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# Aperiodic automorphisms of certain simple $C^*$-algebras

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**Abstract.**

We review recent developments concerning the classification theory of aperiodic automorphisms of certain classes of simple $C^*$-algebras.

## §1. Introduction

In the theory of operator algebras classification of group actions has occupied an important position along with that of algebras themselves, and indeed they have had influenced on each other. In the category of von Neumann algebras, classification of group actions has a long history, which probably began with Dye' s work [9] for measure-preserving ergodic transformations (i.e., a classification result for certain $\mathbb{Z}$-actions on a commutative von Neumann algebra). For non-commutative von Neumann algebras, the first definite result was obtained by Connes [6] for automorphisms (i.e., $\mathbb{Z}$-actions) on the AFD type $\mathrm{II}_1$ factor. Furthermore his breakthrough has been extended by many authors to more general groups and factors (see for example [45]), and in fact quite a general classification theory on discrete group actions on AFD factors was obtained by Katayama, Sutherland and Takesaki [28]. On the other hand, in the category of $C^*$-algebras, some important results were obtained for compact group actions (see [18],[22, 23],[32], [52]) and discrete group actions (see [13],[17], [19],[24, 25],[34, 36],[41]). However, classification programs are far from being satisfactory and there are still many things to be done.

In the last decade we have witnessed remarkable success of classification theories for certain types of simple $C^*$-algebras (especially for approximately homogeneous $C^*$-algebras due to Elliott et al. and for purely infinite simple $C^*$-algebras due to Kirchberg and Phillips). We

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are now in a position to apply them to classification programs for group actions. Here, we will review our recent classification results concerning aperiodic automorphisms (i.e., automorphisms whose non-zero powers are always outer in a suitable sense) on simple A$\mathbb{T}$ $C^*$-algebras and on purely infinite simple $C^*$-algebras. Our main references are [35, 36] for the former and [43] for the latter.

For both of these classes of simple $C^*$-algebras, our argument will proceed along the following common line.

(1) Rohlin type theorem
For an aperiodic automorphism a certain partition of unity (i.e., projections summing up to one) can be chosen where the given automorphism is approximated by several cyclic permutations.

(2) Path lemma
For each of simple A$\mathbb{T}$ $C^*$-algebras or purely infinite simple $C^*$-algebras one can show the following: an almost central unitary path can be replaced by another rectifiable one with the same two end points of length smaller than a universal constant. (this is not a property for automorphisms but one for $C^*$-algebras.)

(3) Stability
Combining the above two properties together with Shapiro's method, we can deduce the stability property of aperiodic automorphisms (roughly speaking, the vanishing of one cohomology for these automorphisms).

(4) Outer conjugacy
The Evans-Kishimoto intertwining argument (i.e., a certain repeated use of the stability property [17]) enables us to show: two asymptotically unitarily equivalent aperiodic automorphisms are actually outer conjugate.

Here are some notations to be used throughout. Let $A_{sa}$, $\mathrm{Proj}(A)$ and $U(A)$ denote the sets of self-adjoint elements, projections and unitary elements of a unital $C^*$-algebra $A$, respectively. Let $1_A$ and $\mathrm{id}_A$ denote the unit and the identity automorphism of $A$, respectively. For $x, y \in A$ and $u \in U(A)$, define $[x, y] = xy - yx$ and $\mathrm{Ad}\, u(x) = uxu^*$.

## §2. Certain finite $C^*$-algebras

A $C^*$-algebra $A$ is called A$\mathbb{T}$ if it is obtained as an inductive limit of finite direct sums of matrix algebras over $C(\mathbb{T})$. In the rapid growth of the classification theory of $C^*$-algebras in the 1990's, the first breakthrough is Elliott's classification of simple A$\mathbb{T}$ $C^*$-algebras of real rank zero by $K$-theoritic data [10] (where a $C^*$-algebra $A$ is said to be *real* *rank zero* if the elements in $A_{sa}$ with finite spectra are dense in $A_{sa}$, and such an algebra is considered as a non-commutative analogue of a 0-dimensional topological space). This class includes AF $C^*$-algebras, irrational rotation $C^*$-algebras [11] (and most of their higher dimensional analogues called non-commutative tori [3, 14, 15, 40]) and a certain class of inductive limits of finite direct sums of matrix algebras over products of spheres [12, Theorem 5.17.] and of 1-dimensional compact metrizable spaces [39, Theorem 2.23.]. A crucial difference between AF $C^*$-algebras and A$\mathbb{T}$ $C^*$-algebras is the presence of the infinite dimensional circle algebra $C(\mathbb{T})$ in each of building blocks of the latter . But fortunately, since $C(\mathbb{T})$ is singly generated (by the canonical unitary element), we can overcome this difference in many stages of our arguments as long as A$\mathbb{T}$ $C^*$-algebras are of real rank zero.

