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Stable rank and real rank of graph C^*-algebras

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Abstract.

For any row finite directed graph E there exists a universal C^*-algebra $C^*(E)$ ([KPR, KPRR]) generated by projections and partial isometries satisfying the Cuntz-Krieger E-relations. This class of graph algebras includes the Cuntz-Krieger algebras and all AF algebras up to stable isomorphisms([D]). In this paper we give conditions for E under which the algebra $C^*(E)$ has stable rank one or real rank zero. A simple graph C^*-algebra is either AF or purely infinite, hence it is always extremally rich. We discuss the extremal richness of some graph C^*-algebras and present several examples of prime ones with finitely many closed ideals.

§1. Introduction

As a generalization of Cuntz-Krieger algebras, a class of C^*-algebras generated by projections and partial isometries subject to the relations determined by directed graphs has been studied in [KPRR], [KPR] and later in [BPRS], and these algebras are called graph C^*-algebras. Since they are basically generated by partial isometries and projections one may expect that most of them must have real rank zero like AF algebras or Cuntz algebras. In fact if the associated graph C^*-algebra for a row finite graph E is simple then $C^*(E)$ always has real rank zero since it is either AF or purely infinite ([KPR]). On the other hand, the Toeplitz algebra can occur as a graph C^*-algebra but its real rank is not zero, hence we want to know when the graph algebra has real rank zero. We will answer the question in terms of the loop structure of a graph in Theorem 3.2 and Theorem 3.4.

Recall that a projection p in a C^*-algebra A is said to be *infinite* if it is Murray-von Neumann equivalent to its proper subprojection. We call a unital C^*-algebra A *infinite* if the unit projection is infinite, and *finite* otherwise. An infinite C^*-algebra whose every nonzero hereditary

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subalgebra contains an infinite projection is called *purely infinite.* If a unital C^*-algebra A has stable rank one ($sr(A) = 1$, see [Rf]), that is, the set A^{-1} of all invertible elements is dense in A, then one can see that A should be finite. All AF-algebras ([Rf]), irrational rotation algebras([Pt]) are those known to have stable rank one. We will give a sufficient and necessary condition for a graph E that $C^*(E)$ has stable rank one in Theorem 3.1.

As an attempt to extend notions and results for finite C^*-algebras to infinite cases Brown and Pedersen ([BP2]) considered the quasi-invertible elements A_q^{-1} in a unital C^*-algebra A and call A *extremally rich* if the set A_q^{-1} is dense in A since it turns out in [BP2] that this condition is equivalent to say that the closed unit ball A_1 contains enough extreme points so that the convex hull of its extreme points coincides with the whole A_1;

$$conv(\mathcal{E}(A)) = A_1,$$

where $\mathcal{E}(A)$ denotes the extreme points of A_1. Since $A^{-1} \subset A_q^{-1}$ for any unital C^*-algebra A we see that a unital C^*-algebra A with $sr(A) = 1$ is always extremally rich. On the other hand it is a nontrivial fact that purely infinite simple C^*-algebras (for example, Cuntz algebras) are also extremally rich (see [LO], [Pd]). Therefore a simple graph C^*-algebra is always extremally rich. Recall that a graph C^*-algebra $C^*(E)$ is simple if and only if E is cofinal and satisfies condition (L). We show the cofinality of a graph E is in fact a sufficient condition for the algebra $C^*(E)$ to be extremally rich and also provide an example of non-extremally rich prime graph C^*-algebra that has only three proper ideals and has real rank zero.

§2. Preliminaries

We recall definitions and results from [KPR], [KPRR], and [BPRS] on directed graphs and graph C^*-algebras. A *directed graph* $E = (E^0, E^1, r, s)$ (or simply $E = (E^0, E^1)$) consists of countable vertices E^0, edges E^1 and the range, source maps $r, s : E^1 \to E^0$. E is *row finite* if each vertex $v \in E^0$ emits at most finitely many edges, and a row finite graph is *locally finite* if each vertex receives only finitely many edges. If $e_1, \dots, e_n$ ($n \geq 2$) are edges with $r(e_i) = s(e_{i+1})$, $1 \leq i \leq n-1$, then one can form a (finite) path $\alpha = (e_1, \dots, e_n)$ of *length* $|\alpha| = n$, and extend the maps r, s by $r(\alpha) = r(e_n), s(\alpha) = s(e_1)$. Similarly one can think of infinite paths.

