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Global and Local Properties of Pencils of Algebraic Curves

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§0. Introduction

Let S be a non-singular projective surface over $\mathbb{C}$, and let $f : S \to B$ be a relatively minimal fibration of curves of genus g over a non-singular projective curve B of genus b. In this article, we discuss some recent developments in the area where its global and local properties interact each other, with comments on several interesting open questions.

From the global point of view, our motivation comes from the study of minimal surfaces of general type. In the birational sense, any algebraic surface has a fibration over a curve, because it has an algebraic function. For surfaces of small Kodaira dimension, we can choose among various pencils a "preassigned" pencil such as Mori fibrations or Iitaka fibrations. On the other hand, there seems to be no canonical way in finding a pencil which reflects well the structure of a surface of general type. However, we often see that a pencil structure appears naturally for them as well. In the series of papers [42], Horikawa showed that most surfaces which are geographically close to the Noether line, $K_S^2 = 2\chi(\mathcal{O}_S) - 6$, have a pencil of curves of genus 2 which is induced on S from a ruling of its canonical image via the canonical map. Similar phenomena can be observed for canonical surfaces close to the Castelnuovo line $K_S^2 = 3\chi(\mathcal{O}_S) - 10$ ([22], [10]). In this case, the quadric hull of the canonical image is a threefold of small degree and its ruling usually induces on S a pencil of non-hyperelliptic curves of genus 3 (see [52], [44]). One can find a lot of such observations in literatures (e.g., Xiao's works) indicating the importance of a systematic study of fibered surfaces in the study of surfaces of general type.

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In §1, we describe the zone of existence for fibrations of genus g with respect to two relative numerical invariants χ_f and $K_{S/B}^2$, where we put $\chi_f = \chi(\mathcal{O}_S) - (g-1)(b-1)$ and $K_{S/B} = K_S - f^*K_B$ for a fibration $f : S \to B$. Recall that we have the fundamental inequality in surface geography

$$K_{S/B}^2 \geq \frac{4(g-1)}{g}\chi_f$$

called the *slope inequality*. It was first discovered by Horikawa and Persson for hyperelliptic pencils (Part V of [42], [81]) and proved by Xiao [96] in general. In §1.2, we give two different proofs due respectively to Xiao [96] and Moriwaki [73]. A general philosophy in geography of fibered surfaces is that, if we impose a certain condition on a general fiber, then the zone of existence of such pencils is restricted further, and, as a result, a sharper slope inequality holds. For example, consider the case $g = 3$. If a general fiber is non-hyperelliptic, then we have $K_{S/B}^2 \geq 3\chi_f$ ([53]), while the bound of genus 3 implied by the slope inequality is $8/3$.

Now, as an ideal model, we recall here Horikawa's works [41] and [43]. Let $f : S \to B$ be a fibration of genus 2, and let $F_1, \cdots, F_l$ be singular fibers of f. Then:

(i) One can define a nonnegative integer $\mathrm{Ind}(F_i)$ for each fiber germ of F_i so that the equality

$$K_{S/B}^2 = 2\chi_f + \sum_{i=1}^{l} \mathrm{Ind}(F_i)$$

holds. We call $\mathrm{Ind}(F_i)$ the Horikawa index of F_i.

(ii) One can classify the fiber germs with positive Horikawa index.

(iii) A germ F_i with $\mathrm{Ind}(F_i) = k > 0$ has a local splitting deformation to k fibers with $\mathrm{Ind} = 1$ modulo fibers with $\mathrm{Ind} = 0$.

How to generalize (i), (ii) and (iii) is the motivation of the following discussions of the slope equality, classification of degenerations and Morsification, respectively.

Our first aim is to generalize the equality in (i). Let $f : S \to B$ be a pencil with a certain condition on its general fiber. If there exists a rational number λ, a finite set of fibers $F_1, \cdots, F_l$ and well-defined non-negative rational numbers $\mathrm{Ind}(F_i)$ satisfying

$$K_{S/B}^2 = \lambda\chi_f + \sum_{i=1}^{l} \mathrm{Ind}(F_i),$$

we call it the slope equality which, once obtained, has nice geographic applications. We describe two known examples, hyperelliptic fibrations and Clifford general fibrations of odd genus, in §2.2 and §2.3, respectively. Note that F_i may be a smooth fiber in general. Indeed, if f is a non-hyperelliptic pencil of genus 3, then the generic fiber with Ind $= 1$ is a smooth hyperelliptic curve (e.g., [52]).