## (2.1) Rohlin type theorem

For analysis of aperiodic automorphisms, our fundamental tool is a so-called Rohlin property. It has been well-known among ergodic theorists, but its non-commutative version was for the first time shown for aperiodic automorphisms of finite factors in the above-mentioned work of Connes. For A$\mathbb{T}$ $C^*$-algebras, the "Rohlin property" appears in the following form.

**Theorem 1.** [35, 36] *Let $A$ be a unital simple A$\mathbb{T}$ $C^*$-algebra of real rank zero with a unique tracial state $\tau$ and let $\alpha$ be an approximately inner automorphism of $A$. Then the following five conditions are equivalent:*

(1) *$\alpha$ is aperiodic, that is, for each non-zero integer $m$, $\alpha^m$ is not weakly inner in the GNS representation $\pi_\tau$ associated with $\tau$. More precisely, for any $m \in \mathbb{Z} \setminus \{0\}$, the canonical extension of $\alpha^m$ to $\pi_\tau(A)''$ is not implemented by any unitary element in $\pi_\tau(A)''$;*

(2) *$\alpha$ has the Rohlin property, that is, for any positive integer $M$, there exist a finite number of positive integers $k_1, \dots, k_m \geq M$ which satisfy the following condition: For any finite subset $F$ of $A$ and any $\varepsilon > 0$, there are projections $e_j^{(l)}$ ($l = 1, \dots, m, j = 0 \dots, k_l - 1$) in $A$ such that*

$$\sum_{l=1}^{m} \sum_{j=0}^{k_l-1} e_j^{(l)} = 1 ,$$

$$\| [e_j^{(l)}, x] \| < \varepsilon ,$$

$$\|\alpha(e_j^{(l)}) - e_{j+1}^{(l)}\| < \varepsilon$$

*for* $l = 1, \dots, m$, $j = 0, \dots, k_l - 1$ *and all* $x \in F$, *where* $e_{k_l}^{(l)}$ *denotes* $e_0^{(l)}$;

(3) *For each non-zero integer* $m$, $\alpha^m$ *is uniformly outer. That is, for any* $m \in \mathbb{Z} \setminus \{0\}$, *any* $a \in A$, *any projection* $p$ *in* $A$ *and any* $\varepsilon > 0$, *there exist a finite number of projections* $p_1, \dots, p_n$ *in* $A$ *such that*

$$\sum_{i=1}^{n} p_i = p,$$

$$\|p_i a \alpha^m(p_i)\| < \varepsilon \quad (i = 1, \dots, n).$$

(4) $A \rtimes_\alpha \mathbb{Z}$ *has a unique tracial state;*

(5) $A \rtimes_\alpha \mathbb{Z}$ *is real rank zero.*

*Furthermore if* $\alpha$ *is homotopically inner automorphism which satisfies that there is* $\varphi \in \mathrm{Hom}(K_1(A), \mathbb{R})$ *with the dense range such that* $\tilde{\eta}_0(\alpha) = [\varphi]$ *(where* $\tilde{\eta}_0(\alpha)$ *and* $[\varphi]$ *are elements of* $\mathrm{OrderExt}(K_1(A), K_0(A))$, *see* [38, Section 1 and 2] *for the precise definition of these objects) then the following condition is also equivalent to the above five conditions:*

(6) $A \rtimes_\alpha \mathbb{Z}$ *is a unital simple* $A\mathbb{T}$ $C^*$*-algebra of real rank zero.*

