Infinite Systems of Non-Colliding Brownian Particles

Makoto Katori, Taro Nagao and Hideki Tanemura

Abstract.

Non-colliding Brownian particles in one dimension is studied. N Brownian particles start from the origin at time 0 and then they do not collide with each other until finite time T. We derive the determinantal expressions for the multitime correlation functions using the self-dual quaternion matrices. We consider the scaling limit of the infinite particles $N \to \infty$ and the infinite time interval $T \to \infty$. Depending on the scaling, two limit theorems are proved for the multitime correlation functions, which may define temporally inhomogeneous infinite particle systems.

§1. Introduction

We consider the process $X(t)$, which represents the system of N Brownian motions in one dimension all started from the origin and conditioned never to collide with each other up to time T. If we take the limit $T \to \infty$, the system becomes a temporally homogeneous diffusion process $Y(t)$, which is the Doob h-transform [3] of the absorbing Brownian motion in a Weyl chamber

$$\mathbf{R}_<^N = \left\{ \mathbf{x} = (x_1, x_2, \dots, x_N) \in \mathbf{R}^N ; x_1 < x_2 < \cdots < x_N \right\},$$

with harmonic function $h_N(\mathbf{x}) = \prod_{1 \le i < j \le N}(x_j - x_i)$ [8]. By virtue of the Karlin-McGregor formula [12, 13], its transition density $f_N(t, \mathbf{x}, \mathbf{y})$ from the state $\mathbf{x}$ to $\mathbf{y}$ in $\mathbf{R}_<^N$ in time period $t > 0$ is given by

$$f_N(t, \mathbf{x}, \mathbf{y}) = \det_{1 \le i, j \le N} \left(p_t(x_i, y_j) \right),$$

where $p_t(x, y) = \frac{1}{\sqrt{2\pi t}} e^{-(x-y)^2/2t}$. On the other hand, if the non-colliding time interval T remains finite, the process $X(t), 0 \le t \le T$, is temporally inhomogeneous [15].

We notice an integral formula found in Harish-Chandra [9], Itzykson and Zuber [10], and Mehta [16],

$$\frac{\det\limits_{1\leq i,j\leq N}(p_t(x_i,y_j))}{h_N(\mathbf{x})h_N(\mathbf{y})} = c \int dU \, \exp\left[-\frac{1}{2t}\mathrm{tr}(X - U^\dagger YU)^2\right]$$

with $c^{-1} = (2\pi)^{N/2}t^{N^2/2}\prod_{i=1}^{N+1}\Gamma(i)$, where X and Y are the $N \times N$ diagonal matrices, $X_{ij} = x_i\delta_{ij}, Y_{ij} = y_i\delta_{ij}$, and the integral is taken over the group of unitary matrix U of size N. This equality implies that the non-colliding Brownian motions such as $X(t)$ and $Y(t)$ can be described by using the eigenvalue-statistics of Hermitian random matrices in Gaussian ensembles [18]. In earlier papers[14, 15], it was shown that $Y(t)$ is identified with Dyson's Brownian motion model with $\beta = 2$ [4] and the particle distribution is expressed by the probability density of eigenvalues of random matrices in the Gaussian unitary ensemble (GUE) with variance t, while $\sqrt{\frac{T}{t(2T-t)}}X(t)$ coincides with the distribution of eigenvalues of random matrices in the Pandey-Mehta ensemble [19, 25] with $\alpha = \sqrt{\frac{T-t}{T}}$, and this temporally inhomogeneous process exhibits a transition from the GUE statistics to the Gaussian orthogonal ensemble (GOE) statistics as the time t goes on from 0 to T.

It is known that the eigenvalue distributions of Hermitian random matrices have determinantal expressions. For instance, in the GUE, the probability density of N eigenvalues is expressed by

$$\rho_N(x_1, x_2, \ldots, x_N) = \frac{1}{N!}\det_{1\leq i,j\leq N}(K_N(x_i, x_j)),$$

with $K_N(x, y) = \sum_{\ell=0}^{N-1}\varphi_\ell(x)\varphi_\ell(y)$, where

(1.1)
$$\varphi_\ell(x) = \frac{1}{\sqrt{h_\ell}}e^{-x^2/2}H_\ell(x)$$

with the ℓ-th Hermite polynomial $H_\ell(x)$ and $h_\ell = \sqrt{\pi}\,2^\ell\,\ell!$. By the orthogonality of $\varphi_k(x)$, we can prove the equality

(1.2)
$$\int \det_{1\leq i,j\leq N'} K_N(x_i, x_j)\, dx_{N'}$$
$$= (N - N' + 1)\det_{1\leq i,j\leq N'-1} K_N(x_i, x_j)$$

for any $1 \leq N' \leq N$. Such integral property enables us not only to obtain determinantal expressions for correlation functions, but also to argue the $N \to \infty$ limit of the system by studying the large N asymptotic of

the function $K_N(x, y)$. With proper scaling limit, determinantal point processes with sine-kernel and Airy-kernel are derived. See [27] and references therein.

In the present paper, we derive the determinantal expressions of the multitime correlation functions for the process $X(t)$. Our aim is to prove limit theorems of the multitime correlation functions in the scaling limits of infinite particles $N \to \infty$ and infinite time interval $T \to \infty$. Depending on the scaling, we derive two kinds of limit theorems, one of which provides a spatially homogeneous but temporally inhomogeneous infinite particle system (Theorem 1), and other of which does the system with inhomogeneity both in space and time (Theorem 2). We remark that it is easier to prove the limit theorems for Dyson's Brownian motion model $Y(t)$. Corresponding to Theorem 1, we will obtain the multitime correlation functions of the homogeneous system, which coincides with the system studied by Spohn [28], Osada [24], and Nagao and Forrester [21]. Similarly, corresponding to Theorem 2, an infinite system with spatial inhomogeneity will be derived, which is related with the Airy process recently studied by Prähofer and Spohn [26] and Johansson [11].

One of the key points of our arguments is that, in order to give the determinantal expressions for the correlation functions for the present processes, we shall prepare matrices with the elements, which are neither real nor complex numbers, but quaternions

$$q = q_0 + q_1 e_1 + q_2 e_2 + q_3 e_3 \in \mathbf{Q}$$

with $q_i \in \mathbf{C}, 0 \leq i \leq 3$, in which the four basic units $\{1, e_1, e_2, e_3\}$ have the following 2×2 matrix representations, $C : \mathbf{Q} \mapsto \mathrm{Mat}_2(\mathbf{C})$;

$$C(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad C(e_1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

$$C(e_2) = \begin{pmatrix} 0 & -\sqrt{-1} \\ -\sqrt{-1} & 0 \end{pmatrix}, \quad C(e_3) = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}.$$

The dual of a quaternion q is defined by $q^\dagger = q_0 - \sum_{i=1}^{3} q_i e_i$, and for a quaternion matrix $Q = (q_{ij}), q_{ij} \in \mathbf{Q}$, its dual matrix $Q^\dagger = ((Q^\dagger)_{ij})$ is defined to have the elements $(Q^\dagger)_{ij} = q_{ji}^\dagger$. Following Dyson's definition of the quaternion determinant for self-dual matrices [5, 17, 18], we can give the quaternion determinantal expressions having the similar properties to (1.2) for arbitrary multitime correlation functions for $X(t)$ (Theorem 3). As briefly reported in [23], the present results can be regarded as simple applications of the results given in Nagao and Forrester [22] and Nagao [20] for multimatrix models, and in Forrester,

Nagao and Honner [6] for the asymptotic of quaternion determinantal systems, here we give, however, a self-contained explanation for all the formulae and calculus developed in the random matrix theory, which are used to prove our limit theorems.

The theorems proved here mean the convergence of processes in the sense of finite dimensional distributions. As argued in Prähofer and Spohn [26] and in Johansson [11], tightness in time should be confirmed.

§2.   Statement of Results

For a given $T > 0$, we define

$$(2.1) \qquad g_N^T(s, \mathbf{x}; t, \mathbf{y}) = \frac{f_N(t - s, \mathbf{x}, \mathbf{y}) \mathcal{N}_N(T - t, \mathbf{y})}{\mathcal{N}_N(T - s, \mathbf{x})}$$

for $0 \leq s \leq t \leq T, \mathbf{x}, \mathbf{y} \in \mathbf{R}_<^N$, where $\mathcal{N}_N(t, \mathbf{x}) = \int_{\mathbf{R}_<^N} f_N(t, \mathbf{x}, \mathbf{y}) d\mathbf{y}$, which is the probability that a Brownian motion started at $\mathbf{x} \in \mathbf{R}_<^N$ does not hit the boundary of $\mathbf{R}_<^N$ up to time $t > 0$. The function $g_N^T(s, \mathbf{x}; t, \mathbf{y})$ can be regarded as the transition probability density from the state $\mathbf{x} \in \mathbf{R}_<^N$ at time s to the state $\mathbf{y} \in \mathbf{R}_<^N$ at time t, and associated with the temporally inhomogeneous diffusion process, which is the N Brownian motions conditioned not to collide with each other in a time interval $[0, T]$. In [14, 15] it was shown that as $|\mathbf{x}| \to 0$, $g_N^T(0, \mathbf{x}, t, \mathbf{y})$ converges to

$$(2.2) \qquad g_N^T(0, \mathbf{0}, t, \mathbf{y}) \equiv C(N, T, t) h_N(\mathbf{y}) \mathcal{N}_N(T - t, \mathbf{y}) \prod_{i=1}^N p_t(0, y_i),$$

where $C(N, T, t) = \pi^{N/2} \left(\prod_{j=1}^N \Gamma(j/2) \right)^{-1} T^{N(N-1)/4} t^{-N(N-1)/2}$. Then the diffusion process $X(t)$ starting from $\mathbf{0}$ can be constructed.