The essence of the slope equality is the local concentration of the global invariants of the surface on a finite number of fiber germs. In §2.1, we define the local signature $\sigma(F_i)$ for fiber germs by using the Horikawa index and the Euler contribution, which enables us to restate the slope equality as the "local concentration formula" of the global signature:

$$\mathrm{Sign}(S) = \sum_{i=1}^{l} \sigma(F_i).$$

This itself has a certain topological meaning. For example, consider hyperelliptic fibrations. The Horikawa index, and therefore the local signature, is explicitly calculated in terms of the datum of the singularities of the branch curve of the double covering as in §2.2. On the other hand, Endo [31] defined the topological local signature $\sigma_{top}(F_i)$ by using the Meyer cocycle [68]. Then Terasoma [89] showed that our $\sigma(F_i)$ coincides with $\sigma_{top}(F_i)$. Furuta [33] defined the topological local signature in more general situations. It is interesting to establish relations with the Horikawa index.

The second aim is to develop the classification theory of singular fiber germs. Kodaira [51] and Namikawa-Ueno [76], [77] studied genus one and two cases, respectively. We recall the classification of genus three case obtained in [9], and discuss the method in §3. Here our central tool is Matsumoto-Montesinos' theorem ([64], [65]) which characterizes the monodromy in the mapping class group Γ_g of genus g. To be more precise, let $f : S \to \Delta$ be a degeneration of curves of genus $g \geq 2$ over a 1-dimensional unit disk Δ with a unique singular fiber $F = f^{-1}(0)$. If we fix $t_0 \in \Delta^*$, then the monodromy action of $\pi_1(\Delta^*, t_0) \simeq \mathbb{Z}$ induces modulo isotopy an orientation-preserving homeomorphism of the fiber $\phi_f : F_0 \to F_0$, where $F_0 = f^{-1}(t_0)$. Since the change of the base point t_0 corresponds to the conjugation in Γ_g, f gives us a uniquely determined element $[\phi_f]$ in the set $\widehat{\Gamma}_g$ of all conjugacy classes of Γ_g. We call $[\phi_f]$ the *topological monodromy* of f. Matsumoto-Montesinos' theorem [64] states that an element of $\widehat{\Gamma}_g$ is realizable as the topological monodromy of a degeneration if and only if it is the class of a pseudo-periodic map of negative twist. Moreover this class is completely determined by certain invariants called valencies, screw numbers and the action on the extended partition graph ([78], [79], [65]). Therefore, the classification of degenerations is reduced to determining these invariants. In §3, we describe the method to carry it out. By this method, for any genus in theory, we can classify singular fibers, topological monodromies and the topological structure of the stable curve corresponding to the moduli point of the given degeneration at the same time.

The last aim is to consider Morsification of degenerations [83]. Let $f : S \to \Delta$ be a degeneration with a unique singular fiber $F = f^{-1}(0)$. If their exists a relative deformation $\{f_u \colon S_u \to \Delta\}_{u\in\Delta'}$ with $f_0 = f$ such that f_u has $l \geq 2$ singular fibers $F_{u,1}, \cdots, F_{u,l}$, then we say that F splits into $F_{u,1}, \cdots, F_{u,l}$. Starting from a given germ $F_{u,i}$ $(1 \leq i \leq l)$, we seek for its splittings successively. Such a reduction will terminate after a finite number of steps. Then we say that F splits into atomic fibers via several splitting families. One of the central problems is Xiao Gang's Morsification conjecture [83] that any atomic fiber has a simple description (see §4.1 for the precise statement). There are two steps to be considered for that: The first is to construct splitting families of a given germ F. The second is to determine all the atomic fibers in a certain category.

In §4.2, we recall the construction of hyperelliptic splitting families via the splitting deformation of singularities of the branch curve of the double cover ([2], [3]). As a result, any hyperelliptic singular fiber is reduced to very simple classes of fiber germs via several splitting families. Moreover we can give the list of hyperelliptic atomic fibers of genus 3 [3]. Since hyperelliptic splitting families satisfy the conservation law of the Horikawa index, it is also considered as the algebraic Morsification defined in §4.1.

In §4.3, we give and discuss seven open questions concerning the Morsification of degenerations.

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§1. Geography of fibered surfaces

1.1. Geography

Though our main interest is in surfaces fibered over a curve, we would like to start from recalling some results about surfaces of general type [82].

