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Zeta Functions and Functional Equations Associated with the Components of the Gelfand-Graev Representations of a Finite Reductive Group

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§0. Introduction

Zeta functions and functional equations associated with them for representations of finite groups were first discussed by Springer [18] and Macdonald [14] for certain representations over the complex field $\mathbb{C}$ of $GL_n(k)$ for a finite field $k = \mathbb{F}_q$. Their results, with one additional assumption, hold for irreducible representations over $\mathbb{C}$ of an arbitrary finite group G embedded in $GL(V)$, for an n-dimensional vector space V over k. In §1, a related functional equation is obtained for irreducible representations of Hecke algebras (or endomorphism algebras) $\mathcal{H}$ of multiplicity free induced representations of finite groups.

The functional equation 1.2.1 for an irreducible representation π of G involves an ε-factor $\varepsilon(\pi, \chi)$ which is given by

$$\varepsilon(\pi,\chi) = q^{-n^2/2}(\deg \pi)^{-1} \sum_{g \in G} \zeta_{\pi^*}(g)\chi(\mathrm{Tr}\,(g)),$$

where ζ_{π^*} is the character of the contragredient representation π^* of π, χ is a nontrivial additive character of k, and $\mathrm{Tr}\,(g)$ is the trace of g in $GL(V)$. The functional equations satisfied by irreducible representations f_π of $\mathcal{H}$, with π an irreducible component of the induced representation, have the form (see Proposition 1.5, §1)

$$f_\pi(\widetilde{h}) = \varepsilon(\pi,\chi) f_\pi(h),$$

with $h \in \mathcal{H}$, and $\widetilde{h}$ a twisted Fourier transform of h (to be defined in §1). The ε-factor $\varepsilon(\pi,\chi)$ is also given by the formula

$$\varepsilon(\pi,\chi) = f_\pi(\widetilde{e}),$$

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where $\widetilde{e}$ is the twisted Fourier transform of the identity element e of $\mathcal{H}$.

In §2, the results are applied to the representations of the Hecke algebra $\mathcal{H}$ of an arbitrary Gelfand-Graev representation Γ of a finite reductive group $G = \mathbf{G}^F$, for a connected reductive algebraic group $\mathbf{G}$ defined over k, with Frobenius endomorphism F, as in [3]. The Gelfand-Graev representations Γ of G are multiplicity free induced representations parametrized and decomposed into irreducible components by Digne, Lehrer, and Michel [9].

In [3] the irreducible representations of $\mathcal{H}$ were parametrized by pairs $(\mathbf{T}, \theta)$ with $\mathbf{T}$ an F-stable maximal torus in $\mathbf{G}$, and θ an irreducible representation of the finite torus $T = \mathbf{T}^F$. In §2 we review the main theorem of [3], which states that each representation $f_{\mathbf{T},\theta}$ of $\mathcal{H}$ has a factorization $f_{\mathbf{T},\theta} = \hat{\theta} \circ f_{\mathbf{T}}$, with $f_{\mathbf{T}}$ a homomorphism of algebras from $\mathcal{H}$ to the group algebra of $T = \mathbf{T}^F$, and $\hat{\theta}$ an extension of θ to an irreducible representation of the group algebra of the torus T.

For a general finite reductive group, a formula is obtained in §2 for an ε-factor $\varepsilon(\pi, \chi)$ of an irreducible component π of Γ of the form $\pi = (-1)^{\sigma(\mathbf{G})+\sigma(\mathbf{T})} R_{\mathbf{T},\theta}$, where $\sigma(\mathbf{G}), \sigma(\mathbf{T})$ are the k-ranks of the reductive groups $\mathbf{G}$ and $\mathbf{T}$ respectively, and $R_{\mathbf{T},\theta}$ is the virtual representation of G constructed by Deligne and Lusztig [8], with θ a character of T in general position. In this situation, the ε-factor $\varepsilon(\pi, \chi)$ is a Gauss sum of the representation π, and is expressed as a character sum over the finite torus $T = \mathbf{T}^F$ by a result in ([16], Theorem 1.2). Using the known structure of the finite tori, the ε-factors $\varepsilon(\pi, \chi)$ have been computed in [16] and [17] for some classical groups, and for the exceptional groups of type G_2. The formulas obtained in [16] and [17] involve Gauss sums, Kloosterman sums, and unitary Kloosterman sums (cf. [5]) associated with finite extensions of k.

In §3 more complete results concerning ε-factors are obtained for $GL_n(k)$. These are based on a formula for $f_{\mathbf{T},\theta}(c_{\dot{w}})$ as a character sum over the finite torus $T = \mathbf{T}^F$, for certain standard basis elements $c_{\dot{w}}$ of $\mathcal{H}$. Applications of this result include a formula for $f_{\mathbf{T},\theta}(\tilde{e})$ for all pairs $(\mathbf{T}, \theta)$. In the case of $GL_n(k)$, the ε-factors $\varepsilon(\pi, \chi)$ were computed for all irreducible representations by Kondo [11] and Macdonald [15] and expressed as products of Gauss sums of finite fields, using Green's results on the irreducible characters of $GL_n(k)$. Our results give formulas for the ε-factors as character sums over the finite tori $T = \mathbf{T}^F$. The last result in §3 is a formula expressing the twisted Fourier transform of the identity element of $\mathcal{H}$ in terms of the standard basis elements. In §4 another application of the formula for $f_{\mathbf{T},\theta}(c_{\dot{w}})$, in case $G = SL_n(k)$, gives a formula for the Gauss sums of unipotent representations.

In §5 the formula for $f_{\mathbf{T},\theta}(c_{\dot{w}})$ is applied to the computation of the norm map $\Delta : \mathcal{H}' \to \mathcal{H}$ ([6]), where $\mathcal{H}'$ is the Hecke algebra of the Gelfand-Graev representation of $GL_n(k')$, and k' is the extension of k of degree m. The result is that

$$\Delta(\widetilde{e'}) = (-1)^{n(m-1)}\tilde{e}^m.$$

As a corollary, we obtain an extension of the Davenport-Hasse theorem for Gauss sums of field extensions to Gauss sums associated with certain irreducible components of the Gelfand-Graev representation of $GL_n(k')$.

