

Direct limit decomposition for C^* -algebras of minimal diffeomorphisms

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This article outlines the proof that the crossed product $C^*(\mathbf{Z}, M, h)$ of a compact smooth manifold M by a minimal diffeomorphism $h: M \rightarrow M$ is isomorphic to a direct limit of subhomogeneous C^* -algebras belonging to a tractable class. This result is motivated by the Elliott classification program for simple nuclear C^* -algebras [9], and the observation that the known classification theorems in the stably finite case mostly apply to certain kinds of direct limits of subhomogeneous C^* -algebras, or at least to C^* -algebras with related structural conditions. (See Section 1.) This theorem is a generalization, in a sense, of direct limit decompositions for crossed products by minimal homeomorphisms of the Cantor set (Section 2 of [32]), for the irrational rotation algebras ([10]), and for some higher dimensional noncommutative toruses ([13], [14], [24], and [5]). (In [32], only a local approximation result is stated, but the C^* -algebras involved are semiprojective.) Our theorem is not a generalization in the strict sense for several reasons; see the discussion in Section 1.

There are four sections. In the first, we state the theorem and discuss some consequences and expected consequences. In the second section, we describe the basic construction in our proof, a modified Rokhlin tower, and show how recursive subhomogeneous algebras appear naturally in our context. The third section describes how to prove local approximation by recursive subhomogeneous algebras, a weak form of the main theorem. In Section 4, we give an outline of how to use the methods of Section 3 to obtain the direct limit decomposition.

This paper is based on a talk given by the second author at the US–Japan Seminar on Operator Algebras and Applications (Fukuoka, June 1999), which roughly covered Sections 2 and 3, and on a talk given by

Research of the second author partially supported by NSF grants DMS 9400904 and DMS 9706850.

2000 *Mathematics Subject Classification.* Primary 46L55; Secondary 19K14, 46L35, 46L80.

the second author at the 28th Canadian Annual Symposium on Operator Algebras (Toronto, June 2000), which roughly covered Sections 1 and 2. At the time of the first talk, only the local approximation result described in Section 3 had been proved. We refer to the earlier survey paper [25] for earlier parts of the story; this paper reports the success of the project described in Section 6 there.

The first author would like to thank George Elliott, John Phillips, and Ian Putnam for funding him at the University of Victoria where some of this work was carried out. He would particularly like to acknowledge his great gratitude to Ian Putnam for many interesting discussions. The second author would like to thank Larry Brown, Marius Dădărlat, George Elliott, and Ian Putnam for useful discussions and email correspondence. Some of the work reported here was carried out during a sabbatical year at Purdue University, and he would like to thank that institution for its hospitality.

§1. The main theorem, consequences, and conjectured consequences

The main theorem is as follows. Undefined terminology is discussed after the statement.

Theorem 1.1. Let M be a connected compact smooth manifold with $\dim(M) = d > 0$, and let $h: M \rightarrow M$ be a minimal diffeomorphism. Then there exists an increasing sequence

$$A_0 \subset A_1 \subset A_2 \subset \cdots \subset C^*(\mathbf{Z}, M, h)$$

of C^* -subalgebras of $C^*(\mathbf{Z}, M, h)$ such that

$$\overline{\bigcup_{n=0}^{\infty} A_n} = C^*(\mathbf{Z}, M, h)$$

and such that each A_n has a separable recursive subhomogeneous decomposition with topological dimension at most d and strong covering number at most $d(d+2)$.

A recursive subhomogeneous algebra (a C^* -algebra with a recursive subhomogeneous decomposition) is a particularly tractable kind of subhomogeneous C^* -algebra. See [29], [30], and [25], and also see the consequences below. We will explain in Section 2 how recursive subhomogeneous algebras arise, and we will recall (informally) the definition there (after Theorem 2.7). A finite direct sum

$$\bigoplus_{k=0}^l C(X_k, M_{n(k)})$$

of (trivial) homogeneous C*-algebras is a special case of a recursive subhomogeneous algebra, and the topological dimension is simply $\max_{0 \leq k \leq l} \dim(X_k)$. (Dimension is taken to be covering dimension; see Definition 1.6.7 of [15].) The condition in the theorem that A_n have topological dimension at most d for all n thus ensures that the resulting direct limit decomposition $C^*(\mathbf{Z}, M, h) \cong \varinjlim A_n$ has no dimension growth.

In general, it is not possible to find a representation as a direct limit (with no dimension growth) of direct sums of corners of trivial homogeneous C*-algebras. A simple direct limit of this sort must even be approximately divisible in the sense of [4], by Theorem 2.1 of [11]. However, a crossed product by a minimal diffeomorphism may have no nontrivial projections (Corollary 3 and Example 4 of Section 5 of [7]).

We will not define the strong covering number here, although some discussion will be given after Theorem 3.1. We have included it in the conclusion because the proof of Theorem 3.1 suggests that a bound on the strong covering number might be necessary for some classification results.

The requirement that we have a diffeomorphism of a manifold is connected with the appearance of a condition on the strong covering number in the hypotheses of Theorem 3.1. This also will be discussed after that theorem. We certainly expect that the theorem will be true for minimal homeomorphisms of finite dimensional compact metric spaces (even, presumably, compact metric spaces with infinite covering dimension).

We point out here that our theorem does not directly imply the Elliott-Evans direct limit representation for the irrational rotation algebras [10]. Our theorem gives a representation of an irrational rotation algebra as a direct limit of recursive subhomogeneous algebras with topological dimension at most 1, while the Elliott-Evans theorem gives a representation as a direct limit of direct sums of homogeneous C*-algebras with topological dimension at most 1 (in fact, circle algebras). We do not recover the results of [13], [14], and [24] (for certain higher dimensional noncommutative toruses), not only because the algebras in our direct system are more complicated but also because not all the algebras considered there are even crossed products by diffeomorphisms. We also do not recover the direct limit decomposition for crossed products by minimal homeomorphisms of the Cantor set (see Section 2 of [32] for the local approximation result), because the Cantor set is not a manifold. (Our methods do specialize to this case, but that would be silly, since our argument is much more complicated.)

Theorem 1.1 has the following consequences for crossed products by minimal diffeomorphisms. These consequences all hold for an arbitrary

simple unital direct limit of recursive subhomogeneous algebras, assuming no dimension growth and that the maps of the system are unital and injective. (Most don't require the full strength of these hypotheses, but all require some restriction on dimension growth. None require any hypotheses on the strong covering number.) The proofs are in [30], and the statements can be found in Section 4 of [25] (except for the last one, which is actually a consequence of stable rank one). In all of these, M is a connected compact smooth manifold with $\dim(M) > 0$, and $h: M \rightarrow M$ is a minimal diffeomorphism.

Corollary 1.2. (Theorem 3.6 of [30].) The algebra $C^*(\mathbf{Z}, M, h)$ has stable rank one in the sense of [33]. That is, the invertible group $\text{inv}(C^*(\mathbf{Z}, M, h))$ is dense in $C^*(\mathbf{Z}, M, h)$.

Corollary 1.3. (Theorem 2.2 of [30].) The projections in

$$M_\infty(C^*(\mathbf{Z}, M, h)) = \bigcup_{n=1}^{\infty} M_n(C^*(\mathbf{Z}, M, h))$$

satisfy cancellation. That is, if $e, p, q \in M_\infty(C^*(\mathbf{Z}, M, h))$ are projections, and if $p \oplus e \sim q \oplus e$, then $p \sim q$.

