq j} v_j^i, \tag{1.35}$$

since there is only one $(j-i+1)$-tuple of nonintersecting paths relevant to the path representation of this case. Expression (1.34) follows immediately from (1.35), which also implies the uniqueness of decomposition (1.33). It remains to show that M has a decomposition of the form (1.33) under the condition on $Q_{i,j}$. We express M in the form

$$M=\begin{bmatrix} A & B \\ C & D \end{bmatrix}, \tag{1.36}$$

so that B becomes a square matrix of size $n-m+1$: $B=M_{m,m+1,\ldots,n}^{1,\ldots,n-m+1}$. We can apply the Gauss decomposition to the matrix

$$B=\begin{array}{c} 1 \\ \\ m \\ \\ \end{array}\begin{bmatrix} * & 1 & & & \\ \vdots & \ddots & \ddots & & 0 \\ * & & \ddots & \ddots & \\ & \ddots & & \ddots & 1 \\ 0 & & * & \cdots & * \end{bmatrix} \tag{1.37}$$

since

$$\det B^{1,\dots,r}_{1,\dots,r} = \det M^{1,\dots,r}_{m,\dots,m+r-1} = Q_{m,m+r-1}(M) \neq 0 \tag{1.38}$$

for $r = 1, \dots, n-m+1$. It also turns out that the lower and the upper triangular components of the Gauss decomposition $B = B_{<0}B_{\geq 0}$ are in the form

$$B_{<0} = \begin{array}{r} 1 \\ \\ \\ m \\ \\ \\ \end{array} \begin{bmatrix} 1 & & & & & \\ * & \ddots & & & 0 & \\ \vdots & \ddots & \ddots & & & \\ * & & \ddots & \ddots & & \\ & \ddots & & \ddots & \ddots & \\ 0 & & * & \cdots & * & 1 \end{bmatrix}, \quad B_{\geq 0} = \begin{bmatrix} * & 1 & & & \\ & \ddots & \ddots & & 0 \\ & & \ddots & \ddots & \\ 0 & & & \ddots & 1 \\ & & & & * \end{bmatrix}. \tag{1.39}$$

Denoting the diagonal entries of $B_{\geq 0}$ by $v^m_m, \dots, v^m_n$, we introduce the vector $\boldsymbol{v}^m = (1, \dots, 1, v^m_m, \dots, v^m_n)$. Then we have the following decomposition of M:

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B_{<0} \\ C & D' \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & B_{\geq 0} \end{bmatrix} = M' E_m(\boldsymbol{v}^m), \tag{1.40}$$

where $D' = DB^{-1}_{\geq 0}$. The matrix M' thus obtained satisfies the condition (1.30) with m replaced by $m-1$, as can be seen from the decomposition above. Also, if $i \leq m-1$ and $i \leq j$, from $M = M'E_m(\boldsymbol{v})$ it follows that

$$\begin{aligned} Q_{i,j}(M) &= \det M^{1,\dots,j-i+1}_{i,\dots,j} \\ &= \det(M')^{1,\dots,j-i+1}_{i,\dots,j} \det E_m(\boldsymbol{v}^m)^{i,\dots,j}_{i,\dots,j} \\ &= Q_{i,j}(M')\, v^m_m \cdots v^m_j \neq 0. \end{aligned} \tag{1.41}$$

Hence we can apply the descending induction on m to obtain the decomposition (1.33). □

We now translate Proposition 1.5 into a statement concerning the decomposition of type H_U. Let H be an $n \times n$ invertible upper triangular matrix. For the decomposition of type $H = H_U$, it is convenient to use the following notation

$$\tau^i_j = \tau^i_j(H) = \det H^{1,\dots,i}_{j-i+1,\dots,j} \qquad (1 \leq i \leq j \leq n). \tag{1.42}$$

We also define $\tau^0_j = 1$ for any j. Setting $M = DH^{-1}D^{-1}$, $D = \operatorname{diag}((-1)^{i-1})^n_{i=1}$, we compare the minor determinants $\tau^i_j = \tau^i_j(H)$ and $Q_{i,j} = Q_{i,j}(M)$. We remark that

$$\det H^{1,\dots,i}_{j-i+1,\dots,j} = \frac{\det M^{1,\dots,j-i,j+1,\dots,n}_{i+1,\dots,n}}{\det M} = \frac{\det M^{1,\dots,j-i}_{i+1,\dots,j}}{\det M^{1,\dots,j}_{1,\dots,j}}; \tag{1.43}$$

for these particular minor determinants, we have no minus sign. This implies

$$(1.44) \qquad \tau_j^i = \frac{Q_{i+1,j}}{Q_{1,j}}, \qquad Q_{i,j} = \frac{\tau_j^{i-1}}{\tau_j^j} \qquad (i \le j).$$

Note also that $\tau_j^j = Q_{1,j}^{-1}$. Hence we see that the condition (1.30) is equivalent to

$$(1.45) \qquad \tau_j^i = \delta_{i,j}\tau_j^m \qquad (m < i \le j \le n).$$

Proposition 1.6. *Let H be an $n \times n$ upper triangular matrix, and suppose that the minor determinants $\tau_j^i = \tau_j^i(H)$ $(1 \le i \le j \le n)$ satisfy the condition*

$$(1.46) \qquad \tau_j^i \ne 0 \quad (1 \le i \le m), \quad \tau_j^i = \delta_{i,j}\tau_j^m \quad (m < i \le n),$$

for some m $(1 \le m \le n)$. Then the matrix H can be decomposed uniquely in the form

$$(1.47) \qquad H = H_m(\boldsymbol{u}^m) \cdots H_2(\boldsymbol{u}^2) H_1(\boldsymbol{u}^1),$$

where $\boldsymbol{u}^i = (1, \ldots, 1, u_i^i, \ldots, u_n^i)$, $u_j^i \ne 0$, for $i = 1, \ldots, m$. Furthermore u_j^i are determined by

$$(1.48) \qquad u_i^i = \frac{\tau_i^i}{\tau_i^{i-1}}, \qquad u_j^i = \frac{\tau_j^i\,\tau_{j-1}^{i-1}}{\tau_j^{i-1}\tau_{j-1}^i} \quad (i < j,\ i \le m).$$

Under the condition (1.46), from Proposition 1.5 we have

$$(1.49) \qquad DH^{-1}D^{-1} = M = E_1(\overline{\boldsymbol{u}}^1)E_2(\overline{\boldsymbol{u}}^2)\cdots E_m(\overline{\boldsymbol{u}}^m),$$

where we have set $\boldsymbol{u}^i = \overline{\boldsymbol{v}}^i$ $(i = 1, \ldots, m)$. Once we have the decomposition (1.47), the minor determinants of H is expressed in terms of nonintersecting paths in diagram (1.28) for $U = \left(u_j^i\right)_{i,j}$. In particular, each τ_j^i is expressed as

$$(1.50) \qquad \tau_j^i = \det H_{j-i+1,\ldots,j}^{1,\ldots,i} = \prod_{(a,b);\, a \le i,\, b \le j} u_b^a,$$

since there is only one i-tuple of nonintersecting paths relevant to this minor determinant. Expression (1.48) for u_j^i follows immediately from (1.50).

Proposition 1.6 implies the following theorem concerning the path representation of minor determinants of a triangular matrix.

Theorem 1.7. *Let H be an $n \times n$ upper triangular matrix, and suppose that the minor determinants $\tau^i_j = \tau^i_j(H)$ $(i \leq j)$ satisfy the condition*

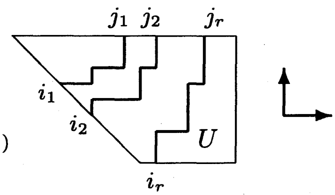
$$\tau^i_j \neq 0 \quad (1 \leq i \leq m), \qquad \tau^i_j = \delta_{i,j}\tau^m_j \quad (m < i \leq n) \tag{1.51}$$

for some m $(1 \leq m \leq n)$. For each (i,j) with $1 \leq i \leq j \leq n$, $i \leq m$, define

$$u^i_i = \frac{\tau^i_i}{\tau^{i-1}_i}, \qquad u^i_j = \frac{\tau^i_j\,\tau^{i-1}_{j-1}}{\tau^{i-1}_j\,\tau^i_{j-1}} \quad (i < j,\ i \leq m). \tag{1.52}$$

Then, for any choice of row indices $i_1 < \ldots < i_r$ and column indices $j_1 < \ldots < j_r$, the minor determinant $\det H^{i_1,\dots,i_r}_{j_1,\dots,j_r}$ is expressed as a sum

$$\det H^{i_1,\dots,i_r}_{j_1,\dots,j_r} = \sum_{(\gamma_1,\dots,\gamma_r)} \tag{1.53}$$



of weights associated with $U = \left(u^i_j\right)_{i,j}$, over all r-tuples of nonintersecting paths $\gamma_k : (\min\{i_k, m\}, i_k) \to (1, j_k)$ from i_k along the lower rim to j_k at the top $(k = 1, \dots, r)$, in diagram (1.28).

Remark 1.8. Proposition 1.6 for $m = n$ can be reformulated as follows. Let us denote

$$\mathcal{B} = \{B = \left(b^i_j\right)^n_{i,j=1} \in GL_n(\mathbb{K}) \mid b^i_j = 0 \quad (i > j)\} \tag{1.54}$$

the group of all $n \times n$ invertible upper triangular matrices. For each $U = \left(u^i_j\right)^n_{i,j=1} \in \mathcal{B}$, we define $H = \left(h^i_j\right)^n_{i,j=1} \in \mathcal{B}$ by setting

$$h^i_j = \sum_{\gamma:(i,i)\to(1,j)} u_\gamma \quad (i \leq j), \quad h^i_j = 0 \quad (i > j). \tag{1.55}$$

We now define two open subsets of $\mathcal{B}$ as follows:

$$\begin{aligned} \mathcal{B}_0 &= \{U = \left(u^i_j\right)^n_{i,j=1} \in \mathcal{B} \mid u^i_j \neq 0 \quad (i \leq j)\}, \\ \mathcal{B}_\tau &= \{H = \left(h^i_j\right) \in \mathcal{B} \mid \tau^i_j(H) \neq 0 \quad (i \leq j)\}. \end{aligned} \tag{1.56}$$

Then the correspondence $U \mapsto H$ induces the isomorphism of affine varieties $h : \mathcal{B}_0 \xrightarrow{\sim} \mathcal{B}_\tau$. The inverse mapping $H \mapsto U$ is given by

$$u^i_i = \frac{\tau^i_i}{\tau^{i-1}_i}, \qquad u^i_j = \frac{\tau^i_i\,\tau^{i-1}_{j-1}}{\tau^{i-1}_i\,\tau^i_{j-1}} \quad (i < j), \tag{1.57}$$

where $\tau^i_j = \tau^i_j(H)$ for $i \leq j$. Under this correspondence $U \leftrightarrow H$, for any choice of row indices $i_1 < \ldots < i_r$ and column indices $j_1 < \ldots < j_r$, the minor determinant $\det H^{i_1,\ldots,i_r}_{j_1,\ldots,j_r}$ of H is expressed as the sum of weights, associated with U, over all r-tuples of nonintersecting paths $\gamma_k : (i_k, i_k) \to (1, j_k)$ $(k = 1, \ldots, r)$.

1.3. Passage from tropical to combinatorial variables

We now assume that $\mathbb{K}$ is a field of characteristic 0. Consider the ring of polynomials $\mathbb{K}[x] = \mathbb{K}[\,x_i\ (i \in I)]$ in a set of variables $x = (x_i)_{i\in I}$. Denoting by

$$\text{(1.58)}\quad \mathbb{N}^{(I)} = \{\alpha = (\alpha_i)_{i\in I} \mid \alpha_i = 0 \ \text{ except for a finite number of } i\text{'s}\}$$

the set of multi-indices, we use the notation of multi-indices $x^\alpha = \prod_{i\in I} x_i^{\alpha_i}$ for the monomials in the x-variables. Note that any polynomial $a(x) \in \mathbb{K}[x]$ is expressed uniquely in the form

$$\text{(1.59)}\qquad a(x) = \sum_{\alpha\in A} a_\alpha\, x^\alpha \qquad (a_\alpha \in \mathbb{K}^*),$$

as a sum of monomials over a finite subset $A \subset \mathbb{N}^{(I)}$ of multi-indices, with nonzero coefficients. In this way, a polynomial $a(x)$ is identified with a pair (A, a) of a finite subset $A \subset \mathbb{N}^{(I)}$ and a mapping $a : A \to \mathbb{K}$. Note that $0 \in \mathbb{K}[x]$ and $c \in \mathbb{K}[x]$ $(c \in \mathbb{K}^*)$ correspond to (ϕ, ϕ) and $(\{0\}, c)$, respectively.

In the following, we fix a multiplicative subgroup $\mathbb{K}_{>0}$ of $\mathbb{K}^*$ such that $c, c' \in \mathbb{K}_{>0} \ \Rightarrow\ c + c' \in \mathbb{K}_{>0}$. We say that a nonzero rational function $f(x) \in \mathbb{K}(x) = \mathbb{K}(x_i\ (i \in I))$ in the x-variables is *subtraction free* (or *tropical*) with respect to the cone $\mathbb{K}_{>0}$ if it is expressed as a ratio

$$\text{(1.60)}\qquad f(x) = \frac{a(x)}{b(x)} \qquad (\,a(x), b(x) \in \mathbb{K}_{>0}[x]\,)$$

of two polynomials with coefficients in $\mathbb{K}_{>0}$. We denote by $\mathbb{K}(x)_{>0}$ the set of all subtraction-free rational functions with respect to $\mathbb{K}_{>0}$. It is clear that $\mathbb{K}(x)_{>0}$ forms again a multiplicative subgroup of $\mathbb{K}(x)^*$ closed under the addition. It is worthwhile to note that all the coefficients of a *polynomial* $f(x) \in \mathbb{K}(x)_{>0}$ may *not* necessarily belong to $\mathbb{K}_{>0}$: Observe the example

$$\text{(1.61)}\qquad f(x, y) = \frac{x^3 + y^3}{x + y} = x^2 - xy + y^2.$$

For a subtraction-free rational function $f = f(x) \in \mathbb{K}(x)_{>0}$ given, choose an expression as (1.60). Expressing $a(x)$ and $b(x)$ as

$$a(x) = \sum_{\alpha\in A} a_\alpha\, x^\alpha, \quad b(x) = \sum_{\beta\in B} b_\beta\, x^\beta \tag{1.62}$$

with coefficients in $\mathbb{K}_{>0}$, we define two piecewise linear functions $M(f)$ and $m(f)$ on $\mathbb{R}^I$ by

$$\begin{aligned} M(f) &= \max\{\langle\alpha, x\rangle \mid \alpha \in A\} - \max\{\langle\beta, x\rangle \mid \beta \in B\},\\ m(f) &= \min\{\langle\alpha, x\rangle \mid \alpha \in A\} - \min\{\langle\beta, x\rangle \mid \beta \in B\}, \end{aligned} \tag{1.63}$$

where $\langle\alpha, x\rangle = \sum_{i\in I} \alpha_i x_i$. In this definition, we have identified $x = (x_i)_{i\in I}$ with the canonical coordinates of $\mathbb{R}^I$. It is easily shown that the definition of $M(f)$ and $m(f)$ does not depend on the choice of expression (1.60). Note also that $M(c) = m(c) = 0$ for any $c \in \mathbb{K}_{>0}$.

Proposition 1.9. (1) *For any subtraction-free rational functions $f, g \in \mathbb{K}(x)_{>0}$, one has*

$$\begin{gathered} M(fg) = M(f) + M(g), \quad M(\frac{f}{g}) = M(f) - M(g),\\ M(f+g) = \max\{M(f), M(g)\}, \end{gathered} \tag{1.64}$$

and

$$\begin{gathered} m(fg) = m(f) + m(g), \quad m(\frac{f}{g}) = m(f) - m(g),\\ m(f+g) = \min\{m(f), m(g)\}. \end{gathered} \tag{1.65}$$

(2) *Let $\iota : \mathbb{K}(x) \to \mathbb{K}(x)$ be the isomorphism defined by $\iota(x_i) = x_i^{-1}$ $(i \in I)$. Then one has*

$$M(f) = m(\iota(f)^{-1}), \quad m(f) = M(\iota(f)^{-1}) \tag{1.66}$$

for any $f \in \mathbb{K}(x)_{>0}$.

This proposition means that the correspondence $f \mapsto M(f)$ is nothing but the simple procedure of replacing the operations

$$a\,b \to a+b, \quad \frac{a}{b} \to a-b, \quad a+b \to \max\{a,b\}. \tag{1.67}$$

Similarly, the correspondence $f \mapsto m(f)$ is the procedure

$$a\,b \to a+b, \quad \frac{a}{b} \to a-b, \quad a+b \to \min\{a,b\}. \tag{1.68}$$

The second part of the proposition implies that one can interchange "max" and "min" freely with each other, by using the operation $f(x) \to \iota(f)^{-1} = f(x^{-1})^{-1}$.

Proposition 1.9 guarantees that these procedures can be applied consistently to arbitrary subtraction-free rational functions to obtain piecewise linear functions. This passage from the subtraction-free rational functions to piecewise linear functions, either by max or min, is called in several ways in the literature; it is called the *ultra-discretization* in the context of discrete integrable systems, and also the *tropicalization* in the context of totally positive matrices. In this paper, we will use the adjective "tropical" for objects and notions concerning subtraction-free rational functions, and "combinatorial" for those concerning piecewise linear functions. It should be noted that there is no *canonical* procedure in the opposite direction; when a combinatorial expression is given, it becomes an interesting problem in many occasions to find a *good* counterpart in the tropical setting.

The passage from the tropical side to the combinatorial side is *functorial* in the following sense. Consider two fields of rational functions $\mathbb{K}(x)$ in the variables $x = (x_i)_{i\in I}$ and $\mathbb{K}(y)$ in the variables $y = (y_j)_{j\in J}$. We say that an isomorphism φ form $\mathbb{K}(y)$ into $\mathbb{K}(x)$ is subtraction free if $\varphi(y_j) \in \mathbb{K}(x)_{>0}$ for all $j \in J$. The set of subtraction-free rational functions $f_j(x) = \varphi(y_j)$ $(j \in J)$ then defines a subtraction-free rational mapping

$$F : y_j = f_j(x) \quad (j \in J) \tag{1.69}$$

from the affine space $\mathbb{K}^I$ with coordinates $x = (x_i)_{i\in I}$ to $\mathbb{K}^J$ with coordinates $y = (y_j)_{j\in J}$. For such a rational mapping F given, we define two piecewise linear mappings $M(F), m(F) : \mathbb{R}^I \to \mathbb{R}^J$ by setting

$$M(F) : y_j = M(f_j(x)) \quad (j \in J), \qquad m(F) : y_j = m(f_j(x)) \quad (j \in J), \tag{1.70}$$

respectively. Then Proposition 1.9 implies

Proposition 1.10. *Consider the two subtraction-free rational mappings*

$$F : y_j = f_j(x) \quad (j \in J), \qquad G : z_k = g_k(y) \quad (k \in K). \tag{1.71}$$

Then the piecewise linear mappings corresponding to the composition $G \circ F$ *are given by*

$$M(G \circ F) = M(G) \circ M(F), \quad m(G \circ F) = m(G) \circ m(F). \tag{1.72}$$

§2. Tropical row insertion and tropical tableaux

In this section we introduce a tropical analogue of row insertion by clarifying the internal structure of bumping. Combining this with the matrix approach of Section 1, we give explicit tropical and combinatorial formulas for describing the tableau obtained from a word by row insertion, and for the Schützenberger involution on the set of column strict tableaux.

2.1. Row insertion

Taking the set of letters $\{1, \ldots, n\}$, we consider a column strict tableau T of shape λ, where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ is a partition with $l(\lambda) \le m$. For each $i = 1, \ldots, m$, we define a weakly increasing word

$$w_i = i^{x_i^i}(i+1)^{x_{i+1}^i} \ldots n^{x_n^i} \qquad (x_i^i + \cdots + x_n^i = \lambda_i) \tag{2.1}$$

by reading the i-th row of T from left to right, where x_j^i stands for the number of j's appearing in the i-th row of T for $i \le j$. For a weakly increasing word $v = 1^{a_1}2^{a_2} \ldots n^{a_n}$ given, consider the tableau $T' = T \leftarrow v$ obtained by the row insertion of v into T; we denote by $w_i' = i^{y_i^i}(i+1)^{y_{i+1}^i} \ldots n^{y_n^i}$ the weakly increasing word representing the i-th row of T' for $i = 1, \ldots, m'$. Our question is : *How can one describe y_j^i explicitly in terms of x_j^i's and a_j's ?*

The bumping procedure $T \leftarrow v$ can be decomposed as follows.

$$\begin{array}{ccc} & v & \\ T & \longrightarrow & T' \\ & \phi & \end{array} \quad : \quad \begin{array}{ccc} & v = v_1 & \\ w_1 & \longrightarrow & w_1' \\ & v_2 & \\ w_2 & \longrightarrow & w_2' \\ & v_3 & \\ & \vdots & \end{array} \tag{2.2}$$

Here $v_1 = v$, and for $i = 2, 3, \ldots$, $v_i = i^{a_i^i}(i+1)^{a_{i+1}^i} \ldots n^{a_n^i}$ stands for the weakly increasing word consisting of the letters that have bumped out from w_{i-1} by the row insertion of v_{i-1}. In what follows, we use the diagram

(2.3)

$$\begin{array}{ccc} & v & \\ w & \longrightarrow & w' \\ & v' & \end{array} \qquad \left(\begin{array}{ll} w = 1^{x_1}2^{x_2} \ldots n^{x_n}, & v = 1^{a_1}2^{a_2} \ldots n^{a_n} \\ w' = 1^{y_1}2^{y_2} \ldots n^{y_n}, & v' = 1^{b_1}2^{b_2} \ldots n^{b_n} \end{array} \right)$$

consisting of four weakly increasing words w, v, w', v', to indicate a procedure of inserting a word v into w; $w' = w \leftarrow v$ denotes the resulting word, and v' is the word of the letters bumped out from w. (We always have $b_1 = 0$ in this setting.) Our question is thus reduced to the problem of describing y_j and b_j in terms of x_j and a_j in this diagram. We also use the diagram of row insertion for the corresponding vectors of integers:

$$\boldsymbol{x} \overset{\boldsymbol{a}}{\underset{\boldsymbol{b}}{\longmapsto}} \boldsymbol{y} \qquad \left(\begin{array}{ll} \boldsymbol{x} = (x_1, \dots, x_n), & \boldsymbol{a} = (a_1, \dots, a_n) \\ \boldsymbol{y} = (y_1, \dots, y_n), & \boldsymbol{b} = (b_1, \dots, b_n) \end{array} \right). \tag{2.4}$$

We now consider the procedure of row insertion as in (2.3). It is convenient to use the variables

$$\xi_j = x_1 + x_2 + \cdots + x_j, \quad \eta_j = y_1 + y_2 + \cdots + y_j \quad (j = 1, \dots, n). \tag{2.5}$$

Assume first that $v = k^a$ $(k = 1, \dots, n)$; in this case, it is easy to see

$$\begin{aligned} &\eta_j = \xi_j \quad (j < k), \qquad \eta_k = \xi_k + a, \\ &\eta_j = \max\{\xi_k + a, \xi_j\} = \max\{\eta_k, \xi_j\} \quad (j > k). \end{aligned} \tag{2.6}$$

Applying this result repeatedly for $k = 1, \dots, n$, we obtain following recurrence relations for the general case $v = 1^{a_1} 2^{a_2} \dots n^{a_n}$:

$$\begin{aligned} &\eta_1 = \xi_1 + a_1, \quad \eta_2 = \max\{\eta_1, \xi_2\} + a_2, \\ &\eta_3 = \max\{\eta_1, \eta_2, \xi_3\} + a_3, \quad \dots . \end{aligned} \tag{2.7}$$

Since $\eta_1 \le \eta_2 \le \cdots \le \eta_n$, it is equivalent to

$$\begin{aligned} \eta_1 &= \xi_1 + a_1, \\ \eta_j &= \max\{\eta_{j-1}, \xi_j\} + a_j \\ &= \max\{\eta_{j-1} + a_j, \xi_j + a_j\} \quad (j = 2, \dots, n). \end{aligned} \tag{2.8}$$

Hence we have

$$\begin{aligned} \eta_j &= \max\{\xi_1 + a_1 + \cdots + a_j, \xi_2 + a_2 + \cdots + a_j, \dots, \xi_j + a_j\} \\ &= \max_{1 \le k \le j} \{x_1 + \cdots + x_k + a_k + \cdots + a_j\} \end{aligned} \tag{2.9}$$

for $j = 1, \dots, n$. Note that

$$y_1 = \eta_1, \quad y_j = \eta_j - \eta_{j-1} \quad (j = 2, \dots, n), \tag{2.10}$$

and that b_j are determined as $b_1 = 0$ and

(2.11)
$$b_j = a_j + x_j - y_j = a_j + \xi_j - \xi_{j-1} - \eta_j + \eta_{j-1} \quad (j = 2, \dots, n),$$

since the number of j's is conserved during the process.