§1. The zeta function of a representation of a finite group

1.1. Let G be a finite group. We consider a faithful representation ρ of G, $\rho : G \to GL(V)$, where V is an n-dimensional vector space over a finite field $k = \mathbb{F}_q$, so that G can be identified with a subgroup of $GL(V)$. We shall identify an element $g \in G$ with the corresponding linear transformation $\rho(g)$. Let $X = \mathrm{End}_k(V)$ and let $\mathbb{C}(X)$ be the space of complex valued functions on X. Following Springer, [18], or Macdonald, [14], we introduce the notion of the Fourier transform and zeta function of complex representations of G as follows. Let χ be a nontrivial additive character of k, which is fixed throughout this paper. Then for $\Phi \in \mathbb{C}(X)$, the Fourier transform $\widehat{\Phi}$ of Φ is defined by

$$\widehat{\Phi}(x) = q^{-n^2/2} \sum_{y \in X} \Phi(y)\chi(\mathrm{Tr}\,(xy)).$$

Then we have $\widehat{\widehat{\Phi}}(x) = \Phi(-x)$ for all $x \in X$. For a finite dimensional complex representation π of G, and for $\Phi \in \mathbb{C}(X)$, define the zeta function $Z(\Phi, \pi)$ by

$$Z(\Phi, \pi) = \sum_{g \in G} \Phi(g)\pi(g);$$

then $Z(\Phi, \pi) = \pi(a_\Phi)$ where $a_\Phi = \sum_{g \in G} \Phi(g)g$ is the element of the group algebra $\mathbb{C}G$ of G over $\mathbb{C}$ with coefficients $\Phi(g)$.

For $x \in X$, define

$$W(\pi, \chi; x) = q^{-n^2/2} \sum_{g \in G} \chi(\mathrm{Tr}\,(gx))\pi(g).$$

Then

$$Z(\Phi,\pi)=\sum_{x\in X}\widehat{\Phi}(-x)W(\pi,\chi;x).$$

For $g\in G$, one has

$$\begin{aligned}W(\pi,\chi;xg) &= \pi(g)^{-1}W(\pi,\chi;x),\\ W(\pi,\chi;gx) &= W(\pi,\chi;x)\pi(g)^{-1}.\end{aligned}$$

Putting $x=1$, these imply that $\pi(g)$ commutes with $W(\pi,\chi;1)$, so if π is irreducible,

$$W(\pi,\chi;1)=w(\pi,\chi)\pi(1),$$

where $w(\pi,\chi)\in\mathbb{C}$. Define the ε-factor $\varepsilon(\pi,\chi)$ by

$$\varepsilon(\pi,\chi)=w(\pi^*,\chi),$$

where π^* is the contragredient representation of π.

Proposition 1.2. *Let π be an irreducible representation of G and let $\Phi\in\mathbb{C}(X)$ vanish outside G. Then*

$$ {}^tZ(\widehat{\Phi},\pi^*)=\varepsilon(\pi,\chi)Z(\Phi,\pi). \tag{1.2.1}$$

Proof.

$$\begin{aligned}{}^tZ(\widehat{\Phi},\pi^*) &= \sum_{x\in X}\widehat{\widehat{\Phi}}(-x)\,{}^tW(\pi^*,\chi;x)\\ &= \sum_{x\in X}\Phi(x)\,{}^tW(\pi^*,\chi;x)\\ &= \sum_{g\in G}\Phi(g)\,{}^tW(\pi^*,\chi;g)\\ &= \sum_{g\in G}\Phi(g)\,{}^t\pi^*(g^{-1})\,{}^tW(\pi^*,\chi;1)\\ &= \sum_{g\in G}\Phi(g)\pi(g)w(\pi^*,\chi).\end{aligned}$$

□

For all irreducible representations π of $GL_n(k)$ having no one component, Macdonald proved that $W(\pi^*,\chi;x)$ has support contained in $GL_n(k)$, so that the functional equation 1.2.1 holds for all functions Φ (see [14], and [18] for the case of an irreducible cuspidal representation of G). With the assumption that Φ has support in G the formula given in

Proposition 1.2 for an arbitrary finite group embedded in $GL(V)$ follows from Macdonald's argument, as given above. In case π_ϕ is an irreducible cuspidal representation of $GL_n(k)$ associated with a regular character ϕ of the multiplicative group $k_n^\times$ of the extension k_n of k of degree n, Springer proved that the ε-factor is a Gauss sum

$$(-1)^n q^{-n/2} \sum_{x \in k_n^\times} \chi(\mathrm{Tr}_{\,k_n/k} x)\phi(x).$$

Springer also gave an example to show that no functional equation of the above form holds for all irreducible representations π of $GL_n(k)$ and all functions Φ. The zeta function is an analogue for finite fields of a concept introduced by Godement and Jacquet (SLN 260).

1.3. Let U be a subgroup of G and ψ a complex linear character of U. We use the notation concerning the Hecke algebra of the induced representation ψ^G introduced in [3, §2B]. In particular, ψ^G is afforded by the left ideal $\mathbb{C}Ge_\psi$ in the group algebra of G generated by the idempotent

$$e_\psi = | U |^{-1} \sum_{u \in U} \psi(u^{-1})u.$$

The Hecke algebra $\mathcal{H}$ associated with the induced representation ψ^G is defined by

$$\mathcal{H} = e_\psi \mathbb{C}Ge_\psi.$$

We assume $\mathcal{H}$ is commutative (so that (G, H, ψ) is a twisted Gelfand pair according to [15, p.397]).