## (2.2) Path lemma

The path lemma claims the following property for $C^*$-algebras: For an almost central unitary path in the given $C^*$-algebra we can take another almost central and rectifiable one of length smaller than a universal constant with the same two end points. In [24] Herman and Ocneanu indicated (probably for the first time) usefulness of such a property to show the stability (which is the next step of our argument) of aperiodic automorphisms. Since they deal with UHF $C^*$-algebras, the path lemma is trivially valid due to the fact that each of building blocks is finite-dimensional. On the other hand, in case of $A\mathbb{T}$ $C^*$-algebras, to establish almost centrality of a unitary, almost commutativity against (in addition to finite dimensional parts) each of canonical unitary generators of $C(\mathbb{T})$ is needed. To this end, we can make use of the following theorem.

**Theorem 2.** [5, Theorem 8.1] *For any* $\varepsilon > 0$ *there exists a* $\delta > 0$ *with the following property: If* $A$ *is a unital simple* $A\mathbb{T}$ $C^*$*-algebra of real rank zero and* $u$, $v$ *are unitary elements in* $A$ *with*

$$[v] = 0 \quad \textit{in } K_1(A),$$

$$\| [u,v] \| < \delta ,$$

$$\mathrm{Isospec}(u,v) = 0,$$

*then there exists a rectifiable element* $v \in C([0,1], U(A))$ *with*

$$v(0) = v, \quad v(1) = 1,$$

$$\mathrm{Length}(v) \leq 5\pi + 1,$$

$$\| [u, v(t)] \| < \varepsilon ,$$

*for all* $t \in [0,1]$, *where* Isospec *is a homotopy-invariant function into* $K_0(A)$ *defined on almost commuting pairs of unitary elements* (*see* [5, Section 6] *for the precise definition*) *and is the same as the Bott function* ( [5, Theorem 9.1]).

As an easy application, we can obtain the following path lemma for A$\mathbb{T}$ $C^*$-algebras of real rank zero.

**Theorem 3.** [36] *Let* $A$ *be a unital simple* A$\mathbb{T}$ $C^*$*-algebra of real rank zero. For any finite subset* $F$ *of* $A$ *and any* $\varepsilon > 0$, *there exist a finite subset* $G$ *of* $A$ *and a* $\delta > 0$ *satisfying the following condition: For any* $u \in C([0,1], U(A))$ *with*

$$\| [u(t), y] \| < \delta$$

*for all* $t \in [0,1]$ *and* $y \in G$, *there is a* $v \in C([0,1], U(A))$ *such that*

$$v(0) = u(0), \quad v(1) = u(1),$$

$$\mathrm{Length}(v) < C,$$

$$\| [v(t), x] \| < \varepsilon$$

*for all* $t \in [0,1]$ *and* $x \in F$, *where* $C$ *is a universal constant positve number. Furthermore if* $F$ *is empty then it is possible that* $G$ *is empty.*

### (2.3) Stability

In [6] Connes shows the stability property of aperiodic automorphisms of certain von Neumann algebras. Assume that $M$ is separable and strongly stable (i.e., $M\bar{\otimes}\mathcal{R} \cong M$ with the AFD type $\mathrm{II}_1$ factor $\mathcal{R}$) and let $M_\omega$ be the $\omega$-centarl sequence von Neumann algebra for some free ultra-filter $\omega$ over $\mathbb{N}$. His result says that an aperiodic automorphism $\theta$ (i.e., the canonical extension $\theta_\omega$ of $\theta$ to $M_\omega$ satisfies $\theta_\omega^m \neq \mathrm{id}$ for each $m \in \mathbb{Z} \setminus \{0\}$) is stable in the sense that for any unitary $u \in U(M_\omega)$ there is a $v \in U(M_\omega)$ such that $u = v\theta_\omega(v)^*$. To prove this he first shows the Rohlin property of $\theta_\omega$, and then he concludes the stability property of $\theta_\omega$ by applying Shapiro's method. In his argument the crucial fact is that each of the trace norms of Rohlin projections (recall that $M_\omega$ is of type $\mathrm{II}_1$) is automatically decreasing as the number of Rohlin projections is increasing. But in case of $C^*$-algebras each of the norms of Rohlin projections is always one. To overcome this difficulty. we make use of the above lemma and the trick appearing in the paper [24] of Hermann and Ocneanu. As a result we obtain the stability property for aperiodic automorphisms of simple $A\mathbb{T}$ $C^*$-algebras of real rank zero as follows.