Example 2.1. Let us consider an example of row insertion

$$w = \underline{2}2\underline{3}4\underline{5}\,\underline{\ } \quad \underset{v' = 235}{\overset{v = 1245}{\longrightarrow}} \quad w' = 122445 \ . \tag{2.12}$$

In terms of the vectors of integers, this procedure is expressed as

$$\boldsymbol{x} = (0, 2, 1, 1, 1) \quad \underset{\boldsymbol{b} = (0, 1, 1, 0, 1)}{\overset{\boldsymbol{a} = (1, 1, 0, 1, 1)}{\longrightarrow}} \quad \boldsymbol{y} = (1, 2, 0, 2, 1) \ . \tag{2.13}$$

The numbers

$$\eta_j = \max_{1 \le k \le j} (x_1 + \cdots + x_k + a_k + \cdots + a_j) \tag{2.14}$$

can be read off from the table

(2.15)

	1	2	3	4	5
$\boldsymbol{x}$:	0	2	1	1	1
$\boldsymbol{a}$:	1	1	0	1	1

as

$$\eta_1 = 1, \quad \eta_2 = 3, \quad \eta_3 = 3, \quad \eta_4 = 5, \quad \eta_5 = 6. \tag{2.16}$$

By taking the first difference of this sequence, we have $\boldsymbol{y} = (1, 2, 0, 2, 1)$.

The general procedure $T' = T \leftarrow v$ of inserting a weakly increasing word v into a column strict tableau T should be described as a superposition of row insertions of type (2.3). We will make use of the tropical analogue of combinatorial (piecewise linear) formulas above in order to systematize the superposition of row insertions.

2.2. Tropical row insertion

We introduce a tropical analogue of combinatorial formulas for the row insertion (2.3). We use the same symbols x_j, a_j, y_j, b_j as in (2.4) for the *tropical variables* (indeterminates). Introducing the auxiliary variables

$$\xi_j = x_1 \cdots x_j, \quad \eta_j = y_1 \cdots y_j \quad (j = 1, \dots, n), \tag{2.17}$$

we define the transformation $(\boldsymbol{x}, \boldsymbol{a}) \mapsto (\boldsymbol{y}, \boldsymbol{b})$ by

$$
\begin{array}{llll}
 & \eta_1 = \xi_1 a_1, & \eta_j = (\eta_{j-1} + \xi_j) a_j & (j = 2, \dots, n), \\
(2.18) & y_1 = \eta_1, & y_j = \dfrac{\eta_j}{\eta_{j-1}} & (j = 2, \dots, n), \\
 & b_1 = 1, & b_j = a_j \dfrac{x_j}{y_j} = a_j \dfrac{\xi_j \eta_{j-1}}{\xi_{j-1} \eta_j} & (j = 2, \dots, n).
\end{array}
$$

We have made these formulas from the combinatorial formulas (2.8), (2.10), (2.11) by the simple rule of replacement

$$
(2.19) \qquad \max\{a, b\} \to a + b, \quad a + b \to a\, b \quad a - b \to \frac{a}{b},
$$

which is sometimes called the *tropical variable change*. From the recurrence relations for η_j above, we easily obtain

$$
\begin{aligned}
\eta_j &= \xi_1 a_1 \cdots a_j + \xi_2 a_2 \cdots a_j + \cdots + \xi_j a_j \\
(2.20) \qquad &= x_1 a_1 a_2 \cdots a_j + x_1 x_2 a_2 \cdots a_j + \cdots + x_1 \cdots x_j a_j.
\end{aligned}
$$

From this formula, we can recover the combinatorial formula (2.9) by the standard procedure as we discussed in Section 1.3.

The tropical transformation $(\boldsymbol{x}, \boldsymbol{a}) \mapsto (\boldsymbol{y}, \boldsymbol{b})$ we have discussed above arises also from the system of algebraic equations *of discrete Toda type*

$$
\begin{array}{llll}
 & a_1 x_1 = y_1, & a_j x_j = y_j b_j & (j = 2, \dots, n), \\
(2.21) & \dfrac{1}{a_1} + \dfrac{1}{x_2} = \dfrac{1}{b_2}, & \dfrac{1}{a_j} + \dfrac{1}{x_{j+1}} = \dfrac{1}{y_j} + \dfrac{1}{b_{j+1}} & (j = 2, \dots, n),
\end{array}
$$

for $(y_1, \dots, y_n)$ and $(b_2, \dots, b_n)$, where we regard x_j, a_j as given variables, and y_j, b_j as unknown functions. (For the relationship between (2.21) and the discrete Toda equation, see Remark 2.3 below.)

Lemma 2.2. *The system of algebraic equations* (2.21) *is equivalent to the recurrence formulas* (2.18) *together with* (2.17).

Proof. In fact, by eliminating b_j $(j = 2, \dots, n)$ from (2.21), and by rewriting the equations in terms of ξ_j and η_j, we obtain

(2.22)
$$
\frac{\eta_n - \eta_{n-1} a_n}{\xi_n a_n} = \frac{\eta_{n-1} - \eta_{n-2} a_{n-1}}{\xi_{n-1} a_{n-1}} = \cdots = \frac{\eta_2 - \eta_1 a_2}{\xi_2 a_2} = \frac{\eta_1}{\xi_1 a_1} = 1,
$$

which is equivalent to (2.18). □

This fact is a key to our matrix approach to tropical combinatorics. Note that (2.21) is written as a matrix equation:

$$\begin{bmatrix} \overline{a}_1 & 1 & & & \\ & \overline{a}_2 & 1 & & \\ & & \ddots & \ddots & \\ & & & \overline{a}_{n-1} & 1 \\ & & & & \overline{a}_n \end{bmatrix} \begin{bmatrix} \overline{x}_1 & 1 & & & \\ & \overline{x}_2 & 1 & & \\ & & \ddots & \ddots & \\ & & & \overline{x}_{n-1} & 1 \\ & & & & \overline{x}_n \end{bmatrix}$$

$$(2.23) \qquad = \begin{bmatrix} \overline{y}_1 & 1 & & & \\ & \overline{y}_2 & 1 & & \\ & & \ddots & \ddots & \\ & & & \overline{y}_{n-1} & 1 \\ & & & & \overline{y}_n \end{bmatrix} \begin{bmatrix} 1 & 0 & & & \\ & \overline{b}_2 & 1 & & \\ & & \ddots & \ddots & \\ & & & \overline{b}_{n-1} & 1 \\ & & & & \overline{b}_n \end{bmatrix},$$

where we have used the notation $\overline{x} = \frac{1}{x}$. By using the notation of Section 1, this equation can be expressed as

$$(2.24) \qquad E(\overline{\boldsymbol{a}})\, E(\overline{\boldsymbol{x}}) = E_1(\overline{\boldsymbol{y}})\, E_2(\overline{\boldsymbol{b}}).$$

It is also equivalent to

$$(2.25) \qquad H(\boldsymbol{x})\, H(\boldsymbol{a}) = H_2(\boldsymbol{b}) H_1(\boldsymbol{y}).$$

Each of these two equations (2.24) and (2.25) can be thought of as a tropical expression of the row insertion

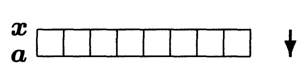
(2.26)

$$\boldsymbol{x} \overset{\boldsymbol{a}}{\underset{\boldsymbol{b}}{\longrightarrow}} \boldsymbol{y} \qquad \left(\begin{array}{ll} \boldsymbol{x} = \big(x_1, \dots, x_n\big), & \boldsymbol{a} = \big(a_1, \dots, a_n\big) \\ \boldsymbol{y} = \big(y_1, \dots, y_n\big), & \boldsymbol{b} = \big(b_1, \dots, b_n\big) \end{array} \right),$$

where $b_1 = 1$.

We show how the matrix equation (2.24) is solved by using the path representations of Section 1. Setting $H = H(\boldsymbol{x})H(\boldsymbol{a})$, we look at the minor determinants $\tau^i_j(H)$ $(i \le j)$. By the argument of Section 1, the minor determinants of H are read off from the following diagram.

(2.27)



The result for $\tau^i_j = \tau^i_j(H)$ is:

$$\text{(2.28)} \qquad \begin{aligned} &\tau^1_j = \sum_{k=1}^{j} x_1 \cdots x_k \, a_k \cdots a_j && (j \geq 1), \\ &\tau^2_j = \tau^j_j = x_1 \cdots x_j a_1 \cdots a_j && (j \geq 2), \\ &\tau^i_j = 0 && (3 \leq i < j \leq n). \end{aligned}$$

By Proposition 1.6, we already know that equation (2.25) has a unique solution such that

(2.29)
$$\tau^1_j = y_1 \dots y_j \quad (j = 1, \dots, n), \quad \tau^2_j = y_1 \dots y_j b_1 \dots b_j \quad (j = 2, \dots, n).$$

Namely, y_j and b_j are determined as

$$\text{(2.30)} \quad y_j = \frac{\tau^1_j}{\tau^1_{j-1}} \quad (j = 1, \dots, n), \quad b_j = \frac{x_j a_j}{y_j} \quad (j = 2, \dots, n),$$

consistently with what we have seen before.

Remark 2.3. The system of algebraic equations (2.21) is closely related to the *discrete Toda equation* ([9], [26]):

$$\text{(2.31)} \qquad I^{t+1}_i V^{t+1}_i = I^t_{i+1} V^t_i, \quad I^{t+1}_i + V^{t+1}_{i-1} = I^t_i + V^t_i$$

where $i \in \mathbb{Z}$ and $t \in \mathbb{Z}$ stand for the discrete coordinates of space and time, respectively, and I^t_i, V^t_i are the dependent variables. If we set

(2.32)
$$a_i = (I^t_{i+1})^{-1}, \quad x_i = (V^t_i)^{-1}, \quad y_i = (V^{t+1}_i)^{-1}, \quad b_i = (I^{t+1}_i)^{-1},$$

we have

$$\text{(2.33)} \qquad a_i x_i = y_i b_i, \quad \frac{1}{a_i} + \frac{1}{x_{i+1}} = \frac{1}{y_i} + \frac{1}{b_{i+1}} \quad (i \in \mathbb{Z}).$$

Note also that the discrete Toda equation can be expressed as the matrix equation

$$\text{(2.34)} \qquad L(t+1)R(t+1) = R(t)L(t)$$

for the $\mathbb{Z} \times \mathbb{Z}$ matrices

$$\text{(2.35)} \quad L(t) = \sum_{i \in \mathbb{Z}} E_{ii} + \sum_{j \in \mathbb{Z}} V^t_i E_{i+1,i}, \quad R(t) = \sum_{i \in \mathbb{Z}} I^t_i E_{ii} + \sum_{i \in \mathbb{Z}} E_{i,i+1}.$$

2.3. Tropical tableaux

In the following we discuss the following question both in the tropical and the combinatorial setting.

Question: For a sequence of weakly increasing words $w_1, \ldots, w_m$ given, find an explicit formula for the column strict tableau

$$P = P(w) = (\cdots(w_1 \leftarrow w_2) \cdots \leftarrow w_m) \tag{2.36}$$

obtained from the word $w = w_1 \ldots w_m$ by the row insertion.

We can employ (2.25) as building blocks for the tropical analogue of various combinatorial algorithms. Let us consider the procedure of successive row insertion

$$P = (\cdots(w_1 \leftarrow w_2) \leftarrow \cdots \leftarrow w_m) \tag{2.37}$$

of weakly increasing word $w_i = 1^{x_1^i} \cdots n^{x_n^i}$ $(i = 1, \ldots, n)$ to obtain a column strict tableau P. This procedure can be described by the following diagram.

$$\begin{array}{ccccccc}
 & \boldsymbol{x}^1 = \boldsymbol{x}^{1,1} & & \boldsymbol{x}^2 = \boldsymbol{x}^{2,1} & & \boldsymbol{x}^3 = \boldsymbol{x}^{3,1} & \\
\phi & \underset{\downarrow}{\longrightarrow} & \boldsymbol{y}^{1,1} & \underset{\downarrow}{\longrightarrow} & \boldsymbol{y}^{2,1} & \underset{\downarrow}{\longrightarrow} & \cdots \\
 & \phi & & \boldsymbol{x}^{2,2} & & \boldsymbol{x}^{3,2} & \\
 & & \phi & \underset{\downarrow}{\longrightarrow} & \boldsymbol{y}^{2,2} & \underset{\downarrow}{\longrightarrow} & \cdots \\
 & & & \phi & & \boldsymbol{x}^{3,3} & \\
 & & & & \phi & \underset{\downarrow}{\longrightarrow} & \cdots
\end{array} \tag{2.38}$$

Passing the tropical variables $\boldsymbol{x}^i = (x_1^i, \ldots, x_n^i)$ $(i = 1, \ldots, m)$, we can compute the row insertion above as

$$\begin{aligned}
H(\boldsymbol{x}^1) &= H_1(\boldsymbol{y}^{1,1}) \\
H(\boldsymbol{x}^1)H(\boldsymbol{x}^2) &= H_1(\boldsymbol{y}^{1,1})H(\boldsymbol{x}^{2,1}) \\
&= H_2(\boldsymbol{y}^{2,2})H_1(\boldsymbol{y}^{2,1}) \\
H(\boldsymbol{x}^1)H(\boldsymbol{x}^2)H(\boldsymbol{x}^3) &= H_2(\boldsymbol{y}^{2,2})H_1(\boldsymbol{y}^{2,1})H(\boldsymbol{x}^3) \\
&= H_2(\boldsymbol{y}^{2,2})H_2(\boldsymbol{x}^{3,2})H_1(\boldsymbol{y}^{3,1}) \\
&= H_3(\boldsymbol{y}^{3,3})H_2(\boldsymbol{y}^{3,2})H_1(\boldsymbol{y}^{3,1}) \\
&\cdots,
\end{aligned} \tag{2.39}$$

where $\boldsymbol{y}^{k,i} = (1, \ldots, 1, y_i^{k,i}, \ldots, y_n^{k,i})$. When $m \leq n$, by setting $\boldsymbol{y}^{m,i} = \boldsymbol{p}^i$, $\boldsymbol{p}^i = (1, \ldots, 1, p_i^i, \ldots, p_n^i)$ $(i = 1, \ldots, m)$, we finally obtain

$$H(\boldsymbol{x}^1)H(\boldsymbol{x}^2) \cdots H(\boldsymbol{x}^m) = H_m(\boldsymbol{p}^m) \cdots H_2(\boldsymbol{p}^2)H_1(\boldsymbol{p}^1), \tag{2.40}$$

In this formula, each p^i_j $(i \leq j)$ denotes the tropical variable corresponding to the number of j's in the i-th row of the tableau P. Namely, we can regard the expression

$$H_P = H_m(\boldsymbol{p}^m) \cdots H_2(\boldsymbol{p}^2) H_1(\boldsymbol{p}^1) \tag{2.41}$$

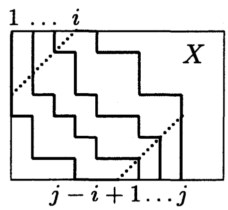
as representing the *tropical tableau* $P = \left(p^i_j\right)_{i \leq j}$; it provides the tropical analogue of a general column strict tableau whose shape is a partition of length m.

The argument above shows that our question can be answered by considering the matrix equation

$$\begin{aligned} &H(\boldsymbol{x}^1)H(\boldsymbol{x}^2)\cdots H(\boldsymbol{x}^m) = H_m(\boldsymbol{p}^m)\cdots H_2(\boldsymbol{p}^2)H_1(\boldsymbol{p}^1) \quad (m \leq n), \\ (2.42)\, &H(\boldsymbol{x}^1)H(\boldsymbol{x}^2)\cdots H(\boldsymbol{x}^m) = H_n(\boldsymbol{p}^n)\cdots H_2(\boldsymbol{p}^2)H_1(\boldsymbol{p}^1) \quad (m \geq n) \end{aligned}$$

for the unknowns $\boldsymbol{p}^i = (1, \ldots, 1, p^i_i, \ldots, p^i_n)$ $(i = 1, \ldots, \min\{m, n\})$. In the following we regard x^i_j $(i = 1, \ldots,\ j = 1, \ldots, n)$ as indeterminates, and look for solutions p^i_j of (2.42) in the filed of rational functions in the x-variables.

Denoting the left-hand side by H, consider the minor determinants $\tau^i_j(H) = \det H^{1,\ldots,i}_{j-i+1,\ldots,j}$ for $1 \leq i \leq j \leq n$. By Proposition 1.3, $\tau^i_j(H)$ is expressed in terms of the nonintersecting paths in the $m \times n$ rectangle associated with the matrix $X = \left(x^i_j\right)_{i,j}$:

(2.43) $\qquad \tau^i_j(H) = \sum$ 

When $m < i \leq j \leq n$, we have $\tau^i_j(H) = \delta_{i,j}\tau^m_j(H)$, $\tau^m_j(H) = \prod_{(a,b);\, b \leq j} x^a_b$. Hence, by Theorem 1.7, we see that the matrix equation (2.42) has a unique rational solution in the x-variables; the solution is expressed by $\tau^i_j(H)$ above. To summarize, we have

Theorem 2.4. *For $m \times n$ indeterminates x^i_j $(1 \leq i \leq m,\ 1 \leq j \leq n)$ given, consider the following matrix equation for unknown variables p^i_j $(1 \leq i \leq l, 1 \leq j \leq n, i \leq j)$, $l = \min\{m, n\}$:*

$$H(\boldsymbol{x}^1)H(\boldsymbol{x}^2)\cdots H(\boldsymbol{x}^m) = H_l(\boldsymbol{p}^l)\cdots H_2(\boldsymbol{p}^2)H_1(\boldsymbol{p}^1), \tag{2.44}$$

where $\boldsymbol{x}^i = (x_1^i, \dots, x_n^i)$ *and* $\boldsymbol{p}^i = (1, \dots, 1, p_i^i, \dots, p_n^i)$*. This equation has a unique rational solution in the x-variables ; it is given explicitly as*

$$(2.45)\qquad p_i^i = \frac{\tau_i^i}{\tau_i^{i-1}}, \qquad p_j^i = \frac{\tau_j^i\,\tau_{j-1}^{i-1}}{\tau_j^{i-1}\tau_{j-1}^i} \quad (i < j)$$

for $1 \le i \le l$, $1 \le j \le n$, *with* τ_j^i $(i \le j)$ *defined as the sum*

$$(2.46)\qquad \tau_j^i = \sum_{(\gamma_1, \dots, \gamma_i)} x_{\gamma_1} \cdots x_{\gamma_i}$$

of monomials over all i-tuples of nonintersecting paths $\gamma_k : (1, k) \to (m, j - i + k)$ $(k = 1, \dots, i)$ *in the* $m \times n$ *rectangle. Here, the weight* x_γ *of a path* γ *is the product*

$$(2.47)\qquad x_\gamma = \prod_{(a,b) \in \gamma} x_b^a$$

of all x_b^a*'s corresponding to the vertices on* γ*.*

The explicit formula for p_j^i above is formulated by A.N. Kirillov [14], Theorem 4.23.

By the standard passage from subtraction-free rational functions to piecewise-linear functions, we obtain the following combinatorial formula ([14], Theorem 3.5) for the column strict tableaux P obtained from a word $w = w_1 w_2 \dots w_m$ by the row insertion.

Theorem 2.5. *Taking the set of letters* $\{1, \dots, n\}$*, let* $w_1, \dots, w_m$ *be a sequence of weakly increasing words* $w^i = 1^{x_1^i} \cdots n^{x_n^i}$ $(i = 1, \dots, m)$*. Consider the column strict tableau*

$$(2.48)\qquad P = P(w) = (\cdots (w_1 \leftarrow w_2) \leftarrow \cdots \leftarrow w_m)$$

obtained from the word $w = w_1 w_2 \cdots w_m$ *by row insertion, and denote by* $i^{p_i^i} \cdots n^{p_n^i}$ *the weakly increasing word representing the i-th row of P, for* $i = 1, \dots, l$, $l = \min\{m, n\}$*. Then, for each* (i, j)*, the number* p_j^i *of the letter j in the i-th row of P is determined explicitly as*

$$(2.49)\qquad p_i^i = \tau_i^i - \tau_i^{i-1}, \quad p_j^i = \tau_j^i - \tau_j^{i-1} - \tau_{j-1}^i + \tau_{j-1}^{i-1} \quad (i < j)$$

with τ_j^i $(i \le j)$ *defined as the maximum*

$$(2.50)\qquad \tau_j^i = \max_{(\gamma_1, \dots, \gamma_i)} (x_{\gamma_1} + \cdots + x_{\gamma_i})$$

of weights over all i-tuples of nonintersecting paths $\gamma_k : (1,k) \to (m, j-i+k)$ $(k = 1, \dots, i)$ in the $m \times n$ rectangle; we set $\tau_j^0 = 0$ for all j. Here, the weight x_γ of a path γ is the sum

$$x_\gamma = \sum_{(a,b)\in\gamma} x_b^a \tag{2.51}$$

of all x_b^a's corresponding to the vertices on γ.

Note also that each τ_j^i represents the sum of p_b^a's in the region $a \le i$, $b \le j$:

$$\tau_j^i = \sum_{(a,b):\, a\le i,\, b\le j} p_b^a \qquad (i \le j). \tag{2.52}$$

Example 2.6. We give an example with $m = 3$, $n = 4$ by taking the word

$$w = 2234134411224 = 2234|1344|11224, \tag{2.53}$$

which corresponds to the column strict tableau

$$P = P(w) = \begin{array}{l} 1\,1\,1\,2\,2\,4\,4 \\ 2\,2\,3\,3\,4 \\ 4 \end{array}. \tag{2.54}$$

From w, we first construct the 3×4 matrix $X = \left(x_j^i\right)_{i,j}$ by counting the number of j's in the i-th block of w:

$$X = \begin{bmatrix} 0 & 2 & 1 & 1 \\ 1 & 0 & 1 & 2 \\ 2 & 2 & 0 & 1 \end{bmatrix}. \tag{2.55}$$

The numbers τ_j^i of (2.50) are determined from X by statistics of nonintersecting paths:

$$\tau = \left(\tau_j^i\right)_{i,j} = \begin{bmatrix} 3 & 5 & 5 & 7 \\ & 7 & 9 & 12 \\ & & 9 & 13 \end{bmatrix}. \tag{2.56}$$

For instance, τ_3^2 is computed as the maximum of weights over three pairs of nonintersecting paths (γ_1, γ_2) such that $\gamma_1 : (1,1) \to (3,2)$ and $\gamma_2 : (1,2) \to (3,3)$:

$$X = \begin{bmatrix} 0 & 2 & 1 & 1 \\ 1 & 0 & 1 & 2 \\ 2 & 2 & 0 & 1 \end{bmatrix}. \tag{2.57}$$

From

$$\begin{array}{c}(0+1+2+2)+(2+0+1+0)=8\\(0+1+2+2)+(2+1+1+0)=9\\(0+1+0+2)+(2+1+1+0)=7\end{array}\tag{2.58}$$

we get

$$\tau^2_3 = \max\{8,9,7\} = 9.\tag{2.59}$$

According to (2.49), we compute p^i_j by taking the *discrete Laplacian* of τ^i_j:

$$\boldsymbol{p} = \left(p^i_j\right)_{i,j} = \begin{bmatrix} 3 & 2 & 0 & 2 \\ & 2 & 2 & 1 \\ & & 0 & 1 \end{bmatrix}.\tag{2.60}$$

Then p^i_j gives the number of j's in the i-th row of the tableau P above.