Lemma 1.4. *Let $\Phi \in \mathbb{C}(X)$ and assume that Φ vanishes outside G. Then $\sum_{g\in G} \Phi(g)g \in \mathcal{H}$ implies $\sum_{g\in G} \widehat{\Phi}(g)g^{-1} \in \mathcal{H}$.*

Proof. First we notice that $\sum_{g\in G} \Phi(g)g \in \mathcal{H}$ if and only if $\Phi(ug) = \Phi(gu) = \psi(u^{-1})\Phi(g)$ for $u \in U$, $g \in G$. So we have to prove that Ψ satisfies these conditions where $\Psi(g) = \widehat{\Phi}(g^{-1})$. We have, using the assumption that Φ is supported on G,

$$\Psi(ug) = \widehat{\Phi}(g^{-1}u^{-1}) = q^{-n^2/2} \sum_{y \in G} \Phi(y)\chi(\mathrm{Tr}\,(g^{-1}u^{-1}y)).$$

Putting $z = g^{-1}u^{-1}y$, the right hand side becomes

$$\begin{aligned} q^{-n^2/2}\sum_{z\in G}\Phi(ugz)\chi(\mathrm{Tr}\,(z)) &= \psi(u^{-1})q^{-n^2/2}\sum_{z\in G}\Phi(gz)\chi(\mathrm{Tr}\,(z)) \\ &= \psi(u^{-1})q^{-n^2/2}\sum_{y\in X}\Phi(y)\chi(\mathrm{Tr}\,(g^{-1}y)) \\ &= \psi(u^{-1})\widehat{\Phi}(g^{-1}) = \psi(u^{-1})\Psi(g) \end{aligned}$$

as required. The formula $\Psi(gu) = \lambda(u^{-1})\Psi(g)$ follows similarly. □

We remark that the converse holds if $-1 \in G$, since $\widehat{\widehat{\Phi}}(x) = \Phi(-x)$. For $h = \sum_{g\in G}\Phi(g)g \in \mathcal{H}$ with Φ supported on G, the element $\widetilde{h} = \sum_{g\in G}\widehat{\Phi}(g)g^{-1} \in \mathcal{H}$ will sometimes be called the twisted Fourier transform of h.

Proposition 1.5. *Let π be an irreducible constituent in ψ^G, and let f_π be the corresponding representation of $\mathcal{H}$. Then*

$$f_\pi(\widetilde{h}) = \varepsilon(\pi,\chi)f_\pi(h)$$

where $h = \sum_{g\in G}\Phi(g)g \in \mathcal{H}, \widetilde{h} = \sum_{g\in G}\widehat{\Phi}(g)g^{-1}$, and Φ vanishes outside G, so $\widetilde{h} \in \mathcal{H}$.

Proof. Taking traces of (1.2.1), one has

$$\sum_{g\in G}\widehat{\Phi}(g)\,\mathrm{Tr}\,(\pi(g^{-1})) = \varepsilon(\pi,\chi)\sum_{g\in G}\Phi(g)\,\mathrm{Tr}\,(\pi(g)).$$

Then the Proposition follows from the previous Lemma. □

We note that $\widetilde{\widetilde{h}}$ is not related to $\widehat{\widehat{\Phi}}$, since $\widehat{\Phi}$ is not supported by G in general, even if Φ is supported by G.

Corollary 1.6. $f_\pi(\widetilde{e}_\psi) = \varepsilon(\pi,\chi)$ *and* $\widetilde{h} = \widetilde{e}_\psi h$.

Proof. Putting $h = e_\psi$ in the above Proposition, we have the first assertion. Then we have

$$f_\pi(\widetilde{h}) = f_\pi(\widetilde{e}_\psi h),$$

for every irreducible representation f_π of the semisimple algebra $\mathcal{H}$, which proves the second. □

§2. Zeta functions and Gelfand-Graev representation of a finite reductive group

2.1. Let $\mathbf{G}$ be a connected reductive algebraic group defined over a finite field $k = \mathbb{F}_q$ with Frobenius map F, and let $G = \mathbf{G}^F$ be the finite group consisting of elements in $\mathbf{G}$ fixed by F. We choose an F-stable Borel subgroup $\mathbf{B}_0$ and an F-stable maximal torus $\mathbf{T}_0$ contained in $\mathbf{B}_0$; and denote by $\mathbf{U}_0$ the unipotent radical of $\mathbf{B}_0$. We put $B_0 = \mathbf{B}_0^F$, $T_0 = \mathbf{T}_0^F$, and $U_0 = \mathbf{U}_0^F$.

Let ρ be a faithful representation of $\mathbf{G}$,

$$\rho : \mathbf{G} \to GL_n(\overline{k}),$$

with $\overline{k}$ the algebraic closure of k. We assume that ρ commutes with Frobenius maps as follows: $\rho \circ F = F' \circ \rho$, where $F'(x) = x^{(q)} = (x_{ij}^q)$ for $x = (x_{ij}) \in GL_n(\overline{k})$. Thus G can be identified with a subgroup of $GL_n(k)$.

2.2. Before discussing representations, it is necessary to change the field from $\mathbb{C}$ to $\overline{\mathbb{Q}}_\ell$, the algebraic closure of the field of ℓ-adic numbers with ℓ a prime different from the characteristic of k, as in the Deligne-Lustzig paper [8].

As for Gelfand-Graev representations of G, we shall follow the notation and preliminary discussion from [3]. We also carry over the notation from the preceding section. In particular, $\Gamma = \psi^G$ denotes a fixed Gelfand-Graev representation of G, parametrized by an element $z \in H^1(F, Z(\mathbf{G}))$ as in [3]; while $\mathcal{H}$ denotes the Hecke algebra of Γ, $e = e_\psi$ the identity element of $\mathcal{H}$, etc. As in [3], $f_{\mathbf{T},\theta}$ denotes the irreducible representation of the Hecke algebra $\mathcal{H}$ associated with the pair consisting of an F-stable maximal torus $\mathbf{T}$ and a character θ of $T = \mathbf{T}^F$. We recall the following factorization theorem ([3, Theorem (4.2)]).

Theorem 2.3. *For each pair $(\mathbf{T}, \theta)$ as above, the corresponding representation $f_{\mathbf{T},\theta} : \mathcal{H} \to \overline{\mathbb{Q}}_l$ can be factored,*

$$f_{\mathbf{T},\theta} = \widehat{\theta} \circ f_{\mathbf{T}},$$

with $f_{\mathbf{T}}$ a homomorphism of algebras from $\mathcal{H}$ to $\overline{\mathbb{Q}}_\ell T$, independent of θ. Let $f_{\mathbf{T}}(c) = \sum f_{\mathbf{T}}(c)(t)t \in \overline{\mathbb{Q}}_l T$, for $c \in \mathcal{H}$. Then the value of the coefficient function $f_{\mathbf{T}}(c_n)(t)$, for a standard basis element c_n of $\mathcal{H}$ and