Corollary 1.4. (Theorem 2.3 of [30].) The algebra $C^*(\mathbf{Z}, M, h)$ satisfies Blackadar's Second Fundamental Comparability Question ([2], 1.3.1). That is, if $p, q \in M_\infty(C^*(\mathbf{Z}, M, h))$ are projections, and if $\tau(p) < \tau(q)$ for every normalized trace τ on $C^*(\mathbf{Z}, M, h)$, then $p \precsim q$.

Corollary 1.5. (Theorem 2.4 of [30].) The group $K_0(C^*(\mathbf{Z}, M, h))$ is unperforated for the strict order. That is, if $\eta \in K_0(C^*(\mathbf{Z}, M, h))$ and if there is $n > 0$ such that $n\eta > 0$, then $\eta > 0$.

(In the simple case, this is the same as saying that $K_0(C^*(\mathbf{Z}, M, h))$ is weakly unperforated in the sense of 2.1 of [8].)

Corollary 1.6. (Theorem 2.1 of [30].) The canonical map

$$U(C^*(\mathbf{Z}, M, h))/U_0(C^*(\mathbf{Z}, M, h)) \rightarrow K_1(C^*(\mathbf{Z}, M, h))$$

is an isomorphism.

A small part of these results could already be obtained using the weaker (and much simpler) methods described in Sections 1 and 5 of [25]. It had already been shown that the order on $K_0(C^*(\mathbf{Z}, M, h))$ is determined by traces (a weak form of Corollary 1.4), and hence that $K_0(C^*(\mathbf{Z}, M, h))$ is unperforated for the strict order (Corollary 1.5). Also, surjectivity in Corollary 1.6 (but not injectivity) was known.

The criterion in [3], for when a simple direct limit of direct sums of trivial homogeneous C*-algebras with slow dimension growth has real rank zero, is known to fail for simple direct limits of recursive subhomogeneous algebras with no dimension growth. (Indeed, it even fails for crossed products by minimal diffeomorphisms; see Example 5.7 of [25].) Nevertheless, it appears likely that a suitable strengthening of the condition will be equivalent to real rank zero for such direct limits, and that the proof will not be difficult. Specializing (for simplicity) to the case of a unique trace, we obtain the following, which we state as a conjecture.

Conjecture 1.7. Let M be a connected compact smooth manifold with $\dim(M) > 0$, and let $h: M \rightarrow M$ be a uniquely ergodic minimal diffeomorphism. Let

$$\tau: C^*(\mathbf{Z}, M, h) \rightarrow \mathbf{C}$$

be the trace induced by the unique invariant probability measure. Then $C^*(\mathbf{Z}, M, h)$ has real rank zero ([6]) if and only if $\tau_*(K_0(C^*(\mathbf{Z}, M, h)))$ is dense in \mathbf{R} .

For methods for computing the ranges of traces on the K-theory of crossed products by \mathbf{Z} , we refer to [16].

It might not be terribly difficult to prove that if a simple C*-algebra A is a direct limit of a system of recursive subhomogeneous algebras with no dimension growth, and possibly also assuming that the maps of the system are unital and injective, then real rank zero implies tracial rank zero in the sense of H. Lin [20]. If so, then the following result of H. Lin (Theorem 3.9 of [23]) implies classifiability:

Theorem 1.8. Suppose A and B are separable simple unital C*-algebras with tracial rank zero in the sense of [20]. Suppose that each has local approximation by subalgebras with bounded dimensions of irreducible representations. That is, for every finite subset $F \subset A$ and every $\varepsilon > 0$, there is a C*-subalgebra $D \subset A$ and an integer N such that every element of F is within ε of an element of D and every irreducible representation of D has dimension at most N ; and similarly for B . Then

$$(K_0(A), K_0(A)_+, [1_A], K_1(A)) \cong (K_0(B), K_0(B)_+, [1_B], K_1(B))$$

implies $A \cong B$.

In particular, one would have a proof of the following conjecture:

Conjecture 1.9. Let M be a connected compact smooth manifold with $\dim(M) > 0$, and let $h: M \rightarrow M$ be a uniquely ergodic minimal

diffeomorphism. Let

$$\tau: C^*(\mathbf{Z}, M, h) \rightarrow \mathbf{C}$$

be the trace induced by the unique invariant probability measure, and assume that $\tau_*(K_0(C^*(\mathbf{Z}, M, h)))$ is dense in \mathbf{R} . Then the crossed product C^* -algebra $C^*(\mathbf{Z}, M, h)$ is classifiable.

We will not give a precise definition of “classifiable” here.

We note that H. Lin’s classification theorem has no hypotheses involving slow dimension growth, and does not even require a direct limit representation; only local approximation is needed, and the condition on the approximating algebras is weak. (Indeed, H. Lin has other classification theorems which don’t even require local approximation, but do require further restrictions on the K-theory.) However, at least with our current state of knowledge, the direct limit representation in Theorem 1.1, including the no dimension growth condition, seems to be needed to verify the other hypotheses of Theorem 1.8. For example, simple direct limits that don’t have slow dimension growth need not even have stable rank one [35].

Since $C^*(\mathbf{Z}, M, h)$ always has stable rank one, if it doesn’t have real rank zero then it has real rank one. However, most of the currently known general classification theorems apply only to algebras with many projections, and those that don’t are much too restrictive in other ways (such as assuming trivial K-theory). In particular, the C^* -algebras covered by [17] and [12] are approximately divisible (as discussed above), and the theorems of H. Lin (see [21] and [22]) require a finite value of the tracial rank, the definition of which again requires the existence of many nontrivial projections. However, as mentioned above, the example of Connes shows that $C^*(\mathbf{Z}, M, h)$ may have no nontrivial projections. There is a classification theorem [19] for a special class of direct limits which includes simple C^* -algebras with no nontrivial projections, but the building blocks there are much more special than those appearing in our theorem.

We are hopeful that the approach of [17] and [12], which now covers simple direct limits, with no dimension growth, of direct sums of homogeneous C^* -algebras (actually, a slightly larger class), can be generalized to cover simple direct limits, with no dimension growth, of recursive subhomogeneous algebras, possibly with the added restriction of no growth of the strong covering number. One reason for optimism (as well as for the belief that conditions on the strong covering number might be necessary) is the successful generalization of exponential length results

from the case of trivial homogeneous C*-algebras to recursive subhomogeneous algebras; see Theorem 3.1 below. The related results for the trivial homogeneous case (see Theorems 3.3 and 4.5 of [28]) depended heavily on the existence of many projections, but in the proof of Theorem 3.1 we had to learn to handle situations with no nontrivial projections at all. However, we do not know whether Theorem 3.1 is even true without the condition on the strong covering number. (See the discussion after the statement of that theorem.) We included the bound on the strong covering number in Theorem 1.1 because of the possibility that it might be necessary for our suggested approach to proving a classification result, or perhaps even for a classification result to hold.

In any case, a generalization of the methods of [17] and [12] is likely to be very difficult. Possibly the situation will be improved by a generalization of H. Lin’s methods that is strong enough to apply to simple C*-algebras which contain no nontrivial projections.