2.4. Tropical Schützenberger involution

We now recall the Schützenberger involution on the set of column strict tableaux. Taking the set $\{1,2\dots,n\}$ of letters as before, we define an involution $k \mapsto k^*$ on $\{1,2\dots,n\}$ by $k^* = n-k+1$ for $k = 1,\dots,n$. For a word $w = k_1\,k_2\dots k_l$ consisting of letters in $\{1,\dots,n\}$ given, we define the word w^* by

$$w^* = k^*_l \dots k^*_2\,k^*_1,\tag{2.61}$$

by applying $k \mapsto k^*$ to each letter, and then by reversing the order. Let us denote by $P = P(w)$ the column strict tableau obtained from a word w. Since the involution $w \mapsto w^*$ on the set of words preserve the Knuth equivalence, it induces an involution $P \mapsto P^{\mathrm{s}}$ on the set of column strict tableaux such that

$$P(w^*) = P(w)^{\mathrm{s}}\tag{2.62}$$

for any word w; we call this involution $P \mapsto P^{\mathrm{s}}$ the *Schützenberger involution.* It is well known that P and P^{s} has the same shape, and that the column strict tableau P^{s} is obtained from P as the evacuation tableau by a successive application of *jeu de taquin.* We remark that, when the word $w = k_1\dots k_n$ represents a permutation in $\boldsymbol{S}_n$, w^* is the conjugation of w by the longest element of $\boldsymbol{S}_n$. (Schützenberger's algorithm for column strict tableaux can be obtained essentially from [25], Theorem 3.9.4, for instance. In [25], Schützenberger's algorithm is formulated for permutations, but it is not difficult to extend it to that for words.)

Example 2.7. Consider the word $w = 42213132$ with $n = 4$. In this case, we have $w^* = 32424331$.

$$P = P(w) = \begin{matrix} 1\,1\,2\,3 \\ 2\,2\,3 \\ 4 \end{matrix}\;, \quad P^{\mathrm{s}} = P(w^*) = \begin{matrix} 1\,2\,3\,3 \\ 2\,4\,4 \\ 3 \end{matrix}\;. \tag{2.63}$$

Let us decompose a given word w into a chain of weakly increasing words:

$$w = w_1 w_2 \dots w_m, \qquad w_i = 1^{x_1^i} 2^{x_2^i} \dots n^{x_n^i} \quad (i = 1, \dots, m). \tag{2.64}$$

Then we have

$$w^* = w_n^* \dots w_2^* \, w_1^*, \qquad w_i^* = 1^{x_n^i} 2^{x_{n-1}^i} \dots n^{x_1^i} \quad (i = 1, \dots, m). \tag{2.65}$$

Setting $P = P(w)$, $P^{\mathrm{s}} = P(w^*)$, we denote by p_j^i and by $\widetilde{p}_j^i$ the number of j's in the i-th row of P and P^{s}, respectively. Passing to the tropical variables, we have the matrix equation

$$\begin{aligned} &H(\boldsymbol{x}^1)H(\boldsymbol{x}^2)\cdots H(\boldsymbol{x}^m) = H_P = H_l(\boldsymbol{p}^n)\cdots H_2(\boldsymbol{p}^2)H_1(\boldsymbol{p}^1), \\ &H(\boldsymbol{x}_*^m)\cdots H(\boldsymbol{x}_*^2)H(\boldsymbol{x}_*^1) = H_{P^{\mathrm{s}}} = H_l(\widetilde{\boldsymbol{p}}^n)\cdots H_2(\widetilde{\boldsymbol{p}}^2)H_1(\widetilde{\boldsymbol{p}}^1), \end{aligned} \tag{2.66}$$

where $l = \min\{m, n\}$ and, for a vector $\boldsymbol{x} = (x_1, x_2, \dots, x_n)$ given, $\boldsymbol{x}_* = (x_n, \dots, x_2, x_1)$ denotes the vector obtained by reversing the order. In the following, we denote by

$$J_n = \left(\delta_{i+j,n+1}\right)_{i,j=1}^n \tag{2.67}$$

the permutation matrix representing the longest element of $\boldsymbol{S}_n$. Since $J_n \Lambda J_n = \Lambda^{\mathrm{t}}$, from

$$H(\boldsymbol{x}) = (\mathrm{diag}(\overline{\boldsymbol{x}}) - \Lambda)^{-1}, \quad \boldsymbol{x} = (x_1, x_2, \dots, x_n) \tag{2.68}$$

we have

$$J_n H(\boldsymbol{x})^{\mathrm{t}} J_n = (\mathrm{diag}(\overline{\boldsymbol{x}}_*) - \Lambda)^{-1}, \quad \boldsymbol{x}_* = (x_n, \dots, x_2, x_1), \tag{2.69}$$

namely, $J_n H(\boldsymbol{x})^{\mathrm{t}} J_n = H(\boldsymbol{x}_*)$. Hence, (2.66) implies $J_n H_P^{\mathrm{t}} J_n = H_{P^{\mathrm{s}}}$, namely,

$$J_n H_1(\boldsymbol{p}^1)^{\mathrm{t}}\, H_2(\boldsymbol{p}^2)^{\mathrm{t}} \cdots H_n(\boldsymbol{p}^n)^{\mathrm{t}} J_n = H_n(\widetilde{\boldsymbol{p}}^n)\cdots H_2(\widetilde{\boldsymbol{p}}^2)H_1(\widetilde{\boldsymbol{p}}^1). \tag{2.70}$$

Supposing that $m \leq n$, we take two general tropical tableaux

$$H_U = H_m(\boldsymbol{u}^m)\cdots H_1(\boldsymbol{u}^1), \quad H_V = H_m(\boldsymbol{v}^m)\cdots H_1(\boldsymbol{v}^1), \tag{2.71}$$

where $\boldsymbol{u}^i = (u^i_i, \dots, u^i_n)$ and $\boldsymbol{v}^i = (v^i_i, \dots, v^i_n)$ for $i = 1, \dots, m$. In view of (2.70), we define the *tropical Schützenberger involution* to be the birational transformation $U \mapsto V$ defined through the matrix equation $J_n H_U^{\mathrm{t}} J_n = H_V$:

$$J_n H_1(\boldsymbol{u}^1)^{\mathrm{t}} \cdots H_m(\boldsymbol{u}^m)^{\mathrm{t}} J_n = H_m(\boldsymbol{v}^m) \cdots H_1(\boldsymbol{v}^1). \tag{2.72}$$

We now propose to solve this matrix equation for v^i_j $(i \le j)$, regarding u^i_j $(i \le j)$ as indeterminates. Note that

$$J_n \Lambda^{\mathrm{t}}_{\ge k} J_n = \Lambda_{\le n-k+1}, \quad \Lambda_{\le k} = \sum_{i=1}^{k-1} E_{i,i+1}, \tag{2.73}$$

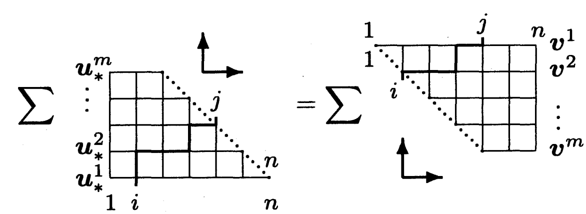
for $k = 1, \dots, n$. Hence, (2.72) can be written as

(2.74)

$$(\mathrm{diag}(\overline{\boldsymbol{u}}^1_*) - \Lambda_{\le n})^{-1} (\mathrm{diag}(\overline{\boldsymbol{u}}^2_*) - \Lambda_{\le n-1})^{-1} \cdots (\mathrm{diag}(\overline{\boldsymbol{u}}^m_*) - \Lambda_{\le n-m+1})^{-1}$$
$$= (\mathrm{diag}(\overline{\boldsymbol{v}}^m) - \Lambda_{\ge m})^{-1} \cdots (\mathrm{diag}(\overline{\boldsymbol{v}}^2) - \Lambda_{\ge 2})^{-1} (\mathrm{diag}(\overline{\boldsymbol{v}}^1) - \Lambda_{\ge 1})^{-1}.$$

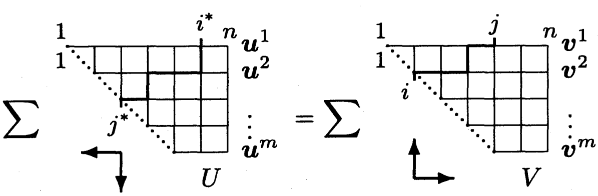
This equality is expressed graphically as follows. For each $1 \le i \le j \le n$,

(2.75)



or equivalently,

(2.76)



where $i^* = n - i + 1$ and $j^* = n - j + 1$. Hence we have

Theorem 2.8. *For the indeterminates u^i_j $(1 \le i \le j \le n)$ given, consider the matrix equation*

(2.77)

$$J_n H_1(\boldsymbol{u}^1)^{\mathrm{t}}\, H_2(\boldsymbol{u}^2)^{\mathrm{t}} \cdots H_m(\boldsymbol{u}^m)^{\mathrm{t}} J_n = H_m(\boldsymbol{v}^m) \cdots H_2(\boldsymbol{v}^2) H_1(\boldsymbol{v}^1)$$

for v^i_j $(1 \le i \le j \le n)$*, where* $m \le n$*, and* $\boldsymbol{u}^i = (u^i_i, \dots, u^i_n)$*,* $\boldsymbol{v}^i = (v^i_i, \dots, v^i_n)$ *for* $i = 1, \dots, m$*. This equation has a unique rational solution; it is given by*

$$v^i_i = \frac{\sigma^i_i}{\sigma^{i-1}_i}, \quad v^i_j = \frac{\sigma^i_j\,\sigma^{i-1}_{j-1}}{\sigma^{i-1}_j\sigma^i_{j-1}} \quad (i < j), \tag{2.78}$$

with σ^i_j $(i \le j)$ *defined as the sum*

$$\sigma^i_j = \sum_{(\gamma_1, \dots, \gamma_i)} u_{\gamma_1} \cdots u_{\gamma_i} \tag{2.79}$$

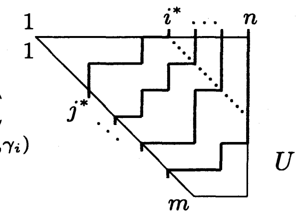
of weights over all i-tuples of nonintersecting paths

$$\gamma_k : (1, n - i + k) \to (\min\{m, n - j + k\}, n - j + k) \tag{2.80}$$

$(k = 1, \dots, i)$*, where the weight* u_γ *of a path* γ *is the product of all* u^a_b*'s corresponding to the vertices of* γ*.*

Graphically, σ^i_j $(i \le j)$ in the explicit formula above is expressed as follows.

$$\sigma^i_j = \sum_{(\gamma_1, \dots, \gamma_i)}$$



(2.81)

The explicit formula above for the tropical Schützenberger involution is proposed by A.N. Kirillov [14], Theorem 4.18.

By returning to the combinatorial variables, we obtain the following explicit formula for the Schützenberger involution (with $m = n$), due to H. Knight and A. Zelevinsky [16] (see also [1], [30]).

Theorem 2.9. *Let P be a column strict tableau and denote by* p^i_j *the number of j's in the i-th row of P for* $1 \le i \le j \le n$*. Let* P^{s} *be the column strict tableau obtained from P by applying the Schützenberger involution, and denote by* $\widetilde{p}^i_j$ *the number of j's in the i-th row of* P^{s} *for* $1 \le i \le j \le n$*. Then* $\widetilde{p}^i_j$ *are determined from* p^i_j *by the following explicit formula:*

$$\widetilde{p}^i_i = \sigma^i_i - \sigma^{i-1}_i, \quad \widetilde{p}^i_j = \sigma^i_j - \sigma^{i-1}_j - \sigma^i_{j-1} + \sigma^{i-1}_{j-1} \quad (i < j), \tag{2.82}$$

with σ_j^i $(1 \le i \le j \le n)$ defined as the maximum

$$\sigma_j^i = \max_{(\gamma_1,\dots,\gamma_i)} (p_{\gamma_1} + \cdots + p_{\gamma_i}) \tag{2.83}$$

of weights over all i-tuples of nonintersecting paths $\gamma_k : (1, n-i+k) \to (n-j+k, n-j+k)$ $(k = 1, \dots, i)$, where the weight p_γ of a path γ is the sum of all p_b^a's corresponding to the vertices of γ.

Example 2.10. In the case of P and P^{s} of Example 2.7, $\boldsymbol{p} = \left(p_j^i\right)_{i\le j}$ and $\widetilde{\boldsymbol{p}} = \left(\widetilde{p}_j^i\right)_{i\le j}$ are given by

$$\boldsymbol{p} = \begin{bmatrix} 2 & 1 & 1 & 0 \\ & 2 & 1 & 0 \\ & & 0 & 1 \\ & & & 0 \end{bmatrix}, \quad \widetilde{\boldsymbol{p}} = \begin{bmatrix} 1 & 1 & 2 & 0 \\ & 1 & 0 & 2 \\ & & 1 & 0 \\ & & & 0 \end{bmatrix}. \tag{2.84}$$

The table $\widetilde{\boldsymbol{p}}$ can be determined through $\boldsymbol{\sigma} = \left(\sigma_j^i\right)_{i\le j}$:

$$\boldsymbol{\sigma} = \begin{bmatrix} 1 & 2 & 4 & 4 \\ & 3 & 5 & 7 \\ & & 6 & 8 \\ & & & 8 \end{bmatrix}. \tag{2.85}$$

Remark 2.11. With the notation of Remark 1.8, the tropical Schützenberger involution can be formulated as follows. We define the involution $\theta : \mathcal{B} \to \mathcal{B}$ by setting

$$\theta(H) = J_n H^{\mathrm{t}} J_n \qquad (H \in \mathcal{B}). \tag{2.86}$$

Then the isomorphism $h : \mathcal{B}_0 \xrightarrow{\sim} \mathcal{B}_\tau$ induces a birational involution $U \mapsto U^{\mathrm{s}}$ on $\mathcal{B}_0$ such that

$$h(U^{\mathrm{s}}) = \theta(h(U)) = J_n h(U)^{\mathrm{t}} J_n \tag{2.87}$$

for generic $U \in \mathcal{B}_0$. We already gave the explicit formula for $V = U^{\mathrm{s}}$ in Theorem 2.8. Note also that the inverse correspondence $V \mapsto U$ is given by the same formula.

§3. Tropical RSK correspondence

3.1. Variations of RSK correspondence

Let $A = \left(a^i_j\right)_{i,j} \in \mathrm{Mat}_{m,n}(\mathbb{N})$ be an $m \times n$ matrix of nonnegative integers.

$$A = \begin{bmatrix} a^1_1 & a^1_2 & \dots & a^1_n \\ a^2_1 & a^2_2 & \dots & a^2_n \\ \vdots & \vdots & & \vdots \\ a^m_1 & a^m_2 & \dots & a^m_n \end{bmatrix} \tag{3.1}$$

Setting $w_i = 1^{a^i_1} \dots n^{a^i_n}$ for $i = 1, \dots, m$, we denote by $P = P(w)$ the column strict tableau obtained from the word $w = w_1 \dots w_m$ by row insertion. Similarly, setting $w'_j = 1^{a^1_j} \dots m^{a^m_j}$ for $j = 1, \dots, n$, we denote by $Q = P(w')$ the column strict tableau obtained from the word $w' = w'_1 \dots w'_n$ by row insertion. The two tableaux P and Q have the same shape, and the correspondence $A \mapsto (P, Q)$ induces a bijection between the set of all $m \times n$ matrices of nonnegative integers and the set of pairs (P, Q) of column strict tableaux of a same shape, P with contents in $\{1, \dots, n\}$ and Q with contents in $\{1, \dots, m\}$. This bijection $A \mapsto (P, Q)$ is called the *Robinson-Schensted-Knuth correspondence* (*RSK correspondence*, for short). In this context, the matrix A is sometimes called the *transportation matrix*. By combining this *standard* RSK correspondence with the Schützenberger involution, we have the following four variations of RSK correspondences:

$$\begin{aligned} A &\mapsto (P, Q), & A &\mapsto (P, Q^{\mathrm{s}}), \\ A &\mapsto (P^{\mathrm{s}}, Q), & A &\mapsto (P^{\mathrm{s}}, Q^{\mathrm{s}}). \end{aligned} \tag{3.2}$$

Let us denote by p^i_j, q^i_j, $\widetilde{p}^i_j$ and $\widetilde{q}^i_j$ the number of j's in the i-th row of the tableaux P, Q, P^{s} and Q^{s}, respectively. The common shape $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$, $l = \min\{m, n\}$, of these four tableaux is given by

(3.3)
$$\lambda_i = p^i_i + \dots + p^i_n = q^i_i + \dots + q^i_m = \widetilde{p}^i_i + \dots + \widetilde{p}^i_n = \widetilde{q}^i_i + \dots + \widetilde{q}^i_m$$

for $i = 1, \dots, l$.

Example 3.1. Consider the transportation matrix

$$A = \begin{bmatrix} 0 & 2 & 1 & 1 \\ 1 & 0 & 1 & 2 \\ 2 & 2 & 0 & 1 \end{bmatrix}. \tag{3.4}$$

The tableaux P, Q for A, and their counterparts under the Schützenberger involution are determined as follows:

$$P = P(2234|1344|11224) = \begin{array}{l} 1\,1\,1\,2\,2\,4\,4 \\ 2\,2\,3\,3\,4 \\ 4 \end{array}, \quad \boldsymbol{p} = \begin{bmatrix} 3&2&0&2 \\ &2&2&1 \\ &&0&1 \end{bmatrix}$$

$$Q = P(233|1133|12|1223) = \begin{array}{l} 1\,1\,1\,1\,2\,2\,3 \\ 2\,2\,3\,3\,3 \\ 3 \end{array}, \quad \boldsymbol{q} = \begin{bmatrix} 4&2&1 \\ &2&3 \\ &&1 \end{bmatrix}$$

$$P^{\mathrm{s}} = P(13344|1124|1233) = \begin{array}{l} 1\,1\,1\,1\,2\,3\,3 \\ 2\,3\,4\,4\,4 \\ 3 \end{array}, \quad \widetilde{\boldsymbol{p}} = \begin{bmatrix} 4&1&2&0 \\ &1&1&3 \\ &&1&0 \end{bmatrix}$$

$$Q^{\mathrm{s}} = P(1223|23|1133|112) = \begin{array}{l} 1\,1\,1\,1\,1\,2\,3 \\ 2\,2\,2\,3\,3 \\ 3 \end{array}, \quad \widetilde{\boldsymbol{q}} = \begin{bmatrix} 5&1&1 \\ &3&2 \\ &&1 \end{bmatrix} \tag{3.5}$$

In the arguments of this section, the correspondence $A \mapsto (P, Q^{\mathrm{s}})$ will play the essential role, rather than the ordinary RSK correspondence. For this reason, we use the following convention. Let $X = \left(x_j^i\right)_{i,j}$ be an $m \times n$ matrix of nonnegative integers. We denote the i-th row of X by $\boldsymbol{x}^i$, and the j-th column of X by $\boldsymbol{x}_j$:

$$X = \begin{bmatrix} x_1^1 & x_2^1 & \dots & x_n^1 \\ x_1^2 & x_2^2 & \dots & x_n^2 \\ \vdots & \vdots & & \vdots \\ x_1^m & x_2^m & \dots & x_n^m \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}^1 \\ \boldsymbol{x}^2 \\ \vdots \\ \boldsymbol{x}^m \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}_1 \ \boldsymbol{x}_2 \ \dots \ \boldsymbol{x}_n \end{bmatrix}. \tag{3.6}$$

From X, we construct the column strict tableau

$$U = P(w_m w_{m-1} \cdots w_1), \quad w_i = 1^{x_1^i} 2^{x_2^i} \cdots n^{x_n^i} \tag{3.7}$$

by reading the rows $\boldsymbol{x}^m, \dots, \boldsymbol{x}^1$ of X *from bottom to top*, and

$$V = P(w'_n w'_{n-1} \cdots w'_1), \quad w'_j = 1^{x_j^1} 2^{x_j^2} \cdots m^{x_j^m}, \tag{3.8}$$

by reading the columns $\boldsymbol{x}_n, \dots, \boldsymbol{x}_1$ *from right to left*. If we set $X = J_m A$, these U and V correspond to P and Q^{s}, determined from A, so that

$$U = P, \quad V = Q^{\mathrm{s}}, \quad U^{\mathrm{s}} = P^{\mathrm{s}}, \quad V^{\mathrm{s}} = Q. \tag{3.9}$$

In what follows, we refer to this particular correspondence $X \mapsto (U, V)$ as the *RSK* correspondence.*

Before the discussion of tropical RSK correspondences, we formulate an isomorphism theorem concerning the path representation of generic matrices.

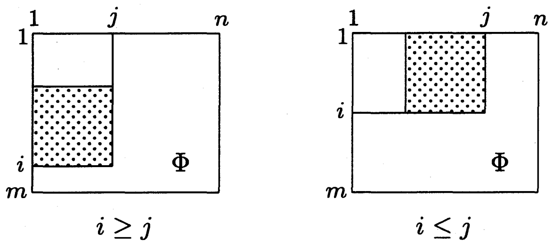
3.2. A fundamental isomorphism

We first fix a notation of *special* minor determinants. For an $m \times n$ matrix $\Phi = \left(\varphi^i_j\right)_{i,j}$ given, we introduce the notation

$$\tau^i_j(\Phi) = \begin{cases} \det \Phi^{i-j+1,\ldots,i}_{1,\ldots,j} & (i \geq j), \\ \det \Phi^{1,\ldots,i}_{j-i+1,\ldots,j} & (i \leq j), \end{cases} \tag{3.10}$$

of the minor determinant of Φ corresponding to the largest square in the rectangle $\{1,\ldots,m\} \times \{1,\ldots,n\}$ whose right-bottom corner is located at (i,j).

$\tau^i_j(\Phi)$:

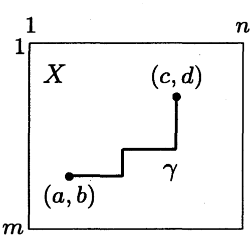


(3.11)

For convenience, we define $\tau^0_k(\Phi) = \tau^k_0(\Phi) = 0$ for any k. We define the subset $\mathrm{Mat}_{m,n}(\mathbb{K})_\tau$ of $\mathrm{Mat}_{m,n}(\mathbb{K})$ by

$$\mathrm{Mat}_{m,n}(\mathbb{K})_\tau = \{\Phi \in \mathrm{Mat}_{m,n}(\mathbb{K}) \mid \tau^i_j(\Phi) \neq 0 \quad \text{for all} \quad (i,j)\}. \tag{3.12}$$

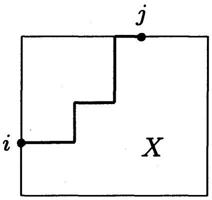
For an $m \times n$ matrix $X = \left(x^i_j\right)_{i,j}$ given, we now construct an $m \times n$ matrix $\Phi = \left(\varphi^i_j\right)_{i,j}$ by using the paths on the lattice $\{1,\ldots m\} \times \{1,\ldots,n\}$. When we refer to a path in the rectangular lattice, we mean a shortest path joining two vertices, without specifying the orientation of edges. As before, for each path $\gamma : (a,b) \to (c,d)$, we define the weight x_γ of γ, associated with X, to be the product of all x^i_j's corresponding the vertices on γ.