$t \in T$, is given by the following formula:

$$f_{\mathbf{T}}(c_n)(t) = \text{ind } n < Q^{\mathbf{G}}_{\mathbf{T}}, \Gamma >^{-1} | U_0 |^{-1} | C_{\mathbf{G}}(t)^{\circ F} |^{-1} \times \sum_{\substack{g \in G, u \in U_0 \\ (gung^{-1})_{ss} = t}} \psi(u^{-1}) Q^{C_{\mathbf{G}}(t)^{\circ}}_{\mathbf{T}}((gung^{-1})_{uni}). \tag{2.3.1}$$

2.3.2. *Remark* In what follows, we shall denote $(-1)^{\sigma(\mathbf{G})-\sigma(\mathbf{T})}$ by $\varepsilon(\mathbf{T})$. In case the center of $\mathbf{G}$ is connected, we have $< Q^{\mathbf{G}}_{\mathbf{T}}, \Gamma > = \varepsilon(\mathbf{T})$ from §10 of [8]. In the case of $GL_n(k)$ and if $\mathbf{T}$ corresponds to $w \in S_n$, we have $\varepsilon(\mathbf{T}) = \text{sgn}(w)$.

Theorem 2.4. *Let π be an irreducible representation of G.*

(i) *The ε-factor corresponding to π is given by*

$$\begin{aligned} \varepsilon(\pi, \chi) &= \frac{1}{\deg \pi} \text{Tr } W(\pi^*, \chi; 1) \\ &= \frac{q^{-n^2/2}}{\deg \pi} \sum_{g \in G} \zeta_{\pi^*}(g) \chi(\text{Tr}(g)), \end{aligned}$$

where ζ_{π^*} is the character of the contragredient representation π^*.

(ii) *In case π is a component of Γ corresponding to the representation $f_{\mathbf{T},\theta}$ of $\mathcal{H}$, we have*

$$f_{\mathbf{T},\theta}(\widetilde{h}) = \varepsilon(\pi, \chi) f_{\mathbf{T},\theta}(h),$$

for all $h \in \mathcal{H}$, $h = \sum \Phi(g) g$, with Φ vanishing outside G.

(iii) *In case the irreducible representation π has the form $\varepsilon(\mathbf{T}) R_{\mathbf{T},\theta}$ with θ in general position, one has*

$$\varepsilon(\pi, \chi) = \varepsilon(\mathbf{T}) q^{-n^2/2} | G |_p \sum_{t \in T} \theta^{-1}(t) \chi(\text{Tr}(t)).$$

Proof. The first statement follows from the definition of $\varepsilon(\pi, \chi)$ in §1.1. Part (ii) follows from (1.5), while (iii) follows from ([16], Theorem 1.2) and the fact that $R^*_{\mathbf{T},\theta} = R_{\mathbf{T},\theta^{-1}}$. □

Corollary 2.5. *With π corresponding to $f_{\mathbf{T},\theta}$ as in part* (ii) *of the Theorem, we have by* (1.6)

$$f_{\mathbf{T},\theta}(\widetilde{e}) = \varepsilon(\pi, \chi).$$

Remarks 2.6. (i) For any irreducible representation π of G, the sum

$$\tau(\pi) = \sum_{g \in G} \mathrm{Tr}\,(\pi(g))\chi(\mathrm{Tr}\,(g))$$

is called a Gauss sum of G associated with (π, χ). These have been computed in the case of $G = GL_n(k)$ for all irreducible representations ([11], [15]). In the situation of part (iii) of the Theorem, and also for unipotent representations, the Gauss sums have been computed for several other classical groups and for G_2 ([16], [17]).

(ii) Let $\phi(g) = \chi(\mathrm{Tr}\,(g))$ for $g \in G$ and let $<,>_G$ be the inner product of class functions on G. Then we have

$$\begin{aligned} \tau(\pi) &= |G| < \zeta_{\pi^*}, \phi >_G \\ \varepsilon(\pi, \chi) &= (\deg \pi)^{-1} q^{-n^2/2} |G| < \zeta_\pi, \phi >_G . \end{aligned}$$

We also notice that since the value of ϕ depends only on the semisimple part of the element $g \in G$, ϕ is expressed as a linear combination of the virtual characters of Deligne-Lusztig by [8, (7.12.1)] (see also [1, Proposition 7.6.4]).

§3. ε-Factors for $GL_n(k)$

In this section, let $G = GL_n(k)$ and let U be the upper triangular unipotent subgroup of G. Then $G = \mathbf{G}^F$ for $\mathbf{G} = GL_n(\bar{k})$ with the usual Frobenius endomorphism F. In this case there is, up to equivalence, just one Gelfand-Graev representation $\Gamma = \psi^G$, for the linear character ψ of U given by $\psi(u) = \chi(u_{12} + \cdots + u_{n-1n})$ with $u = (u_{ij}) \in U$.

We begin with some computations of the homomorphisms f_{T} on standard basis elements of $\mathcal{H}$.

Lemma 3.1. *For $a \in k^*$, let*

$$\dot{w}(a) = \begin{pmatrix} & & & a \\ -1 & & & \\ & \ddots & & \\ & & -1 & \end{pmatrix} \in G. \tag{3.1.1}$$

Then for all $u \in U$, $u\dot{w}(a)$ is a regular element, i.e. $(u\dot{w}(a))_{uni}$ is a regular unipotent element in $C_G((u\dot{w}(a))_{ss})$.

Proof. It is enough to show that the minimal polynomial of $u\dot{w}(a)$ is the characteristic polynomial of $u\dot{w}(a)$ and for that it is enough to show that

(3.1.2)
$$(x_1 I - u\dot{w}(a)) \cdots (x_{n-1} I - u\dot{w}(a)) \neq 0, \quad \text{for all } x_1, ..., x_{n-1} \in \bar{k},$$

where I is the identity matrix in G. Let $u = (u_{ij})$ and $A = u\dot{w}(a)$. Thus

$$A = \begin{pmatrix} -u_{12} & -u_{13} & \cdots & -u_{1n} & a \\ -1 & -u_{23} & \cdots & -u_{2n} & \\ & -1 & \cdots & -u_{3n} & \\ & & \ddots & \vdots & \\ & & & -1 & \end{pmatrix}.$$

Let $A_i = x_i I - A, (i = 1, ..., n-1)$, then it is easy to see that the $(n,1)$-entry of $A_1 \cdots A_{n-1}$ is nonzero, which proves (3.1.2). □

Lemma 3.2. *We have*
(i) ${}^{\dot{w}(a)}\psi = \psi$ *on* $U \cap {}^{\dot{w}(a)} U$, *and*
(ii) $[U : U \cap {}^{\dot{w}(a)} U] = q^{n-1}$.
for all nonzero elements $a \in k$.