**Lemma 4.** [36] *Let $A$ be a unital simple $A\mathbb{T}$ $C^*$-algebra of real rank zero and let $\alpha$ be an automorphism of $A$. If $\alpha$ has the Rohlin propety then $\alpha$ has the following property: For any finite subset $F$ of $A$ and any $\varepsilon > 0$, there exist a finite subset $G$ of $A$ and a $\delta > 0$ which satisfy the following condition: If $u$ is an element in $C([0,1], U(A))$ such that*

$$u(0) = 1, \quad \| [u(t), y] \| < \delta$$

*for all $t \in [0,1]$ and $y \in G$, then there exists a unitary element $v$ in $A$ such that*

$$\|u(1) - v\alpha(v)^*\| < \varepsilon \ ,$$

$$\| [v, x] \| < \varepsilon$$

*for all $x \in F$. Furthermore if $F$ is empty then it is possible that $G$ is empty.*

## (2.4) Outer conjugacy

Let $\alpha, \beta$ be automorphisms of a unital $C^*$-algebra $A$. $\alpha$ is said to be *asymptotically unitarily equivalent* to $\beta$ if there exists a continuous mapping $u$ from $[0, \infty)$ into $U(A)$ such that $\alpha = \lim_{t \to \infty} \mathrm{Ad}\, u(t) \circ \beta$ (pointwisely). $\alpha$ is said to be *outer conjugate* to $\beta$ if there exist a unitary element $u$ in $A$ and an automorphism $\gamma$ of $A$ such that $\alpha = \mathrm{Ad}\, u \circ \gamma \circ \beta \circ \gamma^{-1}$. As stated in the introduction, if we apply the stability property successively to infinitely many unitary paths which are parts of the infinite unitary path appearing in the definition of asymptotic uniatry equivalence, we have the following main theorem of this section.

**Theorem 5.** [36] *Let $A$ be a unital simple $A\mathbb{T}$ $C^*$-algebra of real rank zero with a unique tracial state and let $\alpha$ and $\beta$ be approximately inner and aperiodic automorphisms of $A$. If $\alpha$ is asymptotically unitarily equivalent to $\beta$ then $\alpha$ is outer conjugate to $\beta$.*

## §3. Certain infinite $C^*$-algebras

A unital $C^*$-algebra $A$ is called *purely infinite* if every non-zero hereditary $C^*$-subalgebra of $A$ contains an infinite projection. As was shown by Cuntz [8], for a simple $C^*$-algebra $A$ ($\neq \mathbb{C}$) this condition is equivalent to the following: for each $a \in A \setminus \{0\}$ there exist $x, y \in A$ satisfying $xay = 1$. In the history of the classification theory of $C^*$-algebras (or maybe of the theory of $C^*$-algebras) the classification of separable nuclear unital purely infinite simple $C^*$-algebras owing to Kirchberg and Phillips [29, 48] is one of the most outstanding results. Motivated by their remarkable success, we classify aperiodic automorphisms of a such $C^*$-algebra in terms of Kasparov's KK-theory [27].

### (3.1) Rohlin type Theorem

As in case of $A\mathbb{T}$ $C^*$-algebras we first mention a Rohlin type theorem for automorphisms of nuclear purely infinite simple $C^*$-algebras. This theorem is for the first time verified by Izumi [26]. A proof is based on Kishimoto's method [33] in finite case together with the following result [30, Proposition 3.4.]: if $A$ is a unital purely infinite simple $C^*$-algebra then so is the $\omega$-central sequence $C^*$-algebra $A_\omega$.