Let E^n be the set of all finite paths of length n (so vertices in E^0 are regarded as finite paths of length zero) and let E^* be the set of all finite paths, and E^∞ the set of infinite paths. A vertex $v \in E^0$ with $s^{-1}(v) = \emptyset$ is called a *sink*.

For a row finite directed graph E, a *Cuntz-Krieger E-family* consists of mutually orthogonal projections $\{P_v \,|\, v \in E^0\}$ and partial isometries $\{S_e \,|\, e \in E^1\}$ satisfying the Cuntz-Krieger relations

$$S_e{}^* S_e = P_{r(e)}, \ e \in E^1, \text{ and } \ P_v = \sum_{s(e)=v} S_e S_e{}^*, \ v \in s(E^1).$$

From these relations, it can be shown that every non-zero word in S_e, P_v and S_f^* reduces to a partial isometry of the form $S_\alpha S_\beta^*$ for some $\alpha, \beta \in E^*$ with $r(\alpha) = r(\beta)$ ([KPR], Lemma 1.1).

Theorem 2.1. ([KPR], Theorem 1.2) *For a row finite directed graph $E = (E^0, E^1)$, there exists a C^*-algebra $C^*(E)$ generated by a Cuntz-Krieger E-family $\{s_e, p_v \,|\, v \in E^0, e \in E^1\}$ of non-zero elements such that for any Cuntz-Krieger E-family $\{S_e, P_v \,|\, v \in E^0, e \in E^1\}$ of partial isometries acting on a Hilbert space $\mathcal{H}$, there exists a representation $\pi : C^*(E) \to B(\mathcal{H})$ such that*

$$\pi(s_e) = S_e, \ \textit{and} \ \ \pi(p_v) = P_v$$

for all $e \in E^1, v \in E^0$.

Let $\{s_e, p_v \mid e \in E^1, v \in E^0\}$ be a Cuntz-Krieger E-family generating the C^*-algebra $C^*(E)$. Then for each $z \in \mathbb{T}$ we have another Cuntz-Krieger E-family $\{zs_e, p_v \mid e \in E^1, \ v \in E^0\}$ in $C^*(E)$, and by the universal property of $C^*(E)$ there exists an isomorphism $\gamma_z : C^*(E) \to C^*(E)$ such that $\gamma_z(s_e) = zs_e$ and $\gamma_z(p_v) = p_v$. In fact, $\gamma : z \mapsto \gamma_z \in \mathrm{Aut}(C^*(E))$ is a strongly continuous action of $\mathbb{T}$ on $C^*(E)$ and called the *gauge action* ([BPRS]).

A finite path α with $|\alpha| > 0$ is called a *loop* at v if $s(\alpha) = r(\alpha) = v$. It turns out that the distribution of loops in a graph E is very important to understand the structure of a graph C^*-algebra $C^*(E)$, in particular if E has no loops then $C(E)$ is AF.

A graph E is said to satisfy *condition* (L) if every loop in E has an exit, and *condition* (K) if for any vertex v on a loop there exist at least two distinct loops based at v. Note that condition (K) is stronger than (L) and if E has no loops then the two conditions are trivially satisfied.

For two vertices v, w we simply write $v \geq w$ if there is a path $\alpha \in E^*$ from v to w. A subset H of E^0 is said to be *hereditary* if $v \geq w$ and $v \in H$ imply $w \in H$, and a hereditary set H is *saturated* if $s^{-1}(v) \neq \emptyset$ and $\{r(e) \mid s(e) = v\} \subset H$ imply $v \in H$. The *saturation* of a hereditary set H is the smallest saturated subset of E^0 containing H.

Let H be a saturated hereditary subset of E^0. Then the ideal $I(H) = \overline{span}\{s_\alpha s_\beta^* \mid \alpha, \beta \in E^*, r(\alpha) = r(\beta) \in H\}$ is clearly gauge-invariant and $I(H)$ is generated by $\{p_v \mid v \in H\}$.