§2. Modified Rokhlin towers

Throughout this section, M is a compact metric space and $h: M \rightarrow M$ is a minimal homeomorphism. (The requirement that M be a manifold will not be needed until the next section.) We let u denote the implementing unitary in $C^*(\mathbf{Z}, M, h)$, so that $ufu^* = f \circ h^{-1}$ for $f \in C(M)$.

We start with a definition.

Definition 2.1. Let $Y \subset M$, and let $x \in Y$. The *first return time* $\lambda_Y(x)$ (or $\lambda(x)$ if Y is understood) of x to Y is the smallest integer $n \geq 1$ such that $h^n(x) \in Y$. We set $\lambda(x) = \infty$ if no such n exists.

The following result is well known in the area, and is easily proved:

Lemma 2.2. If $\text{int}(Y) \neq \emptyset$, then $\sup_{x \in Y} \lambda(x) < \infty$.

Let $Y \subset M$ with $\text{int}(Y) \neq \emptyset$. Let $n(0) < n(1) < \dots < n(l)$ (or, if the dependence on Y must be made explicit, $n_Y(0) < n_Y(1) < \dots < n_Y(l_Y)$) be the distinct values of $\lambda(x)$ for $x \in Y$. The Rokhlin tower based on a subset $Y \subset M$ with $\text{int}(Y) \neq \emptyset$ consists of the partition

$$Y = \bigsqcup_{k=0}^l \{x \in Y : \lambda(x) = n(k)\}$$

of Y (the sets here are the base sets), and the corresponding partition

$$M = \bigsqcup_{k=0}^l \bigsqcup_{j=0}^{n(k)-1} h^j(\{x \in Y : \lambda(x) = n(k)\})$$

of M . Note that h acts like a cyclic shift except on the top space

$$h^{n(k)-1}(\{x \in Y : \lambda(x) = n(k)\})$$

of each “tower”

$$\prod_{j=0}^{n(k)-1} h^j(\{x \in Y : \lambda(x) = n(k)\}).$$

Actually, for our purposes it is more convenient to use the partition

$$M = \prod_{k=0}^l \prod_{j=1}^{n(k)} h^j(\{x \in Y : \lambda(x) = n(k)\}).$$

Note that

$$Y = \prod_{k=0}^l h^{n(k)}(\{x \in Y : \lambda(x) = n(k)\}),$$

so that h now acts like a cyclic shift on the towers, except on Y itself.

We will be interested in arbitrarily small choices for Y , in particular with arbitrarily small diameter and for which the smallest first return time $n_Y(0)$ is arbitrarily large. If M is totally disconnected, then we may choose Y to be both closed and open. In this case, the sets

$$Y_k = \{x \in Y : \lambda(x) = n(k)\}$$

are all closed, and there is a composite homomorphism γ_0 given by

$$C(M) \longrightarrow \bigoplus_{k=0}^l \bigoplus_{j=1}^{n(k)} C(h^j(Y_k)) \xrightarrow{\cong} \bigoplus_{k=0}^l C(Y_k)^{n(k)},$$

which is in fact an isomorphism. The formula is

$$\gamma_0(f) = ((f \circ h|_{Y_0}, \dots, f \circ h^{n(0)}|_{Y_0}), \dots, (f \circ h|_{Y_l}, \dots, f \circ h^{n(l)}|_{Y_l})).$$

See [31] for the exploitation of this idea.

In order to have a C^* -algebraically sensible codomain for γ_0 , we must insist that the sets Y_k be closed. However, the spaces M we are interested in are connected, so we are forced to choose

$$Y_k = \overline{\{x \in Y : \lambda(x) = n(k)\}}$$

instead. The sets $h^j(Y_k)$ are no longer disjoint (although they certainly cover M), so our map

$$\gamma_0: C(M) \rightarrow \bigoplus_{k=0}^l C(Y_k)^{n(k)},$$

while still injective, is no longer an isomorphism.

Next, define

$$s_k \in M_{n(k)} \subset C(Y_k, M_{n(k)})$$

by

$$s_k = \begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 & 0 & 1 \\ 1 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & & \vdots & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 1 & 0 \end{pmatrix},$$

and define

$$s = (s_0, s_1, \dots, s_l) \in \bigoplus_{k=0}^l C(Y_k, M_{n(k)}).$$

Then s is unitary. Identifying $C(Y_k)^{n(k)}$ with the diagonal matrices in $C(Y_k, M_{n(k)})$ in the obvious way, one can check that if $f \in C(M)$ vanishes on Y , then

$$\gamma_0(ufu^*) = \gamma_0(f \circ h^{-1}) = s\gamma_0(f)s^*.$$

The calculation uses the fact that

$$Y = \bigcup_{k=0}^l h^{n(k)}(Y_k),$$

and in fact our choice to start our towers at $h(Y_k)$ rather than at Y_k was made to have this formula work correctly when f vanishes on Y (rather than when f vanishes on $h^{-1}(Y)$).

This relation allows us to extend γ_0 to the following subalgebra of $C^*(\mathbf{Z}, M, h)$:

Definition 2.3. For any closed subset $Y \subset M$, we define

$$A(Y) = C^*(C(M), uC_0(M \setminus Y)) \subset C^*(\mathbf{Z}, M, h),$$

the C^* -subalgebra of $C^*(\mathbf{Z}, M, h)$ generated by $C(M)$ and $uC_0(M \setminus Y)$. Here, we identify $C_0(M \setminus Y)$ in the obvious way with the subalgebra of $C(M)$ consisting of those functions vanishing on Y . We use the analogous convention throughout the paper.

Proposition 2.4. Suppose M is a compact metric space, and let $h: M \rightarrow M$ be a minimal homeomorphism. Let $Y \subset M$ be closed with $\text{int}(Y) \neq \emptyset$. Then there exists a unique homomorphism

$$\gamma_Y: A(Y) \rightarrow \bigoplus_{k=0}^l C(Y_k, M_{n(k)})$$

such that if $f \in C(M)$, then

$$\gamma_Y(f)_k = \text{diag}(f \circ h|_{Y_k}, f \circ h^2|_{Y_k}, \dots, f \circ h^{n(k)}|_{Y_k})$$

and if $f \in C_0(M \setminus Y)$, then

$$(\gamma_Y(uf))_k = s_k \gamma_Y(f)_k.$$

Moreover, γ_Y is unital and injective.

We now introduce a slight twist on these ideas.

Definition 2.5. Let $Y \subset M$ be closed with $\text{int}(Y) \neq \emptyset$. Let $S \subset \text{int}(Y_0)$ be closed. Define

$$\text{ex}(S) = \{h(S), h^2(S), \dots, h^{n(0)}(S)\},$$

which is a collection of disjoint closed subsets of M . Define $C(M)_{\text{ex}(S)}$ to be the set of all $f \in C(M)$ such that f is constant on T for every $T \in \text{ex}(S)$. (The constant value is allowed to depend on T .) Define $A(Y, S)$ to be the C^* -subalgebra of $C^*(\mathbf{Z}, M, h)$ given by

$$A(Y, S) = C^*(C(M)_{\text{ex}(S)}, u[C_0(M \setminus Y) \cap C(M)_{\text{ex}(S)}]) \subset A(Y).$$

As we will see below, the point of this definition is that (when $\text{int}(S) \neq \emptyset$) we can construct useful unitaries in $C^*(\mathbf{Z}, M, h)$ which commute with (most of) $A(Y, S)$. (See Step 9 in the proof outline in Section 3.)