We denote by $\mathfrak{X}$ the space of countable subset ξ of $\mathbf{R}$ satisfying $\sharp(\xi \cap K) < \infty$ for any compact subset K. We introduce the map γ from $\bigcup_{n=1}^\infty \mathbf{R}^n$ to $\mathfrak{X}$ defined by $\gamma(x_1, x_2, \ldots, x_n) = \{x_i\}_{i=1}^n$. Then $\Xi^N(t) = \gamma X(t)$ is the diffusion process on the set $\mathfrak{X}$ with transition density function $\widetilde{g}_N^T(s, \xi; t, \eta)$, $0 \leq s \leq t \leq T$:

$$\widetilde{g}_N^T(s, \xi; t, \eta) = \begin{cases} g_N^T(s, \mathbf{x}; t, \mathbf{y}), & \text{if } s > 0, \ \sharp\xi = \sharp\eta = N, \\ g_N^T(0, \mathbf{0}; t, \mathbf{y}), & \text{if } s = 0, \ \xi = \{0\}, \ \sharp\eta = N, \\ 0, & \text{otherwise,} \end{cases}$$

where $\mathbf{x}$ and $\mathbf{y}$ are the elements of $\mathbf{R}_<^N$ with $\xi = \gamma\mathbf{x}$, $\eta = \gamma\mathbf{y}$. For $\mathbf{x}_N^{(m)} \in \mathbf{R}_<^N$, $1 \leq m \leq M + 1$, and $N' = 1, 2, \ldots, N$, we put $\mathbf{x}_{N'}^{(m)} =$

$\left(x_1^{(m)}, x_2^{(m)}, \ldots, x_{N'}^{(m)}\right)$ and $\xi_m^{N'} = \gamma \mathbf{x}_{N'}^{(m)}$. For a given time interval $[0, T]$, we consider the M intermediate times $0 < t_1 < t_2 < \cdots < t_M < T$. Then the multitime transition density function of the process $\Xi^N(t)$ is given by

$$(2.3) \qquad \rho_N^T(t_1, \xi_1^N; \ldots; t_{M+1}, \xi_{M+1}^N) = \prod_{m=0}^{M} \widetilde{g}_N^T(t_m, \xi_m^N; t_{m+1}, \xi_{m+1}^N),$$

where, for convenience, we set $t_0 = 0$, $t_{M+1} = T$ and $\xi_0^N = \{0\}$. From (2.1) and (2.2) we have

$$(2.4) \qquad \rho_N^T(t_1, \xi_1^N; t_2, \xi_2^N; \ldots; t_{M+1}, \xi_{M+1}^N)$$

$$= C(N, T, t_1) h_N\left(\mathbf{x}_N^{(1)}\right) \operatorname{sgn}\left(h_N\left(\mathbf{x}_N^{(M+1)}\right)\right)$$

$$\times \prod_{i=1}^{N} p_{t_1}\left(0, x_i^{(1)}\right) \prod_{m=1}^{M} \det_{1 \le i,j \le N}\left(p_{t_{m+1}-t_m}\left(x_i^{(m)}, x_j^{(m+1)}\right)\right).$$

For a sequence $\{N_m\}_{m=1}^{M+1}$ of positive integers less than or equal to N, we define the $(N_1, N_2, \ldots, N_{M+1})$-multitime correlation function by

$$(2.5) \qquad \rho_N^T\left(t_1, \xi_1^{N_1}; t_2, \xi_2^{N_2}; \ldots; t_{M+1}, \xi_{M+1}^{N_{M+1}}\right)$$

$$= \int_{\prod_{m=1}^{M+1} \mathbf{R}^{N-N_m}} \prod_{m=1}^{M+1} \frac{1}{(N-N_m)!} \prod_{i=N_m+1}^{N} dx_i^{(m)}$$

$$\times \rho_N^T(t_1, \xi_1^N; t_2, \xi_2^N; \ldots; t_{M+1}, \xi_{M+1}^N).$$

We will study limit theorems of the correlation functions $\rho_N^{T_N}$ as $N \to \infty$. First, we consider the case $T_N = 2N$. Let

$$\widetilde{\mathbb{S}}(s, x; t, y)$$

$$= \begin{cases} \dfrac{1}{\pi} \displaystyle\int_0^1 d\lambda \, \cos(\lambda(x-y)) e^{-\lambda^2(t-s)/2} & \text{if } s > t \\[2ex] \dfrac{\sin(x-y)}{\pi(x-y)} & \text{if } s = t \\[2ex] -\dfrac{1}{\pi} \displaystyle\int_1^\infty d\lambda \, \cos(\lambda(x-y)) e^{-\lambda^2(t-s)/2} & \text{if } s < t \end{cases}$$

$$\mathbb{D}(s, x; t, y) = -\frac{1}{\pi} \int_0^1 d\lambda \; \lambda \sin(\lambda(x - y)) e^{-(s+t)\lambda^2/2}$$

$$\widetilde{\mathbb{I}}(s, x; t, y) = -\frac{1}{\pi} \int_1^\infty d\lambda \; \frac{1}{\lambda} \sin(\lambda(x - y)) e^{(s+t)\lambda^2/2}.$$

And let $\mathbf{q}^{m,n}(x, y)$ be the quaternion, whose 2×2 matrix expression is given by

$$C(\mathbf{q}^{m,n}(x, y)) = \begin{pmatrix} \widetilde{\mathbb{S}}(s_m, x; s_n, y) & \widetilde{\mathbb{I}}(s_m, x; s_n, y) \\ \mathbb{D}(s_m, x; s_n, y) & \widetilde{\mathbb{S}}(s_n, y; s_m, x) \end{pmatrix}.$$

Let $M \geq 1$ and $\{N_m\}_{m=1}^{M+1}$ be a sequence of positive integers. We denote by $\mathbb{Q}\left(\mathbf{x}_{N_1}^{(1)}, \mathbf{x}_{N_2}^{(2)}, \ldots, \mathbf{x}_{N_{M+1}}^{(M+1)}\right)$ the self-dual $\sum_{m=1}^{M+1} N_m \times \sum_{m=1}^{M+1} N_m$ quaternion matrix whose elements are $\mathbf{q}^{m,n}\left(x_i^{(m)}, x_j^{(n)}\right)$, $1 \leq i \leq N_m$, $1 \leq j \leq N_n$, $1 \leq m, n \leq M + 1$, that is,

$$
\mathbb{Q}\left(\mathbf{x}_{N_1}^{(1)}, \mathbf{x}_{N_2}^{(2)}, \ldots, \mathbf{x}_{N_{M+1}}^{(M+1)}\right)
$$
$$
= \begin{bmatrix}
\mathbb{Q}^{1,1}\left(\mathbf{x}_{N_1}^{(1)}, \mathbf{x}_{N_1}^{(1)}\right) & \cdots & \mathbb{Q}^{1,M+1}\left(\mathbf{x}_{N_1}^{(1)}, \mathbf{x}_{N_{M+1}}^{(M+1)}\right) \\
\mathbb{Q}^{2,1}\left(\mathbf{x}_{N_2}^{(2)}, \mathbf{x}_{N_1}^{(1)}\right) & \cdots & \mathbb{Q}^{2,M+1}\left(\mathbf{x}_{N_2}^{(2)}, \mathbf{x}_{N_{M+1}}^{(M+1)}\right) \\
\cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots \\
\mathbb{Q}^{M+1,1}\left(\mathbf{x}_{N_{M+1}}^{(M+1)}, \mathbf{x}_{N_1}^{(1)}\right) & \cdots & \mathbb{Q}^{M+1,M+1}\left(\mathbf{x}_{N_{M+1}}^{(M+1)}, \mathbf{x}_{N_{M+1}}^{(M+1)}\right)
\end{bmatrix}
$$

with blocks of $N_m \times N_n$ quaternion matrices

$$\mathbb{Q}^{m,n}\left(\mathbf{x}_{N_m}^{(m)}, \mathbf{x}_{N_n}^{(n)}\right) = \left(\mathbf{q}^{m,n}\left(x_i^{(m)}, x_j^{(n)}\right)\right)_{1 \leq i \leq N_m, 1 \leq j \leq N_n},$$

for $1 \leq m, n \leq M + 1$.

For an $N \times N$ self-dual quaternion matrix Q, the quaternion determinant $\mathrm{Tdet}Q$ is defined by Dyson [5] as

$$\mathrm{Tdet}Q = \sum_{\pi \in S_N} (-1)^{N - \ell(\pi)} \prod_1^{\ell(\pi)} q_{ab} q_{bc} \cdots q_{da},$$

where $\ell(\pi)$ denotes the number of exclusive cycles of the form $(a \to b \to c \to \cdots \to d \to a)$ included in a permutation $\pi \in S_N$.