Proof. For $u = (u_{ij}) \in U$, we have

$${}^{\dot{w}(a)}u = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ -a^{-1}u_{1n} & 1 & u_{12} & \cdots & u_{1n-1} \\ -a^{-1}u_{2n} & 0 & 1 & \cdots & u_{2n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a^{-1}u_{n-1n} & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Thus the condition for ${}^{\dot{w}(a)}u \in U$ is $u_{1n} = u_{2n} = \cdots = u_{n-1n} = 0$, which proves (ii).

Take any $u_0 \in U \cap {}^{\dot{w}(a)} U$, then there exists $u = (u_{ij}) \in U$ such that $u_0 = {}^{\dot{w}(a)} u$. Therefore, using the first part of the proof,

$$\begin{aligned} {}^{\dot{w}(a)}\psi(u_0) &= \psi({}^{\dot{w}(a)^{-1}}u_0) = \psi(u) \\ &= \chi(u_{12} + \cdots + u_{n-2,n-1}) = \psi(u_0), \end{aligned}$$

which proves the first assertion. □

Theorem 3.3. *Let $G = GL_n(k)$ and let $\dot{w}(a)$ be defined as in (3.1). Then $c_{\dot{w}(a)}$ is a standard basis element of $\mathcal{H}$. For each F-stable maximal torus $\mathbf{T}$ of $\mathbf{G}$, we have, for all $t \in T$,*

$$(3.3.1) \qquad f_{\mathbf{T}}(c_{\dot{w}(a)})(t) = \delta_{\det t, a}\varepsilon(\mathbf{T})\chi(\mathrm{Tr}\ t),$$

where $\delta_{\det t,a} = 1$, if $\det t = a$, and $= 0$, otherwise. Therefore

$$(3.3.2) \qquad f_{\mathbf{T},\theta}(c_{\dot{w}(a)}) = \varepsilon(\mathbf{T}) \sum_{t\in T, \det t = a} \chi(\mathrm{Tr}\ t)\theta(t).$$

Proof. By Theorem 2.3, Lemma 3.1, and Lemma 3.2 (2), together with the fact that $Q^{\mathbf{G}}_{\mathbf{T}}(u) = 1$ if u is regular unipotent by [8, Theorem 9.16], we have

$$f_{\mathbf{T}}(c_{\dot{w}(a)})(t) = q^{n-1}\varepsilon(\mathbf{T}) \mid U \mid^{-1} \mid C_G(t) \mid^{-1} \sum_{\substack{g\in G, u\in U \\ (gu\dot{w}(a)g^{-1})_{ss}=t}} \psi(u^{-1}).$$

Two semisimple elements, $(u\dot{w}(a))_{ss}$ and t are conjugate if and only if their characteristic polynomials are the same. Let t be conjugate to $\mathrm{diag}(\alpha_1, \alpha_2, ..., \alpha_n)$ in $\mathbf{G} = GL_n(\bar{k})$, and let $u = (u_{ij})$, where $u_{ij} = 0$, if $i > j$ and $u_{ii} = 1$. Regarding u_{ij} $(i < j)$ as variables and defining polynomials $p_m(u) = p_m(u_{12}, u_{13}, ...)$ over k by $\det(xI - u\dot{w}(a)) = \sum_{m=0}^{n} p_m(u)x^{n-m}$ we can show easily that

$$p_m(u) = (-1)^{m+1}u_{1,m+1} + q_m(u), \quad \text{for } m = 1, ..., n-1,$$

where $q_m(u)$ is a polynomial in the variables u_{1j} $(1 < j < m+1)$ and u_{ij} $(1 < i < j)$. In particular $p_1(u) = \sum_{i=1}^{n-1} u_{ii+1}$.

Thus $(u\dot{w}(a))_{ss}$ and t are conjugate if and only if

$$(3.3.3) \quad (-1)^m p_m(u) = \sum_{1\le i_1 < i_2 \cdots < i_m \le n} \alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_m}, \text{ for } m = 1, ..., n.$$

These simultaneous equations have solutions if $\det t = a$ and in this case the number of solutions is $q^{(n-1)(n-2)/2}$ since for any values of u_{ij} $(2 \le i < j \le n)$, u_{1j} $(2 \le j \le n)$ are uniquely determined by the equations (3.3.3). Notice that $\mathrm{Tr}\ t = -\sum_{i=1}^{n-1} u_{ii+1}$. Moreover if $(u\dot{w}(a))_{ss}$ and t are conjugate, then the set $\{g \in G \mid g(u\dot{w}(a))_{ss}g^{-1} = t\}$ is a coset of $C_G(t)$. Putting these facts together we have the equations in the theorem. □

Corollary 3.4. *If* $(\mathbf{T},\theta)$ *and* $(\mathbf{T}',\theta')$ *are geometrically conjugate, we have*

$$\varepsilon(\mathbf{T}) \sum_{t\in T,\det t=a} \chi(\mathrm{Tr}\ t)\theta(t) = \varepsilon(\mathbf{T}') \sum_{t\in T',\det t=a} \chi(\mathrm{Tr}\ t)\theta'(t).$$

Proof. If $(\mathbf{T},\theta)$ and $(\mathbf{T}',\theta')$ are geometrically conjugate, we have $f_{\mathbf{T},\theta} = f_{\mathbf{T}',\theta'}$ (cf. [3]). By evaluating them on $c_{\dot{w}(a)}$, the assertion follows. □

We remark that the corollary is a generalization of [2, Lemma (5.1)]. In particular if we apply (3.4) to $GL_2(q)$, $(\mathbf{T}_1,1)$ and $(\mathbf{T}_w,1)$ (cf. the notation in [5]), we have

$$\sum_{x\in k^\times} \chi(x+ax^{-1}) = -\sum_{y\in k_2^\times, N_{2,1}y=a} \chi(y+y^q),$$

which is (1.3) of [2].