It is not obvious what the image

$$\gamma_Y(A(Y, S)) \subset \bigoplus_{k=0}^l C(Y_k, M_{n(k)})$$

looks like, and working with it directly threatens to be very complicated. Fortunately, the essential properties can be abstracted in a tractable way; the result is what we call a recursive subhomogeneous algebra. (The definition of a recursive subhomogeneous algebra was in fact invented for exactly this purpose.) First, we recall the notion of a pullback.

Definition 2.6. Let A and B be C^* -algebras, and let a third C^* -algebra C and homomorphisms $\varphi: A \rightarrow C$ and $\psi: B \rightarrow C$ be given. The *pullback* (also called fibered product or restricted direct sum) is

$$A \oplus_C B = A \oplus_{C, \varphi, \psi} B = \{(a, b) \in A \oplus B : \varphi(a) = \psi(b)\}.$$

If the maps φ and ψ are understood, we will write $A \oplus_C B$.

Theorem 2.7. Let M be a compact metric space, and let $h: M \rightarrow M$ be a minimal homeomorphism. Let $Y \subset M$ be closed with $\text{int}(Y) \neq \emptyset$. Let $S \subset \text{int}(Y_0)$ be closed. Then there exist closed subsets

$$Y_k^{(0)} \subset \partial Y_k \subset Y_k$$

for $1 \leq k \leq l$, and homomorphisms φ_k and ψ_k (with ψ_k being just the restriction map) such that the image $\gamma_Y(A(Y, S))$ is equal to the subalgebra

$$\begin{aligned} & \left[\cdots \left[[C(Y_0, M_{n(0)})_S \oplus_{C(Y_1^{(0)}, M_{n(1)}), \varphi_1, \psi_1} C(Y_1, M_{n(1)})] \right. \right. \\ & \qquad \qquad \qquad \left. \oplus_{C(Y_2^{(0)}, M_{n(2)}), \varphi_2, \psi_2} C(Y_2, M_{n(2)}) \right] \cdots \left. \right] \\ & \oplus_{C(Y_l^{(0)}, M_{n(l)}), \varphi_l, \psi_l} C(Y_l, M_{n(l)}) \end{aligned}$$

of $\bigoplus_{k=0}^l C(Y_k, M_{n(k)})$. Here, by analogy with Definition 2.5, we set

$$C(Y_0, M_{n(0)})_S = \{f \in C(Y_0, M_{n(0)}) : f \text{ is constant on } S\}.$$

A C^* -algebra given as an iterated pullback as in the conclusion of this theorem, in which the algebras have the form $C(X_k, M_{n(k)})$, the maps φ_k are unital, and the maps ψ_k are unital and surjective, is called a *recursive subhomogeneous algebra*. We refer to Section 2 of [25] for a more careful definition, for some useful associated terminology, and

examples; to Section 3 of [25] for a discussion of the proof of Theorem 2.7 (in the case $S = \emptyset$); and to Section 4 of [25] for a discussion of why the concept of a recursive subhomogeneous decomposition is useful and what can be done with it. We recall here that the topological dimension is the largest dimension $\dim(X_k)$. Unfortunately, it depends on the particular decomposition; see Example 2.9 of [25]. We will always have a decomposition in mind, usually coming from Theorem 2.7.

The next difficulty we face is that the unitary

$$s = (s_0, s_1, \dots, s_l) \in \bigoplus_{k=0}^l C(Y_k, M_{n(k)})$$

is not in the image of $A(Y)$. (When M is totally disconnected and Y is both closed and open, there is no problem: the image of γ_Y is all of $\bigoplus_{k=0}^l C(Y_k, M_{n(k)})$.) The cure for this problem is the following lemma, which however requires that we look at two nested subsets Y and Z , along with the associated subalgebras $A(Y)$ and $A(Z)$.

Lemma 2.8. Let M be a compact metric space with finite covering dimension d , and let $h: M \rightarrow M$ be a minimal homeomorphism. Let $Y \subset M$ be closed with $\text{int}(Y) \neq \emptyset$. Then every point of $\text{int}(Y)$ has a neighborhood $U \subset \text{int}(Y)$ such that for every closed set $Z \subset U$ with $\text{int}(Z) \neq \emptyset$, and every closed subset $S \subset \text{int}(Z_0)$, there is a unitary $v \in A(Z, S)$ such that $vf = uf$ in $C^*(\mathbf{Z}, M, h)$ whenever $f \in C(M)$ vanishes on Y .

The condition on U used in the proof is that there are at least $\max(1, \frac{1}{2}d)$ images of \bar{U} under positive powers h^r of h , with r less than the smallest first return time of \bar{U} to itself, which are contained in $\text{int}(Y)$. Under this condition, the first step in the construction of v is an approximate polar decomposition, in the recursive subhomogeneous algebra $\gamma_Z(A(Z, S))$, of ug for a suitable function $g \in C(M)_{\text{ex}_Z(S)}$ which, in particular, is required to be equal to 1 on $M \setminus \text{int}(Y)$ and to vanish on Z .

It isn't in general true that $\text{int}(Z) \neq \emptyset$ implies $\text{int}(Z_0) \neq \emptyset$, although it happens that the sets we use in the diffeomorphism case automatically have $\text{int}(Z_k) \neq \emptyset$ for all k .

To sum up: We have what might be called the "basic construction" for weak approximation in $C^*(\mathbf{Z}, M, h)$ (not to be confused with the basic construction of subfactor theory), namely a triple (Y, Z, v) (or a quadruple (Y, Z, S, v)) consisting of closed subsets with

$$S \subset \text{int}(Z_0) \subset Z \subset \text{int}(Y) \subset Y \subset M$$

(or, if S is not present, at least $\text{int}(Z) \neq \emptyset$), and a unitary $v \in A(Z, S)$ ($A(Z)$ if S is not present) such that $vf = uf$ in $C^*(\mathbf{Z}, M, h)$ whenever $f \in C(M)$ vanishes on Y . We say weak approximation here because we have not approximated u in norm; rather, we have a unitary $v \in A(Z, S)$ which “acts like u ” (that is, like h) on most of the space M . In particular, this construction is not the same as what we call a “basic approximation” in [26]. The basic approximation, of which we describe an easier form in the next section, does permit the norm approximation of u , but requires two nested basic constructions and an additional unitary.

§3. An outline of the proof of local approximation

In this section, we outline the proof of a weak form of Theorem 1.1, namely that if $h: M \rightarrow M$ is a minimal diffeomorphism of a connected compact smooth manifold M with $\dim(M) > 0$, and if $F \subset C^*(\mathbf{Z}, M, h)$ is a finite set and $\varepsilon > 0$, then there is a recursive subhomogeneous algebra $A \subset C^*(\mathbf{Z}, M, h)$ which approximately contains F to within ε . This result requires most of the machinery needed for the proof of the full direct limit decomposition result.

The crucial ingredient not yet mentioned is related to Loring’s version [27] of Berg’s technique [1]. This method (described in Step 7 below) requires a priori bounds on the lengths of paths connecting certain elements in the unitary groups of hereditary subalgebras of recursive subhomogeneous algebras. This is an exponential length problem in the sense of [34]. We therefore begin by stating our exponential length result; we require some terminology.