Since Bogomolov-Miyaoka-Yau's inequality was established, the geography of surfaces of general type, introduced by Ulf Persson [81], has been one of the main subjects in the surface theory. Recall that the numerical characters of a minimal surface S of general type satisfy

$$\chi(\mathcal{O}_S) > 0, \quad K_S^2 > 0, \quad 2\chi(\mathcal{O}_S) - 6 \leq K_S^2 \leq 9\chi(\mathcal{O}_S).$$

These inequalities enable us to plot the pair $(\chi(\mathcal{O}_S), K_S^2)$ as a lattice point in the area defined by them in the xy- plane. The upper bound $y = 9x$ is the Bogomolov-Miyaoka-Yau line, and it is famous that surfaces (whose numerical characters are) on this line can be obtained as a quotient of the complex ball. The lower bound $y = 2x - 6$ is the Noether line, and regular surfaces near this line were studied in detail by Horikawa [42]. In the intermediate area, we can also find some important lines. We would like to recall, among others, the Castelnuovo line $y = 3x - 10$. Castelnuovo's second inequality [15] says that $K_S^2\ geq 3\chi(\mathcal{O}_S) - 10$ holds for canonical surfaces S, where a minimal surface of general type is called canonical if its canonical map is birational onto the image. Furthermore, it is known that the canonical map of surfaces in the region $2x - 6 \leq y < 3x - 10$ gives a double covering of a ruled surface (cf. [15]). In other words, those who live in this area are of hyperelliptic type. Almost all regular canonical surfaces on $y = 3x - 10$, $3x - 9$ have a pencil of non-hyperelliptic curves of genus three (see [22], [10], [52], [44]). Therefore, surfaces fibered over curves appear quite naturally through the canonical map. This is one of the main reasons why we are interested in pencils of curves. Note also that Persson [81] and Chen [24] constructed fibered surfaces whose invariants can almost fill the zone of existence $2x - 6 \leq y \leq 9x$ (see also [7]).

Let $f : S \to B$ be a surjective morphism of a non-singular projective surface S onto a non-singular projective curve B of genus b with connected fibers. We say that f is a *relatively minimal* fibration if there are no (-1)-curves (i.e., a non-singular rational curve with self-intersection number -1) contained in fibers. We denote by g the genus of a general fiber F of f. If $g = 0$, then f is a $\mathbb{P}^1$-bundle and there exists a vector bundle $\mathcal{E}$ of rank two with $S \simeq \mathbb{P}_B(\mathcal{E})$. If $g = 1$, then it is so called an elliptic surface whose structure was studied by Kodaira [51] extensively.

In what follows, a fibration means a relatively minimal fibration of genus $g \geq 2$, unless otherwise stated explicitly.

Let $K_{S/B}$ be the relative canonical bundle $K_S - f^*K_B$. By Arakelov's theorem (see [16]), it is a nef line bundle,

that is, the intersection number $K_{S/B}C$ is non-negative for any irreducible curve C on S. Furthermore, it is known that $K_{S/B}C = 0$ if and only if C is a (-2)-curve contained in a fiber. Hence one can regard the relative canonical bundle as a candidate of the canonical bundle of minimal surfaces of general type. Now, we can introduce three numerical invariants associated to f.

1. $K_{S/B}^2$
2. $\chi_f := \deg f_*\omega_{S/B} = \chi(\mathcal{O}_S) - (g-1)(b-1)$
3. $e_f := e(S) - e(F)e(B)$, where F is a general fiber of f, and $e(X)$ denotes the topological Euler number of the space X.

Note that e_f has the following "localization"

$$e_f = \sum_{P \in B} e_f(f^{-1}P), \tag{1.1.1}$$

where $e_f(f^{-1}P) := e(f^{-1}P) + 2g - 2$ (Euler contribution).

The three invariants are non-negative integers related by Noether's formula:

$$K_{S/B}^2 + e_f = 12\chi_f \tag{1.1.2}$$

Hence one can choose any two of them as basic invariants. We choose here $K_{S/B}^2$ and χ_f as basics, and will consider relations among them, usually assuming that f is not a locally trivial fibration (i.e., not an analytic fiber bundle). This condition is equivalent to assuming that $\chi_f > 0$. In such a case, the ratio

$$\lambda_f := K_{S/B}^2/\chi_f \tag{1.1.3}$$

is called the *slope* of the fibration [96].