Theorem 1. *Let $T_N = 2N$. For any $M \geq 1$, any sequence $\{N_m\}_{m=1}^{M+1}$ of positive integers, and any strictly increasing sequence $\{s_m\}_{m=1}^{M+1}$ of nonpositive numbers with $s_{M+1} = 0$,*

$$\rho\left(s_1, \xi_1^{N_1}; s_2, \xi_2^{N_2}; \ldots; s_M, \xi_M^{N_M}; 0, \xi_{M+1}^{N_{M+1}}\right)$$

$$\equiv \lim_{N \to \infty} \rho_N^{T_N}\left(T_N + s_1, \xi_1^{N_1}; T_N + s_2, \xi_2^{N_2}; \ldots; T_N, \xi_{M+1}^{N_{M+1}}\right)$$

$$= \mathrm{Tdet}\,\mathbb{Q}\left(\mathbf{x}_{N_1}^{(1)}, \mathbf{x}_{N_2}^{(2)}, \ldots, \mathbf{x}_{N_{M+1}}^{(M+1)}\right).$$

Remark 1. The above system is spatially homogeneous, since all elements of the quaternion determinant are functions of difference of positions, $x_i^{(m)} - x_j^{(n)}$. This expresses the bulk property of our infinite particle system. When $M = 1$, the present system is equivalent with the $N \to \infty$ limit of the two-matrix model reported by Pandey and Mehta [19, 25]. In the system defined by Theorem 1, if we take the further limit such that $s_m \to -\infty$ with the time difference $s_n - s_m$ fixed, $1 \leq m, n \leq M$, then $\mathbb{D}(s_m, x; s_n, y) \to \infty$, $\widetilde{\mathbb{I}}(s_m, x; s_n, y) \to 0$, while the product $\mathbb{D}(s_m, x; s_n, y)\widetilde{\mathbb{I}}(s_m, x; s_n, y) \to 0$. Therefore, we may replace $\mathbb{D}$ and $\widetilde{\mathbb{I}}$ by zeros in this limit, and the quaternion determinant $\mathrm{Tdet}\,\mathbb{Q}\left(\mathbf{x}_{N_1}^{(1)}, \mathbf{x}_{N_2}^{(2)}, \ldots, \mathbf{x}_{N_M}^{(M)}\right)$ will be reduced to an ordinary determinant $\det \mathbb{A}\left(\mathbf{x}_{N_1}^{(1)}, \mathbf{x}_{N_2}^{(2)}, \ldots, \mathbf{x}_{N_M}^{(M)}\right)$ with the elements $\mathbf{a}^{m,n}\left(x_i^{(m)}, x_j^{(n)}\right) = \widetilde{\mathbb{S}}\left(s_m, x_i^{(m)}; s_n, x_j^{(n)}\right)$. Hence, we obtain a temporally and spatially homogeneous system, whose correlation functions are given by

$$\rho'\left(s_1, \xi_1^{N_1}; s_2, \xi_2^{N_2}; \ldots; s_M, \xi_M^{N_M}\right) = \det \mathbb{A}\left(\mathbf{x}_{N_1}^{(1)}, \mathbf{x}_{N_2}^{(2)}, \ldots, \mathbf{x}_{N_M}^{(M)}\right).$$

Such a homogeneous system was studied by Spohn [28], Osada [24] and Nagao and Forrester [21] as an infinite particle limit of Dyson's Brownian motion model[4].

Next, we consider the case that $T_N = 2N^{1/3}$. In order to state the result, we have to introduce the following functions. Let $\mathrm{Ai}(z)$ be the Airy function:

$$(2.6) \qquad \mathrm{Ai}(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\sqrt{-1}(zt+t^3/3)}dt.$$

For $s, t < 0$ and $x, y \in \mathbf{R}$, we put

$$
\begin{aligned}
\mathcal{D}(s, x; t, y) &= \frac{1}{4} \left[\int_0^\infty d\lambda \, e^{s\lambda/2} \mathrm{Ai}(x + \lambda) \frac{d}{d\lambda} \left\{ e^{t\lambda/2} \mathrm{Ai}(y + \lambda) \right\} \right. \\
&\quad \left. - \int_0^\infty d\lambda \, e^{t\lambda/2} \mathrm{Ai}(y + \lambda) \frac{d}{d\lambda} \left\{ e^{s\lambda/2} \mathrm{Ai}(x + \lambda) \right\} \right], \\
\widetilde{\mathcal{I}}(s, x; t, y) &= \int_0^\infty d\lambda \, e^{t\lambda/2} \mathrm{Ai}(y - \lambda) \int_\lambda^\infty d\lambda' \, e^{s\lambda'/2} \mathrm{Ai}(x - \lambda') \\
&\quad - \int_0^\infty d\lambda \, e^{s\lambda/2} \mathrm{Ai}(x - \lambda) \int_\lambda^\infty d\lambda' \, e^{t\lambda'/2} \mathrm{Ai}(y - \lambda'),
\end{aligned}
$$

and

$$
\widetilde{\mathcal{S}}(s, x; t, y) = \mathcal{S}(s, x; t, y) - \mathcal{P}(s, x; t, y) 1(s < t),
$$

with

$$
\begin{aligned}
\mathcal{S}(s, x; t, y) &= \int_0^\infty d\lambda \, e^{(t-s)\lambda/2} \mathrm{Ai}(x + \lambda) \mathrm{Ai}(y + \lambda) \\
&\quad + \frac{1}{2} \mathrm{Ai}(y) \int_0^\infty d\lambda \, e^{s\lambda/2} \mathrm{Ai}(x - \lambda), \\
\mathcal{P}(s, x; t, y) &= \int_{-\infty}^\infty d\lambda \, e^{(t-s)\lambda/2} \mathrm{Ai}(x + \lambda) \mathrm{Ai}(y + \lambda),
\end{aligned}
$$

where $1(s < t) = 1$ if $s < t$, and $= 0$ otherwise. And let $\mathfrak{q}^{m,n}(x, y)$ be the quaternion, whose 2×2 matrix expression is given by

$$
C(\mathfrak{q}^{m,n}(x, y)) = \begin{pmatrix} \widetilde{\mathcal{S}}(s_m, x; s_n, y) & \widetilde{\mathcal{I}}(s_m, x; s_n, y) \\ \mathcal{D}(s_m, x; s_n, y) & \widetilde{\mathcal{S}}(s_n, y; s_m, x) \end{pmatrix}.
$$

Let $M \geq 1$ and $\{N_m\}_{m=1}^{M+1}$ be a sequence of positive integers. We denote by $\mathcal{Q} \left(\mathbf{x}_{N_1}^{(1)}, \mathbf{x}_{N_2}^{(2)}, \ldots, \mathbf{x}_{N_{M+1}}^{(M+1)} \right)$ the self-dual $\sum_{m=1}^{M+1} N_m \times \sum_{m=1}^{M+1} N_m$ quaternion matrix whose elements are $\mathfrak{q}^{m,n} \left(x_i^{(m)}, x_j^{(n)} \right)$, $1 \leq i \leq N_m$, $1 \leq j \leq N_n$, $1 \leq m, n \leq M + 1$.

Theorem 2. *Let $T_N = 2N^{1/3}$ and $a_N(s) = 2N^{2/3} - s^2/4$ for $s \in \mathbf{R}$. For any $M \geq 1$, any sequence $\{N_m\}_{m=1}^{M+1}$ of positive integers, and any strictly increasing sequence $\{s_m\}_{m=1}^{M+1}$ of nonpositive numbers with $s_{M+1} = 0$,*

$$
\begin{aligned}
&\widehat{\rho} \left(s_1, \xi_1^{N_1}; \ldots; s_{M+1}, \xi_{M+1}^{N_{M+1}} \right) \\
&\equiv \lim_{N \to \infty} \rho_N^{T_N} \left(T_N + s_1, \theta_{a_N(s_1)} \xi_1^{N_1}; \ldots; T_N, \theta_{a_N(s_{M+1})} \xi_{M+1}^{N_{M+1}} \right) \\
&= \mathrm{Tdet} \, \mathcal{Q} \left(\mathbf{x}_{N_1}^{(1)}, \ldots, \mathbf{x}_{N_{M+1}}^{(M+1)} \right),
\end{aligned}
$$

where $\theta_u\{x_i\} = \{x_i + u\}$.