(3.13)

With this definition of weight, we define the matrix $\Phi = \left(\varphi^i_j\right)_{i,j}$ by setting

$$\varphi^i_j = \varphi^i_j(X) = \sum_{\gamma:(i,1)\to(1,j)} x_\gamma = \sum_\gamma \tag{3.14}$$



for each $(i,j) \in \{1,\dots,m\} \times \{1,\dots,n\}$, where the summation is taken over all paths from $(i,1)$ to $(1,j)$. The mapping $X \mapsto \Phi$ thus obtained will be denoted by

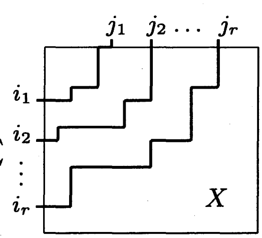
$$\varphi \;:\; \mathrm{Mat}_{m,n}(\mathbb{K}) \to \mathrm{Mat}_{m,n}(\mathbb{K}), \quad \varphi(X) = \left(\varphi^i_j(X)\right)_{i,j}. \tag{3.15}$$

This mapping φ provides us with a device for generating nonintersecting paths on the lattice $\{1,\dots,m\} \times \{1,\dots,n\}$. In fact, from the theorem of Gessel-Viennot [5], it follows that, for any choice of column indices $i_1 < \cdots < i_r$ and row indices $j_1 < \cdots < j_r$, the corresponding minor determinant of Φ is expressed as a sum

$$\det \Phi^{i_1,\dots,i_r}_{j_1,\dots,j_r} = \sum_{(\gamma_1,\dots,\gamma_r)} x_{\gamma_1} \cdots x_{\gamma_r} \tag{3.16}$$

of the product of weights over all r-tuples $(\gamma_1,\dots,\gamma_r)$ of nonintersecting paths $\gamma_k : (i_k,1) \to (1,j_k)$ $(k = 1,\dots,r)$. Graphically, this summation can be expressed as follows.

$$\det \Phi^{i_1,\dots,i_r}_{j_1,\dots,j_r} = \sum \tag{3.17}$$



Let us look at the special minor determinants $\tau^i_j(\Phi)$ introduced above. Notice that, for each (i,j), there is only one r-tuple ($r = \min\{i,j\}$) of nonintersecting paths relevant to the summation, so that the minor determinant $\tau^i_j(\Phi)$ reduces to the product

$$\tau^i_j(\Phi) = \prod_{(a,b);\, a\le i,\, b\le j} x^a_b. \tag{3.18}$$

This implies that, if $x^i_j \neq 0$ for all i, j, then one has $\tau^i_j(\Phi) \neq 0$ for all i, j. From formula (3.18), it is also clear that the entries x^i_j of the matrix X are recovered as the ratios of minor determinants of Φ

$$(3.19) \qquad x^i_j = \frac{\tau^i_j \, \tau^{i-1}_{j-1}}{\tau^{i-1}_j \, \tau^i_{j-1}}, \qquad \tau^i_j = \tau^i_j(\Phi),$$

provided that $x^i_j \neq 0$ for all i, j. The correspondence $X \mapsto \Phi$ defined by (3.14) induces a mapping

$$(3.20) \qquad \varphi \ : \ \mathrm{Mat}_{m,n}(\mathbb{K}^*) \ \to \ \mathrm{Mat}_{m,n}(\mathbb{K})_\tau.$$

As we have seen above, if $\Phi = \varphi(X)$, then the matrix X is recovered by the formula (3.19).

Theorem 3.2. *The correspondence $X \mapsto \Phi$ defined by* (3.14) *induces an isomorphism of affine varieties*

$$(3.21) \qquad \varphi \ : \ \mathrm{Mat}_{m,n}(\mathbb{K}^*) \ \xrightarrow{\sim} \ \mathrm{Mat}_{m,n}(\mathbb{K})_\tau.$$

Proof. Since φ has a left inverse defined by (3.19), we have only to show that φ is surjective. For each $\Phi \in \mathrm{Mat}_{m,n}(\mathbb{K})_\tau$, we construct an $X \in \mathrm{Mat}_{m,n}(\mathbb{K}^*)$ such that $\varphi(X) = \Phi$, by the induction on m. For this purpose, we first investigate the inductive structure of the mapping φ. Assuming that $\varphi(X) = \Phi$, set

$$(3.22) \qquad \psi^i_j = \sum_{\gamma:(i,1)\to(2,j)} x_\gamma \qquad (2 \leq i \leq m, 1 \leq j \leq n).$$

Then φ^i_j are determined as

(3.23)

$$\varphi^1_j = x^1_1 \, x^1_2 \cdots x^1_j, \qquad \varphi^i_j = \sum_{k=1}^{j} \psi^i_k \, x^1_k \, x^1_{k+1} \cdots x^1_j \quad (i = 2, \ldots, m),$$

for all j. In view of this, we consider the $n \times n$ upper triangular matrix $H(\boldsymbol{x}^1)$, $\boldsymbol{x}^1 = \left(x^1_1, \ldots, x^1_n\right)$, associated with the first row of X. Then the condition (3.23) is equivalent to the matrix equation

$$(3.24) \qquad \Phi = \Psi \, H(\boldsymbol{x}^1), \quad \Psi = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ \psi^2_1 & \psi^2_2 & \ldots & \psi^2_n \\ \vdots & \vdots & & \vdots \\ \psi^m_1 & \psi^m_2 & \ldots & \psi^m_n \end{bmatrix}.$$

Let us show that any $\Phi \in \mathrm{Mat}_{m,n}(\mathbb{K})_\tau$ can be decomposed in this form by choosing x^1_j and ψ^i_j appropriately. The condition to be satisfied by the first row $(x^1_1, \dots, x^1_n)$ of X is:

$$(3.25) \qquad \left(\Phi H(\boldsymbol{x}^1)^{-1}\right)^1_j = \delta_{1,j} \qquad (j = 1, \dots, n).$$

Since $H(\boldsymbol{x}^1)^{-1} = \mathrm{diag}(\overline{\boldsymbol{x}}^1) - \Lambda$, it is easily seen that (3.25) is equivalent to

$$(3.26) \qquad x^1_1 = \varphi^1_1, \quad x^1_j = \frac{\varphi^1_j}{\varphi^1_{j-1}} \quad (j = 2, \dots, n).$$

Since $\varphi^1_j = \tau^1_j(\Phi) \neq 0$, we can define x^1_j $(j = 1, \dots, n)$ as above. Then the matrix $\Psi = \Phi H(\boldsymbol{x}^1)^{-1}$ has the first row $(1, 0, \dots, 0)$; hence, $H(\boldsymbol{x}^1)$ and Ψ satisfy the condition (3.24). Define Φ' to be the $(m-1) \times n$ matrix obtained from Ψ by removing the first row. We will verify that $\Phi' \in \mathrm{Mat}_{m-1,n}(\mathbb{K})_\tau$ so that Φ' can be expressed as $\Phi' = \varphi(X')$ by the induction hypothesis. Then, setting

$$(3.27) \qquad X = \begin{bmatrix} x^1_1 & \dots & x^1_n \\ & X' & \end{bmatrix}, \quad X' = \begin{bmatrix} x^2_1 & \dots & x^2_n \\ \vdots & & \vdots \\ x^m_1 & \dots & x^m_n \end{bmatrix},$$

we must have $\varphi(X) = \Phi$, which will complete the proof of Theorem 3.2. We now examine the minor determinants of $\Phi = \Psi H(\boldsymbol{x}^1)$. Let $(i,j) \in \{1, \dots, m\} \times \{1, \dots, n\}$ and assume $i > j$. Then it is clear

$$(3.28) \qquad \tau^i_j(\Phi) = \det \Phi^{i-j+1,\dots,i}_{1,\dots,j} = \det \Psi^{i-j+1,\dots,i}_{1,\dots,j} x^1_1 \dots x^1_j,$$

since H is upper triangular. Hence we have

$$(3.29) \qquad \tau^{i-1}_j(\Phi') = \det \Psi^{i-j+1,\dots,i}_{1,\dots,j} = \frac{1}{x^1_1 \dots x^1_j} \tau^i_j(\Phi) \neq 0 \quad (i > j).$$

Next assume $i \leq j$. In this case, we have

(3.30)
$$\tau^i_j(\Phi) = \det \Phi^{1,\dots,i}_{j-i+1,\dots,j} = \sum_{k_1 < \dots < k_i} \det \Psi^{1,\dots,i}_{k_1,\dots,k_i} \det H(\boldsymbol{x}^1)^{k_1,\dots,k_i}_{j-i+1,\dots,j}.$$

Since the first row of Ψ is $\big(1, 0, \dots, 0\big)$, we have $\det \Psi^{1,\dots,i}_{k_1,\dots,k_i} = 0$ unless $k_1 = 1$. When $k_1 = 1$, from Lemma 1.4 it follows that $\det H(\boldsymbol{x}^1)^{1,k_2,\dots,k_i}_{j-i+1,\dots,j}$

$= 0$ unless $(k_2, \dots, k_i) = (j-i+2, \dots, j)$. Since $\det H(\boldsymbol{x}^1)^{1,j-i+2,\dots,j}_{j-i+1,\dots,j} = x^1_1 x^1_2 \cdots x^1_j$, we finally obtain

$$\tau^i_j(\Phi) = \det \Psi^{1,\dots,i}_{1,j-i+2,\dots,j}\, x^1_1 \cdots x^1_j. \tag{3.31}$$

Hence we have

$$\tau^{i-1}_j(\Phi') = \det \Psi^{1,\dots,i}_{1,j-i+2,\dots,j} = \frac{1}{x^1_1 \cdots x^1_j} \tau^i_j(\Phi) \neq 0 \quad (i \leq j). \tag{3.32}$$

This argument implies $\tau^i_j(\Phi') \neq 0$ for all (i,j) with $1 \leq i \leq m-1$ and $1 \leq j \leq n$ as desired. This completes the proof of Theorem 3.2. □

It is convenient for our purpose to restate Theorem 3.2 as follows.

Theorem 3.3. *Let $\Phi = \left(\varphi^i_j\right)_{i,j}$ be an $m \times n$ matrix with coefficients in $\mathbb{K}$ such that $\tau^i_j(\Phi) \neq 0$ for all (i,j). For such a matrix Φ given, define the $m \times n$ matrix $X = \left(x^i_j\right)_{i,j}$ by setting*

$$x^i_j = \frac{\tau^i_j\, \tau^{i-1}_{j-1}}{\tau^{i-1}_j\, \tau^i_{j-1}}, \qquad \tau^i_j = \tau^i_j(\Phi). \tag{3.33}$$

Then, for any choice of column indices $i_1 < \cdots < i_r$ and row indices $j_1 < \cdots < j_r$, the corresponding minor determinant of Φ is expressed as a sum

$$\det \Phi^{i_1,\dots,i_r}_{j_1,\dots,j_r} = \sum_{(\gamma_1,\dots,\gamma_r)} x_{\gamma_1} \cdots x_{\gamma_r} \tag{3.34}$$

of the product of weights over all r-tuples $(\gamma_1, \dots, \gamma_r)$ of nonintersecting paths $\gamma_k : (i_k, 1) \to (1, j_k)$ $(k = 1, \dots, r)$ in the $m \times n$ rectangle.

Remark 3.4. An $m \times n$ real matrix Φ ($\mathbb{K} = \mathbb{R}$) is said to be *totally positive* if $\det \Phi^{i_1,\dots,i_r}_{j_1,\dots,j_r} > 0$ for any choice of row indices $i_1 < \cdots < i_r$ and column indices $j_1 < \cdots < j_r$. Theorem 3.2 implies that, if $\tau^i_j(\Phi) > 0$ for all (i,j), then Φ is already totally positive. In fact, if this condition is satisfied, all the x^i_j's are positive; hence, any minor determinant $\det \Phi^{i_1,\dots,i_r}_{j_1,\dots,j_r}$ is positive since it is expressed as a sum of weights associated with X over r-tuples of nonintersecting paths as in (3.34). Let us denote by $\mathrm{Mat}_{m,n}(\mathbb{R})_{\text{tot.pos.}}$ the open subset of $\mathrm{Mat}_{m,n}(\mathbb{R})$ consisting of all totally positive matrices. Then Theorem 3.2 also implies that φ induces the isomorphism

$$\varphi \; : \; \mathrm{Mat}_{m,n}(\mathbb{R}_{>0}) \xrightarrow{\sim} \mathrm{Mat}_{m,n}(\mathbb{R})_{\text{tot.pos}}. \tag{3.35}$$

In particular, $\mathrm{Mat}_{m,n}(\mathbb{R})_{\mathrm{tot.pos}}$ is isomorphic to $\mathbb{R}_{>0}^{mn}$ as a real analytic manifold. For the theory of totally positive matrices, we refer the reader to [1].

We apply the fundamental isomorphism of Theorem 3.2 to formulating a prototype of subtraction-free birational involution on the space $\mathrm{Mat}_{m,n}(\mathbb{K}^*)$ of matrices.

For each $\Phi \in \mathrm{Mat}_{m,n}(\mathbb{K})$, we define the matrix $\Phi^\vee$ by setting

$$\Phi^\vee = J_m\, \Phi\, J_n, \tag{3.36}$$

where $J_m = \left(\delta_{i+j,m+1}\right)_{i,j=1}^{m}$ and $J_n = \left(\delta_{i+j,n+1}\right)_{i,j=1}^{n}$ are the permutation matrices representing to the longest element of $\boldsymbol{S}_m$ and $\boldsymbol{S}_n$, respectively. This correspondence $\Phi \mapsto \Phi^\vee$ defines an involution on the space $\mathrm{Mat}_{m,n}(\mathbb{K})$ of $m \times n$ matrices. Then, via the isomorphism

$$\varphi:\ \mathrm{Mat}_{m,n}(\mathbb{K}^*) \xrightarrow{\sim} \mathrm{Mat}_{m,n}(\mathbb{K})_\tau, \tag{3.37}$$

we obtain a birational involution $X \mapsto \iota(X)$ on $\mathrm{Mat}_{m,n}(\mathbb{K}^*)$ such that

$$\varphi(\iota(X)) = \varphi(X)^\vee = J_m\, \varphi(X)\, J_n \tag{3.38}$$

for *generic* $X \in \mathrm{Mat}_{m,n}(\mathbb{K}^*)$.

To be more explicit, let us consider two matrices $X, Y \in \mathrm{Mat}_{m,n}(\mathbb{K}^*)$, and set $\Phi = \varphi(X)$ and $\Psi = \varphi(Y)$. If we impose the relation $\Psi = \Phi^\vee$ between Φ and Ψ, it induces a birational correspondence between X and $Y = \iota(X)$. As we will see below, this correspondence $X \leftrightarrow Y$ provides the essential ingredient of the RSK* correspondence. Recall that $X = \left(x_j^i\right)_{i,j}$ is recovered from $\Phi = \left(\varphi_j^i\right)_{i,j}$ by the formula

$$x_j^i = \frac{\tau_j^i\, \tau_{j-1}^{i-1}}{\tau_j^{i-1}\tau_{j-1}^i}, \qquad \tau_j^i = \tau_j^i(\Phi), \tag{3.39}$$

and $Y = \left(y_j^i\right)_{i,j}$ from $\Psi = \left(\psi_j^i\right)_{i,j}$ by

$$y_j^i = \frac{\sigma_j^i\, \sigma_{j-1}^{i-1}}{\sigma_j^{i-1}\sigma_{j-1}^i}, \qquad \sigma_j^i = \tau_j^i(\Psi). \tag{3.40}$$

We now look at the determinant σ_j^i. Since $\Psi = \Phi^\vee$, we have

$$\sigma_j^i = \tau_j^i(\Psi) = \det \Psi_{j-r+1,\dots,j}^{i-r+1,\dots,i} = \det \Phi_{n-j+1,\dots,n-j+r}^{m-i+1,\dots,m-i+r}, \tag{3.41}$$

where $r = \min\{i, j\}$. Hence, each σ^i_j is expressed as the sum

$$\sigma^i_j = \sum_{(\gamma_1,\dots,\gamma_r)} x_{\gamma_1} \cdots x_{\gamma_r} \tag{3.42}$$

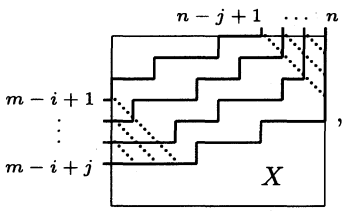
of weights associated with X, over all r-tuples of nonintersecting paths

$$\gamma_k : (m - i + k, 1) \to (1, n - j + k) \quad (l = 1, \dots, r). \tag{3.43}$$

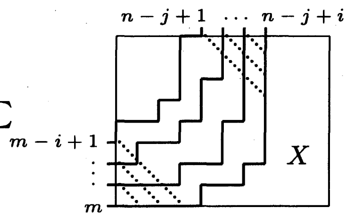
Graphically, σ^i_j can be expressed as follows.

(3.44)

$\sigma^i_j = \sum$



, or $\sum$



$(i \geq j)$ $\qquad$ $(i \leq j)$

From symmetry of the construction, x^i_j are recovered from y^i_j by the same procedure.

Theorem 3.5. *Let $X = \left(x^i_j\right)_{i,j}$, $Y = \left(y^i_j\right)_{i,j}$ be two $m \times n$ matrices such that $x^i_j \neq 0$, $y^i_j \neq 0$ for all i, j. Setting $\Phi = \varphi(X)$, $\Psi = \varphi(Y)$, suppose that Φ and Ψ are related as $\Psi = \Phi^\vee$:*

$$X \xrightarrow{\varphi} \Phi \overset{\vee}{\leftrightarrow} \Psi \xleftarrow{\varphi} Y. \tag{3.45}$$

Then, for each (i, j), y^i_j is expressed as follows in terms of X :

$$y^i_j = \frac{\sigma^i_j \, \sigma^{i-1}_{j-1}}{\sigma^{i-1}_j \sigma^i_{j-1}}, \qquad \sigma^i_j = \sum_{(\gamma_1,\dots,\gamma_r)} x_{\gamma_1} \cdots x_{\gamma_r}, \tag{3.46}$$

where $r = \min\{i, j\}$, and the summation is taken over all r-tuples of nonintersecting paths $\gamma_k : (m - i + k, 1) \to (1, n - j + k)$ $(k = 1, \dots, r)$. Conversely, each x^i_j is expressed as follows in terms of Y :

$$x^i_j = \frac{\tau^i_j \, \tau^{i-1}_{j-1}}{\tau^{i-1}_j \tau^i_{j-1}}, \qquad \tau^i_j = \sum_{(\gamma_1,\dots,\gamma_r)} y_{\gamma_1} \cdots y_{\gamma_r}, \tag{3.47}$$

summed over the same set of r-tuples of nonintersecting paths as above.

Note that the transformation from $X = \left(x_j^i\right)_{i,j}$ to $Y = \left(y_j^i\right)_{i,j}$ in Theorem 3.5 is realized as a subtraction-free birational mapping from $\mathrm{Mat}_{m,n}(\mathbb{K}^*)$ to itself; this birational mapping is in fact an involution on $\mathrm{Mat}_{m,n}(\mathbb{K}^*)$. Passing to the piecewise linear functions, we obtain

Theorem 3.6. *For each $m \times n$ matrix $X = \left(x_j^i\right)_{i,j} \in \mathrm{Mat}_{m,n}(\mathbb{R})$, define an $m \times n$ matrix $Y = \left(y_j^i\right)_{i,j}$ by*

(3.48)
$$y_j^i = \sigma_j^i - \sigma_j^{i-1} - \sigma_{j-1}^i + \sigma_{j-1}^{i-1}, \qquad \sigma_j^i = \max_{(\gamma_1,\dots,\gamma_r)} (x_{\gamma_1} + \cdots + x_{\gamma_r}),$$

where $r = \min\{i,j\}$, and the maximum is taken over all r-tuples of nonintersecting paths $\gamma_k : (m-i+k, 1) \to (1, n-j+k)$ $(k = 1, \dots, r)$; the weight of a path γ is the sum of all x_b^a's corresponding to the vertices of γ. Then the piecewise linear mapping $X \mapsto Y$ is an involution on $\mathrm{Mat}_{m,n}(\mathbb{R})$.

3.3. Tropical RSK* correspondence

Theorem 3.5 is an essential ingredient of the tropical RSK correspondences. Regarding x_j^i as indeterminates, we now work within the field of rational functions $\mathbb{K}(x)$ in mn variables x_j^i $(1 \le i \le m, 1 \le j \le n)$. In what follows, we assume that $m \le n$ to fix the idea.

Consider the $m \times n$ matrix $X = \left(x_j^i\right)_{i,j}$ regarding x_j^i as indeterminates. We denote the i-th row of X by $\boldsymbol{x}^i$, and the j-th column of X by $\boldsymbol{x}_j$:

$$X = \begin{bmatrix} x_1^1 & x_2^1 & \dots & x_n^1 \\ x_1^2 & x_2^2 & \dots & x_n^2 \\ \vdots & \vdots & & \vdots \\ x_1^m & x_2^m & \dots & x_n^m \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}^1 \\ \boldsymbol{x}^2 \\ \vdots \\ \boldsymbol{x}^m \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}_1 \ \boldsymbol{x}_2 \ \dots \ \boldsymbol{x}_n \end{bmatrix}. \tag{3.49}$$

From the matrix $X = \left(x_j^i\right)_{i,j}$, we construct four tropical tableaux

$$U = \left(u_j^i\right)_{i \le j}, \quad V = \left(v_j^i\right)_{i \le j}, \quad U^{\mathsf{s}} = \left(\widetilde{u}_j^i\right)_{i \le j}, \quad V^{\mathsf{s}} = \left(\widetilde{v}_j^i\right)_{i \le j} \tag{3.50}$$

as follows:

$$\begin{aligned} H(\boldsymbol{x}^m) \cdots H(\boldsymbol{x}^2) H(\boldsymbol{x}^1) &= H_m(\boldsymbol{u}^m) \cdots H_2(\boldsymbol{u}^2) H_1(\boldsymbol{u}^1) = H_U, \\ H(\boldsymbol{x}_n) \cdots H(\boldsymbol{x}_2) H(\boldsymbol{x}_1) &= H_m(\boldsymbol{v}^m) \cdots H_2(\boldsymbol{v}^2) H_1(\boldsymbol{v}^1) = H_V, \\ H(\boldsymbol{x}_*^1) H(\boldsymbol{x}_*^2) \cdots H(\boldsymbol{x}_*^m) &= H_m(\widetilde{\boldsymbol{u}}^m) \cdots H_2(\widetilde{\boldsymbol{u}}^2) H_1(\widetilde{\boldsymbol{u}}^1) = H_{U^{\mathsf{s}}}, \\ H(\boldsymbol{x}_1^*) H(\boldsymbol{x}_2^*) \cdots H(\boldsymbol{x}_n^*) &= H_m(\widetilde{\boldsymbol{v}}^m) \cdots H_2(\widetilde{\boldsymbol{v}}^2) H_1(\widetilde{\boldsymbol{v}}^1) = H_{V^{\mathsf{s}}}, \end{aligned} \tag{3.51}$$

where $\boldsymbol{x}_*^i = (x_n^i, \dots, x_2^i, x_1^i)$ and $\boldsymbol{x}_j^* = (x_j^m, \dots, x_j^2, x_j^1)$; H_U, H_{U^s} are $n \times n$ matrices, and H_V, H_{V^s} are $m \times m$ matrices. Note that U and U^s (resp. V and V^s) are transformed into each other by the tropical Schützenberger involution. We also introduce the *tropical Gelfand-Tsetlin pattern* $\boldsymbol{\mu}$ associated with the tropical tableau U as

$$\boldsymbol{\mu} = \begin{bmatrix} \mu_1^{(n)} \quad \mu_2^{(n)} & \cdots & \mu_n^{(n)} \\ \mu_1^{(n-1)} \ \mu_2^{(n-1)} & \cdots & \mu_{n-1}^{(n-1)} \\ & \cdots & \\ & \mu_1^{(1)} & \end{bmatrix}, \tag{3.52}$$

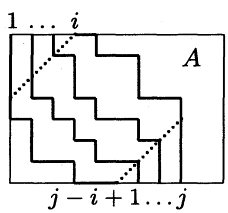
where, for $i \leq j$, we define $\mu_i^{(j)} = u_i^i \cdots u_j^i$ $(i \leq m)$ and $\mu_i^{(j)} = 1$ $(i > m)$.