On one hand, it is easy to get the upper bound for $K_{S/B}^2$ in terms of χ_f: Since $e_f \geq 0$, we get $K_{S/B}^2 \leq 12\chi_f$. The equality holds here if and only if f is a *Kodaira fibration*, that is, f has no singular fibers but with variable moduli. On the other hand, the lower bound is non-trivial, and it is called the *slope inequality*:

$$K_{S/B}^2 \geq \frac{4(g-1)}{g}\chi_f \tag{1.1.4}$$

shown by Xiao [96] (see also [28] for semi-stable fibrations).

All the above enable us to plot $(\chi_f, K^2_{S/B})$ on the plane, quite similarly as in the case of surfaces of general type. The zone of existence is given by

$$x > 0, \quad y > 0, \quad \left(4 - \frac{4}{g}\right) x \leq y \leq 12x$$

Fibrations on the lower bound $y = (4 - 4/g)x$ are of hyperelliptic type with only "simple" singular fibers (see §2.2). Fibrations on the upper bound $y = 12x$ are of non-hyperelliptic type also with only beautiful (non-singular !) fibers. Then one may ask:

- What happens in the intermediate area ? How about singular fibers ?
- Are there any important lines like the Castelnuovo line ?

Remark. For hyperelliptic fibrations, Xiao [97] showed

$$\lambda_f \leq \begin{cases} 12 - \dfrac{8g+4}{g^2} & \text{if } g \text{ is even,} \\[2ex] 12 - \dfrac{8g+4}{g^2-1} & \text{if } g \text{ is odd} \end{cases}$$

See also [66].

1.2. Proofs of the Slope Inequality

Here we outline two proofs of the slope inequality both of which involve an interesting vector bundle argument.

(A) The first proof is due to Xiao [96]. Let $\mathcal{E}$ be a vector bundle (or a locally free sheaf) on a non-singular irreducible curve. We denote by $\mathrm{rk}\mathcal{E}$ the rank of $\mathcal{E}$ and by $\deg \mathcal{E}$ the degree of the determinant line bundle of $\mathcal{E}$. The ratio $\mu(\mathcal{E}) := \deg \mathcal{E}/\mathrm{rk}\mathcal{E}$ is called the slope of $\mathcal{E}$. Recall that $\mathcal{E}$ is said to be semi-stable if $\mu(\mathcal{V}) \leq \mu(\mathcal{E})$ holds for any subbundle $\mathcal{V}$ of $\mathcal{E}$. Even if $\mathcal{E}$ is not semi-stable, we have a uniquely determined filtration, the Harder-Narashimhan filtration [35],

(1.2.1) $$0 =: \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_n = \mathcal{E}$$

satisfying the following properties

(1) Each $\mathcal{E}_i/\mathcal{E}_{i-1}$ is semi-stable

(2) $\mu_1 > \cdots > \mu_n$, where $\mu_i := \mu(\mathcal{E}_i/\mathcal{E}_{i-1})$

If we put $r_i = \mathrm{rk}\mathcal{E}_i$, then

$$\text{(1.2.2)} \qquad \deg \mathcal{E} = \sum_{i=1}^{n} r_i(\mu_i - \mu_{i+1}), \qquad (\text{where } \mu_{n+1} = 0).$$

We consider $f_*\omega_{S/B}$ and its Harder-Narashimhan filtration. Since $f_*\omega_{S/B}$ is nef by Fujita's theorem [32], μ_n is non-negative. The invariant χ_f can be calculated by the formula (1.2.2).

For each i, the composite of the natural sheaf homomorphisms

$$f^*\mathcal{E}_i \hookrightarrow f^*f_*\omega_{S/B} \to \omega_{S/B}$$

induces a rational map $S \to \mathbb{P}_B(\mathcal{E}_i)$. Let $\rho : \tilde{S} \to S$ be a minimal succession of blowing-ups such that the above map becomes a morphism for every i. We denote by M_i the pull-back to $\tilde{S}$ of the tautological line bundle H_i on $\mathbb{P}_B(\mathcal{E}_i)$. Put $M_{n+1} = \rho^*K_{S/B}$. For simplicity, we denote a general fiber of $\tilde{S} \to B$ also by F.

By the construction, we have effective divisors Z_i such that

$$\rho^*K_{S/B} \equiv M_i + Z_i, \quad Z_1 \geq \cdots \geq Z_n \geq Z_{n+1} = 0$$

where the symbol $\equiv$ means numerical equivalence of divisors. According to a theorem of Miyaoka [69], the $\mathbb{Q}$-divisors $H_i - \mu_i\Gamma$ are nef, where Γ denotes a fiber of $\mathbb{P}_B(\mathcal{E}_i) \to B$. It follows that $N_i := M_i - \mu_i F$ is also nef being the pull-back of a nef divisor.