First, if A is a unital C^* -algebra and $B \subset A$ is a hereditary subalgebra, we define the unitary group $U(B)$ to be

$$U(B) = \{u \in U(A) : u - 1 \in B\}.$$

(This is the same as a common definition in terms of the unitization B^+ of B , namely

$$U(B) = \{u \in U(B^+) : u - 1 \in B\}.$$

Moreover, if B is actually a corner, then this group can be canonically identified with the usual unitary group of B .) Further, let

$$A = \left[\cdots \left[\left[C(X_0, M_{n(0)}) \oplus_{C(X_1^{(0)}, M_{n(1)})} C(X_1, M_{n(1)}) \right] \right. \right. \\ \left. \left. \oplus_{C(X_2^{(0)}, M_{n(2)})} C(X_2, M_{n(2)}) \right] \cdots \right] \oplus_{C(X_l^{(0)}, M_{n(l)})} C(X_l, M_{n(l)})$$

be a recursive subhomogeneous algebra. If $B \subset A$ is a hereditary subalgebra and $x \in X_k$ for some k , then we define $\text{rank}_x(B)$ to be the rank of the identity in the image of B in the finite dimensional C^* -algebra $M_{n(k)}$ under the map ev_x given by point evaluation at $x \in X_k$. If $v \in U(A)$, then we say that $\det(v) = 1$ if $\det(\text{ev}_x(v)) = 1$ for all k and all $x \in X_k$. (Although determinants are not well defined in recursive subhomogeneous algebras, one can show that the condition $\det(v) = 1$ is well defined.)

Theorem 3.1. *Let $d, d' \geq 0$ be integers. Then there is an integer R such that the following holds.*

Let A be a recursive subhomogeneous algebra which has a separable recursive subhomogeneous decomposition with topological dimension at most d and strong covering number at most d' . Let $B \subset A$ be a hereditary subalgebra such that $\text{rank}_x(B) \geq R$ for every x in the total space of A . Let $v \in U(B)$ satisfy $\det(v) = 1$ and be connected to 1 by a path $t \mapsto v_t$ in $U(B)$ such that $\det(v_t) = 1$ for all t . Then there is a continuous path from v to 1 in $U(B)$ with length less than $4\pi(d' + 2)$.

At this point, we should give a brief indication of the significance of the strong covering number. We explained in Section 4 of [25] how to use relative versions of the subprojection and cancellation theorems for $C(X, M_n)$ to obtain analogous theorems for recursive subhomogeneous algebras. Theorem 3.1, however, is an exponential length theorem, and, at a crucial step in its proof, we have only been able to prove an approximate relative theorem for $C(X, M_n)$. (See Theorem 6.2 of [25].) Roughly speaking, errors accumulate everywhere that the recursive subhomogeneous decomposition of A specifies that two algebras be glued together. The strong covering number gives a limit on how often a neighborhood of a particular point in one of the base spaces is involved in such a gluing. It is a strengthened version of the most obvious notion (the “covering number”); the more obvious version proved to be technically too weak.

The definition of the strong covering number is somewhat complicated, and is omitted; instead, we illustrate with an example. Let X be a compact metric space, let E be a locally trivial continuous field over X with fiber M_n , and let $\Gamma(E)$ be the corresponding section algebra. Then any finite cover X_0, X_1, \dots, X_l of X by closed subsets, such that $E|_{X_k}$ is trivial for each k , induces a recursive subhomogeneous decomposition of $\Gamma(E)$. (See the proof of Proposition 1.7 of [29] and Example 2.8 of [25].) It can be shown that the strong covering number of this recursive subhomogeneous decomposition is the order (as in Definition 1.6.6 of [15]) of the cover of X by the sets X_0, X_1, \dots, X_l , that is, the largest

number d such that there are distinct r_0, r_1, \dots, r_d for which

$$\bigcap_{j=0}^d X_{r_j} \neq \emptyset.$$

Note the parallel with the definition of the covering dimension (Definition 1.6.7 of [15]).

At this point, we can explain how we use the condition that we have a diffeomorphism of a manifold. Let $Y \subset M$ satisfy $\text{int}(Y) \neq \emptyset$. Our method for bounding the strong covering number requires that there be an integer m such that, for any $m+1$ distinct integers $r_0, r_1, \dots, r_m \in \mathbf{Z}$, we have

$$\bigcap_{j=0}^m h^{r_j}(\partial Y) = \emptyset.$$

When h is a minimal diffeomorphism of a compact manifold, this is arranged as follows. First, require that ∂Y be a smooth submanifold (of codimension 1). Then perturb ∂Y by an arbitrarily small amount, so that all finite sets

$$h^{r_0}(\partial Y), h^{r_1}(\partial Y), \dots, h^{r_m}(\partial Y)$$

of distinct images of ∂Y under powers of h are jointly mutually transverse. This means, first, that $h^{r_0}(\partial Y)$ and $h^{r_1}(\partial Y)$ are transverse (see pages 28–30 of [18]) whenever $r_0 \neq r_1$, so that $h^{r_0}(\partial Y) \cap h^{r_1}(\partial Y)$ is a smooth submanifold (of codimension 2; see the theorem on page 30 of [18]); that $h^{r_2}(\partial Y)$ and $h^{r_0}(\partial Y) \cap h^{r_1}(\partial Y)$ are transverse whenever r_0, r_1 , and r_2 are all distinct, so that $h^{r_0}(\partial Y) \cap h^{r_1}(\partial Y) \cap h^{r_2}(\partial Y)$ is a smooth submanifold (of codimension 3); etc. These conditions guarantee that the intersection of any $\dim(M)+1$ distinct images of ∂Y under powers of h will be empty. (Note, however, that the resulting upper bound on the strong covering number turns out to be $\dim(M)(\dim(M) + 2)$, not $\dim(M)$. The situation is much more complicated than for section algebras of locally trivial continuous fields.) We thus have:

Proposition 3.2. Let M be a connected compact smooth manifold with $\dim(M) = d > 0$, and let $h: M \rightarrow M$ be a minimal diffeomorphism. For every $x \in M$ and open $U \subset M$ with $x \in U$, there is a closed set $Y \subset M$ with $x \in \text{int}(Y) \subset Y \subset U$ such that for every closed set $S \subset \text{int}(Y_0)$ (notation as in Section 2) which is homeomorphic to a closed ball in \mathbf{R}^d , the subalgebra $A(Y, S)$ satisfies the following properties:

- The recursive subhomogeneous decomposition of Theorem 2.7 has topological dimension equal to d .
- The decomposition of Theorem 2.7 has strong covering number at most $d(d + 2)$.
- In the notation of Theorem 2.7, we have $Y_k^{(0)} \subset \partial Y_k$ for all k .

We hope that if h is a minimal homeomorphism of a finite dimensional compact metric space, then one might be able to substitute a dimension theory argument for transversality in the above. We have not yet had time to look into this. What to do about infinite dimensional compact metric spaces (such as $(S^1)^{\mathbf{Z}}$) is less clear.

Now we start the outline of the proof of local approximation. We fix a connected compact smooth manifold M with $\dim(M) > 0$ and a minimal diffeomorphism $h: M \rightarrow M$.

Step 1. It suffices to prove the following: Let

$$f_1, f_2, \dots, f_m \in C(M) \subset C^*(\mathbf{Z}, M, h)$$

be a finite collection of functions, and let $\varepsilon > 0$. Then there is a recursive subhomogeneous algebra $A \subset C^*(\mathbf{Z}, M, h)$ which approximately contains $\{f_1, f_2, \dots, f_m, u\}$ to within ε . (The reason is that $C(M)$ and u generate $C^*(\mathbf{Z}, M, h)$ as a C^* -algebra.)