In case E has no sinks, in [KPRR], an isomorphism of the lattice of saturated hereditary subsets V of E^0 into the lattice of ideals $I(V)$ in $C^*(E)$ was established and it is shown that the quotient algebra $C^*(E)/I(V)$ is isomorphic to a graph algebra $C^*(G)$ for a certain subgraph G of E. More generally, the following was proved in [BPRS].

Theorem 2.2. ([KPRR] [BPRS, Theorem 4.1]) *Let $E = (E^0, E^1, r, s)$ be a row finite directed graph. For each subset H of E^0, let $I(H)$ be the ideal in $C^*(E)$ generated by $\{p_v \mid v \in H\}$.*

(a) *The map $H \mapsto I(H)$ is an isomorphism of the lattice of saturated hereditary subsets of E^0 onto the lattice of closed gauge-invariant ideals of $C^*(E)$.*

(b) *Suppose H is saturated and hereditary. If $G^0 := E^0 \setminus H$, $G^1 := \{e \in E^1 \mid r(e) \notin H\}$, and $G := (G^0, G^1, r, s)$, then $C^*(E)/I(H)$ is canonically isomorphic to $C^*(G)$ and the ideal $I(H)$ is strong Morita equivalent to $C^*(K)$, where $K := (H, \{e \mid s(e) \in H\})$.*

Note that if a graph E satisfies condition (K) then the isomorphism of Theorem 2.2.(a) maps onto the lattice of all closed ideals in $C^*(E)$, that is, every ideal is gauge-invariant. It is known ([BPRS], [JPS]) that for a row-finite graph E, the graph C^*-algebra $C^*(E)$ is simple if and only if E is a cofinal graph satisfying condition (L), here we say that E is *cofinal* if every vertex connects to every infinite path.

Proposition 2.3. ([KPR], Corollary 3.11) *Let E be a locally finite graph which has no sinks, is cofinal, and satisfies condition* (L). *Then $C^*(E)$ is simple, and*

(i) *if E has no loops, then $C^*(E)$ is AF;*

(ii) *if E has a loop, then $C^*(E)$ is purely infinite.*

§3. Stable rank and real rank of graph C^*-algebras

If a graph E has no loops at all then the resulting algebra $C^*(E)$ is AF([KPR, Theorem 2.4]), hence its stable rank is one. To see if a graph with loops can have stable rank one consider the simple graph E consisting of a single vertex v and a single loop at v. Then the graph algebra $C^*(E)$ is the commutative C^*-algebra with the spectrum $\mathbb{T}$, the unit circle, and it also has stable rank one. But if we add an edge ranging at other vertex than v, the resulting graph algebra is the Toeplitz algebra whose stable rank is 2. The following shows precisely when the graph algebra has its stable rank one. Actually if a loop has an exit then there are infinite projections in the graph algebra and so the stable rank is not one anymore.

Theorem 3.1. ([JPS, Theorem 3.3]) *Let $E = (E^0, E^1)$ be a row finite directed graph. Then E has no loop with an exit if and only if $sr(C^*(E)) = 1$.*

Recall from [BP1] that a unital C^*-algebra A (or $\tilde{A}$ if A is non-unital) has real rank zero if the set of invertible self adjoint elements is dense in the whole set of self adjoint elements, or equivalently every non zero hereditary C^*-subalgebra contains a non zero projection. So the C^*-algebras with real rank zero (for example, AF algebras, purely infinite simple C^*-algebras, all von Neumann algebras) have been considered as the ones containing reasonably many projections in some sense.

Theorem 3.2. [JPS, Theorem 4.3] *Let E be a locally finite directed graph with no sinks. If $RR(C^*(E)) = 0$ then E satisfies condition* (K).