Remark 2. This theorem may define an infinite particle system, in which any type of space-time correlation function is given by the above quaternion determinant. This quaternion determinantal system is the same as that derived in Forrester, Nagao and Honner [6], and it is inhomogeneous both in space and time. The spatial inhomogeneity is attributed to the fact that this system expresses the edge property of the infinite non-colliding Brownian particles. Thus, if we take the bulk limit, $x_i^{(m)} \to -\infty$ with the position differences $x_i^{(m)} - x_j^{(n)}$ fixed, then the system should recover spatial homogeneity. It is confirmed by observing that the quaternion determinantal system given in Theorem 1 can be derived as the bulk limit of the system of Theorem 2, if we use the asymptotic expansion of the Airy function (2.6) [1],

$$\text{Ai}(-x) \sim \frac{1}{\pi^{1/2}x^{1/4}} \cos\left(\frac{2}{3}x^{3/2} - \frac{\pi}{4}\right) \quad \text{as } x \to \infty.$$

On the other hand, keeping the spatial inhomogeneity, one can consider the limit $s_m \to -\infty$ with the time difference $s_n - s_m$ fixed, $1 \le m, n \le M$. In this limit, $\mathcal{D}(s_m, x; s_n, y) \to 0$, $\widetilde{\mathcal{I}}(s_m, x; s_n, y) \to 0$, and

$$\mathcal{S}(s_m, x; s_n, y) \to \int_0^\infty d\lambda \, e^{(s_n - s_m)\lambda/2}\text{Ai}(x + \lambda)\text{Ai}(y + \lambda).$$

Hence the off-diagonal elements vanish in the 2×2 matrix expressions of quaternion $\mathfrak{q}^{m,n}(x, y)$ and

$$C(\mathfrak{q}^{m,n}(x, y)) \to \left(\begin{array}{cc} \mathfrak{a}^{m,n}(x, y) & 0 \\ 0 & \mathfrak{a}^{n,m}(y, x) \end{array}\right)$$

for $1 \le m, n \le M$, where

$$\mathfrak{a}^{m,n}(x, y) = \mathfrak{a}(s_m, x; s_n, y)$$
$$= \begin{cases} \displaystyle\int_0^\infty d\lambda \, e^{(s_n - s_m)\lambda/2}\text{Ai}(x + \lambda)\text{Ai}(y + \lambda) & \text{if } m \ge n \\ \displaystyle-\int_{-\infty}^0 d\lambda \, e^{(s_n - s_m)\lambda/2}\text{Ai}(x + \lambda)\text{Ai}(y + \lambda) & \text{if } m < n. \end{cases}$$

Then the quaternion determinant $\text{Tdet}\,\mathcal{Q}\left(\mathbf{x}_{N_1}^{(1)}, \mathbf{x}_{N_2}^{(2)}, \ldots, \mathbf{x}_{N_M}^{(M)}\right)$ is reduced to an ordinary determinant $\det\mathcal{A}\left(\mathbf{x}_{N_1}^{(1)}, \mathbf{x}_{N_2}^{(2)}, \ldots, \mathbf{x}_{N_M}^{(M)}\right)$ with the

elements $\mathfrak{a}^{m,n}\left(x_i^{(m)}, x_j^{(n)}\right)$. In this way, we will obtain the infinite particle system, which is temporally homogeneous but spatially inhomogeneous with the multitime correlation functions

$$\widehat{\rho}'\left(s_1, \xi_1^{N_1}; s_2, \xi_2^{N_2}; \ldots; s_M, \xi_M^{N_M}\right) = \det \mathcal{A}\left(\mathbf{x}_{N_1}^{(1)}, \mathbf{x}_{N_2}^{(2)}, \ldots, \mathbf{x}_{N_M}^{(M)}\right).$$

In particular, if we set $N_1 = N_2 = \cdots = N_M = 1$, then

$$\widehat{\rho}'\left(s_1, \{x^{(1)}\}; \ldots; s_M, \{x^{(M)}\}\right) = \det_{1 \leq m,n \leq M}\left(\mathfrak{a}^{m,n}\left(x^{(m)}, x^{(n)}\right)\right).$$

This is the same as the system called the Airy process by Prähofer and Spohn in [26]. (See also [11].)

§3.   Quaternion determinantal expressions of the correlations

In this section we give quaternion determinantal expressions for the correlation functions defined in (2.5) along the procedure in [20]. From now on we consider the case N is even, for simplicity of notations. See [20], for necessary modifications for odd case. For $1 \leq m, n \leq M + 1$, define

$$(3.1) \quad F^{m,n}(x,y) = \int_{-\infty}^{\infty} dw \int_{-\infty}^{w} dz \begin{vmatrix} p_{T-t_m}(x,z) & p_{T-t_m}(x,w) \\ p_{T-t_n}(y,z) & p_{T-t_n}(y,w) \end{vmatrix},$$

where $p_0(y,x)dy = \delta_x(dy)$. We introduce an antisymmetric inner products

$$\langle f, g \rangle_m = \int_{\mathbf{R}} dx \int_{\mathbf{R}} dy \, F^{m,m}(x,y)f(x)g(y),$$

and

$$\langle f, g \rangle = \int_{\mathbf{R}} dx \int_{\mathbf{R}} dy \, F^{1,1}(x,y)p_{t_1}(0,x)p_{t_1}(0,y)f(x)g(y).$$

For $k = 0, 1, \ldots$ we consider the polynomials in x of degree k defined by

$$(3.2) \qquad R_k(x) = z_1^{-k} \sum_{j=1}^{k} \alpha_{kj} H_j\left(\frac{x}{c_1}\right) z_1^j,$$

where $c_1 = \sqrt{\frac{t_1(2T-t_1)}{T}}$, $z_1 = \sqrt{\frac{2T-t_1}{t_1}}$,

$$(3.3) \qquad \alpha_{kj} = \begin{cases} 2^{-k}c_1^k \delta_{kj}, & \text{if } k \text{ is even,} \\ 2^{-k}c_1^k\left\{\delta_{kj} - 2(k-1)\delta_{k-2\,j}\right\}, & \text{if } k \text{ is odd,} \end{cases}$$

and $H_j(x)$ are the Hermite polynomials. They are monic and satisfy the skew orthogonal relations:

$$\langle R_{2j}, R_{2\ell+1}\rangle = -\langle R_{2\ell+1}, R_{2j}\rangle = r_j \delta_{j\ell},$$
$$\langle R_{2j}, R_{2\ell}\rangle = \langle R_{2j+1}, R_{2\ell+1}\rangle = 0, \quad j, \ell = 0, 1, 2, \ldots,$$

where

$$r_j = \frac{\Gamma(j+\frac{1}{2})\Gamma(j+1)}{\pi}\left(\frac{t_1^2}{T}\right)^{2j+1/2}.$$

For $m = 1, 2, \ldots, M+1$, and $k = 0, 1, \ldots$, put

$$(3.4) \qquad R_k^{(m)}(x) = \int_{\mathbf{R}} dy \ R_k(y) p_{t_1}(0,y) p_{t_m - t_1}(y, x).$$

Then we can prove the skew orthogonal relations

$$\langle R_{2j}^{(m)}, R_{2\ell+1}^{(m)}\rangle_m = -\langle R_{2\ell+1}^{(m)}, R_{2j}^{(m)}\rangle_m = r_j \delta_{j\ell},$$
$$\langle R_{2j}^{(m)}, R_{2\ell}^{(m)}\rangle_m = \langle R_{2j+1}^{(m)}, R_{2\ell+1}^{(m)}\rangle_m = 0, \quad j, \ell = 0, 1, 2, \ldots,$$

for any $m = 1, 2, \ldots, M+1$. For $m = 1, 2, \ldots, M+1$, define

$$(3.5) \qquad \Phi_k^{(m)}(x) = \int_{\mathbf{R}} dy \ R_k^{(m)}(y) F^{m,m}(y, x), \quad k = 0, 1, 2, \ldots.$$

Now we introduce the functions on $\mathbf{R}^2$, $D^{m,n}$, $I^{m,n}$ and $S^{m,n}$, $1 \le m, n \le M+1$, given by

$$(3.6) \ \ D^{m,n}(x, y) = \sum_{k=0}^{(N/2)-1} \frac{1}{r_k}\left[R_{2k}^{(m)}(x) R_{2k+1}^{(n)}(y) - R_{2k+1}^{(m)}(x) R_{2k}^{(n)}(y)\right],$$

$$(3.7) \ \ I^{m,n}(x, y) = -\sum_{k=0}^{(N/2)-1} \frac{1}{r_k}\left[\Phi_{2k}^{(m)}(x)\Phi_{2k+1}^{(n)}(y) - \Phi_{2k+1}^{(m)}(x)\Phi_{2k}^{(n)}(y)\right],$$

$$(3.8) \ \ S^{m,n}(x, y) = \sum_{k=0}^{(N/2)-1} \frac{1}{r_k}\left[\Phi_{2k}^{(m)}(x) R_{2k+1}^{(n)}(y) - \Phi_{2k+1}^{(m)}(x) R_{2k}^{(n)}(y)\right].$$

Further we define

$$(3.9) \qquad \tilde{S}^{m,n}(x, y) = S^{m,n}(x, y) - p_{t_n - t_m}(x, y)1(m < n),$$
$$(3.10) \qquad \tilde{I}^{m,n}(x, y) = I^{m,n}(x, y) + F^{m,n}(x, y).$$

Define the quaternions $q^{m,n}(x,y)$, $1 \leq m,n \leq M+1$, $x,y \in \mathbf{R}$ so that these 2×2 matrix expressions $C(q^{m,n}(x,y))$ are given by

$$C(q^{m,n}(x,y)) = \begin{pmatrix} \widetilde{S}^{m,n}(x,y) & \widetilde{I}^{m,n}(x,y) \\ D^{m,n}(x,y) & \widetilde{S}^{n,m}(y,x) \end{pmatrix}.$$

Let $M \geq 1$ and $\{N_m\}_{m=1}^{M+1}$ be a sequence of positive integers less than or equal to N. For $\mathbf{x}_N^{(m)} \in \mathbf{R}_<^N$, $1 \leq m \leq M+1$, we denote by $Q\left(\mathbf{x}_{N_1}^{(1)}, \mathbf{x}_{N_2}^{(2)}, \ldots, \mathbf{x}_{N_{M+1}}^{(M+1)}\right)$ the self-dual $\sum_{m=1}^{M+1} N_m \times \sum_{m=1}^{M+1} N_m$ quaternion matrix whose elements are $q^{m,n}\left(x_i^{(m)}, x_j^{(n)}\right)$, $1 \leq i \leq N_m$, $1 \leq j \leq N_n$, $1 \leq m,n \leq M+1$. Then we show the following relation.