**Theorem 6.** [26, 43] *Let $A$ be a separable nuclear unital purely infinite simple $C^*$-algebra and let $\alpha$ be an automorphism of $A$. Then the following conditions are equivalent:*

(1) *$\alpha$ is aperiodic, that is, for any non-zero integer $k$, $\alpha^k$ is not implemented by any unitary element of $A$;*

(2) *$\alpha$ has the Rohlin property, that is, for any positive integer $M$, any finite subset $F$ of $A$ and any $\varepsilon > 0$, there exist projections $e_0, \dots, e_{M-1}, f_0, \dots, f_M$ in $A$ such that*

$$\sum_{i=0}^{M-1} e_i + \sum_{j=0}^{M} f_j = 1 ,$$

$$\| [e_i, x] \| < \varepsilon, \ \| [f_j, x] \| < \varepsilon ,$$

$$\|\alpha(e_i) - e_{i+1}\| < \varepsilon, \ \|\alpha(f_j) - f_{j+1}\| < \varepsilon$$

*for* $i = 0, \ldots, M-1$, $j = 0, \ldots, M$ *and all* $x \in F$, *where* $e_M$ *and* $f_{M+1}$ *mean* $e_0$ *and* $f_0$, *respectively.*

### (3.2) Path lemma

In case of AT $C^*$-algebras we make use of a deformation theorem (Theorem 2) for almost commuting unitary elements to prove the path lemma. The proof of Theorem 2 (requiring a detailed analysis on spectral theory of almost commuting unitary elements) is quite complicated. So it is as good as impossible for us to apply the same method to an arbitrary finite number of (or even three) almost commuting unitary elements. But Haagerup and Rørdam find the following (probably deep) result for infinite $C^*$-algebras in the other context. Their proof is essentially based on the infiniteness of the Cuntz algebra $\mathcal{O}_2$(or of $B(H)$) and is surprisingly simpler than that of Theorem 2.

**Lemma 7.** [21, Lemma 5.1.] *Let* $A$ *be a unital* $C^*$*-algebra and let* $B$ *be a unital* $C^*$*-subalgebra of* $A$*, which is isomorphic to the Cuntz algebra* $O_2$*. Then for any unitary elements* $u_0$, $u_1 \in B' \cap A$*, there exists an element* $v \in C([0,1], U(A))$ *such that*

$$v(0) = u_0, \ v(1) = u_1,$$

$$\mathrm{Length}(v) \leq \frac{8\pi}{3},$$

$$\| [v(t), x] \| \leq 4\| [u_1, x] \| + 5\| [u_0, x] \|$$

*for any* $x \in B' \cap A$ *and any* $t \in [0,1]$.

Using Lemma 7 and the technique in [47, Theorem 3.4.], we can prove not only the path lemma but also a kind of its continuous version for separable nuclear unital purely infinite simple $C^*$-algebras. Our proof heavily depends on a deep result of Kirchberg and Phillips [30]: a separable nuclear unital purely infinite simple $C^*$-algebra $A$ satisfies $A \otimes \mathcal{O}_\infty \cong A$.

**Theorem 8.** [43] *Let* $A$ *be a separable nuclear unital purely infinite simple* $C^*$*-algebra. For any finite subset* $F$ *of* $A$ *and any* $\varepsilon > 0$*, there exist a finite subset* $G$ *of* $A$ *and* $\delta > 0$ *satisfying the following condition: For any element* $u$ *in* $C([0,1]^2, U(A))$ *with*

$$\| [u(s,t), y] \| < \delta$$

*for all* $s, t \in [0,1]$ *and* $y \in G$*, there is an element* $v$ *in* $C([0,1]^2, U(A))$ *such that*

$$v(s,0) = u(s,0), \quad v(s,1) = u(s,1),$$

$$\mathrm{Length}(v(s,\cdot)) < 6\pi,$$

$$\| [v(s,t), x] \| < \varepsilon$$

*for all* $s, t \in [0,1]$ *and* $x \in F$. *Furthermore if* $F$ *is empty then it is possible that* $G$ *is empty.*

## (3.3) Stability

Once the above theorem is obtained, the following (continuous version of) stability property can be proved in a similar fashion as in case of A$\mathbb{T}$ $C^*$-algebras.