Corollary 3.3. *Let E be a locally finite directed graph with no sinks. If $sr(C^*(E)) = 1$ and $RR(C^*(E)) = 0$ then $C^*(E)$ is AF.*

Theorem 3.4. [JPS, Theorem 4.6] *Let E be a locally finite directed graph with no sinks which satisfies condition* (K). *If $C^*(E)$ has only finitely many ideals then $RR(C^*(E)) = 0$. In particular, if E is a finite graph then $RR(C^*(E)) = 0$.*

Let A be a $\{0, 1\}$-matrix with no zero row or column. Then A can be viewed as a vertex matrix of a finite graph E with no sinks. If A satisfies Cuntz-Krieger's condition (I) in [CK] then it clearly follows that E satisfies (L) (or, equivalently condition (I) introduced for graphs in [KPR]) from their definitions. By Proposition 4.1 of [KPRR], the graph algebra $C^*(E)$ is also generated by a Cuntz-Krieger A-family of partial isometries, hence the Cuntz-Krieger algebra $\mathcal{O}_A$ is isomorphic to the graph algebra $C^*(E)$. On the other hand, the graph algebra $C^*(E)$ is known to be isomorphic to the Cuntz-Krieger algebra $\mathcal{O}_B$ associated with the edge matrix B of E. Therefore those three algebras are all isomorphic. Furthermore by Theorem 3.2, 3.4, and Lemma 6.1 of [KPRR], we have the following corollary.

Corollary 3.5. *Let A be a $\{0, 1\}$-matrix with no zero row or column. Suppose A satisfies Cuntz-Krieger's condition* (I) *and let E be the finite graph having A as its vertex matrix. Then the following are equivalent:*
(i) $RR(\mathcal{O}_A) = 0$,
(ii) *A satisfies Cuntz's condition* (II),

(iii) *E satisfies condition* (K).

§4. Extremal richness of graph C^*-algebras

Let A be a unital C^*-algebra. Then it is well known that an extreme point v in A_1 is characterized as a partial isometry satisfying $(1-vv^*)A(1-v^*v) = 0$ ([Pd, Proposition 1.4.7]). Let A_+^{-1} be the set of all positive invertible elements of A. We call elements $x \in \mathcal{E}A_+^{-1}(= A^{-1}\mathcal{E}A^{-1})$ *quasi-invertible* ([BP3]) and denotes the set of all quasi-invertible elements in A by A_q^{-1}. If A_q^{-1} is dense in A A is called *extremally rich.* For a non-unital C^*-algebra A, A is said to be extremally rich when its unitization $\tilde{A}$ is so. Obviously a C^*-algebra A with $sr(A) = 1$ is extremally rich since $A^{-1} \subset A_q^{-1}$. In particular, all AF-algebras are extremally rich. Other examples are purely infinite simple C^*-algebras ([Pd, Theorem 10.1], [LO, Lemma 3.3]), the Toeplitz algebra ([Pd, Corollary 9.2]), commutative C^*-algebras $C(X)$ with $\dim(X) \leq 1$ (see [BP3, section 3]), and all von Neumann algebras ([Pd, Theorem 4.2]). Also a simple C^*-algebra A is extremally rich if and only if it is purely infinite or it has stable rank one ([BP2, Corollary 10.5]). Thus from Proposition 2.3 it follows that every simple graph C^*-algebra $C^*(E)$ (hence, E should be cofinal and satisfy condition (L)) is extremally rich. The following shows in fact that every graph C^*-algebra $C^*(E)$ associated to a cofinal graph E is extremally rich.

Proposition 4.1. ([JPS, Proposition 3.7]) *Let G be a locally finite directed graph. If G is cofinal then either $sr(C^*(G)) = 1$ or it is purely infinite and simple.*

There are extremally rich graph C^*-algebras that are associated to graphs which are not cofinal, for example the Toeplitz algebra (see Example 4.6 below) which is neither purely infinite simple nor of stable rank one ($sr(\mathcal{T}) = 2$). These graph algebras will arise from directed graphs containing some loops with exits, so that they should have many infinite projections and hence their stable rank are not one any more.

To this end note the following corollary of Theorem 2.2.

Corollary 4.2. ([JPS, Theorem 3.5]) *Let $E = (E^0, E^1, r, s)$ be a row-finite directed graph with the set V of sinks. Then there is a subgraph $G = (E^0 \setminus H, \{e \in E^1 \mid r(e) \notin H\})$ of E with no sinks such that $C^*(E)/I(V)$ is isomorphic to $C^*(G)$, where H is the saturation of V and $I(V) = \overline{span}\{s_\alpha s_\beta{}^* \mid \alpha, \beta \in E^*,\ r(\alpha) = r(\beta) \in V\}$.*

We also need to review briefly the following useful results on extremally rich C^*-algebras.