Theorem 3. *The multitime correlation function (2.5) is written as*

$$\rho_N^T\left(t_1, \xi_1^{N_1}; \ldots; t_{M+1}, \xi_{M+1}^{N_{M+1}}\right) = \mathrm{Tdet}\, Q\left(\mathbf{x}_{N_1}^{(1)}, \mathbf{x}_{N_2}^{(2)}, \ldots, \mathbf{x}_{N_{M+1}}^{(M+1)}\right).$$

In order to prove the theorem, first we introduce the Pfaffian. For an integer N and an antisymmetric $2N \times 2N$ matrix $A = (a_{ij})$, the Pfaffian is defined as

$$\mathrm{Pf}(A) = \mathrm{Pf}_{1 \leq i < j \leq 2N}(a_{ij})$$
$$= \frac{1}{N!} \sum_\sigma \mathrm{sgn}(\sigma) a_{\sigma(1)\sigma(2)} a_{\sigma(3)\sigma(4)} \cdots a_{\sigma(2N-1)\sigma(2N)},$$

where the summation is extended over all permutations σ of $(1, 2, \ldots, 2N)$ with restriction $\sigma(2k-1) < \sigma(2k)$, $k = 1, 2, \ldots, N$. If Q is an $N \times N$ self-dual quaternion matrix, then

$$(3.11) \qquad\qquad \mathrm{Tdet}\, Q = \mathrm{Pf}\left(JC(Q)\right),$$

where J is an $2N \times 2N$ antisymmetric matrix with only non-zero elements

$$J_{2k+1,2k+2} = -J_{2k+2,2k+1} = 1, \quad k = 0, 1, 2, \ldots, N-1.$$

See, for instance, Mehta [17].

For a function $\psi^{m,n}$ defined on $\mathbf{R}^2$ we denote the $N \times N$-matrices whose (i,j)-entry is $\psi^{m,n}\left(x_i^{(m)}, x_j^{(n)}\right)$ by $\psi^{m,n}\left(\mathbf{x}_N^{(m)}, \mathbf{x}_N^{(n)}\right)$, or simply by $\psi^{m,n}$ for short. And we denote by $R^{(m)}\left(\mathbf{x}_N^{(m)}\right)$ the $N \times N$ matrix with $R^{(m)}\left(\mathbf{x}_N^{(m)}\right)_{i,j} = R_{j-1}^{(m)}(x_i)$, and by $\Phi^{(m)}\left(\mathbf{x}_N^{(m)}\right)$ that with

$\Phi^{(m)}\left(\mathbf{x}_N^{(m)}\right)_{i,j} = \Phi_{j-1}^{(m)}(x_i)$. Let L be the $N \times N$ diagonal matrix with $L_{i,i} = \sqrt{r_{[(i-1)/2]}}$, $i = 1, 2, \ldots, N$, and $\widetilde{R}^{(m)}\left(\mathbf{x}_N^{(m)}\right) = L^{-1}R^{(m)}\left(\mathbf{x}_N^{(m)}\right)$. Then we have

$$(3.12) \qquad \widetilde{R}^{(m)}\left(\mathbf{x}_N^{(1)}\right) J \widetilde{R}^{(n)}\left(\mathbf{x}_N^{(1)}\right)^T = D^{m,n}\left(\mathbf{x}_N^{(m)}, \mathbf{x}_N^{(n)}\right).$$

As the first step of the proof of Theorem 3. We show that the multitime probability density defined in (2.3) is written as

$$(3.13) \qquad \rho_N^T\left(t_1, \xi_1^N; \ldots; t_{M+1}, \xi_{M+1}^N\right) = \mathrm{Tdet}Q\left(\mathbf{x}_N^{(1)}, \ldots, \mathbf{x}_N^{(M+1)}\right).$$

For simplicity of notation, here we give the proof of (3.13) for $M = 2$. It is straightforward to prove (3.13) for general M. Since

$$\mathrm{sgn}\left(h_N\left(\mathbf{x}_N^{(3)}\right)\right) = \mathrm{Pf}_{1 \le i < j \le N}\left(\mathrm{sgn}\left(x_j^{(3)} - x_i^{(3)}\right)\right),$$

and $\mathrm{sgn}(y - x) = F^{3,3}(x, y)$, we have

$$(3.14) \qquad \mathrm{sgn}\left(h_N\left(\mathbf{x}_N^{(3)}\right)\right) = \mathrm{Pf}\left[F^{3,3}\right].$$

Noting that $R_k(x)$ is the monic polynomial of degree k, we have

$$h_N\left(\mathbf{x}^{(1)}\right) = \det_{1 \le i,j \le N}\left(\left(x_j^{(1)}\right)^{i-1}\right) = \det_{1 \le i,j \le N}\left(R_{i-1}\left(x_j^{(1)}\right)\right),$$

and so

$$(3.15) \qquad \prod_{i=1}^N p_{t_1}\left(0, x_i^{(1)}\right) h_N\left(\mathbf{x}^{(1)}\right) = \det\left[R^{(1)}\left(\mathbf{x}_N^{(1)}\right)\right]$$

Since $\det L = \prod_{k=0}^{N/2-1} r_k = C(N, T, t_1)^{-1}$, from (3.12) and (3.15)

$$(3.16) \qquad C(N, T, t_1) \prod_{i=1}^N p_{t_1}\left(0, x_i^{(1)}\right) h_N\left(\mathbf{x}^{(1)}\right) = \det\left[\widetilde{R}^{(1)}\left(\mathbf{x}_N^{(1)}\right)\right]$$

$$= \mathrm{Pf}\left[\widetilde{R}^{(1)}\left(\mathbf{x}_N^{(1)}\right) J \widetilde{R}^{(1)}\left(\mathbf{x}_N^{(1)}\right)^T\right]$$

$$= \mathrm{Pf}\left[D^{1,1}\left(\mathbf{x}_N^{(1)}, \mathbf{x}_N^{(1)}\right)\right].$$

Then from (2.4), (3.14) and (3.16) we have

$$\rho_N^T(t_1, \xi_1^N; t_2, \xi_2^N; t_3, \xi_3^N)$$

$$= \mathrm{Pf}[D^{1,1}]\mathrm{Pf}[F^{3,3}] \prod_{m=1}^{2} \det_{1 \le i,j \le N} \left[p_{t_{m+1}-t_m} \right]$$

$$= (-1)^{3N/2} \mathrm{Pf} \begin{bmatrix} D^{1,1} & O \\ O & -F^{3,3} \end{bmatrix} \prod_{m=1}^{2} \mathrm{Pf} \begin{bmatrix} O & -(p_{t_{m+1}-t_m})^T \\ p_{t_{m+1}-t_m} & O \end{bmatrix}.$$

By basic properties of the Pfaffians, we have

$$\mathrm{Pf} \begin{bmatrix} D^{1,1} & O \\ O & -F^{3,3} \end{bmatrix} \prod_{m=1}^{2} \mathrm{Pf} \begin{bmatrix} O & -(p_{t_{m+1}-t_m})^T \\ p_{t_{m+1}-t_m} & O \end{bmatrix}$$

$$= \mathrm{Pf} \begin{bmatrix} D^{1,1} & O & O & O & O & O \\ O & -F^{3,3} & O & O & O & O \\ O & O & O & -(p_{t_2-t_1})^T & O & O \\ O & O & p_{t_2-t_1} & O & O & O \\ O & O & O & O & O & -(p_{t_3-t_2})^T \\ O & O & O & O & p_{t_3-t_2} & O \end{bmatrix}$$

$$= \mathrm{Pf} \begin{bmatrix} D^{1,1} & O & O & O & O & O \\ O & O & p_{t_2-t_1} & O & O & O \\ O & -(p_{t_2-t_1})^T & O & O & p_{t_3-t_2} & O \\ O & O & O & O & O & O \\ O & O & O & -(p_{t_3-t_2})^T & O & O \\ O & O & O & O & O & -F^{3,3} \end{bmatrix}$$

$$= \mathrm{Pf} \begin{bmatrix} D^{1,1} & O & O & O & O & O \\ O & -F^{1,1} & p_{t_2-t_1} & -F^{1,2} & p_{t_3-t_1} & -F^{1,3} \\ O & -(p_{t_2-t_1})^T & O & O & O & O \\ O & -F^{2,1} & O & -F^{2,2} & p_{t_3-t_2} & -F^{2,3} \\ O & -(p_{t_3-t_1})^T & O & -(p_{t_3-t_2})^T & O & O \\ O & -F^{3,1} & O & -F^{3,2} & O & -F^{3,3} \end{bmatrix}.$$