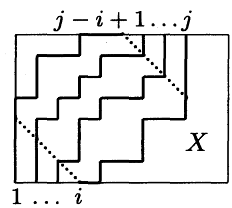
Applying Theorem 2.4 to $A = J_m X$, we already know that the variables u_j^i $(i \leq j)$ are determined by

$$u_i^i = \frac{\tau_i^i(H_U)}{\tau_i^{i-1}(H_U)}, \quad u_j^i = \frac{\tau_j^i(H_U)\, \tau_{j-1}^{i-1}(H_U)}{\tau_j^{i-1}(H_U)\tau_{j-1}^i(H_U)} \quad (i < j), \tag{3.53}$$

with

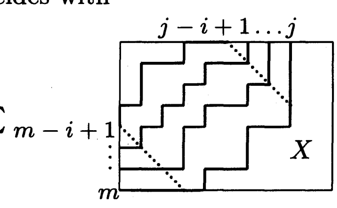
(3.54)

$\tau_j^i(H_U) = \sum$ 

$= \sum$ 

.

Notice that $\tau_j^i(H_U)$ for $i \leq j$ coincides with

$\det \Phi_{j-i+1,\dots,j}^{m-i+1,\dots,m} = \sum$ 

.

(3.55)

Hence we have

$$\tau_j^i(H_U) = \det \Phi_{j-i+1,\dots,j}^{m-i+1,\dots,m} = \det \Psi_{n-j+1,\dots,n-j+i}^{1,\dots,i} = \tau_{n-j+i}^i(\Psi). \tag{3.56}$$

This implies

$$\sigma_j^i = \tau_j^i(\Psi) = \tau_{n-j+i}^i(H_U) = \prod_{(a,b);\, a \leq i,\, b \leq n-j+i} u_b^a \qquad (i \leq j). \tag{3.57}$$

Similarly, for $i \le j$, we have

(3.58)
$$\tau_j^i(H_V) = \det \Phi_{n-i+1,\dots,i}^{j-i+1,\dots,j} = \det \Psi_{1,\dots,i}^{m-j+1,\dots,m-j+i} = \tau_i^{m-j+i}(\Psi),$$

hence, for $i \ge j$,

$$\text{(3.59)}\quad \sigma_j^i = \tau_j^i(\Psi) = \tau_{m-i+j}^j(H_V) = \prod_{(a,b);\, a\le j,\, b\le m-i+j} v_b^a \qquad (i \ge j).$$

Summarizing the argument above, we have

$$\text{(3.60)}\qquad \sigma_j^i = \tau_j^i(\Psi) = \begin{cases} \displaystyle\prod_{(a,b);\, a\le i,\, b\le n-j+i} u_b^a & (i \le j), \\ \displaystyle\prod_{(a,b);\, a\le j,\, b\le m-i+j} v_b^a & (i \ge j), \end{cases}$$

for all (i,j) with $1 \le i \le m$ and $1 \le j \le n$. Conversely, u_j^i and v_j^i are determined as

$$\text{(3.61)}\qquad u_i^i = \frac{\sigma_n^i}{\sigma_{n-1}^{i-1}}, \quad u_j^i = \frac{\sigma_{n-j+i}^i\,\sigma_{n-j+i}^{i-1}}{\sigma_{n-j+i-1}^{i-1}\sigma_{n-j+i+1}^i} \quad (i < j)$$

and

$$\text{(3.62)}\qquad v_i^i = \frac{\sigma_i^m}{\sigma_{i-1}^{m-1}}, \quad v_j^i = \frac{\sigma_i^{m-j+i}\,\sigma_{i-1}^{m-j+i}}{\sigma_{i-1}^{m-j+i-1}\sigma_i^{m-j+i-1}} \quad (i < j).$$

Remark 3.7. It should be noted that the upper (resp. lower) triangular components of the $m \times n$ matrix $S = \left(\sigma_j^i\right)_{i,j}$ are determined from $U = \left(u_j^i\right)_{i\le j}$ (resp. $V = \left(v_j^i\right)_{i\le j}$), and *vice versa*. Formula (3.60) is also equivalent to

$$\text{(3.63)}\qquad \frac{\sigma_j^i}{\sigma_{j-1}^{i-1}} = \begin{cases} u_i^i \cdots u_{n-j+i}^i = \mu_i^{(n-j+i)} & (i \le j), \\ v_j^j \cdots v_{m-i+j}^j = \nu_j^{(m-i+j)} & (i \ge j), \end{cases}$$

where $\mu_i^{(j)}$ and $\nu_i^{(j)}$ are the tropical variables representing the Gelfand-Tsetlin pattern of U and V, respectively. Namely, the $m \times n$ matrix

$$\left(\frac{\sigma_j^i}{\sigma_{j-1}^{i-1}}\right)_{i,j} = \begin{bmatrix} \lambda_1 & \mu_1^{(n-1)} & \cdots\cdots & & \mu_1^{(n-m)} & \cdots & \mu_1^{(1)} \\ \nu_1^{(m-1)} & \lambda_2 & \ddots & & \vdots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & & \vdots \\ \nu_1^{(1)} & \cdots & \nu_{m-1}^{(m-1)} & \lambda_m & \mu_m^{(n-1)} & \cdots & \mu_m^{(m)} \end{bmatrix},$$

(3.64)

defined by the ratios of σ^i_j, is obtained by glueing the two Gelfand-Tsetlin patterns $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ at the main diagonal, where the diagonal entries

$$\lambda_i = \mu^{(n)}_i = \nu^{(m)}_i \quad (i = 1, \dots, m) \tag{3.65}$$

are the tropical variables representing the common shape of U and V.

From (3.60), we obtain the following expression for y^i_j:

$$y^i_j = \frac{\sigma^i_j\,\sigma^{i-1}_{j-1}}{\sigma^{i-1}_j\sigma^i_{j-1}} = \begin{cases} \dfrac{u^1_{n-j+i}\cdots u^{i-1}_{n-j+i}}{u^1_{n-j+i+1}\cdots u^i_{n-j+i+1}} & (i<j), \\ \dfrac{\lambda_i}{u^1_n\cdots u^{i-1}_n v^1\cdots v^{i-1}_m} & (i=j), \\ \dfrac{v^1_{m-i+j}\cdots v^{j-1}_{m-i+j}}{v^1_{m-i+j+1}\cdots u^j_{m-i+j+1}} & (i>j). \end{cases} \tag{3.66}$$

Hence we have

Theorem 3.8. *Under the assumption of Theorem 3.5, let* $U = \left(u^i_j\right)_{i\le j}$, $V = \left(v^i_j\right)_{i\le j}$ *be the tropical tableaux defined by the tropical row insertions*

$$H_U = H(\boldsymbol{x}^m)\cdots H(\boldsymbol{x}^2)H(\boldsymbol{x}^1), \quad H_V = H(\boldsymbol{x}_n)\cdots H(\boldsymbol{x}_2)H(\boldsymbol{x}_1), \tag{3.67}$$

respectively. Then u^i_j *and* v^i_j *are expressed as* (3.61) *and* (3.62), *respectively, in terms of* σ^i_j *defined in Theorem 3.5. Conversely, the matrix* $X = \left(x^i_j\right)_{i,j}$ *is recovered from the tropical tableaux* U *and* V *by the formula* (3.47) *with* y^i_j *defined by* (3.66).

Passing to the combinatorial variables, we obtain the explicit inversion formula for the RSK* correspondence.

Theorem 3.9. *Let* $X = \left(x^i_j\right)_{i,j}$ *be an* $m\times n$ *matrix of nonnegative integers. Consider the two column strict tableaux* U *and* V *obtained by the row insertion*

$$\begin{aligned} U &= (\cdots(w_m \leftarrow w_{m-1}) \leftarrow \cdots \leftarrow w_1), \quad w_i = 1^{x^i_1}2^{x^i_2}\cdots n^{x^i_n} \\ V &= (\cdots(w'_n \leftarrow w'_{n-1}) \leftarrow \cdots \leftarrow w'_1), \quad w'_j = 1^{x^1_j}2^{x^2_j}\cdots m^{x^m_j}. \end{aligned} \tag{3.68}$$

Denote by u^i_j (resp. v^i_j) be the number of j's in the i-th row of U (resp. V). Then u^i_j and v^i_j are expressed as

$$u^i_i = \sigma^i_n - \sigma^{i-1}_{n-1},$$
$$u^i_j = \sigma^i_{n-j+i} - \sigma^{i-1}_{n-j+i-1} - \sigma^i_{n-j+i+1} + \sigma^{i-1}_{n-j+i} \quad (i < j), \tag{3.69}$$
$$v^i_i = \sigma^m_i - \sigma^{m-1}_{i-1},$$
$$v^i_j = \sigma^{m-j+i}_i - \sigma^{m-j+i-1}_{i-1} - \sigma^{m-j+i-1}_i + \sigma^{m-j+i}_{i-1} \quad (i < j) \tag{3.70}$$

in terms of σ^i_j defined in Theorem 3.6. Let $\lambda = (\lambda_1, \dots, \lambda_l)$, $l = \min\{m, n\}$, be the common shape of U and V, and set

$$y^i_j = \begin{cases} \sum_{k=1}^{i-1} u^k_{n-j+i} - \sum_{k=1}^{i} u^k_{n-j+i+1} & (i < j), \\ \lambda_i - \sum_{k=1}^{i-1} u^k_n - \sum_{k=1}^{i-1} v^k_m & (i = j), \\ \sum_{k=1}^{j-1} v^k_{m-i+j} - \sum_{k=1}^{j} v^k_{m-i+j+1} & (i > j), \end{cases} \tag{3.71}$$

for each $i = 1, \dots, m$ and $j = 1, \dots, n$. Then the matrix X is recovered from U and V by the formulas

$$x^i_j = \tau^i_j - \tau^{i-1}_j - \tau^i_{j-1} + \tau^{i-1}_{j-1}, \qquad \tau^i_j = \max_{(\gamma_1, \dots, \gamma_r)} (y_{\gamma_1} + \cdots + y_{\gamma_r}), \tag{3.72}$$

where $r = \min\{i, j\}$, and the maximum is taken over all r-tuples of nonintersecting paths $\gamma_k : (m - i + k, 1) \to (1, n - j + k)$ $(k = 1, \dots, r)$ in the $m \times n$ rectangle; the weight of a path γ is the sum of all y^a_b's corresponding to the vertices of γ.

An explicit inversion formula for the usual RSK correspondence is obtained by combining Theorem 3.9 and the Schützenberger involution. The inversion formula discussed above has an obvious theoretical meaning, but is somewhat indirect. We will discuss in the following subsection a different type of inversion formulas for the four variations of RSK correspondences which recovers the transportation matrix directly from the corresponding tableaux.

3.4. Inverse tropical RSK via the Gauss decomposition

Keeping the notations X, Φ, Ψ as before, we now consider the "Gauss decomposition" of the $m \times n$ matrix Ψ $(m \le n)$:

$$\Psi = \Psi_- \, \Psi_0 \, \Psi_+, \tag{3.73}$$

where Ψ_+, Ψ_0 and Ψ_- are a $m \times n$ upper unitriangular matrix, a $m \times m$ diagonal matrix, and a $m \times m$ lower unitriangular matrix, respectively:

(3.74)
$$(\Psi_+)^i_j = \delta_{i,j} \quad (i \geq j), \quad (\Psi_0)^i_j = 0 \quad (i \neq j), \quad (\Psi_-)^i_j = \delta_{i,j} \quad (i \leq j).$$

Nontrivial entries of these matrices are given explicitly as follows:

$$\begin{aligned}
(\Psi_+)^i_j &= \frac{\det \Psi^{1,\dots,i}_{1,\dots,i-1,j}}{\det \Psi^{1,\dots,i}_{1,\dots,i}} = \frac{\det \Phi^{m-i+1,\dots,m}_{n-j+1,n-i+2,\dots,n}}{\det \Phi^{m-i+1,\dots,m}_{n-i+1,\dots,n}} && (i \leq j), \\
(\Psi_0)^i_i &= \frac{\det \Psi^{1,\dots,i}_{1,\dots,i}}{\det \Psi^{1,\dots,i-1}_{1,\dots,i-1}} = \frac{\det \Phi^{m-i+1,\dots,m}_{n-i+1,\dots,n}}{\det \Phi^{m-i+2,\dots,m}_{n-i+2,\dots,n}} && (i = j), \qquad (3.75) \\
(\Psi_-)^i_j &= \frac{\det \Psi^{1,\dots,j-1,i}_{1,\dots,j}}{\det \Psi^{1,\dots,j}_{1,\dots,j}} = \frac{\det \Phi^{m-i+1,m-j+2,\dots,m}_{n-j+1,\dots,n}}{\det \Phi^{m-j+1,\dots,m}_{n-j+1,\dots,n}} && (i \geq j).
\end{aligned}$$

Comparing the path representations of Φ and H_U, $H_{U^{\mathrm{s}}}$, H_V, $H_{V^{\mathrm{s}}}$, we have

$$\det \Phi^{m-r+1,\dots,m}_{l_1,\dots,l_r} = \det(H_U)^{1,\dots,r}_{l_1,\dots,l_r} = \det(H_{U^{\mathrm{s}}})^{l_1^*,\dots,l_r^*}_{n-r+1,\dots,n}, \qquad (3.76)$$

where $1 \leq l_1 < \ldots < l_r \leq n$ and $l_i^* = n - l_i + 1$, and

$$\det \Phi^{k_1,\dots,k_r}_{n-r+1,\dots,n} = \det(H_V)^{1,\dots,r}_{k_1,\dots,k_r} = \det(H_{V^{\mathrm{s}}})^{k_1^*,\dots,k_r^*}_{m-r+1,\dots,m}, \qquad (3.77)$$

for $1 \leq k_1 < \ldots < k_r \leq m$ with $k_i^* = m - k_i + 1$. From these formulas, we see for instance,

$$(\Psi_0)^i_i = u^i_i \cdots u^i_n = \widetilde{u}^i_i \cdots \widetilde{u}^i_n = v^i_i \cdots v^i_m = \widetilde{v}^i_i \cdots \widetilde{v}^i_m \qquad (3.78)$$

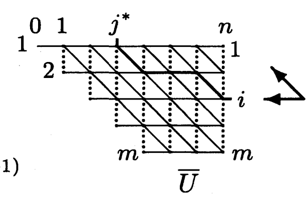
for $i = 1, \dots, m$. We set $\lambda_i = (\Psi_0)^i_i$, so that the vector $\lambda = (\lambda_1, \dots, \lambda_m)$ represents the common shape of U, U^{s}, V and V^{s}. By using formulas (3.76), (3.77), we can represent Ψ_+, Ψ_0, Ψ_- in terms of the tropical tableaux.

Proposition 3.10. (0) *For* $i = 1, \dots, m$,

$$(\Psi_0)^i_i = \lambda_i = u^i_i \cdots u^i_n = \widetilde{u}^i_i \cdots \widetilde{u}^i_n = v^i_i \cdots v^i_m = \widetilde{v}^i_i \cdots \widetilde{v}^i_m. \qquad (3.79)$$

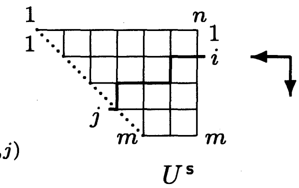
(1) *For* $i \leq j$, $(\Psi_+)^i_j$ *is expressed in terms of* U *as the sum*

$$(\Psi_+)^i_j = \sum_{\gamma:(i,n)\to(1,n-j+1)} \qquad (3.80)$$



of weights over all path $\gamma : (i, n) \to (1, n - j + 1)$*, with weight* $\overline{u}^a_b = \frac{1}{u^a_b}$ *assigned to the horizontal edge connecting* $(a, b-1)$ *and* (a, b)*, for each* $a \le b$*. In terms of* U^{s}*,* $(\Psi_+)^i_j$ *is expressed as*

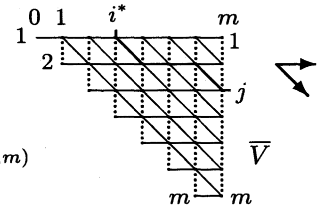
$$(\Psi_+)^i_j = \lambda_i^{-1} \sum_{\gamma:(i,n)\to(\min\{j,m\},j)}$$



(3.81)

(2) *For* $i \ge j$*,* $(\Psi_-)^i_j$ *is expressed in terms of* V *as the sum*

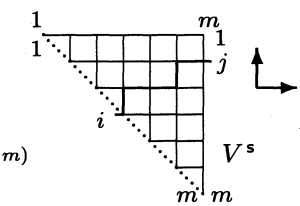
$$(\Psi_-)^i_j = \sum_{\gamma:(1,m-i+1)\to(j,m)}$$



(3.82)

of weights over all path $\gamma : (m - i + 1, 1) \to (j, m)$*, with weight* $\overline{v}^a_b = \frac{1}{v^a_b}$ *assigned to the horizontal edge connecting* $(a, b-1)$ *and* (a, b)*, for each* $a \le b$*. In terms of* V^{s}*,* $(\Psi_-)^i_j$ *is expressed as*

$$(\Psi_-)^i_j = \lambda_j^{-1} \sum_{\gamma:(i,i)\to(j,m)}$$



(3.83)

This proposition implies that the matrix $\Psi = \Psi_- \Psi_0 \Psi_+$, as well as $\Phi = \Psi^{\vee}$, is completely recovered from each of the four pairs of tropical tableaux

$$(U, V), \quad (U, V^{\mathrm{s}}), \quad (U^{\mathrm{s}}, V), \quad (U^{\mathrm{s}}, V^{\mathrm{s}}). \tag{3.84}$$

By combining the graphical representations in Proposition 3.10, we can construct a path representation of Ψ, associated with each pair in (3.84). According to

$$\Psi^i_j = \sum_{k=1}^{m} (\Psi_-)^i_k \, (\Psi_0)^k_k \, (\Psi_+)^k_j, \tag{3.85}$$

we glue the diagrams of (3.80) or (3.81) for Ψ_+ and (3.82) or (3.83) for Ψ_-. We show in Figure 1 the diagrams

$$\Gamma = \Gamma_{U,V}, \quad \Gamma_{U,V^{\mathrm{s}}}, \quad \Gamma_{U^{\mathrm{s}},V}, \quad \Gamma_{U^{\mathrm{s}},V^{\mathrm{s}}} \tag{3.86}$$

obtained in this way. In diagram Γ, the orientation of the edges are indicated by arrows. We assign the weights, associated with the pair of tropical tableaux, to the thick edges and the vertices marked by $\bullet$. For each path γ in Γ, we define the weight $\mathrm{wt}(\gamma)$ to be the product of weights attached to all the edges and the vertices. In Figure 1, $a_1, \ldots, a_m$ and $b_1, \ldots, b_n$ indicate the entrances and the exits for the path representation of Ψ, respectively. Namely, for each (i,j), ψ^i_j is expressed as the sum

$$\psi^i_j = \sum_{\gamma: a_i \to b_j} \mathrm{wt}(\gamma) \tag{3.87}$$

of weights defined as above, over all paths $\gamma : a_i \to b_j$ in Γ. Recall that the matrix $X = \left(x^i_j\right)$ is determined from Φ through the minor determinants $\tau^i_j = \tau^i_j(\Phi)$ of $\Phi = \Psi^\vee$. Hence each x^i_j is determined as

$$x^i_j = \frac{\tau^i_j\, \tau^{i-1}_{j-1}}{\tau^{i-1}_j\, \tau^i_{j-1}}, \qquad \tau^i_j = \sum_{(\gamma_1,\ldots,\gamma_r)} \mathrm{wt}(\gamma_1)\cdots\mathrm{wt}(\gamma_r), \tag{3.88}$$

where the summation is taken over all r-tuples ($r = \min\{i,j\}$) of nonintersecting paths

$$\gamma_k : a_{m-i+k} \to b_{n-j+k} \qquad (k = 1, \ldots, r) \tag{3.89}$$

in Γ. For each pair in (3.84), we have thus obtained an explicit inversion formula of the corresponding tropical RSK correspondence in terms of nonintersecting paths. The corresponding combinatorial formula for the inverse RSK correspondence is obtained simply by the standard procedure:

(3.90)
$$x^i_j = \tau^i_j - \tau^{i-1}_j - \tau^i_{j-1} + \tau^{i-1}_{j-1}, \qquad \tau^i_j = \max_{(\gamma_1,\ldots,\gamma_r)} (\mathrm{wt}(\gamma_1) + \cdots + \mathrm{wt}(\gamma_r)),$$

where the weight of a path is defined as the sum of weights attached to the edges and the vertices; read the weights $\bar{u}^i_j$ and $\bar{v}^i_j$ in Γ as $-u^i_j$ and $-v^i_j$ in the combinatorial setting.

In the case of the pair (U,V), we can give a rectangular diagram as well, by deforming the diagram $\Gamma_{U,V}$. Note first that the diagram $\Gamma_{U,V}$

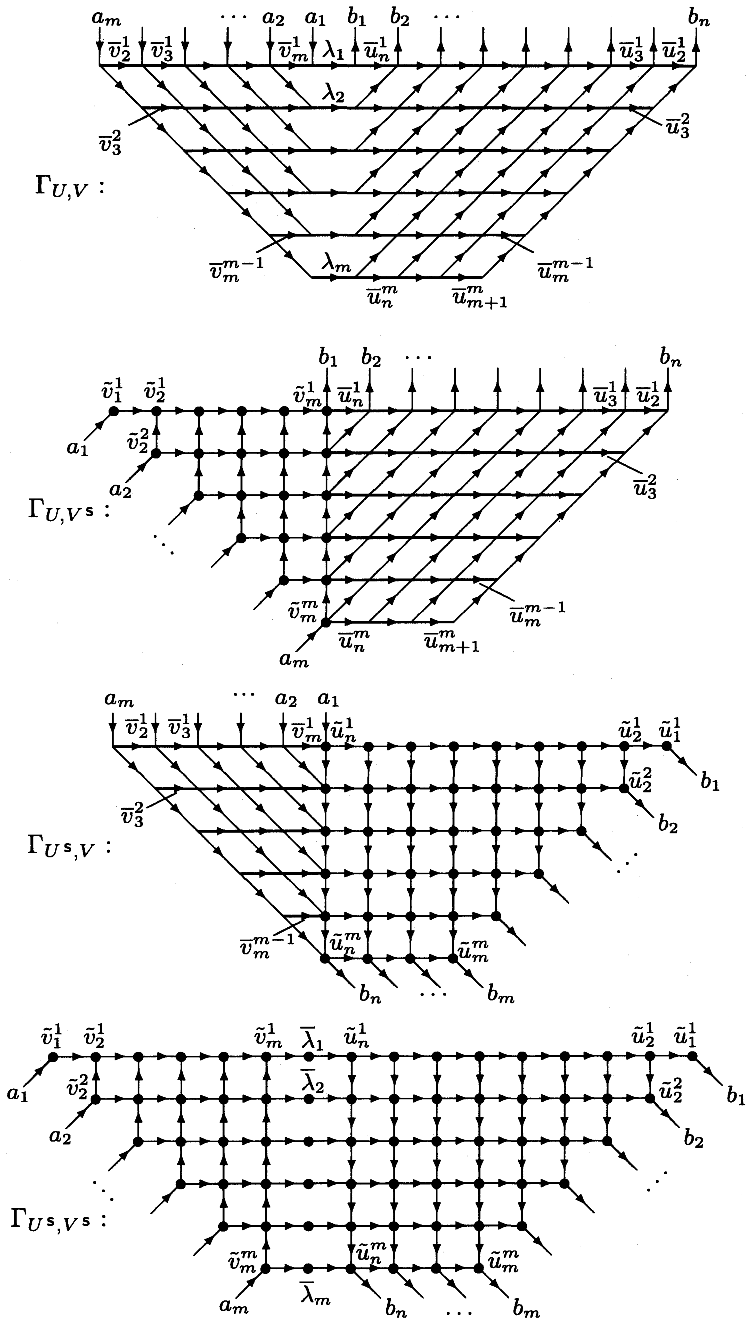
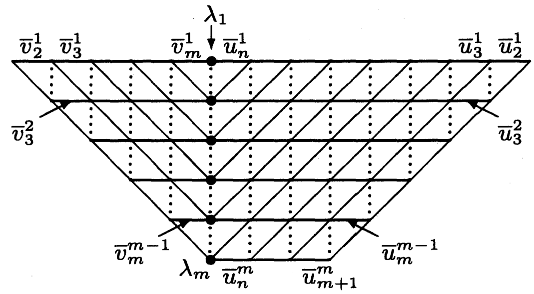


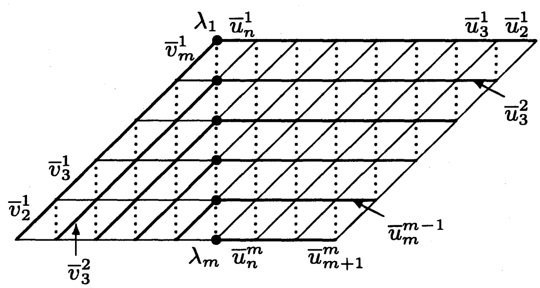
Fig. 1. Path representations of Ψ

is equivalent to the following.