**Lemma 9.** [43] *Let* $A$ *be a separable nuclear unital purely infinite simple* $C^*$*-algebra and let* $\alpha$ *be an automorphism of* $A$. *If* $\alpha$ *is aperiodic then* $\alpha$ *has the property: For any finite subset* $F$ *of* $A$ *and any* $\varepsilon > 0$, *there exist a finite subset* $G$ *of* $A$ *and a* $\delta > 0$ *which satisfy the following condition: If* $u$ *is an element in* $C([0,1]^2, U(A))$ *such that*

$$u(s,0) = u(0,t) = 1, \quad \| [u(s,t), y] \| < \delta$$

*for all* $s, t \in [0,1]$ *and* $y \in G$, *then there exists an element* $v$ *in* $C([0,1], U(A))$ *such that*

$$\|u(s,1) - v(s)\alpha(v(s))^*\| < \varepsilon \ ,$$

$$v(0) = 1, \quad \| [v(s), x] \| < \varepsilon$$

*for all* $s \in C[0,1]$ *and* $x \in F$. *Furthermore if* $F$ *is empty then it is possible that* $G$ *is empty.*

## (3.4) Outer conjugacy

Again by the Evans-Kishimoto intertwining argument, we have the following theorem (with the continuity of the stability property) which is a little bit stronger than Theorem 5.

**Theorem 10.** [43] *Let* $A$ *be a separable nuclear unital purely infinite simple* $C^*$*-algebra and let* $\alpha$, $\beta$ *be aperiodic automorphisms of* $A$. *If* $\alpha$ *is asymptotically unitarily equivalent to* $\beta$ *then* $\alpha$ *is outer conjugate to* $\beta$ *with a conjugation* $\gamma$ *isotopic to the identity mapping* $\mathrm{id}_A$ *of* $A$. *In other words, we can find a unitary* $u$ *in* $A$ *satisfying* $\alpha = \mathrm{Ad}\, u \circ \gamma \circ \beta \circ \gamma^{-1}$ *and a homotopy consisting of automorphisms of* $A$ *between* $\gamma$ *and* $\mathrm{id}_A$.

Combining Theorem 10 and the classification theory due to Phillips, we reach our classification theorem of aperiodic automorphisms of a nuclear purely infinite simple $C^*$-algebra.

**Theorem 11.** [43] *Let $A$ be a separable nuclear unital purely infinite simple $C^*$-algebra and let $\alpha$ and $\beta$ be aperiodic automorphisms of $A$. Then the following three conditions are equivalent:*

(1) $[\alpha] = [\beta]$ *in* $KK(A, A)$*;*

(2) $\alpha$ *is asymptotically unitarily equivalent to* $\beta$*;*

(3) $\alpha$ *is outer conjugate to* $\beta$ *with a conjugation* $\gamma$ *which is homotopic to the identity mapping of* $A$*.*

*Moreover under the same assumption the following three conditions are also equivalent to each other:*

(1') *There exists a* $KK$*-equivalence* $\eta$ *such that* $[\alpha] = \eta \cdot [\beta] \cdot \eta^{-1}$ *and* $[1_A] \cdot \eta = [1_A]$*;*

(2') *There exist a continuous mapping* $u$ *from* $[0, \infty)$ *into* $U(A)$ *and an automorphism* $\gamma$ *of* $A$ *such that* $\alpha = \lim_{t\to\infty} \mathrm{Ad}\, u(t) \circ \gamma \circ \beta \circ \gamma^{-1}$ *pointwisely;*

(3') $\alpha$ *is outer conjugate to* $\beta$*.*

**Remark 12.** Theorem 11 is still valid for a separable non-unital nuclear purely infinite simple $C^*$-algebra $A$, if one removes the condition $[1_A] \cdot \eta = [1_A]$ from (1') and regards the unitaries considering there as the elements of the unitization of $A$.

The following corollary is an direct consequence of [30, Theorem 3.14.] and Theorem 11 above, and the author thanks Masaki Izumi for pointing out this fact.

**Corollary 13.** [26, 43] *Let $A$ be a separable nuclear unital purely infinite simple $C^*$-algebra and let $\alpha$ be an automorphism of $A$. If $\alpha$ is aperiodic then $\alpha$ absorbs any automorphism $\beta$ of $O_\infty$, that is, there exist a unitary element $u$ in $O_\infty \otimes A$ and an isomorphism $\gamma$ from $A$ onto $O_\infty \otimes A$ such that*

$$\beta \otimes \alpha = \mathrm{Ad}\, u \circ \gamma \circ \alpha \circ \gamma^{-1} .$$

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