Since $\mathbf{x}_N^{(1)} \in \mathbf{R}_<^N$, $h_N\left(\mathbf{x}_N^{(1)}\right) \ne 0$, and so $\det\left[R^{(1)}\left(\mathbf{x}_N^{(1)}\right)\right] \ne 0$ by (3.15). Hence we can define matrices

$$U^{(m)} = R^{(m)}\left(\mathbf{x}_N^{(m)}\right) R^{(1)}\left(\mathbf{x}_N^{(1)}\right)^{-1}, \quad V^{(m)} = \Phi^{(m)}\left(\mathbf{x}_N^{(m)}\right) R^{(1)}\left(\mathbf{x}_N^{(1)}\right)^{-1},$$

which satisfies

$$U^{(m)} D^{1,1}(U^{(n)})^T = D^{m,n}, \quad V^{(m)} D^{1,1}(V^{(n)})^T = -I^{m,n},$$
$$V^{(m)} D^{1,1}(U^{(n)})^T = S^{m,n}, \quad U^{(m)} D^{1,1}(V^{(n)})^T = -(S^{n,m})^T.$$

By repeating elementary operations, we see that the last Pfaffian equals to

$$\mathrm{Pf}\begin{bmatrix} D^{1,1} & (S^{1,1})^T & D^{1,2} & (S^{2,1})^T & D^{1,3} & (S^{3,1})^T \\ -S^{1,1} & -\tilde{I}^{1,1} & -\tilde{S}^{1,2} & -\tilde{I}^{1,2} & -\tilde{S}^{1,3} & -\tilde{I}^{1,3} \\ D^{2,1} & (\tilde{S}^{1,2})^T & D^{2,2} & (S^{2,2})^T & D^{2,3} & S^{3,2} \\ -S^{2,1} & -\tilde{I}^{2,1} & -\tilde{S}^{2,2} & -\tilde{I}^{2,2} & -\tilde{S}^{2,3} & -\tilde{I}^{2,3} \\ D^{3,1} & (\tilde{S}^{1,3})^T & D^{3,2} & (\tilde{S}^{2,3})^T & D^{3,3} & (S^{3,3})^T \\ -S^{3,1} & -\tilde{I}^{3,1} & -S^{3,2} & -\tilde{I}^{3,2} & -S^{3,3} & -\tilde{I}^{3,3} \end{bmatrix}$$

$$= (-1)^{3N/2}\mathrm{Pf}\begin{bmatrix} A^{1,1} & A^{1,2} & A^{1,3} \\ A^{2,1} & A^{2,2} & A^{2,3} \\ A^{3,1} & A^{3,2} & A^{3,3} \end{bmatrix},$$

where each $A^{m,n} = (A^{m,n}_{ij})$ is a $2N \times 2N$ matrix which consists of 2×2 blocks

$$A^{m,n}_{ij} = \begin{pmatrix} D^{m,n}_{ij} & \tilde{S}^{n,m}_{ji} \\ -\tilde{S}^{m,n}_{ij} & -\tilde{I}^{m,n}_{ij} \end{pmatrix}.$$

We can see that the above matrix $A = (A^{m,n}_{ij})$ satisfies the relation $A = JC(Q)$. Therefore, (3.13) is derived from (3.11).

For square integrable functions ϕ and ψ defined on $\mathbf{R}^2$, put $\phi * \psi(x,y) = \int_{\mathbf{R}} \phi(x,z)\psi(z,y)dz$. Then we have

$$S^{m,p} * S^{p,m} = I^{m,p} * D^{p,n} = D^{m,p} * F^{p,n} = S^{m,p},$$

$$D^{m,p} * S^{p,n} = D^{m,n}, \quad S^{m,p} * I^{p,n} = S^{m,p} * F^{p,n} = I^{m,n},$$

$$S^{m,p} * p_{t_n - t_p} = S^{m,n}, \quad D^{m,p} * p_{t_n - t_p} = D^{m,n}, \quad \text{if } p < n.$$

Hence by simple calculation we see that

$$\int_{\mathbf{R}} q^{m,m}(z,z)dz = N,$$

$$\int_{\mathbf{R}} q^{m,p}(x,z)q^{p,n}(z,y)dz = q^{m,n}(x,y)$$
$$+ q^{m,n}(x,y)\kappa(n,p) - \kappa(p,m)q^{m,n}(x,y),$$

where $\kappa(n,p)$ is a quaternion with

$$C(\kappa(n,p)) = \begin{pmatrix} 1 - 1(p < n) & 0 \\ 0 & -1(n < p) \end{pmatrix}.$$

Then by slight modification of Theorem 6 in [22] we have the following integral formula for any $1 \leq N_m \leq N, m = 1, 2, \ldots, M + 1$,

$$
\int_{\mathbf{R}} \mathrm{T} \det Q \left(\mathbf{x}_{N_1}^{(1)}, \ldots, \mathbf{x}_{N_m}^{(m)}, \ldots, \mathbf{x}_{N_{M+1}}^{(M+1)} \right) dx_{N_m}^{(m)}
$$
$$
= (N - N_m + 1) \mathrm{T} \det Q \left(\mathbf{x}_{N_1}^{(1)}, \ldots, \mathbf{x}_{N_m-1}^{(m)}, \ldots, \mathbf{x}_{N_{M+1}}^{(M+1)} \right),
$$

which is the generalization of the formula (1.2) given in Introduction of the present paper. Successive application of the above relation yields Theorem 3.

§4.   Expansion using Hermite polynomials

In this section we show expansions of functions $p_{t_n-t_m}$, $R_k^{(m)}$ and $\Phi_k^{(m)}$ by using Hermite polynomials H_k. Put

$$
c_n = \sqrt{\frac{t_n(2T - t_n)}{T}}, \quad \gamma_n = -\frac{T - t_n}{T}, \quad z_n = \sqrt{\frac{2T - t_n}{t_n}},
$$

and $\tau^{(n)} = -\log z_n$. By simple calculation we have

$$
p_{t_n-t_m}(x, y) = \frac{e^{-(t_m/2T)(x/c_m)^2} e^{(t_n/2T)(y/c_n)^2}}{\sqrt{2\pi(t_n - t_m)}}
$$
$$
\times \exp\left(-\frac{\left\{ (y/c_n) - e^{-(\tau^{(n)} - \tau^{(m)})}(x/c_m) \right\}^2}{1 - e^{-2(\tau^{(n)} - \tau^{(m)})}} \right)
$$

for $1 \leq m < n < M + 1$. Using Mehler's formula [2]

$$
\exp\left(-\frac{(y - xz)^2}{1 - z^2} \right) = e^{-y^2} \sqrt{\pi(1 - z^2)} \sum_{k=0}^{\infty} \frac{z^k}{h_k} H_k(x) H_k(y),
$$

we will have the following expansions using the Hermite polynomials. For $1 \leq m < n \leq M + 1$,

$$
(4.1) \quad p_{t_n-t_m}(x, y) = \frac{\sqrt{T} e^{-\frac{1}{2}(1+\gamma_m)(x/c_m)^2} e^{-\frac{1}{2}(1-\gamma_n)(y/c_n)^2}}{\sqrt{t_n(2T - t_m)}}
$$
$$
\times \sum_{k=0}^{\infty} \frac{e^{-k(\tau^{(n)} - \tau^{(m)})}}{h_k} H_k\left(\frac{x}{c_m}\right) H_k\left(\frac{y}{c_n}\right),
$$

and for $1 < m \le M + 1$,

$$
p_{t_1}(0, x) p_{t_m - t_1}(x, y) = \frac{\sqrt{T} e^{-(x/c_1)^2} e^{-\frac{1}{2}(1-\gamma_m)(y/c_m)^2}}{\sqrt{2\pi t_1 t_m (2T - t_1)}}
$$
$$
\times \sum_{k=0}^{\infty} \frac{e^{-k(\tau^{(m)} - \tau^{(1)})}}{h_k} H_k \left(\frac{x}{c_1} \right) H_k \left(\frac{y}{c_m} \right).
$$

Then from (3.2), (3.4) and the orthogonal relation of the Hermitian polynomials, we obtain

$$
(4.2) \quad R_k^{(m)}(x) = \frac{e^{-\frac{1}{2}(1-\gamma_m)(x/c_m)^2} e^{k\tau^{(1)}}}{\sqrt{2\pi t_m}} \sum_{j=0}^{k} \alpha_{kj} e^{-j\tau^{(m)}} H_j \left(\frac{x}{c_m} \right).
$$