To obtain the value of $f_{\mathbf{T}}$ on $c_{{}^t\dot{w}(a)}$, we consider the following automorphism α on G. Let $w_0 = (w_{0,ij})$ be the matrix in G, with $w_{0,ij} = \delta_{i+j,n+1}(-1)^{i-1}$ and put $\alpha(g) = ({}^tg^{-1})^{w_0}$ for $g\in\mathbf{G}$. Then α is an involutive automorphism of $\mathbf{G}$, G, and U. It can be checked easily that $\psi\circ\alpha=\psi$. The extension of α to an automorphism of $\mathbb{C}G$ induces an automorphism of $\mathcal{H}$.

Noting that for an F-stable maximal torus $\mathbf{T}$, $\mathbf{T}$ and $\alpha(\mathbf{T})$ are G-conjugate, and using Theorem 2.3, we obtain without difficulty that

(3.4.1) $$f_{\mathbf{T}}(c_{\alpha(n)})(t) = f_{\alpha(\mathbf{T})}(c_n)(\alpha(t)), \text{ and}$$

(3.4.2) $$f_{\mathbf{T},\theta}(c_{\alpha(n)}) = f_{\alpha(\mathbf{T}),\theta\circ\alpha}(c_n).$$

Lemma 3.5. *We have*

$$f_{\mathbf{T},\theta}(c_{\alpha(\dot{w}(a))}) = f_{\mathbf{T},\bar{\theta}}(c_{\dot{w}(a)}),$$

where $\bar{\theta}=\theta^{-1}$. *Therefore*

(3.5.1) $$f_{\mathbf{T},\theta}(c_{-{}^t\dot{w}(a)}) = \varepsilon(\mathbf{T}) \sum_{t\in T,\det t=(-1)^n a^{-1}} \chi(\mathrm{Tr}\ t)\theta(t^{-1}).$$

Proof. From the preceding discussion, we have

$$\begin{aligned} f_{\mathbf{T},\theta}(c_{\alpha(\dot{w}(a))}) &= f_{\alpha(\mathbf{T}),\theta\circ\alpha}(c_{\dot{w}(a)}) \quad \text{(by the equation (3.4.2))} \\ &= \varepsilon(\alpha(\mathbf{T})) \sum_{t'\in\alpha(T),\det t'=a} \chi(\mathrm{Tr}\ t')\theta(\alpha(t')) \\ &= \varepsilon(\mathbf{T}) \sum_{t\in T,\det t=a^{-1}} \chi(\mathrm{Tr}\ t^{-1})\theta(t) \\ &= \varepsilon(\mathbf{T}) \sum_{t\in T,\det t=a} \chi(\mathrm{Tr}\ t)\theta(t^{-1}) \\ &= f_{\mathbf{T},\bar{\theta}}(c_{\dot{w}(a)}), \end{aligned}$$

by Theorem 3.3. The second assertion follows from this and $\alpha(\dot{w}(a)) = -{}^t(\dot{w}((-1)^n a^{-1}))$.

□

We remark that the equations (3.3.2) and (3.5.1), together with Theorem 4.2 in [3], generalize Theorem 4.1 in [2] to $GL_n(q)$.

The following theorem was proved by Kondo [11] for all irreducible characters of $G = GL_n(k)$, using the results of J. A. Green on the irreducible characters of G. Kondo stated the theorem in terms of Gauss sums of field extensions of k. Our theorem is stated in terms of character sums over a torus, and is proved using the Deligne-Lusztig theory [8].

Theorem 3.6. *Let ζ be an irreducible character of $G = GL_n(k)$ and let ζ be a component of $R_{\mathbf{T},\theta}$. Then the Gauss sum of the character ζ is given by*

$$\tau(\zeta) = \sum_{g\in G} \zeta(g)\chi(\mathrm{Tr}\,(g)) = \deg\zeta \mid G\mid_p \varepsilon(\mathbf{T}) \sum_{t\in T} \chi(\mathrm{Tr}\,(t))\theta(t).$$

Proof. We shall denote by $\rho_{\mathbf{T},\theta}$ the character of the virtual representation $R_{\mathbf{T},\theta}$. From ([13], §3) and ([8], Prop. 5.11) we have

$$\zeta = \sum_{[(\mathbf{T}',\theta')]} c_{(\mathbf{T}',\theta')}\rho_{\mathbf{T}',\theta'},$$

for some $c_{(\mathbf{T}',\theta')} \in \mathbb{Q}$, where $(\mathbf{T}',\theta')$ runs over members of the geometric conjugacy class of $(\mathbf{T},\theta)$. Since τ is additive (cf. [16]), we have

$$\tau(\zeta) = \sum_{[(\mathbf{T}',\theta')]} c_{(\mathbf{T}',\theta')}\tau(\rho_{\mathbf{T}',\theta'}).$$

By [loc.cit.,(1.2)], the Gauss sums of the virtual characters $\rho_{\mathbf{T}',\theta'}$ are given by

$$\tau(\rho_{\mathbf{T}',\theta'}) = \frac{|G|}{|T'|} \sum_{t' \in T'} \theta'(t')\chi(\mathrm{Tr}\,(t')).$$

Then by (3.4) we have

$$\varepsilon(\mathbf{T}) \sum_{t \in T} \chi(\mathrm{Tr}\ t)\theta(t) = \varepsilon(\mathbf{T}') \sum_{t \in T'} \chi(\mathrm{Tr}\ t)\theta'(t).$$

for pairs $(\mathbf{T}, \theta)$ and $(\mathbf{T}', \theta')$ in the same geometric conjugacy class. Therefore

$$\tau(\zeta) = \{\varepsilon(\mathbf{T}) \sum_{t \in T} \theta(t)\chi(\mathrm{Tr}\ t)\}\{ \sum_{[(\mathbf{T}',\theta')]} c_{(\mathbf{T}',\theta')}\varepsilon(\mathbf{T}')\frac{|G|}{|T'|}\}.$$

Since

$$\deg \zeta = \sum_{[(\mathbf{T}',\theta')]} c_{(\mathbf{T}',\theta')}\varepsilon(\mathbf{T}')\frac{|G|_{p'}}{|T'|},$$

the result follows. □

Corollary 3.7. *Let $\pi_{\mathbf{T},\theta}$ be an irreducible component of the Gelfand-Graev representation, associated with the representation $f_{\mathbf{T},\theta}$ of $\mathcal{H}$, for an arbitrary pair $(\mathbf{T}, \theta)$ as in ([8], §10). Then we have*

$$f_{\mathbf{T},\theta}(\widetilde{e}) = \varepsilon(\pi_{\mathbf{T},\theta}, \chi) = q^{-n/2}\varepsilon(\mathbf{T}) \sum_{t \in T} \theta^{-1}(t)\chi(\mathrm{Tr}\ t).$$