Theorem 4.3. ([BP2],[BP3], [LO]) (a) *Every quotient, every direct sum or direct product and every hereditary C^*-subalgebra of an extremally rich C^*-algebra is again extremally rich.*
(b) *If A is strong Morita equivalent (or stably isomorphic) to an extremally rich C^*-algebra B then A is also extremally rich.*

Let A be a unital C^*-algebra and I be a closed two-sided ideal.
(c) *Suppose $sr(I) = 1$. Then A is extremally rich if and only if A/I is extremally rich and extreme partial isometries lift.*
(d) *$sr(A) = 1$ if and only if $sr(I) = sr(A/I) = 1$ and every invertible elements lifts, that is, $(\tilde{A}/I)^{-1} = \tilde{A}^{-1}/I$.*
(e) *If I and A/I are purely infinite simple C^*-algebras then A is extremally rich.*

For a C^*-algebra A and projections P, Q in A, the extreme points $\mathcal{E}(PAQ)$ of the closed convex set PA_1Q consists of elements $u \in PA_1Q$ which is a partial isometry such that $(P - uu^*)A(Q - u^*u) = \{0\}$. We say that the space PAQ is *extremally rich* if either $\mathcal{E}(PAQ) = \emptyset$ or $\mathcal{E}(PAQ) \neq \emptyset$ and $(PAP)^{-1}\mathcal{E}(PAQ)(QAQ)^{-1}$ is dense in PAQ. If $\mathcal{E}(PAQ) \neq \emptyset$ then PAQ is extremally rich if and only if $PA_1Q = conv(\mathcal{E}(PAQ))$ (see [BP2]).

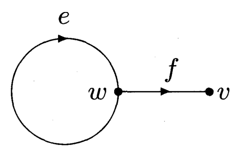
For any non-zero projections P, Q acting on a Hilbert space $\mathcal{H}$, one can show that $\mathcal{E}(PB(\mathcal{H})Q) \neq \emptyset$ and the space $PB(\mathcal{H})Q$ is extremally rich by Proposition 11.4 of [BP2]; if A is a C^*-algebra with real rank zero and $\mathcal{E}(PAQ) \neq \emptyset$ for every pair of projections P, Q in A, then every such a space PAQ is, in fact, extremally rich.

Proposition 4.4. ([BP2, Proposition 11.7]) *Let I be a closed ideal with real rank zero in a unital C^*-algebra A, such that PIQ is extremally rich for any pair of projections such that $P \in A$ and $Q \in I$. If A/I is extremally rich and $\mathcal{E}(A/I)$ consists only of isometries and co-isometries then A is extremally rich.*

Note that the C^*-algebra $B(\mathcal{H})$ and its closed ideal $\mathcal{K}(\mathcal{H})$ of compact operators are known to have real rank zero. The following generalization of Proposition 4.1 can be proved by applying proposition 4.4.

Theorem 4.5. ([J]) *Let $E = (E^0, E^1)$ be a locally finite directed graph and V the set of sinks. If the subgraph G in Corollary 4.2 is cofinal then $C^*(E)$ is extremally rich.*

Example 4.6. Consider the following graph E.

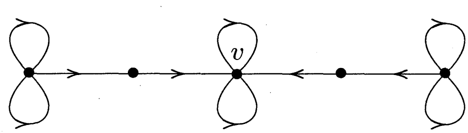


The sink v generates an ideal I which is isomorphic to $\mathcal{K}$, the compact operators acting on an infinite dimensional separable Hilbert space. Set $S = s_e + s_f$. Then $S^*S = 1$ and $SS^* = p_w < 1 = p_w + p_v$. Thus S is a proper isometry. Let $\mathcal{T}$ be the C^*-subalgebra of $C^*(E)$ generated by S. Since $S(1 - SS^*) = s_f$, it follows that $s_f \in \mathcal{T}$, hence $C^*(E) = \mathcal{T}$ is the Toeplitz algebra. Note that the subgraph G in Corollary 4.2 consists of the simple loop e, and so cofinal and by Theorem 4.5 we see that $\mathcal{T}$ is extremally rich, which is known in [Pd]. More generally, if a graph E consists of a simple loop with n vertices and each of the vertices emits an edge then we can conclude that the resulting graph algebra $C^*(E)$ is extremally rich.