From the definition (3.1) and the expansion (4.1) we can obtain

$$
(4.3) \quad F^{m,n}(x, y) = \frac{e^{-\frac{1}{2}(1+\gamma_m)(x/c_m)^2} e^{-\frac{1}{2}(1+\gamma_n)(y/c_n)^2}}{\sqrt{(2T - t_m)(2T - t_n)}}
$$
$$
\times \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \frac{e^{k\tau^{(m)}} e^{\ell\tau^{(n)}}}{h_k h_\ell} H_k \left(\frac{x}{c_m} \right) H_\ell \left(\frac{y}{c_n} \right)
$$
$$
\times \left\langle H_k \left(\frac{\cdot}{\sqrt{T}} \right), H_\ell \left(\frac{\cdot}{\sqrt{T}} \right) \right\rangle_*,
$$

where $\langle \cdot, \cdot \rangle_*$ is the antisymmetric inner product defined by

$$
\langle f, g \rangle_* = \int_{-\infty}^{\infty} dw \int_{-\infty}^{w} dz \, e^{-(z^2 + w^2)/2T} \left[f(z)g(w) - f(w)g(z) \right].
$$

Put $R_k^*(x) = \sum_{j=0}^{k} \alpha_{kj} H_j \left(\frac{x}{c_{M+1}} \right)$. Then $\{R_k^*(x)\}$ satisfy the following skew orthogonal relations

$$
(4.4) \quad \langle R_{2j}^*, R_{2\ell+1}^* \rangle_* = -\langle R_{2\ell+1}^*, R_{2j}^* \rangle_* = r_j^* \delta_{j\ell},
$$
$$
\langle R_{2j}^*, R_{2\ell}^* \rangle_* = 0, \quad \langle R_{2j+1}^*, R_{2\ell+1}^* \rangle_* = 0, \quad \text{for } j, \ell = 0, 1, 2, \ldots,
$$

where $r_\ell^* = 4h_{2\ell} T (c_1/2)^{4\ell+1}$. We put

$$
\beta_{kj} = \begin{cases} 2^k c_1^{-k} \delta_{jk}, & \text{if } k \text{ is even,} \\ 2^k \left(\frac{k-1}{2} \right)! \left\{ c_1^j \left(\frac{j-1}{2} \right)! \right\}^{-1}, & \text{if } k, j \text{ are odd and } k \ge j, \\ 0, & \text{otherwise,} \end{cases}
$$

for nonnegative integers k and j. Then $\sum_{j=s}^{k} \beta_{kj}\alpha_{js} = \delta_{ks}$, if $0 \leq s \leq k$, and

$$(4.5) \qquad H_k\left(\frac{x}{\sqrt{T}}\right) = \sum_{j=0}^{k} \beta_{kj} R_j^*(x).$$

From the definition (3.5) and the equations (4.2) and (4.3) we have

$$\Phi_k^{(m)}(x) = \frac{c_m}{\sqrt{2\pi t_m}\,(2T - t_m)} e^{-\frac{1}{2}(1+\gamma_m)(x/c_m)^2} e^{k\tau^{(1)}}$$
$$\times \sum_{\ell=0}^{\infty} \sum_{j=0}^{k} \frac{e^{\ell\tau^{(m)}}}{h_\ell} H_\ell\left(\frac{x}{c_m}\right) \alpha_{kj} \left\langle H_j\left(\frac{\cdot}{\sqrt{T}}\right), H_\ell\left(\frac{\cdot}{\sqrt{T}}\right)\right\rangle_*$$
$$= \frac{e^{-\frac{1}{2}(1+\gamma_m)(x/c_m)^2}}{\sqrt{2\pi T(2T - t_m)}} e^{k\tau^{(1)}} \sum_{j=0}^{\infty} \langle R_k^*, R_j^* \rangle_* \sum_{\ell=j}^{\infty} \frac{e^{\ell\tau^{(m)}}}{h_\ell} H_\ell\left(\frac{x}{c_m}\right) \beta_{\ell j}.$$

Using the skew orthogonal relations (4.4), we show that for $k = 0, 1, 2, \ldots$

$$(4.6) \qquad \Phi_{2k}^{(m)}(x) = \frac{e^{-\frac{1}{2}(1+\gamma_m)(x/c_m)^2} r_k^*}{\sqrt{2\pi T(2T - t_m)}} e^{2k\tau^{(1)}}$$
$$\times \sum_{\ell=2k+1}^{\infty} \frac{e^{\ell\tau^{(m)}}}{h_\ell} \beta_{\ell 2k+1} H_\ell\left(\frac{x}{c_m}\right),$$

$$(4.7) \qquad \Phi_{2k+1}^{(m)}(x) = -\frac{e^{-\frac{1}{2}(1+\gamma_m)(x/c_m)^2} r_k^*}{\sqrt{2\pi T(2T - t_m)}} e^{(2k+1)\tau^{(1)}}$$
$$\times \sum_{\ell=2k}^{\infty} \frac{e^{\ell\tau^{(m)}}}{h_\ell} \beta_{\ell 2k} H_\ell\left(\frac{x}{c_m}\right).$$

Using above expansions we show the following lemma.

Lemma 4. *For* $1 \leq m, n \leq M + 1$,

$$(4.8) \quad F^{m,n}(x,y) = \sum_{k=0}^{\infty} \frac{1}{r_k} \left[\Phi_{2k}^{(m)}(x)\Phi_{2k+1}^{(n)}(y) - \Phi_{2k+1}^{(m)}(x)\Phi_{2k}^{(n)}(y)\right],$$

$$(4.9) \quad \widetilde{I}^{m,n}(x,y) = \sum_{k=N/2}^{\infty} \frac{1}{r_k} \left[\Phi_{2k}^{(m)}(x)\Phi_{2k+1}^{(n)}(y) - \Phi_{2k+1}^{(m)}(x)\Phi_{2k}^{(n)}(y)\right].$$

Proof. By (4.6), (4.7) and the relation

$$(4.10) \qquad r_k = \frac{1}{2\pi T}\left(\frac{t_1}{2T - t_1}\right)^{2k+1/2} r_k^*,$$

we have

$$\sum_{k=0}^{\infty} \frac{1}{r_k} \left[-\Phi_{2k+1}^{(m)}(x)\Phi_{2k}^{(n)}(y) + \Phi_{2k}^{(m)}(x)\Phi_{2k+1}^{(n)}(y) \right]$$

$$= \frac{e^{-\frac{1}{2}(1+\gamma_m)(x/c_m)^2} e^{-\frac{1}{2}(1+\gamma_n)(y/c_n)^2}}{\sqrt{(2T-t_m)(2T-t_n)}} \sum_{k=0}^{\infty} r_k^*$$

$$\times \left\{ \sum_{j=2k}^{\infty} \frac{e^{j\tau^{(m)}}}{h_j} \beta_{j\,2k} H_j\left(\frac{x}{c_m}\right) \times \sum_{\ell=2k+1}^{\infty} \frac{e^{\ell\tau^{(n)}}}{h_\ell} \beta_{\ell\,2k+1} H_\ell\left(\frac{y}{c_n}\right) \right.$$

$$\left. - \sum_{j=2k+1}^{\infty} \frac{e^{j\tau^{(m)}}}{h_j} \beta_{j\,2k+1} H_j\left(\frac{x}{c_m}\right) \times \sum_{\ell=2k}^{\infty} \frac{e^{\ell\tau^{(n)}}}{h_\ell} \beta_{\ell\,2k} H_\ell\left(\frac{y}{c_n}\right) \right\}.$$

By (4.4) and (4.5) the right hand side of the above equation equals to

$$\frac{e^{-\frac{1}{2}(1+\gamma_m)(x/c_m)^2} e^{-\frac{1}{2}(1+\gamma_n)(y/c_n)^2}}{\sqrt{(2T-t_m)(2T-t_n)}} \sum_{\mu=0}^{\infty}\sum_{\nu=0}^{\infty} \langle R_\mu^*, R_\nu^* \rangle_*$$

$$\times \sum_{j=\mu}^{\infty}\sum_{\ell=\nu}^{\infty} \frac{e^{j\tau^{(m)}} e^{\ell\tau^{(n)}}}{h_j h_\ell} \beta_{j\mu} H_j\left(\frac{x}{c_m}\right) \beta_{\ell\nu} H_\ell\left(\frac{y}{c_n}\right)$$

$$= F^{m,n}(x,y),$$

where we have used (4.3). From the definitions (3.7) and (3.10), (4.9) is derived from (4.8).

§5. Proof of Theorems

The following formulae are known for (1.1) [2, 29]. For $u \in \mathbf{R}$,

$$(5.1) \qquad \lim_{\ell\to\infty} (-1)^\ell \ell^{1/4} \varphi_{2\ell}\left(\frac{u}{2\sqrt{\ell}}\right) = \frac{1}{\sqrt{\pi}} \cos u,$$

$$(5.2) \qquad \lim_{\ell\to\infty} (-1)^\ell \ell^{1/4} \varphi_{2\ell+1}\left(\frac{u}{2\sqrt{\ell}}\right) = \frac{1}{\sqrt{\pi}} \sin u,$$

$$(5.3) \qquad \lim_{\ell\to\infty} 2^{-\frac{1}{4}} \ell^{\frac{1}{12}} \varphi_\ell\left(\sqrt{2\ell} - \frac{u}{\sqrt{2}\,\ell^{1/6}}\right) = \mathrm{Ai}(-u).$$

Here we give the proof of Theorem 2 by using (5.3). The proof of Theorem 1 will be easier and given by the similar argument using (5.1) and (5.2).