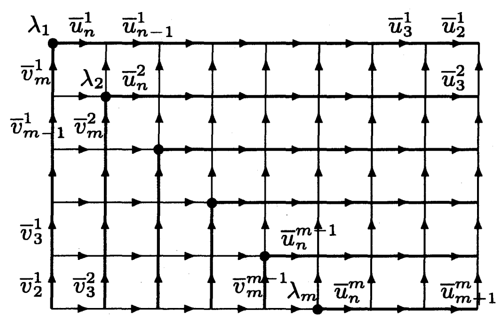
(3.91)

We deform this diagram to



(3.92)

and finally to the $m \times n$ rectangle:



(3.93)

By this rectangle, ψ_j^i is expressed as the sum

$$\psi_j^i = \sum_{\gamma:(i,1)\to\gamma(1,j)} \mathrm{wt}(\gamma) \tag{3.94}$$

of weights defined as above, over all paths $\gamma : (i,1) \to (1,j)$. (This representation is similar to that by $Y = \left(y_j^i\right)_{i,j}$, although the weights are defined in a different way.) Hence we have

$$x_j^i = \frac{\tau_j^i\,\tau_{j-1}^{i-1}}{\tau_j^{i-1}\tau_{j-1}^i},\quad \tau_j^i = \sum_{(\gamma_1,\dots,\gamma_r)} \mathrm{wt}(\gamma_1)\cdots\mathrm{wt}(\gamma_r), \tag{3.95}$$

where $r = \min\{i,j\}$ and the summation is taken over all r-tuples of nonintersecting paths $\gamma_k : (m-i+k,1) \to (1,n-j+k)$ $(k=1,\dots,r)$ in the $m \times n$ rectangle. This inversion formula is essentially equivalent to the inverse RSK* correspondence discussed in the previous subsection.

Remark 3.11. As we have seen above, the RSK correspondence can be thought of as the Gauss (or LR) decomposition of *ultra-discretized* matrices with respect to the product defined by

$$(XY)_j^i = \max_k\,(X_k^i + Y_j^k). \tag{3.96}$$

§4. Birational Weyl group actions

In this section, we introduce a subtraction-free birational affine Weyl group action on the space of tropical transportation matrices. It induces an action of the symmetric group on the space of tropical tableaux through the tropical RSK correspondence. In this section, we work with the generic $m \times n$ matrix $X = \left(x_j^i\right)_{i,j}$, regarding x_j^i as indeterminates.

4.1. Affine Weyl group action on the matrix space

In what follows, we consider the following two (extended) affine Weyl groups $\widetilde{W}^m$ and $\widetilde{W}_n$ of type $A_{m-1}^{(1)}$ and $A_{n-1}^{(1)}$, respectively. We denote by

$$\widetilde{W}^m = \langle r_0, r_1, \dots, r_{m-1}, \omega\rangle \tag{4.1}$$

the group generated by the *simple reflections* $r_0, r_1, \dots, r_{m-1}$ and the *diagram rotation* ω subject to the fundamental relations

$$\begin{aligned} &r_i^2 = 1, \\ &r_i r_j = r_j r_i && (j \not\equiv i, i\pm 1 \mod m), \\ &r_i r_j r_i = r_j r_i r_j && (j \equiv i \pm 1 \mod m), \\ &\omega r_i = r_{i+1}\omega, \end{aligned} \tag{4.2}$$

where we understand the indices for r_i as elements of $\mathbb{Z}/m\mathbb{Z}$. Notice that we have not imposed the relation $\omega^m = 1$. This version of extended affine Weyl group is isomorphic to the semidirect product of the lattice $\mathbb{Z}^m$ of rank m (*not* of rank $m-1$) and the symmetric group $\boldsymbol{S}_m$ acting on it; the subgroup $\langle r_1, \dots, r_{m-1}\rangle$ of $\widetilde{W}^m$ is identified with $\boldsymbol{S}_m$ by mapping each r_i to the adjacent transposition $\sigma_i = (i, i+1)$ $(i = 1, \dots, m-1)$. We define $\widetilde{W}_n = \langle s_0, s_1, \dots, s_{n-1}, \pi\rangle$ similarly to be the group generated by simple reflections $s_0, s_1, \dots, s_{n-1}$ and the diagram rotation π:

$$
\begin{aligned}
&s_i^2 = 1, \qquad s_i s_j = s_j s_i \quad (j \not\equiv i, i\pm 1 \mod n),\\
&s_i s_j s_i = s_j s_i s_j \quad (j \equiv i \pm 1 \mod n), \qquad \pi s_i = s_{i+1}\pi.
\end{aligned}
\tag{4.3}
$$

The subgroup $\langle s_1, \dots, s_{n-1}\rangle$ of $\widetilde{W}_n$ is identified with the symmetric group $\boldsymbol{S}_n$.

We now propose to realize these two affine Weyl groups as a group of automorphisms of the field of rational functions $\mathbb{K}(x)$ in mn variables $x = (x^i_j)_{i,j}$. With two extra parameters p, q, we take the field of rational functions $\mathbb{K} = \mathbb{Q}(p, q)$ in (p, q) as the ground field. In our realization, the groups $\widetilde{W}^m$ and $\widetilde{W}_n$ concern the *nontrivial* permutation of rows and columns of the matrix $X = \left(x^i_j\right)_{i,j}$, respectively. We first extend the indexing set $\{1, \dots, m\} \times \{1, \dots, n\}$ for the matrix $X = \left(x^i_j\right)_{i,j}$ to $\mathbb{Z} \times \mathbb{Z}$ by imposing the periodicity condition

$$
x^{i+m}_j = q^{-1} x^i_j, \quad x^i_{j+n} = p^{-1} x^i_j \qquad (i, j \in \mathbb{Z}).
\tag{4.4}
$$

We define the automorphism r_k $(k \in \mathbb{Z}/m\mathbb{Z})$ and ω of $\mathbb{K}(x)$ by

$$
\begin{aligned}
&r_k(x^i_j) = p\, x^{i+1}_j \frac{P^i_j}{P^i_{j-1}}, \quad r_k(x^{i+1}_j) = p^{-1} x^i_j \frac{P^i_{j-1}}{P^i_j} \quad (i \equiv k \mod m),\\
&r_k(x^i_j) = x^i_j \quad (i \not\equiv k, k+1 \mod m), \quad \omega(x^i_j) = x^{i+1}_j
\end{aligned}
\tag{4.5}
$$

for $i, j \in \mathbb{Z}$, where P^i_j is the sum

$$
P^i_j = \sum_{k=1}^{n} x^{i+1}_{j+1} x^{i+1}_{j+2} \cdots x^{i+1}_{j+k} x^i_{j+k} x^i_{j+k+1} \cdots x^i_{j+n}
\tag{4.6}
$$

over all paths $\gamma : (i+1, j+1) \to (i, j+n)$ in the lattice $\mathbb{Z} \times \mathbb{Z}$. We define s_l $(l \in \mathbb{Z}/n\mathbb{Z})$ and π by interchanging the roles of rows and columns, and of p and q:

$$
\begin{aligned}
&s_l(x^i_j) = q\, x^i_{j+1} \frac{Q^i_j}{Q^{i-1}_j}, \quad s_l(x^i_{j+1}) = q^{-1} x^i_j \frac{Q^{i-1}_j}{Q^i_j} \quad (j \equiv l \mod n),\\
&s_l(x^i_j) = x^i_j \quad (j \not\equiv l, l+1 \mod n), \quad \pi(x^i_j) = x^i_{j+1}
\end{aligned}
\tag{4.7}
$$

for $i, j \in \mathbb{Z}$, where

$$(4.8) \qquad Q_j^i = \sum_{k=1}^{m} x_{j+1}^{i+1} x_{j+1}^{i+2} \cdots x_{j+1}^{i+k} x_j^{i+k} x_j^{i+k+1} \cdots x_j^{i+m}$$

summed over all paths $\gamma : (i+1, j+1) \to (i+m, j)$. It is directly seen that these definitions are consistent with the periodicity conditions on x_j^i. Also, it is clear that r_k and s_l have rotational symmetry

$$(4.9) \qquad \begin{aligned} &\omega\, r_k = r_{k+1}\omega, \qquad &&\pi\, r_k = r_k \pi \qquad &&(k \in \mathbb{Z}/m\mathbb{Z}),\\ &\omega\, s_l = s_l \omega, \qquad &&\pi\, s_l = s_{l+1}\pi \qquad &&(l \in \mathbb{Z}/n\mathbb{Z}), \end{aligned}$$

respectively.

Remark 4.1. The polynomials P_j^i and Q_j^i above are characterized by the recurrence relations

$$(4.10) \qquad \begin{aligned} x_j^{i+1} P_j^i - P_{j-1}^i x_{j+n}^i &= x_j^{i+1}(x_{j+1}^{i+1} \cdots x_{j+n}^{i+1} - x_j^i \cdots x_{j+n-1}^i) x_{j+n}^{i+1},\\ x_{j+1}^i Q_j^i - Q_j^{i-1} x_j^{i+m} &= x_{j+1}^i (x_{j+1}^{i+1} \cdots x_{j+1}^{i+m} - x_j^i \cdots x_j^{i+m-1}) x_j^{i+m}, \end{aligned}$$

and the periodicity conditions $P_{j+n}^i = p^{-n-1} P_j^i$, $Q_j^{i+m} = q^{-m-1} Q_j^i$.

Theorem 4.2. *The automorphisms r_k ($k \in \mathbb{Z}/m\mathbb{Z}$), ω, s_l ($l \in \mathbb{Z}/n\mathbb{Z}$), π of $\mathbb{K}(x)$ defined as above give a realization of the direct product $\widetilde{W}^m \times \widetilde{W}_n$ of two extended affine Weyl groups. In particular, the actions of $\widetilde{W}^m = \langle r_0, \ldots, r_{m-1}, \omega\rangle$ and $\widetilde{W}_n = \langle s_0, \ldots, s_{n-1}, \pi\rangle$ commute with each other.*

In the next two subsections, we give a proof of this theorem by using two characterizations of birational actions of r_k and s_l.

Remark 4.3. The realization of $\widetilde{W}^m \times \widetilde{W}_n$ mentioned above is the same as the one we gave in [11] ($p = q = 1$), and [12]; the variables x_j^i above correspond to x_{ij}^{-1} in [12]. When $p = q = 1$, it coincides with the birational realization of $\widetilde{W}^m \times \widetilde{W}_n$ constructed in [14], Theorem 4.12.

4.2. First characterization

By introducing the spectral parameter z, for an n-vector $\boldsymbol{x} = (x_1, \ldots, x_n)$ with $x_i \neq 0$ given, we introduce the following two matrices:

$$(4.11) \qquad E(\boldsymbol{x}; z) = \mathrm{diag}(\boldsymbol{x}) + \Lambda(z), \quad H(\boldsymbol{x}; z) = (\mathrm{diag}(\overline{\boldsymbol{x}}) - \Lambda(z))^{-1},$$

where

$$\Lambda(z) = \sum_{k=1}^{n-1} E_{k,k+1} + zE_{n,1}. \tag{4.12}$$

Note that the definition of $H(\boldsymbol{x}; z)$ makes sense since

$$\det(\mathrm{diag}(\overline{\boldsymbol{x}}) - \Lambda(z)) = \overline{x}_1 \cdots \overline{x}_n - z. \tag{4.13}$$

When $z = 0$, these matrices reduce to $E(\boldsymbol{x})$ and $H(\boldsymbol{x})$ used in previous sections. Note also that $H(\boldsymbol{x}; z) = DE(\overline{\boldsymbol{x}}; z)^{-1}D^{-1}$, $D = \mathrm{diag}((-1)^{i-1})_{i=1}^{n}$. We remark that the entries of the matrix $H(\boldsymbol{x}; z)$ are expressed explicitly as

$$H(\boldsymbol{x}; z)^i_j = \begin{cases} \dfrac{x_i x_{i+1} \cdots x_j}{1 - x_1 \cdots x_n z} & (i \le j), \\[2ex] \dfrac{x_1 \cdots x_j x_i \cdots x_n z}{1 - x_1 \cdots x_n z} & (i > j). \end{cases} \tag{4.14}$$

For two n-vectors $\boldsymbol{x} = (x_1, \dots, x_n)$, $\boldsymbol{y} = (y_1, \dots, y_n)$ of indeterminates given, we consider the following matrix equation for unknown vectors $\boldsymbol{u} = (u_1, \dots, u_n)$, $\boldsymbol{v} = (v_1, \dots, v_n)$ such that $u_j \ne 0$, $v_j \ne 0$:

$$H(\boldsymbol{y}; z)H(\boldsymbol{x}; pz) = H(\boldsymbol{v}; z)H(\boldsymbol{u}; pz), \tag{4.15}$$

or equivalently,

$$E(\overline{\boldsymbol{x}}; pz)E(\overline{\boldsymbol{y}}; z) = E(\overline{\boldsymbol{u}}; pz)E(\overline{\boldsymbol{v}}; z). \tag{4.16}$$

As before we extend the indexing set for $x_j, y_j, \dots$ to $\mathbb{Z}$ by setting $x_{j+n} = p^{-1}x_j$, $y_{j+n} = p^{-1}y_j, \dots$. Then the matrix equation (4.16) is equivalent to the system of algebraic equations of *discrete Toda type*

$$x_j y_j = u_j v_j, \quad \frac{1}{x_j} + \frac{1}{y_{j+1}} = \frac{1}{u_j} + \frac{1}{v_{j+1}} \quad (j \in \mathbb{Z}). \tag{4.17}$$

(For the discrete Toda equation, see Remark 2.3.) The next lemma is fundamental in the following argument.

Lemma 4.4. *The matrix equation* (4.15) *has the following two solutions:*

(4.18)

$$\begin{array}{llll} (1) & u_j = x_j, & v_j = y_j & (j = 1, \dots, n), \\[1ex] (2) & u_j = p\, y_j \dfrac{P_j}{P_{j-1}}, & v_j = p^{-1} x_j \dfrac{P_{j-1}}{P_j} & (j = 1, \dots, n), \end{array}$$

where

$$P_j = \sum_{k=1}^{n} y_{j+1} \cdots y_{j+k} x_{j+k} \cdots x_{j+n} \quad (j = 0, 1, \ldots, n). \tag{4.19}$$

Proof. If $v_{j+1} = y_{j+1}$ for some $j \in \mathbb{Z}$, form (4.17) it follows that $u_j = x_j$, and $v_j = y_j$. Hence we have $u_j = x_j$, $v_j = y_j$ for all $j \in \mathbb{Z}$. Assuming that $v_j \neq y_j$ for any $j \in \mathbb{Z}$, we introduce the variable h_j ($j \in \mathbb{Z}$) such that

$$\frac{1}{v_{j+1}} = \frac{1}{y_{j+1}} + \frac{1}{h_j} \qquad (j \in \mathbb{Z}), \tag{4.20}$$

so that $h_{j+n} = p^{-1} h_j$. Then by eliminating u_j in (4.17), we obtain the recurrence relations

$$\frac{h_j}{x_j} = 1 + \frac{h_{j-1}}{y_j}, \quad \text{i.e.,} \quad h_j = x_j + \frac{x_j}{y_j} h_{j-1} \qquad (j \in \mathbb{Z}) \tag{4.21}$$

for h_j. Hence we have

(4.22)
$$\begin{aligned} h_{j+n} &= x_{j+n} + \frac{x_{j+n-1} x_{j+n}}{y_{j+n}} + \cdots + \frac{x_{j+1} \cdots x_{j+n}}{y_{j+2} \cdots y_{j+n}} + \frac{x_{j+1} \cdots x_{j+n}}{y_{j+1} \cdots y_{j+n}} h_j \\ &= \frac{P_j + x_{j+1} \cdots x_{j+n} h_j}{y_{j+1} \cdots y_{j+n}}. \end{aligned}$$

Since $h_{j+n} = p^{-1} h_j$, this equation determines h_j as

$$h_j = \frac{P_j}{p^{-1} y_{j+1} \cdots y_{j+n} - x_{j+1} \cdots x_{j+n}}. \tag{4.23}$$

In fact, these h_j satisfy the recurrence relations above, since

$$y_j P_j - P_{j-1} x_{j+n} = y_j y_{j+1} \cdots y_{j+n} x_{j+n} - y_j x_j x_{j+1} \cdots x_{j+n}, \tag{4.24}$$

Hence we have

$$\frac{1}{v_j} = \frac{1}{y_j} + \frac{1}{h_{j-1}} = \frac{h_j}{x_j h_{j-1}} = \frac{p P_j}{x_j P_{j-1}}, \tag{4.25}$$

and

$$\frac{1}{u_j} = \frac{1}{x_j} - \frac{1}{h_j} = \frac{h_{j-1}}{y_j h_j} = \frac{P_{j-1}}{y_j p P_j}, \tag{4.26}$$

which gives the solution (2). □

We remark that the two solutions above are characterized by the conditions

$$(4.27)\qquad \begin{array}{lll} (1) & u_1\cdots u_n = x_1\cdots x_n, & v_1\cdots v_n = y_1\cdots y_n, \\ (2) & u_1\cdots u_n = p^{-1}y_1\cdots y_n, & v_1\cdots v_n = p x_1\cdots x_n, \end{array}$$

respectively. Note here that $P_n = p^{-n-1}P_0$.

Returning to the setting of the previous subsection, we consider the matrix $X = \left(x^i_j\right)_{i,j}$. We denote the row vectors and the column vectors of $X = \left(x^i_j\right)_{i,j}$ by $\boldsymbol{x}^i = (x^i_1, \dots, x^i_n)$ and $\boldsymbol{x}_j = (x^1_j, \dots, x^m_j)$, respectively. Then Lemma 4.4 implies

$$(4.28)\qquad H(\boldsymbol{x}^{k+1}; z)H(\boldsymbol{x}^k; pz) = H(r_k(\boldsymbol{x}^{k+1}); z)H(r_k(\boldsymbol{x}^k); pz),$$

where we have used the notation $r_k(\boldsymbol{x}) = (r_k(x_1), \dots, r_k(x_n))$ for $\boldsymbol{x} = (x_1, \dots, x_n)$. Since $r_k(\boldsymbol{x}^i) = \boldsymbol{x}^i$ for $i \not\equiv k \mod m$, we have

$$(4.29)\qquad \begin{aligned} &H(\boldsymbol{x}^m; z)H(\boldsymbol{x}^{m-1}; pz)\cdots H(\boldsymbol{x}^1; p^{m-1}z) \\ &= H(r_k(\boldsymbol{x}^m); z)H(r_k(\boldsymbol{x}^{m-1}); pz)\cdots H(r_k(\boldsymbol{x}^1); p^{m-1}z) \end{aligned}$$

for $k = 1, \dots, m-1$. Namely, the product of matrices in the left-hand side is invariant under the action of r_k $(k = 1, \dots, m-1)$. Hence we see that

$$(4.30)\qquad \begin{aligned} &H(\boldsymbol{x}^m; z)H(\boldsymbol{x}^{m-1}; pz)\cdots H(\boldsymbol{x}^1; p^{m-1}z) \\ &= H(w(\boldsymbol{x}^m); z)H(w(\boldsymbol{x}^{m-1}); pz)\cdots H(w(\boldsymbol{x}^1); p^{m-1}z) \end{aligned}$$

for any composition $w = r_{k_1}r_{k_2}\cdots r_{k_l}$ with $k_1, \dots, k_l \in \{1, \dots, m-1\}$. In the following, we set

$$(4.31)\qquad H(X; z) = H(\boldsymbol{x}^m; z)H(\boldsymbol{x}^{m-1}; pz)\cdots H(\boldsymbol{x}^1; p^{m-1}z)$$

and

$$(4.32)\qquad M(X; z) = E(\overline{\boldsymbol{x}}^1; p^{m-1}z)E(\overline{\boldsymbol{x}}^2; p^{m-2}z)\cdots E(\overline{\boldsymbol{x}}^m; z)$$

so that $H(X; z) = DM(X; z)^{-1}D^{-1}$. Then we have

$$(4.33)\qquad H(X; z) = H(w(X); z), \qquad M(X; z) = M(w(X); z)$$

for any $w = r_{k_1}r_{k_2}\cdots r_{k_l}$ $(k_1, \dots, k_l \in \{1, \dots, m-1\})$, where $w(X) = \left(w(x^i_j)\right)_{i,j}$ denotes the matrix obtained from X by applying w to its entries.

Proposition 4.5. *All the entries of the matrices $H(X;z)$ and $M(X;z)$ are invariant under the action of $r_1, \dots, r_{m-1}$.*

Considering $X = \left(x_j^i\right)_{i,j}$ as given, we now investigate in general the matrix equation $H(X;z) = H(Y;z)$ for an $m \times n$ unknown matrix $Y = (y_j^i)_{i,j}$, $y_j^i \neq 0$:

$$
\begin{aligned}
&H(\boldsymbol{x}^m;z)H(\boldsymbol{x}^{m-1};pz)\cdots H(\boldsymbol{x}^1;p^{m-1}z)\\
&= H(\boldsymbol{y}^m;z)H(\boldsymbol{y}^{m-1};pz)\cdots H(\boldsymbol{y}^1;p^{m-1}z).
\end{aligned} \tag{4.34}
$$

Note that this equation is equivalent to $M(X;z) = M(Y;z)$:

$$
\begin{aligned}
&E(\overline{\boldsymbol{x}}^1;p^{m-1}z)E(\overline{\boldsymbol{x}}^2;p^{m-2}z)\cdots E(\overline{\boldsymbol{x}}^m;z)\\
&= E(\overline{\boldsymbol{y}}^1;p^{m-1}z)E(\overline{\boldsymbol{y}}^2;p^{m-2}z)\cdots E(\overline{\boldsymbol{y}}^m;z).
\end{aligned} \tag{4.35}
$$

Since $\det H(\boldsymbol{x};z) = (\overline{x}_1\cdots\overline{x}_n - z)^{-1}$, by comparing the determinants of the both sides of (4.34), we see that, for any solution of (4.34), there exists a unique permutation $\sigma \in \boldsymbol{S}_m$ such that

$$
p^{m-i}y_1^i\cdots y_n^i = p^{m-\sigma(i)}x_1^{\sigma(i)}\cdots x_n^{\sigma(i)} \qquad (i=1,\dots,m). \tag{4.36}
$$

Theorem 4.6. *For each permutation $\sigma \in \boldsymbol{S}_m$, the matrix equation* (4.34) *has a unique solution satisfying the condition* (4.36). *For any choice of expression $\sigma = \sigma_{k_1}\cdots\sigma_{k_l}$ of σ as a product of adjacent transpositions $\sigma_k = (k,k+1)$ $(k=1,\dots,m-1)$, the solution corresponding to σ is given by*

$$
y_j^i = w(x_j^i) \qquad (i=1,\dots,m;\ j=1,\dots,n), \tag{4.37}
$$

where $w = r_{k_1}\cdots r_{k_l}$.