Recall from Theorem 4.3(e) that if I is a purely infinite and simple closed ideal of a unital C^*-algebra A such that the quotient algebra A/I is also purely infinite and simple then A is extremally rich. Now suppose a C^*-algebra B has two proper ideals $I_1 \subset I_2$ such that every possible simple quotient is purely infinite. Then one cannot conclude the extremal richness of B. In fact, it is known in [LO, Remark 4.10] that there exists a non-extremally rich unital C^*-algebra B (non-separable) with two proper ideals $I_1 \subset I_2$ such that $I_1, I_2/I_1$ and B/I_2 are all purely infinite and simple.

In the following we give a separable unital graph C^*-algebra B with $RR(B) = 0$ which has exactly three proper ideals and every possible quotient is purely infinite and extremally rich, but B is not.

Example 4.7. Consider the following finite directed graph $E = (E^0, E^1)$.



Since E satisfies condition (K) we see that the graph algebra $C^*(E)$ has real rank zero by Theorem 3.4 and $C^*(E)$ has exactly three proper ideals by Theorem 2.2. Let H be the smallest hereditary saturated vertex subset containing v. Then the ideal $I(H)$ corresponding to H is stably isomorphic to the graph algebra $C^*(G)$, where G is a subgraph of

E with three vertices in the middle of E and four edges connecting them (Theorem 2.2). Since G is cofinal and satisfies (K) (hence (L)) $C^*(G)$ is purely infinite and simple by Proposition 2.3. Thus $I(H)$ is purely infinite and simple since it is well known that being purely infinite and simple is a stable property under a stable isomorphism. Moreover note that $I(H)$ is essential in $C^*(E)$, that is, it has nonzero intersection with every other nonzero closed ideal. Thus the graph algebra $C^*(E)$ is prime and hence its extreme point set of the unit ball consists of isometries or co-isometries. Now consider the quotient algebra $C^*(E)/I(H)$, then it is isomorphic to the graph C^*-algebra $C^*(F)$ by Theorem 2.2, where $F = (E^0 \setminus H, \{e \mid r(e) \notin H\})$. Since $C^*(F)$ is isomorphic to the direct sum $\mathcal{O}_2 \oplus \mathcal{O}_2$ of the Cuntz algebra $\mathcal{O}_2$ the quotient algebra is extremally rich. Let s_1, s_2 be two isometries generating the Cuntz algebra $\mathcal{O}_2$. If $C^*(E)$ were extremally rich then by [BP2, Corollary 9.3] every extreme partial isometry of $C^*(E)/I(H)$ should lift. But the partial isometry $u = s_i \oplus s_i^*$ $(i = 1, 2)$ is extremal in the quotient algebra $C^*(E)/I(H)$ and cannot lift to an isometry or a co-isometry. This proves the assertion. Note that $C^*(E)$ (and so every ideal) is purely infinite since E satisfies (L) and every vertex connects to a loop ([BPRS, Proposition 5.3]).

Example 4.8. *Let $E = (E^0, E^1)$ be a finite graph with $E^0 = \{1, 2, 3\}$ and $E^1 = \{e_{ij} \mid s(e_{ij}) = r(e_{ij}) = i,\ i = 1, 2, 3,\ j = 1, 2\} \cup \{f_i \mid s(f_i) = i, r(f_i) = i + 1, i = 1, 2\}$. Then the ideal generated by the vertex set $\{3\}$ is purely infinite (in fact, isomorphic to the Cuntz algebra $\mathcal{O}_2$) and essential in $C^*(E)$. The quotient (prime) algebra by the ideal is extremally rich by Theorem 4.3 (e) and has isometries and co-isometries as extreme points in its closed unit ball. Thus by* [LO, Theorem 3.6] *we conclude that $C^*(E)$ is extremally rich. More generally one can deduce by induction that for each n there is an extremally rich prime graph C^*-algebra B with precisely n proper ideals $I_1 \subset I_2 \subset \cdots \subset I_n$ and every possible quotient is purely infinite.*

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