Proof. We have

$$f_{\mathbf{T},\theta}(\widetilde{e}) = \varepsilon(\pi_{\mathbf{T},\theta}, \chi) = \frac{q^{-n^2/2}}{\deg \pi} \sum_{g \in G} \chi^*_{\mathbf{T},\theta}(g)\chi(\mathrm{Tr}\,(g)),$$

by (2.4), where $\chi^*_{\mathbf{T},\theta}$ is the character of the contragredient representation $\pi^*_{\mathbf{T},\theta}$. By ([3], Theorem (2.1)), $\pi_{\mathbf{T},\theta}$ is a component of $R_{\mathbf{T},\theta}$, and is associated with the geometric conjugacy class $[(\mathbf{T}, \theta)]$. Then $\chi_{\mathbf{T},\theta}$ is a linear combination of Deligne-Lusztig characters, so $\chi^*_{\mathbf{T},\theta} = \chi_{\mathbf{T},\theta^{-1}}$ as this is true for the Deligne-Lusztig characters. The Corollary now follows from the preceding Theorem. □

As an application of Lemma 3.5 and Corollary 3.7, we give a formula for the twisted Fourier transform of the identity element e of $\mathcal{H}$ in terms of the standard basis elements of $\mathcal{H}$. It would be interesting to know a version of this formula for other types of finite reductive groups.

We recall the notation for the twisted Fourier transform

$$\widetilde{h} = \sum_{g \in G} \widehat{\Phi}(g) g^{-1} \in \mathcal{H} \text{ for } h = \sum \Phi(g) g \in \mathcal{H},$$

with Φ vanishing outside G.

Theorem 3.8. *We have*

$$\widetilde{e} = q^{-n/2} \sum_{a \in k^\times} c_{-{}^t\dot{w}(a)},$$

and

$$\widetilde{h} = q^{-n/2} \left(\sum_{a \in k^\times} c_{-{}^t\dot{w}(a)} \right) h,$$

for all $h \in \mathcal{H}$.

Proof. By the above Corollary together with equation (3.5.1), it follows that

$$f_{\mathbf{T},\theta}(\widetilde{e}) = q^{-n/2} f_{\mathbf{T},\theta}\Big(\sum_{a \in k^\times} c_{-{}^t\dot{w}(a)} \Big),$$

for all pairs $(\mathbf{T}, \theta)$, and the first equation follows. The second equation follows from (1.6). □

§4. Gauss sums of unipotent characters of $SL_n(k)$

For the definitions and notation we refer to [16]. We first notice that by Theorem 3.3 above and Theorem 1.2 of [16] we have

$$\tau(R_{\mathbf{T},\theta}) = [G_0 : T] \varepsilon(\mathbf{T}) f_{\mathbf{T},\theta}(c_{\dot{w}}),$$

where $G_0 = SL_n(k)$ and $\dot{w} = \dot{w}(1)$. Let

$$S = \sum_{\substack{x_1, x_2, \ldots, x_n \in k \\ x_1 \cdots x_n = 1}} \chi(x_1 + \cdots + x_n).$$

Then we have

Theorem 4.1. *Let ρ be any irreducible character of $W = S_n$. For the unipotent character R_ρ of $SL_n(k)$ defined by*

$$R_\rho = \frac{1}{|W|} \sum_{w \in W} \mathrm{Tr}\rho(w) R_{\mathbf{T}_w,1},$$

we have

$$w(R_\rho) = q^{n(n-1)/2} S.$$

Proof. If $\mathbf{T}_0$ is a maximal split torus and $\mathbf{T}$ is an arbitrary F-stable maximal torus in $\mathbf{G}_0$, then the pairs $(\mathbf{T}_0, 1)$ and $(\mathbf{T}, 1)$ are geometrically conjugate. Corollary 3.4 holds for G_0, and we have $S = f_{\mathbf{T},1}(c_{\dot{w}})$, since $S = f_{\mathbf{T}_0,1}(c_{\dot{w}})$. Therefore, by the additivity of τ, we have

$$\begin{aligned} \tau(R_\rho) &= \frac{1}{|W|} \sum_{w \in W} \mathrm{Tr}\,\rho(w) \tau(R_{\mathbf{T}_w,1}) \\ &= \frac{1}{|W|} \sum_{w \in W} \mathrm{Tr}\,\rho(w) [G_0 : T_w] \varepsilon(\mathbf{T}_w) S \\ &= \frac{q^{n(n-1)/2} S}{|W|} \sum_{w \in W} \mathrm{Tr}\,\rho(w) R_{\mathbf{T}_w,1}(1) \\ &= q^{n(n-1)/2} S R_\rho(1). \end{aligned}$$

Since $w(R_\rho) = R_\rho(1)^{-1} \tau(R_\rho)$, we have proved the assertion in the theorem. □

We remark that if ρ is the trivial representation, the above result is proved in [12].

§5. On the norm map $\Delta : \mathcal{H}' \to \mathcal{H}$

We mention here another application of the preceding results to a computation of the norm map $\Delta : \mathcal{H}' \to \mathcal{H}$ on $\tilde{e}' \in \mathcal{H}'$, in the case of $\mathbf{G} = GL_n(\bar{k})$. In this case the norm map is a homomorphism of algebras from the Hecke algebra $\mathcal{H}'$ of a Gelfand-Graev representation of $G' = GL_n(k')$, $k' = k_m = \mathbb{F}_{q^m}$, to the Hecke algebra $\mathcal{H}$ of a Gelfand-Graev representation of $G = GL_n(k)$ (cf. [6]) and it is known to be surjective. Moreover it gives a correspondence of representations of Hecke algebras (or spherical functions) $f_{\mathbf{T},\theta} \to f_{\mathbf{T},\theta} \circ \Delta$. Let $\mathbf{T}$ be an F-stable maximal torus, $T = \mathbf{T}^F, T' = \mathbf{T}^{F^m}$, $N_{\mathbf{T}} : T' \to T$ be the (usual) norm map, and let $\widetilde{N}_{\mathbf{T}}$ be the extension of $N_{\mathbf{T}}$ to a homomorphism of group algebras of T' and T. Then the norm map Δ is characterized as the unique linear map $\Delta : \mathcal{H}' \to \mathcal{H}$ with the property that for each F-stable maximal torus $\mathbf{T}$, one has

$$f_{\mathbf{T}} \circ \Delta = \widetilde{N}_{\mathbf{T}} \circ f'_{\mathbf{T}}.$$