Step 2. Choose $\delta > 0$ so small that the functions f_1, f_2, \dots, f_m are all approximately constant to within $\frac{1}{2}\varepsilon$ on every subset of M with diameter less than δ . Choose an integer R following Theorem 3.1 for the number $d = \dim(M)$ and for $d' = d(d + 2)$, and also with $R \geq \max(1, \frac{1}{2}d)$. Choose an integer N so large that

$$\frac{4\pi(d' + 2)}{N} < \varepsilon.$$

Step 3. Choose a quadruple $(Y^{(1)}, Z^{(1)}, S, v_1)$, as described at the end of the previous section, consisting of closed subsets with

$$\emptyset \neq \text{int}(S) \subset S \subset \text{int}(Z_0^{(1)}) \subset Z^{(1)} \subset \text{int}(Y^{(1)}) \subset Y^{(1)} \subset M$$

and a unitary $v_1 \in A(Z^{(1)}, S)$ such that $v_1 f = u f$ in $C^*(\mathbf{Z}, M, h)$ whenever $f \in C(M)$ vanishes on $Y^{(1)}$. We also require that the conclusions of Proposition 3.2 be satisfied. Let $n_1(0) < n_1(1) < \dots < n_1(l_1)$ be the first return times $n_{Y^{(1)}}(0) < n_{Y^{(1)}}(1) < \dots < n_{Y^{(1)}}(l_{Y^{(1)}})$. We then further require that the sets involved be so small that:

- The sets $Y^{(1)}, h^{-1}(Y^{(1)}), \dots, h^{-N}(Y^{(1)})$ are pairwise disjoint (whence $n_1(0) > N$).

- The sets $Y^{(1)}, h^{-1}(Y^{(1)}), \dots, h^{-N}(Y^{(1)})$ all have diameter less than δ .
- The sets $h(S), h^2(S), \dots, h^{n_1(0)}(S)$ all have diameter less than δ .
- Each of the sets $h(S), h^2(S), \dots, h^{n_1(0)}(S)$ is either contained in one of $Y^{(1)}, h^{-1}(Y^{(1)}), \dots, h^{-N}(Y^{(1)})$ or is disjoint from all of them.

(Note that we choose S after having chosen $Y^{(1)}$.)

Step 4. Choose a triple $(Y^{(2)}, Z^{(2)}, v_2)$, as described at the end of the previous section, consisting of closed subsets with

$$\emptyset \neq \text{int}(Z^{(2)}) \subset Z^{(2)} \subset \text{int}(Y^{(2)}) \subset Y^{(2)} \subset \text{int}(S)$$

and a unitary $v_2 \in A(Z^{(2)}, S)$ such that $v_2 f = u f$ in $C^*(\mathbf{Z}, M, h)$ whenever $f \in C(M)$ vanishes on $Y^{(2)}$. Again, we also require that the conclusions of Proposition 3.2 be satisfied. Let $n_2(0) < n_2(1) < \dots < n_2(l_2)$ be the first return times $n_{Y^{(2)}}(0) < n_{Y^{(2)}}(1) < \dots < n_{Y^{(2)}}(l_{Y^{(2)}})$. Let $B \subset A(Z^{(2)})$ be the hereditary subalgebra generated by $C_0(\text{int}(Y^{(1)})) \subset C(M)$. We then further require that $Z^{(2)}$ be so small that $\gamma_{Z^{(2)}}(B)$, as a hereditary subalgebra of the recursive subhomogeneous algebra $\gamma_{Z^{(2)}}(A(Z^{(2)}))$, satisfies $\text{rank}_x(\gamma_{Z^{(2)}}(B)) \geq R$ for all x (in the sense discussed before Theorem 3.1). (This is accomplished by requiring that there be at least R images of $Z^{(2)}$ under positive powers h^r of h , with $r < n_2(0)$, which are contained in $\text{int}(Y^{(1)})$.)

Step 5. Observe that the relations $v_j f = u f$ in $C^*(\mathbf{Z}, M, h)$ whenever $f \in C(M)$ vanishes on $Y^{(j)}$ imply that $v_1^* v_2 f = f$ whenever $f \in C(M)$ vanishes on $Y^{(1)}$. From this one can deduce that $v_1^* v_2 \in U(B)$. With the help of the condition $\text{rank}_x(\gamma_{Z^{(2)}}(B)) \geq \max(1, \frac{1}{2}d)$, it is possible to alter the choice of v_2 so that, in addition to the conditions we already have, also $z = \gamma_{Z^{(2)}}(v_1^* v_2) \in U(\gamma_{Z^{(2)}}(B))$ satisfies $\det(z) = 1$ and is connected to 1 by a path $t \mapsto z_t$ in $U(\gamma_{Z^{(2)}}(B))$ such that $\det(z_t) = 1$ for all t . (For the meaning of these conditions, see the discussion before Theorem 3.1.) Then also $v_2^* v_1 = (v_1^* v_2)^*$ satisfies these properties.

Step 6. Apply Theorem 3.1 to find a path in $U(B)$ from $v_1^* v_2$ to 1 with total length less than $4\pi(d' + 2)$. Using a suitable subdivision of the domain of this path, find unitaries

$$v_2^* v_1 = w_0, w_1, \dots, w_{N-1}, w_N = 1 \in U(B)$$

such that

$$\|w_j - w_{j-1}\| < \frac{4\pi(d' + 2)}{N} < \varepsilon$$

for $1 \leq j \leq N$.

Step 7. Define

$$w = w_0(u^{-1}w_1u)(u^{-2}w_2u^2)\cdots(u^{-N}w_Nu^N).$$

Then w is a unitary in $C^*(\mathbf{Z}, M, h)$ with the following properties:

- (1) w commutes with every $f \in C(M)$ which is constant on each of the sets

$$Y^{(1)}, h^{-1}(Y^{(1)}), \dots, h^{-N}(Y^{(1)}).$$

- (2) w commutes with uv_2^* .

- (3) $\|wv_1w^* - v_2\| < \varepsilon$.

We will say something below about how these results follow. Some of the ideas are related to calculations in Section 6 of [31] and Section 2 of [32].

Step 8. Set

$$D = C^*(uv_2^*, A(Z^{(1)}, S)) \subset C^*(\mathbf{Z}, M, h) \quad \text{and} \quad A = wDw^*.$$

We show that A approximately contains f_1, f_2, \dots, f_m , and u to within ε .

Let T_1, T_2, \dots, T_r be the sets $Y^{(1)}, h^{-1}(Y^{(1)}), \dots, h^{-N}(Y^{(1)})$, together with all of the sets $h(S), h^2(S), \dots, h^{n_1(0)}(S)$ which are not contained in any of the images of $Y^{(1)}$ listed above. By the construction in Step (3), the sets T_1, T_2, \dots, T_r are pairwise disjoint and have diameter less than δ . The functions f_1, f_2, \dots, f_m are all approximately constant to within $\frac{1}{2}\varepsilon$ on every subset of M with diameter less than δ (by Step 2), so there exist functions $g_1, g_2, \dots, g_m \in C(M)$ which are actually constant on the sets T_1, T_2, \dots, T_r and satisfy $\|g_i - f_i\| < \varepsilon$ for $1 \leq i \leq m$. These functions are then constant on all of

$$Y^{(1)}, h^{-1}(Y^{(1)}), \dots, h^{-N}(Y^{(1)}) \quad \text{and} \quad h(S), h^2(S), \dots, h^{n_1(0)}(S).$$

Now $g_i \in A(Z^{(1)}, S) \subset D$ and (by Step 7 (1)) w commutes with g_1, g_2, \dots, g_m , so $g_1, g_2, \dots, g_m \in wDw^* = A$.