Proof. Since $P_n^i = p^{-n-1}P_0^i$ for any i, we have

(4.38)
$$
\begin{aligned}
&r_k(x_1^k\dots x_n^k) = p^{-1}x_1^{k+1}\dots x_n^{k+1}, \quad r_k(x_1^{k+1}\dots x_n^{k+1}) = p\,x_1^k\dots x_n^k\\
&r_k(x_1^i\dots x_n^i) = x_1^i\dots x_n^i \qquad (i=1,\dots,k-1,k+2,\dots,n)
\end{aligned}
$$

for $k=1,\dots,m-1$. Hence,

$$
r_k(x_1^i\dots x_n^i) = p^{i-\sigma_k(i)}x^{\sigma_k(i)}\dots x_m^{\sigma_k(i)} \qquad (i=1,\dots,m). \tag{4.39}
$$

This implies furthermore that, for any composition $w = r_{k_1}\dots r_{k_l}$ of r_k $(k=1,\dots,m-1)$, we have

$$
w(x_1^i\dots x_n^i) = p^{i-\sigma(i)}x^{\sigma(i)}\dots x_n^{\sigma(i)} \qquad (i=1,\dots,m). \tag{4.40}
$$

where $\sigma = \sigma_{k_1} \cdots \sigma_{k_l}$. Namely, $w(\boldsymbol{x}^1), \ldots, w(\boldsymbol{x}^m)$ give a solution of (4.34) satisfying the condition (4.36). In order to complete the proof of the theorem, we show that any solution $\boldsymbol{y}^1, \ldots, \boldsymbol{y}^m$ satisfying (4.36) must coincide with this solution. In the following we denote by $\xi_i = p^{-m+i}\,\overline{x}_1^i \cdots \overline{x}_n^i$ the pole of $H(\boldsymbol{x}^i; p^{m-i}z)$. Consider the equality

$$\begin{aligned} &H(w(\boldsymbol{x}^m); z)H(w(\boldsymbol{x}^{m-1}); pz) \cdots H(w(\boldsymbol{x}^1); p^{m-1}z) \\ &= H(\boldsymbol{y}^m; z)H(\boldsymbol{y}^{m-1}; pz) \cdots H(\boldsymbol{y}^1; p^{m-1}z), \end{aligned} \tag{4.41}$$

and multiply the both sides by $H(\boldsymbol{y}^m; z)^{-1}$ from the left to get

$$\begin{aligned} &H(\boldsymbol{y}^m; z)^{-1}H(w(\boldsymbol{x}^m); z)H(w(\boldsymbol{x}^{m-1}); pz) \cdots H(w(\boldsymbol{x}^1); p^{m-1}z) \\ &= H(\boldsymbol{y}^{m-1}; pz) \cdots H(\boldsymbol{y}^1; p^{m-1}z). \end{aligned} \tag{4.42}$$

Since the right-hand side is regular at $z = \xi_{\sigma(m)} = \overline{y}_1^m \cdots \overline{y}_n^m$, the residue of the left-hand side at $z = \xi_{\sigma(m)}$ must vanish. It implies

$$\left(\mathrm{diag}(\overline{\boldsymbol{y}}^m) - \Lambda(\xi_{\sigma(m)})\right) \mathrm{Res}_{z=\xi_{\sigma(m)}}(H(w(\boldsymbol{x}^m); z)dz) = 0 \tag{4.43}$$

since the matrices $H(w(\boldsymbol{x}^i); \xi_{\sigma(m)})$ $(i = 1, \ldots, m-1)$ are all invertible. If we set $\widetilde{H}(\boldsymbol{x}; z) = (\overline{x}_1 \cdots \overline{x}_n - z)H(\boldsymbol{x}; z)$, it is equivalent to

$$(\mathrm{diag}(\overline{\boldsymbol{y}}^m) - \Lambda(\xi_{\sigma(m)}))\widetilde{H}(w(\boldsymbol{x}^m); \xi_{\sigma(m)}) = 0. \tag{4.44}$$

This equation determines $\boldsymbol{y}^m$ uniquely since $\widetilde{H}(w(\boldsymbol{x}^m); \xi_{\sigma(m)})^i_j \neq 0$ for any i, j. Since $(\mathrm{diag}(w(\overline{\boldsymbol{x}}^m)) - \Lambda(\xi_{\sigma(m)}))\widetilde{H}(w(\boldsymbol{x}^m); \xi_{\sigma(m)}) = 0$, we have $\boldsymbol{y}^m = w(\boldsymbol{x}^m)$, and also

(4.45)
$$H(w(\boldsymbol{x}^{m-1}); pz) \cdots H(w(\boldsymbol{x}^1); p^{m-1}z) = H(\boldsymbol{y}^{m-1}; pz) \cdots H(\boldsymbol{y}^1; p^{m-1}z).$$

By repeating the same procedure, we finally obtain $\boldsymbol{y}^i = w(\boldsymbol{x}^i)$ for all $i = 1, \ldots, m$, as desired. □

Corollary 4.7. *Let $i_1, \ldots, i_k$ and $j_1, \ldots, j_l$ be two sequences of elements of $\{1, \ldots, m-1\}$ such that*

$$\sigma_{i_1} \cdots \sigma_{i_k} = \sigma_{j_1} \cdots \sigma_{j_l}. \tag{4.46}$$

Then the automorphisms $r_1, \ldots, r_{m-1}$ satisfies the relation

$$r_{i_1} \cdots r_{i_k} = r_{j_1} \cdots r_{j_l}. \tag{4.47}$$

By this corollary and the rotational symmetry of r_k, we see that the automorphisms $r_0, r_1, \ldots, r_{m-1}, \omega$ satisfy the fundamental relations for the generators of $\widetilde{W}^m$. The same statement is valid for $s_0, s_1, \ldots, s_{n-1}$, π and $\widetilde{W}_n$ by the symmetry under the transposition of the matrix X.

Remark 4.8. Lemma 4.4 implies that the action of r_k ($k = 1, \ldots, m-1$) is characterized by the system of algebraic equations of discrete Toda type

$$x^i_j x^{i+1}_j = y^i_j y^{i+1}_j, \quad \frac{1}{x^i_j} + \frac{1}{x^{i+1}_{j+1}} = \frac{1}{y^i_j} + \frac{1}{y^{i+1}_{j+1}} \tag{4.48}$$

for $y^i_j = r_k(x^i_j)$ ($i, j \in \mathbb{Z}$) with periodicity condition $y^{i+m}_j = q^{-1} y^i_j$, $y^i_{j+n} = p^{-1} y^i_j$, and an extra constraint

$$y^i_1 \cdots y^i_n = p^{i-\sigma_k(i)} x^{\sigma_k(i)}_1 \cdots x^{\sigma_k(i)}_n \quad (i = 1, \ldots, m). \tag{4.49}$$

4.3. Second characterization

We give another characterization of r_k and s_l, and use it for proving the commutativity of the actions of $\langle r_0, r_1, \ldots, r_{m-1}, \omega \rangle$ and $\langle s_0, s_1, \ldots, s_{n-1}, \pi \rangle$.

We define the $n \times n$ matrices $G_l(u; z)$, depending on a parameter u, by setting

(4.50)
$$G_0(u; z) = 1 + \frac{1}{u} E_{1,n} z^{-1}, \quad G_l(u; z) = 1 + \frac{1}{u} E_{l+1,l} \quad (l = 1, \ldots, n-1).$$

For $l = 1, \ldots, n-1$, we also use the notation $G_l(u) = G_l(u; z)$ since they do *not* depend on z. Fix an index $l = 0, 1, \ldots, n-1$, and consider the system of matrix equations

$$G_l(g_{i-1}; pz) E(\overline{\boldsymbol{x}}^i; z) = E(\overline{\boldsymbol{y}}^i; z) G_l(g_i; z) \quad (i = 1, \ldots, m) \tag{4.51}$$

for unknown variables $\boldsymbol{y}^i$ ($i = 1, \ldots, m$) and g_i ($i = 0, 1, \ldots, m$).

Theorem 4.9. *Under the periodicity condition $g_m = q^{-1} g_0$, the system of algebraic equations* (4.51) *has a unique solution. It is given explicitly as*

$$\begin{aligned} g_i &= \frac{Q^i_l}{q^{-1} x^{i+1}_{l+1} \cdots x^{i+m}_{l+1} - x^{i+1}_l \cdots x^{i+m}_l} \quad (i = 0, 1, \ldots, m), \\ y^i_j &= s_l(x^i_j) \qquad (i = 1, \ldots, m;\ j = 1, \ldots, n). \end{aligned} \tag{4.52}$$

Proof. It is easily seen that the matrix equation (4.51) is equivalent to the recurrence relations

$$g_i = x_l^i + \frac{x_l^i}{x_{l+1}^i} g_{i-1} \qquad (i = 1, \ldots, m), \tag{4.53}$$

together with

$$\begin{aligned} &\frac{1}{y_l^i} = \frac{1}{x_l^i} - \frac{1}{g_i}, \quad \frac{1}{y_{l+1}^i} = \frac{1}{x_{l+1}^i} + \frac{1}{g_{i-1}}, \\ &y_j^i = x_j^i \qquad (j \not\equiv l, l+1 \mod n). \end{aligned} \tag{4.54}$$

These are the same recurrence relations as we have discussed in Lemma 4.4. As we already know, (4.53) determines g_i, and (4.54) gives rise to the expressions we have used in defining s_l. □

Note that the matrix equation (4.51) implies

$$G_l(g_0; p^m z) M(X; z) = M(Y; z) G_l(q^{-1} g_0; z). \tag{4.55}$$

Hence, by Theorem 4.9, we see that the action of s_l on $M(X; z)$ is described by

$$M(s_l(X); z) = G_l(g_0; p^m z) M(X; z) G_l(q^{-1} g_0; z)^{-1}. \tag{4.56}$$

In terms of the matrix $H(X; z)$, this formula can be written as

$$H(s_l(X); z) = G_l(q^{-1} g_0; (-1)^n z)^{-1} H(X; z) G_l(g_0; (-1)^n z). \tag{4.57}$$

We remark that the rational function g_0 can be determined only from $M(X; z)$. It is an easy exercise to show

Lemma 4.10. *Let $\mathfrak{b}$ be the space of all $n \times n$ matrix $M(z)$ with coefficients in $\mathbb{K}(x)[z]$ such that $M(0)$ is upper triangular. For a matrix*

$$M(z) = M_0 + M_1 z + \cdots + M_d z^d \in \mathfrak{b} \tag{4.58}$$

given, set

$$\begin{aligned} \varepsilon_i &= (M_0)_i^i, & (i = 1, \ldots, n), \\ \varphi_0 = (M_1)_1^n, \quad \varphi_i &= (M_0)_{i+1}^i & (i = 1, \ldots, n-1). \end{aligned} \tag{4.59}$$

Then we have

(4.60)

$$G_0(u; az) M(z) G_0(q^{-1} u; z)^{-1} \in \mathfrak{b} \iff (q^{-1} \varepsilon_n - a \varepsilon_1) u = \varphi_0,$$

and

$$(4.61) \qquad G_l(u)M(z)G_l(q^{-1}u)^{-1} \in \mathfrak{b} \iff (q^{-1}\varepsilon_l - \varepsilon_{l+1})\,u = \varphi_l,$$

for $l = 1, \ldots, n-1$. In particular, the parameter u is determined uniquely from $M(z)$ if ε_i and φ_i are generic.

This lemma implies that g_0 is expressed as a rational function of entries of $M(X;z)$. Hence, by Proposition 4.5, we see that g_0 in invariant under the action of $r_1, \ldots, r_{m-1}$.

We now prove the commutativity of the actions of $\langle r_0, r_1, \ldots, r_{m-1}\rangle$ and $\langle s_0, s_1, \ldots, s_{n-1}\rangle$. By the rotational symmetry of r_k, it suffices to prove $s_l w = w s_l$ ($l = 0, 1, \ldots, n-1$), assuming that $w \in \langle r_1, \ldots, r_{m-1}\rangle$. Applying w to (4.56), we have

$$(4.62) \quad M(ws_l(X);z) = G_l(w(g_0);p^m z)M(w(X);z)G_l(q^{-1}w(g_0);z)^{-1}.$$

By Proposition 4.5, we have $M(w(X);z) = M(X;z)$, and also, $w(g_0) = g_0$ as we remarked above. This implies that $M(ws_l(X);z) = M(s_l(X);z)$. By applying s_l again, we obtain

$$(4.63) \qquad M(s_l w s_l(X);z) = M(s_l^2(X);z) = M(X;z).$$

Note that $s_l(x_1^i \ldots x_n^i) = x_1^i \ldots x_n^i$ for any $i = 1, \ldots, m$. Hence, for $Y = s_l w s_l(X)$, we have

$$(4.64) \qquad y_1^i \cdots y_n^i = p^{i-\sigma(i)} x_1^{\sigma(i)} \ldots x_n^{\sigma(i)} \quad (i = 1, \ldots, m),$$

where $\sigma \in \boldsymbol{S}_m$ is the permutation corresponding to w. Then, by Theorem 4.6, we obtain $Y = w(X)$. This means that $s_l w s_l(X) = w(X)$, namely, $s_l w s_l = w$. This completes the proof of Theorem 4.2.

Recall that the roles of r_k, s_l are interchanged with each other by the transposition of the matrix $X = \left(x_j^i\right)_{i,j}$. Accordingly, the two characterization we have discussed so far can be applied to both r_k and s_l.

4.4. Passage to the tropical tableaux

In what follows we set $\boldsymbol{S}^m = \langle r_1, \ldots, r_{m-1}\rangle$ and $\boldsymbol{S}_n = \langle s_1, \ldots, s_{n-1}\rangle$.

Let us consider the tropical RSK* correspondence $X \mapsto (U, V)$ with the notation as in the previous section:

$$\begin{aligned} &H(\boldsymbol{x}^m)\cdots H(\boldsymbol{x}^2)H(\boldsymbol{x}^1) = H_m(\boldsymbol{u}^m)\cdots H_2(\boldsymbol{u}^2)H_1(\boldsymbol{u}^1) = H_U, \\ (4.65)\, &H(\boldsymbol{x}_n)\cdots H(\boldsymbol{x}_2)H(\boldsymbol{x}_1) = H_m(\boldsymbol{v}^m)\cdots H_2(\boldsymbol{v}^2)H_1(\boldsymbol{v}^1) = H_V. \end{aligned}$$

As before we assume that $m \leq n$. We regard now the variables u^i_j and v^i_j $(i \leq j)$ as elements of $\mathbb{K}(x)$. Note that by specializing the spectral parameter z to zero, we have

$$(4.66) \qquad H(X;0) = H(\boldsymbol{x}^m) \cdots H(\boldsymbol{x}^1) = H_U.$$

By Proposition 4.5, we already know that $H(X;z)$, hence $H(X;0)$ is invariant under the action of the symmetric group $\boldsymbol{S}^m = \langle r_1, \dots, r_{m-1} \rangle$. Since the variables u^i_j are determined uniquely from the matrix $H_U = H(X;0)$, we conclude that all u^i_j are invariant under the action of $\boldsymbol{S}^m = \langle r_1, \dots, r_{m-1} \rangle$.

We now consider the action of $\boldsymbol{S}_n = \langle s_1, \dots, s_{n-1} \rangle$. For $l = 1, \dots, n-1$, from (4.57), we have

$$(4.67) \qquad H(s_l(X);0) = G_l(q^{-1}g_0)^{-1} H(X;0) G_l(g_0),$$

hence

$$(4.68) \qquad s_l(H_U) = G_l(q^{-1}g_0)^{-1} H_U \, G_l(g_0),$$

where

$$(4.69) \qquad g_0 = \frac{\displaystyle\sum_{k=1}^{m} x^1_{l+1} \cdots x^k_{l+1} x^k_l \cdots x^m_l}{q^{-1} x^1_{l+1} \cdots x^m_{l+1} - x^1_l \cdots x^m_l}.$$

This formula is equivalent to

$$(4.70) \qquad s_l(M_U) = G_l(g_0) M_U G_l(q^{-1}g_0),$$

where

$$(4.71) \qquad M_U = D H_U^{-1} D^{-1} = E_1(\overline{\boldsymbol{u}}^1) \cdots E_m(\overline{\boldsymbol{u}}^m).$$

By applying Lemma 4.10 to (4.70), we see that g_0 is expressed as follows in terms of the u-variables:

$$(4.72) \qquad g_0 = \begin{cases} \dfrac{\displaystyle\sum_{k=1}^{l} u^1_{l+1} \cdots u^k_{l+1} u^k_l \cdots u^l_l}{q^{-1} u^1_{l+1} \cdots u^{l+1}_{l+1} - u^1_l \cdots u^l_l} & (l = 1, \dots, m-1), \\[2ex] \dfrac{\displaystyle\sum_{k=1}^{m} u^1_{l+1} \cdots u^k_{l+1} u^k_l \cdots u^m_l}{q^{-1} u^1_{l+1} \cdots u^m_{l+1} - u^1_l \cdots u^m_l} & (l = m, \dots, n-1). \end{cases}$$

Hence, formula (4.70) as well as (4.68) determines completely the action of s_l on the u-variables.

In order to describe the action of s_l on the u-variables, for each $0 \le i \le \min\{l, m\}$, we define

(4.73)
$$\begin{aligned} A_l^i &= u_l^1 \cdots u_l^i \sum_{k=i+1}^{l} u_{l+1}^{i+1} \cdots u_{l+1}^k u_l^k \cdots u_l^l \\ &\quad + q^{-1} u_{l+1}^{i+1} \cdots u_{l+1}^{l+1} \sum_{k=1}^{i} u_{l+1}^1 \cdots u_{l+1}^k u_l^k \cdots u_l^i \qquad (1 \le l \le m-1), \\ A_l^i &= u_l^1 \cdots u_l^i \sum_{k=i+1}^{m} u_{l+1}^{i+1} \cdots u_{l+1}^k u_l^k \cdots u_l^m \\ &\quad + q^{-1} u_{l+1}^{i+1} \cdots u_{l+1}^{m} \sum_{k=1}^{i} u_{l+1}^1 \cdots u_{l+1}^k u_l^k \cdots u_l^i \qquad (m \le l \le n-1). \end{aligned}$$

Theorem 4.11. *Under the tropical RSK* correspondence $X \mapsto (U, V)$, the variables u_j^i $(1 \le i \le m;\ i \le j \le n)$ are invariant with respect to the action of $\boldsymbol{S}^m = \langle r_1, \ldots, r_{m-1} \rangle$. The action of s_l $(l = 1, \ldots, n-1)$ on u_j^i is described as follows:*

$$s_l(u_l^i) = u_{l+1}^i \frac{A_l^i}{A_l^{i-1}}, \quad s_l(u_{l+1}^i) = u_l^i \frac{A_l^{i-1}}{A_l^i}, \quad s_l(u_j^i) = u_j^i \quad (j \ne l, l+1) \tag{4.74}$$

for $1 \le i \le \min\{l, m\}$ and $s_l(u_j^i) = u_j^i$ for $\min\{l, m\} + 1 \le i \le m$.

Proof. Fixing the index $l = 1, \ldots, n-1$, we consider the system of matrix equations

$$G_l(a_{i-1}) E_i(\overline{\boldsymbol{u}}^i) = E_i(\overline{\boldsymbol{t}}^i) G_l(a_i) \quad (i = 1, \ldots, m) \tag{4.75}$$

for unknown variables $\boldsymbol{t}^i = (1, \ldots, 1, t_i^i, \ldots, t_n^i)$ $(i = 1, \ldots, m)$ and a_i $(i = 0, 1, \ldots, m)$. We will construct below a solution of this system such that $a_m = q^{-1} a_0$, so that

$$G_l(a_0) M_U = M_T\, G_l(q^{-1} a_0); \tag{4.76}$$

this equation must imply $a_0 = g_0$ and $T = s_l(U)$. The system of matrix equations (4.75) gives the recurrence relations

$$(4.77)\qquad \begin{aligned} &a_i = u_l^i + \frac{u_l^i}{u_{l+1}^i} a_{i-1} \qquad (1 \le i \le l),\\ &a_{l+1} = \frac{1}{u_{l+1}^{l+1}} a_l, \quad a_i = a_{i-1} \quad (l+2 \le i \le m). \end{aligned}$$

for a_i, and also

$$(4.78)\quad \frac{1}{t_l^i} = \frac{1}{u_l^i} - \frac{1}{a_i}, \quad \frac{1}{t_{l+1}^i} = \frac{1}{u_{l+1}^i} + \frac{1}{a_{i-1}}, \quad t_j^i = u_j^i \quad (j \ne l, l+1).$$

for $1 \le i \le l$ and $t_j^i = u_j^i$ for $l+1 \le i \le m$. Under the condition $a_m = q^{-1} a_0$, the recurrence relations (4.77) for a_i are solved by

$$(4.79)\quad a_i = \frac{A_l^i}{q^{-1} u_{l+1}^1 \cdots u_{l+1}^{\min\{l+1,m\}} - u_l^1 \cdots u_l^l} \qquad (0 \le i \le \min\{l, m\}).$$

Hence we obtain the expression for $t_j^i = s_l(u_j^i)$ as (4.74). □

By eliminating a_i in (4.77), (4.78), we obtain

Proposition 4.12. *The action of s_l $(l = 1, \dots, n-1)$ on the tropical tableau $U = \left(u_j^i\right)_{i \le j}$ is characterized by the following system of algebraic equations of discrete Toda type for $t_j^i = s_l(u_j^i)$:*

$$(4.80)\qquad \begin{aligned} &t_l^i t_{l+1}^i = u_l^i u_{l+1}^i \quad (i = 1, \dots, l), \qquad t_{l+1}^{l+1} = u_{l+1}^{l+1},\\ &\frac{1}{t_l^i} + \frac{1}{t_{l+1}^{i+1}} = \frac{1}{u_l^i} + \frac{1}{u_{l+1}^{i+1}} \qquad (i = 1, \dots, l-1),\\ &\frac{1}{t_l^l} + \frac{q}{t_{l+1}^{l+1} t_{l+1}^1} = \frac{1}{u_l^l} + \frac{q}{u_{l+1}^{l+1} u_{l+1}^1} \end{aligned}$$

for $l = 1, \dots, m-1$, and

$$(4.81)\qquad \begin{aligned} &t_l^i t_{l+1}^i = u_l^i u_{l+1}^i \qquad (i = 1, \dots, m),\\ &\frac{1}{t_l^i} + \frac{1}{t_{l+1}^{i+1}} = \frac{1}{u_l^i} + \frac{1}{u_{l+1}^{i+1}} \qquad (i = 1, \dots, m-1),\\ &\frac{1}{t_l^m} + \frac{q}{t_{l+1}^1} = \frac{1}{u_l^m} + \frac{q}{u_{l+1}^1} \end{aligned}$$

for $l = m, \dots, n-1$, together with the constraint

$$(4.82)\qquad \begin{aligned} &t_l^1 \cdots t_l^{\min\{l,m\}} = u_{l+1}^1 \cdots u_{l+1}^{\min\{l+1,m\}},\\ &t_{l+1}^1 \cdots t_{l+1}^{\min\{l+1,m\}} = u_l^1 \cdots u_l^{\min\{l,m\}}. \end{aligned}$$

The action of the tropical Schützenberger involution on the tropical tableau $U = \left(u_j^i\right)_{i\le j}$ plays the role of reversing the indices of the transformations $s_1, \dots, s_{n-1}$ and interchanging q and q^{-1}. Denoting by $\mathbb{K}(u)$ the field of rational functions in the variables $u = (u_j^i)_{i,j}$, we define the involutive automorphism $\mathsf{s} : \mathbb{K}(u) \to \mathbb{K}(u)$ by using the tropical Schützenberger involution of Theorem 2.8 :

$$\mathsf{s}(u_i^i) = \frac{\sigma_i^i}{\sigma_i^{i-1}}, \quad \mathsf{s}(u_j^i) = \frac{\sigma_j^i\,\sigma_{j-1}^{i-1}}{\sigma_j^{i-1}\sigma_{j-1}^i} \quad (i<j), \tag{4.83}$$

where

$$\sigma_j^i = \sum_{(\gamma_1,\dots,\gamma_i)} u_{\gamma_1}\cdots u_{\gamma_i} \tag{4.84}$$

is the sum of weights associated with U, over all i-tuples of nonintersecting paths $\gamma_k : (1, n-i+k) \to (\min\{m, n-j+k\}, n-j+k)$ $(k = 1, \dots, i)$.