Theorem 5.1. *Let e' be the identity element of $\mathcal{H}'$. Then*

$$\Delta(\widetilde{e'}) = (-1)^{n(m-1)} \widetilde{e}^m.$$

Proof. In the discussion to follow, we shall use the notation k_m for the extension of k of degree m, along with $\mathrm{Tr}_{\,a,b} = \mathrm{Tr}_{\,k_a/k_b}$ and $N_{a,b} = N_{k_a/k_b}$ for trace and norm maps of field extensions, as in [5], where b is a divisor of a.

By the definition of the norm map, it is enough to show that

$$\widetilde{N}_{\mathbf{T}}(f'_{\mathbf{T}}(\widetilde{e'})) = f_{\mathbf{T}}((-1)^{n(m-1)} \widetilde{e}^m),$$

for each F-stable maximal torus $\mathbf{T}$. From the known structure of the F-stable maximal tori, it is not difficult to verify that it is enough to prove the above formula in case $\mathbf{T}$ is isomorphic to $\{\mathrm{diag}(a_1, ..., a_n) \mid a_i \in \overline{k}^{\times}\}$ where the Frobenius map F acts as $F(\mathrm{diag}(a_1, ..., a_n)) = \mathrm{diag}(a_2^q, ..., a_n^q, a_1^q)$. Hence T is isomorphic to $k_n^{\times}$ and T' is isomorphic to $(k_{nm/d}^{\times})^d$, with $d = g.c.d.(m, n)$. Under this identification of T and T', we have

$$\mathrm{Tr}\,(t') = \mathrm{Tr}_{\,nm/d,m}(a'_1 + \cdots + a'_d)$$

and

$$N_{\mathbf{T}}(t') = N_{nm/d,n}(a'_1 {a'_2}^q \cdots {a'_d}^{q^{d-1}})$$

with $t' = (a'_1, ..., a'_d) \in (k_{nm/d}^{\times})^d$. Let $\chi' = \chi \circ \mathrm{Tr}_{\,m,1}$ and $\chi_n = \chi \circ \mathrm{Tr}_{\,n,1}$. Finally, we note that $\varepsilon'(\mathbf{T}) = (-1)^{\sigma'(\mathbf{G}) - \sigma'(\mathbf{T})} = (-1)^{n-d}$, where $\sigma'(\mathbf{G})$, $\sigma'(\mathbf{T})$ are the k'-ranks of $\mathbf{G}$ and $\mathbf{T}$, and $\varepsilon(\mathbf{T}) = (-1)^{n-1}$. Then for each irreducible representation θ of T we have by Corollary 3.7,

$$\begin{aligned}
&\widetilde{\theta}(\widetilde{N}_{\mathbf{T}}(f'_{\mathbf{T}}(\widetilde{e'}))) \\
= \;& q^{-nm/2} \varepsilon'(\mathbf{T}) \sum_{t' \in T'} \theta^{-1}(N_{\mathbf{T}}(t')) \chi'(\mathrm{Tr}\,(t')) \\
= \;& q^{-nm/2} (-1)^{n-d} \sum_{a'_1, \dots, a'_d} \theta^{-1}(N_{nm/d,n}(a'_1 {a'_2}^q \cdots)) \\
& \qquad \times \chi_n(\mathrm{Tr}_{\,nm/d,n}(a'_1 + \cdots + a'_d)) \\
= \;& q^{-nm/2} (-1)^{n-d} \prod_{i=0}^{d-1} G(\chi_n \circ \mathrm{Tr}_{\,nm/d,n}, \theta^{-1} \circ N_{nm/d,n} \circ F_q^i) \\
= \;& q^{-nm/2} (-1)^{n-d} G(\chi_n \circ \mathrm{Tr}_{\,nm/d,n}, \theta^{-1} \circ N_{nm/d,n})^d,
\end{aligned}$$

where $F_q(a) = a^q$ for $a \in k^{\times}_{nm/d}$ and $G(\chi_n \circ \mathrm{Tr}_{\,nm/d,n}, \theta \circ N_{nm/d,n})$ is the Gauss sum over $k_{nm/d}$ with $\chi_n \circ \mathrm{Tr}_{\,nm/d,n}$ (resp. $\theta \circ N_{nm/d,n}$) as its additive (resp. multiplicative) character. Now the Davenport-Hasse theorem implies

$$-G(\chi_n \circ \mathrm{Tr}_{\,nm/d,n}, \theta^{-1} \circ N_{nm/d,n}) = (-G(\chi_n, \theta^{-1}))^{m/d}.$$

Thus we have

$$\widetilde{\theta}(\widetilde{N}_{\mathbf{T}}(f'_{\mathbf{T}}(\widetilde{e}'))) = q^{-nm/2}(-1)^{m+n}G(\chi_n, \theta^{-1})^m.$$

On the other hand we have $f_{\mathbf{T},\theta}(\widetilde{e}) = q^{-n/2}(-1)^{n-1}G(\chi_n, \theta^{-1})$, and the result follows. □

As a corollary we obtain what may be viewed as an extension of the Davenport-Hasse relation for Gauss sums of field extensions to Gauss sums of irreducible components of the Gelfand-Graev representation of $GL_n(k')$ and $GL_n(k)$.

Corollary 5.2. *Keep the notation of the previous theorem and Corollary 3.7. For each irreducible representation θ of T, we have*

$$\varepsilon(\pi'_{\mathbf{T},\theta\circ\widetilde{N}_{\mathbf{T}}}, \chi') = (-1)^{n(m-1)}\varepsilon(\pi_{\mathbf{T},\theta}, \chi)^m,$$

for components of the Gelfand-Graev representations of $GL_n(k')$ and $GL_n(k)$ respectively which correspond by the norm map Δ.

The proof is immediate by the previous Theorem and Corollary 3.7.

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