We also have $w(uv_2^* \cdot v_1)w^* \in A$. Using the relations $w(uv_2^*)w^* = uv_2^*$ and $\|wv_1w^* - v_2\| < \varepsilon$ from Step 7, we get

$$\|w(uv_2^* \cdot v_1)w^* - u\| = \|w(uv_2^*)w^* \cdot wv_1w^* - uv_2^* \cdot v_2\| < \varepsilon.$$

So u is approximately in A .

Step 9. The algebra D , and hence $A = wDw^*$, is a recursive subhomogeneous algebra with topological dimension d and strong covering number at most $d' = d(d + 2)$ (that is, no more complicated than $A(Z^{(1)}, S)$). This step is where S is used in an essential way.

Let's assume for simplicity that $\text{sp}(uv_2^*)$ is the whole unit circle S^1 . Then it turns out that D is a pullback

$$D \cong A(Z^{(1)}, S) \oplus_{M_{n_1(0)}, \varphi, \psi} C(S^1, M_{n_1(0)}).$$

The map $\psi: C(S^1, M_{n_1(0)}) \rightarrow M_{n_1(0)}$ is evaluation at $1 \in S^1$. The map $\varphi: A(Z^{(1)}, S) \rightarrow M_{n_1(0)}$ is the evaluation on the set S in the recursive subhomogeneous decomposition described in Theorem 2.7. (This is really a point evaluation, because the elements of $A(Z^{(1)}, S)$ are constant on S .) The unitary uv_2^* corresponds to the pair

$$(1, \text{diag}(z, 1, \dots, 1))$$

in which z is the identity function $\zeta \mapsto \zeta$ in $C(S^1)$.

The key relation here is that uv_2^* acts as 1 off $h(S)$. Thus, if $f \in C(M)$ vanishes on $h(S)$, then $(uv_2^*)f = f(uv_2^*) = f$. If in addition f vanishes on $Z^{(1)}$, then $(uv_2^*)(uf) = (uf)(uv_2^*) = uf$. These relations imply, for example, that uv_2^* commutes with all elements of $\text{Ker}(\varphi) \subset A(Z^{(1)}, S)$.

The verification of the isomorphism with the pullback requires lots of functional calculus. For example, one needs to define suitable homomorphisms with domain $D = C^*(uv_2^*, A(Z^{(1)}, S))$, or at least determine somehow all the elements of this C^* -algebra. We omit further discussion, except to note that it is much easier to demonstrate that there is an exact sequence

$$0 \longrightarrow \text{Ker}(\varphi) \longrightarrow D \longrightarrow C(S^1, M_{n_1(0)}) \longrightarrow 0,$$

as should certainly happen for a pullback with surjective maps. This exact sequence implies (using Theorem 2.16 of [29]) that D is a recursive subhomogeneous algebra with topological dimension d , but doesn't give anything about the strong covering number.

This finishes the outline of the proof of local approximation.

Let us now return to the explanation of Step 7. We first explain the significance of w , in a greatly simplified context—so much simplified that it does not satisfy the hypotheses of this section. Then we give an outline of how to prove the claimed properties in our case.

For the simple context, let us assume that

$$Z^{(1)} = Y^{(1)} \quad \text{and} \quad M = \prod_{j=1}^n h^j(Z^{(1)}).$$

(We ignore S , since it is not relevant for this step.) In this case, note that $Z_0^{(1)} = Z^{(1)}$, that $n = n_1(0)$, and that $\gamma_{Z^{(1)}}$ induces an isomorphism $A(Z^{(1)}) \cong M_n(C(Z^{(1)}))$, under which functions constant on each of the sets

$$Y^{(1)}, h^{-1}(Y^{(1)}), \dots, h^{-N}(Y^{(1)})$$

are sent to the diagonal matrices in $M_n(C(Z^{(1)}))$, the last $N + 1$ diagonal entries of which are constants. (Our simplifying assumptions imply that $h^{-j}(Z^{(1)}) = h^{n-j}(Z^{(1)})$.)

Let us further assume we have an h -invariant Borel probability measure μ on M , and that $C^*(\mathbf{Z}, M, h)$ is represented faithfully on $L^2(M, \mu)$ with $C(M)$ acting as multiplication operators and u acting as $u\xi = \xi \circ h^{-1}$. There is a direct sum decomposition

$$L^2(M, \mu) = \bigoplus_{j=1}^n L^2(h^j(Z^{(1)})),$$

which determines an identification of $L(L^2(M, \mu))$ with $M_n(L^2(Z^{(1)}))$ which is compatible in a suitable sense with the isomorphism $\gamma_{Z^{(1)}}$. Further let e_j be the projection onto $L^2(h^j(Z^{(1)}))$. With respect to this identification, we can write

$$u = \begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 & 0 & u^{(0)} \\ 1 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & & \vdots & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 1 & 0 \end{pmatrix},$$

with $u^{(0)} \in e_1 L(L^2(M, \mu)) e_n$. (Note that it is equal to the shift matrix s_0 considered in Section 2, except for the upper right corner.) Similarly,

we can write

$$v_j = \begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 & 0 & v_j^{(0)} \\ 1 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & & \vdots & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

Again, the difference is in the the upper right corner, but note that v_1 and v_2 are now in $A(Z^{(2)})$.

In this situation, we let $w'_i = e_0 w_i e_0$, and identify w as

$$w = \text{diag} (1, 1, \dots, 1, w'_N, w'_{N-1}, \dots, w'_1, w'_0).$$

(We have used the fact that $n \geq N + 1$.) Now Condition (1) of Step 7 follows from the fact that w is block diagonal and that functions in $C(M)$ constant on each of the sets

$$Y^{(1)}, h^{-1}(Y^{(1)}), \dots, h^{-N}(Y^{(1)})$$

are diagonal matrices, the last $N + 1$ diagonal entries of which are constants. For Condition (2) of Step 7, we calculate:

$$wv_2^* = \text{diag} (u^{(0)}(v_2^{(0)})^*, 1, 1, \dots, 1).$$

This element clearly commutes with w . (The worst case is $n = N + 1$; then, recall that $w_N = 1$.) For Condition (3) of Step 7, we estimate instead $\|w - v_2 w v_1^*\|$. (This is easily seen to be equivalent.) A computation shows that

$$\begin{aligned} v_2 w v_1^* &= \text{diag} (v_2^{(0)} w'_0 (v_1^{(0)})^*, 1, \dots, 1, 1, w'_N, \dots, w'_2, w'_1) \\ &= \text{diag} (1, 1, \dots, 1, 1, w'_N, \dots, w'_2, w'_1). \end{aligned}$$

(The entries of w have all been moved one space down the diagonal. In addition, the new first entry has been modified. Since $w_0 = v_2^* v_1$, we have $w'_0 = (v_2^{(0)})^* v_1^{(0)}$.) Therefore, using $w_N = 1$, we get

$$\|w - v_2 w v_1^*\| = \max_{1 \leq j \leq N} \|w_j - w_{j-1}\| < \varepsilon.$$