Theorem 4.13. *For each $l = 1, \dots, n-1$, let $s_l^q : \mathbb{K}(u) \to \mathbb{K}(u)$ the automorphisms defined as in Theorem 4.11. Then we have $\mathsf{s}\, s_l^q = s_{n-l}^{q^{-1}}\mathsf{s}$ for $(l = 1, \dots, n-1)$.*

Proof. The tropical tableau $\mathsf{s}(U) = \left(\mathsf{s}(u_j^i)\right)_{i\le j}$ is characterized by the condition

$$H_{\mathsf{s}(U)} = \theta(H_U) = J_n H_U^{\mathrm{t}} J_n, \tag{4.85}$$

or equivalently, by

$$M_{\mathsf{s}(U)} = \theta(M_U) = J_n M_U^{\mathrm{t}} J_n. \tag{4.86}$$

Hence, by applying s_l^q to this equality, we have

$$M_{s_l^q\,\mathsf{s}(U)} = \theta(M_{s_l^q(U)}). \tag{4.87}$$

Recall that the action of s_l $(l = 1, \dots, n-1)$ is characterized by

$$M_{s_l^q(U)} = G_l(a_0)\; M_U\; G_l(q^{-1}a_0)^{-1}. \tag{4.88}$$

Since $\theta(G_l(a)) = G_{n-l}(-a) = G_{n-l}(a)^{-1}$, we obtain

$$\begin{aligned} \theta(M_{s_l^q(U)}) &= G_{n-l}(q^{-1}a_0)\;\theta(M_U)\;G_{n-l}(a_0)^{-1} \\ &= G_{n-l}(q^{-1}a_0)\;M_{\mathsf{s}(U)}\;G_{n-l}(a_0)^{-1} \end{aligned} \tag{4.89}$$

By combining this with (4.87), we obtain

$$M_{s_l^q \mathsf{s}(U)} = G_{m-l}(q^{-1}a_0)\, M_{\mathsf{s}(U)}\, G_{n-l}(a_0)^{-1}. \tag{4.90}$$

By applying s again, we have

$$\begin{aligned} M_{\mathsf{s}\, s_l^q \mathsf{s}(U)} &= G_{m-l}(q^{-1}\mathsf{s}(a_0))\, M_{\mathsf{s}^2(U)}\, G_{n-l}(\mathsf{s}(a_0))^{-1} \\ &= G_{m-l}(q^{-1}\mathsf{s}(a_0))\, M_U\, G_{n-l}(\mathsf{s}(a_0))^{-1} \\ &= M_{s_l^{q^{-1}}(U)}. \end{aligned} \tag{4.91}$$

The last equality is a consequence of Lemma 4.10. This implies $\mathsf{s}\, s_l^q \mathsf{s}(u_j^i)$ $= s_l^{q^{-1}}(u_j^i)$ for all $i \le j$, as desired. □

4.5. Combinatorial formulas for the Weyl group action

By the standard procedure of Section 1.3, we can derive the piecewise linear action of the direct product $\widetilde{W}^m \times \widetilde{W}_n$ of affine Weyl groups on the space of transportation matrices X. Also, via the RSK* correspondence, we obtain the piecewise linear action of $\boldsymbol{S}_n = \langle s_1, \dots, s_{n-1} \rangle$ (resp. $\boldsymbol{S}^m = \langle r_1, \dots, r_{m-1} \rangle$) on the space of tableaux $U = \left(u_j^i\right)_{i \le j}$ (resp. $V = \left(v_j^i\right)_{i \le j}$).

Consider the space $\mathrm{Mat}_{m,n}(\mathbb{R})$ of real $m \times n$ matrices $X = \left(x_j^i\right)_{i,j}$, regarding $x = (x_j^i)_{i,j}$ as the canonical coordinates. For each multi-index $\alpha = (\alpha_j^i)_{i,j} \in \mathbb{N}^{mn}$ and $a, b \in \mathbb{N}$, we define the linear function $\ell_{\alpha,a,b}(x)$ on $\mathrm{Mat}_{m,n}(\mathbb{R})$ by

$$\ell_{\alpha,a,b}(x) = \sum_{i,j} \alpha_j^i x_j^i + ap + bq, \tag{4.92}$$

where p, q are parameters. Note that $\ell_{\alpha,a,b}(x) = M(x^\alpha p^a q^b)$ in the notation of Section 1.3. We denote by $\mathcal{F}_X$ the set of all piecewise linear functions $f = f(x)$ in the form

(4.93)
$$f(x) = \max\{\ell_{\alpha,a,b}(x) \mid (\alpha, a, b) \in A\} - \max\{\ell_{\beta,c,d}(x) \mid (\beta, c, d) \in B\},$$

where A, B are nonempty finite sets of triples (α, a, b) of $\alpha \in \mathbb{N}^{mn}$ and $a, b \in \mathbb{N}$. Since $\mathcal{F}_X = M(\mathbb{Q}(x, p, q)_{>0})$, $\mathcal{F}_X$ is closed under the addition, the subtraction and "max"; it is also closed under "min" since $\min\{f, g\} = f + g - \max\{f, g\}$. We say that an isomorphism $w : \mathcal{F}_X \to \mathcal{F}_X$ of $\mathbb{Z}$-modules is *combinatorial* if $w(\max\{f, g\}) = \max\{w(f), w(g)\}$ for any $f, g \in \mathcal{F}_X$.

We first extend the indexing set for x^i_j by setting

$$x^{i+m}_j = x^i_j - q, \quad x^i_{j+n} = x^i_j - p \qquad (i,j \in \mathbb{Z}). \tag{4.94}$$

We define the action of r_k $(k = 0,1,\dots,m)$, ω and s_l $(l = 0,1,\dots,n)$, π on the variables x^i_j as follows:

$$\begin{aligned}
&\left.\begin{aligned} r_k(x^i_j) &= x^{i+1}_j + P^i_j - P^i_{j-1} + p \\ r_k(x^{i+1}_j) &= x^i_j + P^i_{j-1} - P^i_j - p \end{aligned}\right. \qquad (i \equiv k \mod m),\\
&r_k(x^i_j) = x^i_j \quad (i \not\equiv k, k+1 \mod m), \quad \omega(x^i_j) = x^{i+1}_j,\\
&\left.\begin{aligned} s_l(x^i_j) &= x^i_{j+1} + Q^i_j - Q^{i-1}_j + q \\ s_l(x^i_{j+1}) &= x^i_j + Q^{i-1}_j - Q^i_j - q \end{aligned}\right. \qquad (j \equiv l \mod n),\\
&s_l(x^i_j) = x^i_j \quad (j \not\equiv l, l+1 \mod n), \quad \pi(x^i_j) = x^i_{j+1},
\end{aligned} \tag{4.95}$$

for $i,j \in \mathbb{Z}$, where

$$\begin{aligned}
P^i_j &= \max_{1\le k\le n} \Big(\sum_{a=1}^{k} x^{i+1}_{j+a} + \sum_{a=k}^{n} x^i_{j+a} \Big),\\
Q^i_j &= \max_{1\le k\le m} \Big(\sum_{a=1}^{k} x^{i+a}_{j+1} + \sum_{a=k}^{m} x^{i+a}_j \Big).
\end{aligned} \tag{4.96}$$

These formulas can also be written in terms of "min"; for instance,

$$\begin{aligned} r_k(x^i_j) &= x^{i+1}_j - R^i_j + R^i_{j-1} + p \\ r_k(x^{i+1}_j) &= x^i_j - R^i_{j-1} + R^i_j - p \end{aligned} \qquad (i \equiv k \mod m), \tag{4.97}$$

where

$$R^i_j = \min_{1\le k\le n} \Big(\sum_{a=1}^{k-1} x^i_{j+a} + \sum_{a=k+1}^{n} x^{i+1}_{j+a} \Big). \tag{4.98}$$

Theorem 4.14. *Define the mappings r_k $(k \in \mathbb{Z}/m\mathbb{Z})$, ω and s_l $(l \in \mathbb{Z}/n\mathbb{Z})$, π from the set of variables x^i_j to $\mathcal{F}_X$ as above. Then each of them extends uniquely to a combinatorial isomorphism $\mathcal{F}_X \to \mathcal{F}_X$. Furthermore, they give a realization of the direct product $\widetilde{W}^m \times \widetilde{W}_n$ of affine Weyl groups as a group of combinatorial isomorphisms of $\mathcal{F}_X$.*

Proposition 4.15. *By using the action of r_k $(k = 1,\dots,m-1)$, set $y^i_j = r_k(x^i_j)$ for all $i,j \in \mathbb{Z}$. Then we obtain a solution to the ultra-discrete equation of Toda type*

$$x^i_j + x^{i+1}_j = y^i_j + y^{i+1}_j, \quad \min\{x^i_j, x^{i+1}_{j+1}\} = \min\{y^i_j, y^{i+1}_{j+1}\} \tag{4.99}$$

with periodicity condition $y_j^{i+m} = y_j^i - q$, $y_{j+n}^i = y_j^i - p$, *satisfying the constraint*

(4.100)
$$y_1^i + \cdots + y_n^i = x_1^{\sigma_k(i)} + \cdots + x_n^{\sigma_k(i)} + (i - \sigma_k(i))p \quad (i = 1, \ldots, m).$$

Remark 4.16. Let $B = \bigcup_{l=0}^{\infty} B_l$ the crystal basis of the symmetric tensor representation $S(V) = \bigoplus_{l=0}^{\infty} S_l(V)$ of $\mathfrak{gl}(n)$ associated with the vector representation $V = \mathbb{C}^n$. Then B is identified with the set of n-vectors $\boldsymbol{x} = (x_1, \ldots, x_n)$ of nonnegative integers. The crystal basis $B^{\otimes m} = B \otimes \cdots \otimes B$ (m times) for the m-th tensor product $S(V)^{\otimes m}$ is parametrized by $\mathbb{N}^{mn}$. We identify the matrix $X = \left(x_j^i\right)_{i,j}$ with the coordinates of $B^{\otimes m} = \mathbb{N}^{mn}$, regarding $\boldsymbol{x}^i = (x_1^i, \ldots, x_n^i)$ as corresponding to the i-th component. When $p = q = 0$, the actions of r_k ($k = 1, \ldots, m-1$) and s_l ($l = 1, \ldots, n-1$) on the variables x_j^i coincide with the combinatorial R-matrix acting on the k-th and $(k+1)$-st components of $B^{\otimes m}$, and Kashiwara's Weyl group actions, respectively (see [6], [29]).

Assuming that $m \leq n$, we consider the variables u_j^i ($1 \leq i \leq m$; $i \leq j \leq n$) and v_j^i ($1 \leq i \leq j \leq m$), associated with the column strict tableaux U and V, through the RSK* correspondence $X \mapsto (U, V)$. Recall that each u_j^i (resp. v_j^i) denotes the number of j's in the i-th row of U (resp. V). Then, by the explicit piecewise linear formulas described in Theorem 3.9, u_j^i and v_j^i are regarded as elements in $\mathcal{F}_X$. We thus obtain a combinatorial action of $\widetilde{W}^m \times \widetilde{W}_n$ on the variables u_j^i and v_j^i. We describe below the action of the subgroups $\boldsymbol{S}^m = \langle r_1, \ldots, r_{m-1} \rangle$ and $\boldsymbol{S}_n = \langle s_1, \ldots, s_{n-1} \rangle$ on u_j^i; their action on v_j^i is given by an obvious modification.

In view of (4.73), we define A_l^i ($0 \leq i \leq \min\{l, m\}$) for the combinatorial version by

(4.101)
$$\begin{aligned} A_l^i = \max\Big\{ &\max_{i+1 \leq k \leq l} (u_l^1 + \cdots + u_l^i + u_{l+1}^{i+1} + \cdots + u_{l+1}^k + u_l^k \cdots + u_l^l), \\ &\max_{1 \leq k \leq i} (u_{l+1}^{i+1} + \cdots + u_{l+1}^{l+1} + u_{l+1}^1 + \cdots + u_{l+1}^k + u_l^k + \cdots + u_l^i - q) \Big\} \end{aligned}$$

for $1 \leq l \leq m-1$, and by

(4.102)
$$\begin{aligned} A_l^i = \max\Big\{ &\max_{i+1 \leq k \leq m} (u_l^1 + \cdots + u_l^i + u_{l+1}^{i+1} + \cdots + u_{l+1}^k + u_l^k \cdots + u_l^m), \\ &\max_{1 \leq k \leq i} (u_{l+1}^{i+1} + \cdots + u_{l+1}^m + u_{l+1}^1 + \cdots + u_{l+1}^k + u_l^k + \cdots + u_l^i - q) \Big\} \end{aligned}$$

for $m \le l \le n-1$. Then, from Theorem 4.11 we obtain

Theorem 4.17. *The variables u^i_j are invariant under the action of the symmetric group $\boldsymbol{S}^m = \langle r_1, \dots, r_{m-1}\rangle$ induced via the RSK* correspondence. The action of s_l $(l = 1, \dots, n-1)$ is given explicitly as follows:*

$$\tag{4.103} \begin{gathered} s_l(u^i_l) = u^i_{l+1} + A^i_l - A^{i-1}_l, \quad s_l(u^i_{l+1}) = u^i_l + A^{i-1}_l - A^i_l, \\ s_l(u^i_j) = u^i_j \quad (j \ne l, l+1) \end{gathered}$$

for $1 \le i \le \min\{l,m\}$ and $s_l(u^i_j) = u^i_j$ for $\min\{l,m\}+1 \le i \le m$.

Note that v^i_j are invariant under the action of $\boldsymbol{S}_n = \{s_1, \dots, s_{n-1}\}$, and that r_k $(k = 1, \dots, m-1)$ act on v^i_j by explicit piecewise linear formulas similar to those described above.

Proposition 4.18. *By using the action of s_l $(l = 1, \dots, n-1)$, set $t^i_j = s_l(u^i_j)$ for $1 \le i \le m$, $i \le j \le n$. Then we obtain a solution to the ultra-discrete equation of Toda type*

$$\tag{4.104} \begin{gathered} t^i_l + t^i_{l+1} = u^i_l + u^i_{l+1} \quad (i = 1, \dots, l), \qquad t^{l+1}_{l+1} = u^{l+1}_{l+1}, \\ \min\{t^i_l, t^{i+1}_{l+1}\} = \min\{u^i_l, u^{i+1}_{l+1}\} \qquad (i = 1, \dots, l-1), \\ \min\{t^l_l, t^{l+1}_{l+1} + t^1_{l+1} - q\} = \min\{u^l_l, u^{l+1}_{l+1} + u^1_{l+1} - q\} \end{gathered}$$

for $l = 1, \dots, m-1$, and

$$\tag{4.105} \begin{gathered} t^i_l + t^i_{l+1} = u^i_l + u^i_{l+1} \qquad (i = 1, \dots, m), \\ \min\{t^i_l, t^{i+1}_{l+1}\} = \min\{u^i_l, u^{i+1}_{l+1}\} \qquad (i = 1, \dots, m-1), \\ \min\{t^m_l, t^1_{l+1} - q\} = \min\{u^m_l, u^1_{l+1} - q\} \end{gathered}$$

for $l = m, \dots, n-1$, satisfying the constraint

$$\tag{4.106} \begin{gathered} t^1_l + \cdots + t^{\min\{l,m\}}_l = u^1_{l+1} + \cdots + u^{\min\{l+1,m\}}_{l+1}, \\ t^1_{l+1} + \cdots + t^{\min\{l+1,m\}}_{l+1} = u^1_l + \cdots + u^{\min\{l,m\}}_l. \end{gathered}$$

Remark 4.19. The variables u^i_j are identified with the coordinates of crystal bases for general finite dimensional irreducible representations of $\mathfrak{gl}(n)$ as in [13]. Then, by using an argument as in [6], it can be shown that the combinatorial action of $\boldsymbol{S}_n = \langle s_1, \dots, s_{n-1}\rangle$ on u^i_j with $q = 0$ provides with the description of Kashiwara's Weyl group actions on the crystal bases. This symmetric group action on the set of column strict tableaux thus coincides with the one introduced earlier by A. Lascoux and M.P. Schützenberger [18](see also [17]). The corresponding piecewise linear action is discussed in [15].

References

[1] A. Berenstein, S. Fomin and A. Zelevinsky: Parametrization of canonical bases and totally positive matrices, Adv. in Math. **122**(1996), 49–149.

[2] A. Berenstein and D. Kazhdan: Geometric and unipotent crystals, GAFA 2000 (Tel Aviv, 1999), Geom. Funct. Anal. 2000, Special Volume, Part I, 188–236.

[3] A. Berenstein and A.N. Kirillov: Domino tableaux, Schützenberger involution, and the symmetric group action, *Formal power series and algebraic combinatorics* (Toronto, ON, 1998). Discrete Math. **225**(2000), 15–24.

[4] S. Fomin and A. Zelevinsky: Double Bruhat cells and total positivity, J. Amer. Math. Soc. **12**(1999), 335–380.

[5] I. Gessel and G. Viennot: Binomial determinants, paths, and hook length formulae, Adv. in Math. **58**(1985), 300–321.

[6] G. Hatayama, K. Hikami, R. Inoue, A. Kuniba, T. Takagi and T. Tokihiro: The $A_M^{(1)}$ automata related to crystals of symmetric tensors, J. Math. Phys. **42**(2001), 274–308.

[7] G. Hatayama, A. Kuniba and T. Takagi: Soliton cellular automata associated with finite crystals, Nucl. Phys. **B 577**(2000), 619–645.

[8] R. Hirota: Nonlinear partial difference equations II. Discrete-time Toda equation, J. Phys. Soc. Japan **43**(1977), 2074–2078.

[9] R Hirota and S. Tsujimoto: Conserved quantities of a class of nonlinear difference-difference equations, J. Phys. Soc. Japan **64**(1995), 3125–3127.

[10] K. Kajiwara, M. Noumi and Y. Yamada: A study on the fourth q-Painlevé equation, J. Phys. A: Math. Gen. **34**(2001), 8563–8581.

[11] K. Kajiwara, M. Noumi and Y. Yamada: Discrete dynamical systems with $W(A^{(1)}_{m-1} \times A^{(1)}_{n-1})$ symmetry, Lett. Math. Phys. **60**(2002), 211-219.

[12] K. Kajiwara, M. Noumi and Y. Yamada: q-Painlevé systems arising from q-KP hierarchy, Lett. Math. Phys. **62**(2002), 250-268.

[13] M. Kashiwara and T. Nakashima: Crystal graphs for representations of the q-analog of classical Lie algebras, J. Algebra, **165**(1994), 295–345.

[14] A.N. Kirillov: Introduction to tropical combinatorics, in *Physics and Combinatorics 2000*, Proceedings of the Nagoya 2000 International Workshop (Eds. A.N. Kirillov and N. Liskova), pp. 82–150, World Scientific, 2001.

[15] A.N. Kirillov and A.D. Berenstein: Groups generated by involutions, Gelfand-Tsetlin patterns, and combinatorics of Young tableaux, St. Petersberg Math. J. **7**(1996), 77–127.

[16] H. Knight and A. Zelevinsky: Representations of quivers of type A and the multisegment duality, Adv. Math. **117**(1996), 273–293.

[17] A. Lascoux, B. Leclerc and J.Y. Thibon: Crystal graphs and q-analogues of weight multiplicities for the root system A_n, Lett. Math. Phys. **35**(1995), 359–374.

[18] A. Lascoux and M.P. Schützenberger: La monoid plaxique, in *Noncommutative Structures in Algebra and Geometric Combinatorics*, pp.129–156, ROMA, CNR, 1981.

[19] G. Lusztig: Total positivity in reductive groups, in *Lie Theory and Geometry*, pp. 531-568, Prog. in Math. **123**, Birkhäuser, 1994.

[20] I.G. Macdonald: *Symmetric Functions and Hall Polynomials*, Second Edition, Oxford University Press, 1995.

[21] J. Nakagawa, M. Noumi, M. Shirakawa and Y. Yamada: Tableau representation for Macdonald's ninth variation of Schur functions, in *Physics and Combinatorics 2000*, Proceedings of the Nagoya International Workshop (Eds. A.N. Kirillov and N. Liskova), pp. 180–195, World Scientific, 2001.

[22] M. Noumi: An introduction to birational Weyl group actions, in *Symmetric Functions 2001: Surveys of Developments and Perspectives* (Ed. S. Fomin), Proceedings of the NATO ASI held in Cambridge, U. K., June 15 - July 6, 2001, pp. 179-222, Kluwer Academic Publishers, 2002.

[23] M. Noumi and Y. Yamada: Affine Weyl groups, discrete dynamical systems and Painlevé equations, Commun. Math. Phys. **199**(1998), 281–295.

[24] M. Noumi and Y. Yamada: Birational Weyl group action arising from a nilpotent Poisson algebra, in *Physics and Combinatorics 1999*, Proceedings of the Nagoya 1999 International Workshop (Eds. A.N. Kirillov, A. Tsuchiya and H. Umemura), pp. 287–319, World Scientific, 2001.

[25] B.E. Sagan: *The Symmetric Group — Representations, Combinatorial Algorithms, and Symmetric Functions*, Second Edition, Graduate Texts in Mathematics 203, Springer, 2000.

[26] S. Saito, N. Saitoh, J. Yamamoto and K. Yoshida: A characterization of discrete time soliton equations, J. Phys. Soc. Japan **70**(2001), 3517-3523.

[27] T. Tokihiro, D. Takahashi, J. Matsukidaira and J. Satsuma: From soliton equations to integrable cellular automata through a limiting procedure, Phys. Rev. Lett. **76**(1996), 3247–3250.

[28] D. Takahashi and J. Satsuma: A soliton cellular automaton, J, Phys. Soc. Japan **59**(1990), 3541–3519.

[29] Y. Yamada: A birational representation of Weyl group, combinatorial *R*-matrix and discrete Toda equation, in *Physics and Combinatorics 2000*, Proceedings of the Nagoya International Workshop (Eds. A.N. Kirillov and N. Liskova), pp. 305–319, World Scientific, 2001.

[30] A. Zelevinsky: Multisegment duality, canonical bases and total positivity, Proceedings of the International Congress of Mathematicians, Berlin 1998 (Doc. Math. J. DMV, Extra Volume ICM 1998), Vol.III, 409–417.

Masatoshi Noumi and Yasuhiko Yamada
Department of Mathematics
Kobe University
Rokko
Kobe 657-8501
Japan