Let $b^m(x) = \sqrt{2T - t_m} \exp\left\{1/2\gamma_m(x/c_m)^2 - N\tau^{(m)}\right\}$ and $\zeta^m(x)$ be the quaternion with

$$C(\zeta^m(x)) = \begin{pmatrix} b^m(x) & 0 \\ 0 & 1/b^m(x) \end{pmatrix}.$$

For $\mathbf{x}_N^{(m)} \in \mathbf{R}_<^N, 1 \leq m \leq M + 1$, we consider the transformation of the quaternions $q^{m,n}\left(x_i^{(m)}, x_j^{(n)}\right) \mapsto \widehat{q}^{m,n}\left(x_i^{(m)}, x_j^{(n)}\right)$ defined by

$$\widehat{q}^{m,n}(x, y) = \zeta^m(x)q^{m,n}(x, y)\zeta^n(y)^{-1}.$$

We denote by $\widehat{Q}\left(\mathbf{x}_{N_1}^{(1)}, \mathbf{x}_{N_2}^{(2)}, \ldots, \mathbf{x}_{N_{M+1}}^{(M+1)}\right)$ the self-dual $\sum_{m=1}^{M+1} N_m \times \sum_{m=1}^{M+1} N_m$ quaternion matrix whose elements are $\widehat{q}^{m,n}\left(x_i^{(m)}, x_j^{(n)}\right), 1 \leq i \leq N_m, 1 \leq j \leq N_n, 1 \leq m, n \leq M + 1$. By the definition of quaternion determinants, the following invariance is established:

$$\text{Tdet } Q\left(\mathbf{x}_{N_1}^{(1)}, \mathbf{x}_{N_2}^{(2)}, \ldots, \mathbf{x}_{N_{M+1}}^{(M+1)}\right) = \text{Tdet } \widehat{Q}\left(\mathbf{x}_{N_1}^{(1)}, \mathbf{x}_{N_2}^{(m)}, \ldots, \mathbf{x}_{N_{M+1}}^{(M+1)}\right).$$

Hence to prove Theorem 2 it is enough to show the following lemma.

Lemma 5. *Let $T_N = 2N^{1/3}$ and $t_m = T_N + s_m$, $1 \leq m, n \leq M + 1$. Then for any $x, y \in \mathbf{R}$,*

$$(5.4) \lim_{N \to \infty} \frac{1}{b^m(x)b^n(y)} D^{m,n}(a_N(s_m) + x, a_N(s_n) + y) = \mathcal{D}(s_m, x; s_n, y),$$

$$(5.5) \lim_{N \to \infty} b^m(x)b^n(y)\tilde{I}^{m,n}(a_N(s_m) + x, a_N(s_n) + y) = \tilde{\mathcal{I}}(s_m, x; s_n, y),$$

$$(5.6) \lim_{N \to \infty} \frac{b^m(x)}{b^n(y)} \tilde{S}^{m,n}(a_N(s_m) + x, a_N(s_n) + y) = \tilde{\mathcal{S}}(s_m, x; s_n, y).$$

We start to prove this lemma by showing

$$(5.7) \quad \lim_{N \to \infty} \frac{b^m(x)}{b^n(y)} p_{t_n - t_m}(a_N(s_m) + x, a_N(s_n) + y) = \mathcal{P}(s_m, x; s_n, y).$$

By (4.1) and the fact

$$\frac{a_N(s_m) + x}{c_m} = \sqrt{2N} + \frac{x}{\sqrt{2}N^{1/6}} + \mathcal{O}(T_N^{-1}),$$

$$\tau^{(n)} = \frac{s_n}{T_N} + \mathcal{O}(T_N^{-2}),$$

for large N, we have

$$\lim_{N\to\infty} \frac{b^m(x)}{b^n(y)} p_{t_n-t_m}(a_N(s_m)+x, a_N(s_n)+y)$$

$$= \lim_{N\to\infty} \sqrt{\frac{1}{T_N}} \sum_{p=-\infty}^{N} e^{\frac{p}{2N^{1/3}}(s_n-s_m)} \varphi_{N-p}\left(\sqrt{2N}+\frac{x}{\sqrt{2}N^{1/6}}\right)$$

$$\times \varphi_{N-p}\left(\sqrt{2N}+\frac{y}{\sqrt{2}N^{1/6}}\right)$$

$$= \lim_{N\to\infty} \frac{1}{N^{1/3}} \sum_{p=-\infty}^{N} e^{\frac{p}{2N^{1/3}}(s_n-s_m)} \operatorname{Ai}\left(x+\frac{p}{N^{1/3}}\right) \operatorname{Ai}\left(y+\frac{p}{N^{1/3}}\right),$$

where we have used (5.3). Then we have (5.7).

From (3.8), (4.2), (4.6) and (4.7), we have

$$S^{m,n}(x,y) = S_1^{m,n}(x,y) + S_2^{m,n}(x,y),$$

with

$$S_1^{m,n}(x,y) = \frac{b^n(y)}{c_n b^m(x)} \sum_{\ell=0}^{N-1} e^{(N-\ell)(\tau^{(n)}-\tau^{(m)})} \varphi_\ell\left(\frac{x}{c_m}\right) \varphi_\ell\left(\frac{y}{c_n}\right),$$

$$S_2^{(mn)}(x,y) = \frac{b^n(y)}{c_n b^m(x)} \varphi_{N-1}(y/c_n)$$

$$\times \sum_{k=N/2}^{\infty} \frac{B(N/2+k)}{B(N/2-1)} e^{-(N-2k-1)\tau^{(m)}} \varphi_{2k+1}\left(\frac{x}{c_m}\right),$$

where $B(k) = \frac{2^k k!}{\sqrt{(2k+1)!}}$. Since

$$\frac{B(k)}{B(\ell)} = \left(\frac{k}{\ell}\right)^{1/4} \left(1+\mathcal{O}\left(\frac{|k-\ell|}{k+\ell}\right)\right),$$

by the same argument to show (5.7) we have

$$(5.8) \quad \lim_{N\to\infty} \frac{b^m(x)}{b^n(y)} S^{m,n}(a_N(s_m)+x, a_N(s_n)+y) = \mathcal{S}(s_m,x;s_n,y).$$

(5.6) is derived from (5.7) and (5.8).

From (4.6) and (4.7), by calculations with (4.10), we have

$$b^m(x)b^n(y)\Phi^{(m)}_{N+2p}(x)\Phi^{(n)}_{N+2p+1}(y)$$

$$= -2^{3/2}r_{N/2+p}\frac{T}{\sqrt{N+2p+1}}e^{(2p)\tau^{(n)}}\varphi_{N+2p}\left(\frac{y}{c_n}\right)$$

$$\times \sum_{k=p}^{\infty}\frac{B(N/2+k)}{B(N/2+p)}e^{(2k+1)\tau^{(m)}}\varphi_{N+2k+1}\left(\frac{x}{c_m}\right).$$

From (4.9) we obtain (5.5) by the same procedure as above.

From (4.2), by calculations with (3.3) and

$$e^{-y^2/2}H_{\ell+1}(y) = -2\frac{d}{dy}\left(e^{-y^2/2}H_\ell(y)\right) + 2\ell e^{-y^2/2}H_{\ell-1}(y),$$

we have

$$\frac{R^{(m)}_{2k}(x)R^{(n)}_{2k+1}(y)}{r_k b^m(x)b^n(y)}$$

$$= \frac{-1}{2\sqrt{t_m t_n(2T-t_m)(2T-t_n)}}e^{(N-2k)\tau^{(m)}}\varphi_{2k}\left(\frac{x}{c_m}\right)e^{(N-2k+1)\tau^{(n)}}$$

$$\times \left\{\varphi'_{2k}\left(\frac{y}{c_n}\right) + \sqrt{2k}\left(1 - e^{-2\tau^{(n)}}\right)\varphi_{2k-1}\left(\frac{y}{c_n}\right)\right\}.$$

Using the fact that

$$\frac{(N-p)^{1/12}}{2^{3/4}N^{1/6}}\varphi'_{N-p}\left(\frac{a_N(s_n)+y}{c_n}\right) = \frac{d}{d\lambda}\text{Ai}(y+\lambda)\bigg|_{\lambda=p/N^{1/3}} + o(1),$$

we obtain (5.4). This completes the proof of Lemma 5.

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Makoto Katori
Department of Physics
Chuo University
Kasuga, Bunkyo-ku
Tokyo 112-8551
Japan
E-mail address: `katori@phys.chuo-u.ac.jp`

Taro Nagao
Department of Physics
Graduate School of Science
Osaka University
Toyonaka
Osaka 560-0043
Japan
E-mail address: `nagao@sphinx.phys.sci.osaka-u.ac.jp`

Hideki Tanemura
Department Mathematics and Informatics
Chiba University
1-33 Yayoi-cho, Inage-ku
Chiba 263-8522
Japan
E-mail address: `tanemura@math.s.chiba-u.ac.jp`