In the actual situation, we work inside $C^*(\mathbf{Z}, M, h)$. Let $B \subset C^*(\mathbf{Z}, M, h)$ be the hereditary subalgebra of Step 4. For the matrix decomposition, we substitute the fact that the hereditary subalgebras

$$B, u^{-1}Bu, u^{-2}Bu^2, \dots, u^{-N}Bu^N$$

are orthogonal in $C^*(\mathbf{Z}, M, h)$. This follows from the fact that the sets

$$Y^{(1)}, h^{-1}(Y^{(1)}), \dots, h^{-N}(Y^{(1)})$$

are pairwise disjoint. As a consequence, the factors

$$w_0, u^{-1}w_1u, u^{-2}w_2u^2, \dots, u^{-N}w_Nu^N$$

of w , which are in the unitary groups of these hereditary subalgebras, all commute with each other, and also with any function $f \in C(M)$ which is constant on each of the sets

$$Y^{(1)}, h^{-1}(Y^{(1)}), \dots, h^{-N}(Y^{(1)}).$$

When proving that w commutes with uv_2^* , it helps to show first that

$$u^{-j}w_jw^j = v_2^{-j}w_jv_2^j$$

for $0 \leq j \leq N$. In fact, this is true if w_j is replaced by any $b \in C^*(\mathbf{Z}, M, h)$ which differs by a scalar from an element of B . For the verification of the norm estimate in Condition (3) of Step 7, one needs in addition the following fact, which is the analog of the estimate on the difference of diagonal matrices above: if C_0, C_1, \dots, C_N are orthogonal hereditary subalgebras in a C^* -algebra A , and if $y_j, z_j \in U(C_j)$ for $0 \leq j \leq N$, then

$$\|y_0y_1 \cdots y_N - z_0z_1 \cdots z_N\| = \max_{0 \leq j \leq N} \|y_j - z_j\|.$$

§4. Direct limit decomposition

We give here a very brief approximate outline of the modifications necessary to achieve the direct limit decomposition of Theorem 1.1, as opposed to merely local approximation. The previous section describes the construction of a (simple version of) a single “basic approximation”, and the problem is to arrange successively better ones so as to obtain an increasing sequence of subalgebras of $C^*(\mathbf{Z}, M, h)$. As will be clear, putting everything together requires complicated notation, and there are interactions between the modifications described below which we do not have room to discuss here.

First, the unitary corresponding to w in each new basic approximation must commute with all elements of the subalgebra $A(Z^{(2)})$ from the previous one. This requires two changes. The old subalgebra $A(Z^{(2)})$ must be replaced by $A(Z^{(2)}, T)$ for some suitable T , and the new set $Y^{(1)}$ must be contained in T . Also, the sequence

$$v_2^* v_1 = w_0, w_1, \dots, w_{N-1}, w_N = 1 \in U(B)$$

used to construct the new w must now consist of constant subsequences, the lengths of which are certain return times associated with the old $Z^{(2)}$.

Second, having constructed one approximating subalgebra, say A_0 , the next one, say A_1 , will be slightly “twisted” with respect to A_0 , even with the adjustment above. To straighten this out, it is necessary to modify A_0 by replacing v_2 in the construction by a nearby unitary. Then, after constructing A_2 , one must further modify the unitaries v_2 associated with both A_1 and A_0 , etc. Enough control must be maintained that the sequences of modifications converge to unitaries not too far from the original choices.

Third, even apart from the “twisting” referred to in the previous paragraph, the use of the subsets S leads to problems with the expected inclusion relations between subalgebras. Suppose, for example, we have closed subsets Y and Z , satisfying the conclusions of Proposition 3.2, with associated first return times

$$n_Y(0) < n_Y(1) < \dots < n_Y(l_Y) \quad \text{and} \quad n_Z(0) < n_Z(1) < \dots < n_Z(l_Z),$$

and with corresponding subsets

$$Y_0, Y_1, \dots, Y_{l_Y} \subset Y \quad \text{and} \quad Z_0, Z_1, \dots, Z_{l_Z} \subset Z.$$

Suppose that

$$\emptyset \neq S \subset \text{int}(Z_0) \subset Z \subset \text{int}(Y_0)$$

(in particular, $Z \subset Y$), and that $n_Z(0) > n_Y(0)$ (this is the relevant situation, because arbitrarily good approximations require arbitrarily large values of the smallest first return time). We have $A(Y) \subset A(Z)$, because every function in $C(M)$ which vanishes on Y also vanishes on Z . However, it is not true that $A(Y, S) \subset A(Z, S)$. In fact, $C(M) \cap A(Z, S)$ consists of those functions in $C(M)$ that are constant on the sets

$$h(S), h^2(S), \dots, h^{n_Z(0)}(S),$$

$C(M) \cap A(Y, S)$ consists of those functions in $C(M)$ that are constant on the sets

$$h(S), h^2(S), \dots, h^{n_Y(0)}(S),$$

and $n_Y(0) < n_Z(0)$, so $C(M) \cap A(Y, S) \subsetneq C(M) \cap A(Z, S)$.

To fix this problem, it is necessary to replace the single set S in the construction of $A(Y, S)$ by a whole family of subsets. One must require that whenever $h^j(S) \subset Y$, with $0 < j < n_Z(0)$, then there is k with $h^j(S) \subset \text{int}(Y_k)$. Then one uses the collection of all such $h^j(S)$, rather than just S itself, with the obvious modification to account for the fact that they are no longer all subsets of $\text{int}(Y_0)$. The resulting subalgebra is a proper subalgebra of $A(Y, S)$.

In the inductive construction of an increasing sequence of approximating subalgebras of $C^*(\mathbf{Z}, M, h)$, this works out as follows. First, one constructs an approximating algebra $A_0^{(0)}$. Then one constructs an approximating algebra $A_1^{(1)}$, incorporating the first two modifications discussed above, and using a sufficiently small set S . Next, one replaces $A_0^{(0)}$ by a smaller algebra $A_0^{(1)}$, using the approach outlined in the previous paragraph on the algebra $A(Z^{(1)}, S)$ appearing in the definition of $A_0^{(0)}$, but with the set S from the construction of $A_1^{(1)}$. That done, one constructs $A_2^{(2)}$. Then it is necessary to go back and replace both $A_1^{(1)}$ and $A_0^{(1)}$ (in that order) by smaller subalgebras $A_1^{(2)}$ and $A_0^{(2)}$, in a similar way. This procedure continues for all n .

There are two problems. First, $\bigcap_{k=n}^{\infty} A_n^{(k)}$ must still be large enough to approximate not too badly the finite set that the first algebra $A_n^{(n)}$ was constructed to approximate. Second, $\bigcap_{k=n}^{\infty} A_n^{(k)}$ must still be a recursive subhomogeneous algebra with topological dimension at most d and strong covering number at most $d(d+2)$. Since subalgebras of recursive subhomogeneous algebras need not even be recursive subhomogeneous algebras (see Example 3.6 of [29]), this requires work. The construction of the subalgebra $A(Y, S)$ can be viewed as identifying the subset S of Y to a point. By the time the inductive process of the previous paragraph is complete, one must identify infinitely many subsets of Y to (distinct) points, in such a way that the resulting space is not only Hausdorff (there is trouble even here) but in fact has dimension no greater than $\dim(Y)$. The details are quite messy.

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