Put $d_i = N_iF = M_iF$. Since $M_i|_F$ is a special divisor on F which induces a map into $\mathbb{P}^{r_i-1}$ (a fiber of $\mathbb{P}_B(\mathcal{E}_i) \to B$), we have $d_i \geq 2r_i - 2$ by Clifford's theorem. Note that we have $d_n = d_{n+1} = 2g - 2$.

For $i > j$, we have $N_i = N_j + (\mu_i - \mu_j)F + (Z_j - Z_i)$ and

$$\begin{aligned} N_i^2 &= N_iN_j + d_i(\mu_i - \mu_j) + N_i(Z_j - Z_i) \\ &= N_j^2 + (d_i + d_j)(\mu_i - \mu_j) + (N_i + N_j)(Z_j - Z_i) \\ &\geq N_j^2 + (d_i + d_j)(\mu_i - \mu_j) \end{aligned}$$

by the nefness of the N_i's. Hence we have:

Lemma 1.1 ([96]). *Let $\{i_1, \cdots, i_m\}$ be a sequence of indices with $1 \leq i_1 < \cdots < i_m \leq n$. Then*

$$K_{S/B}^2 \geq \sum_{p=1}^{m} (d_{i_p} + d_{i_{p+1}})(\mu_{i_p} - \mu_{i_{p+1}})$$

where $i_{m+1} = n + 1$.

We can now show the slope inequality. First, suppose that $f_*\omega_{S/B}$ is semi-stable. We apply Lemma 1.1 to the sequence $\{1\}$ to get

$$K_{S/B}^2 \geq (4g-4)\mu_1 = (4-4/g)\chi_f$$

which is what we want. We can assume that $f_*\omega_{S/B}$ is not semi-stable. Applying Lemma 1.1 to $\{1, 2, \cdots, n\}$ and to $\{1, n\}$, we respectively get

$$\begin{aligned} K_{S/B}^2 &\geq \textstyle\sum_{i=1}^{n}(d_i + d_{i+1})(\mu_i - \mu_{i+1}) \\ &\geq \textstyle\sum(2r_i + 2r_{i+1} - 4)(\mu_i - \mu_{i+1}) \\ &\geq \textstyle\sum_{i=1}^{n-1}(4r_i - 2)(\mu_i - \mu_{i+1}) + (4g-4)\mu_n \\ &= 4\chi_f - 2(\mu_1 + \mu_n) \end{aligned}$$

and

$$K_{S/B}^2 \geq (0 + 2g - 2)(\mu_1 - \mu_n) + (4g-4)\mu_n = 2(g-1)(\mu_1 + \mu_n)$$

From these two, we get (1.1.4) by eliminating $\mu_1 + \mu_n$.

(B) The second proof is taken from Moriwaki [73]. We need the following two results:

Lemma 1.2 ([80]). *For a general fiber F, the kernel of the evaluation map $H^0(F, K_F) \otimes \mathcal{O}_F \to \mathcal{O}_F(K_F)$ is a semi-stable vector bundle.*

Lemma 1.3 (Bogomolov instability theorem [20]). *Let $\mathcal{F}$ be a torsion free sheaf on a non-singular projective surface S and put*

$$\delta(\mathcal{F}) = 2\mathrm{rk}(\mathcal{F})c_2(\mathcal{F}) - (\mathrm{rk}(\mathcal{F}) - 1)c_1^2(\mathcal{F}).$$

If $\delta(\mathcal{F}) < 0$, then there exists a non-zero saturated subsheaf $\mathcal{G}$ of $\mathcal{F}$ such that

$$D := \mathrm{rk}(\mathcal{F})c_1(\mathcal{G}) - \mathrm{rk}(\mathcal{G})c_1(\mathcal{F})$$

is in the positive cone, that is, $D^2 > 0$ and $DH > 0$ for any ample divisor H.

We let $\mathcal{F}$ be the kernel of the natural sheaf homomorphism $\phi : f^*f_*\omega_{S/B} \to \omega_{S/B}$. Since a general fiber F is a non-singular curve of genus $g \geq 2$, K_F is generated by its global sections. Hence ϕ is generically surjective. Its image L is torsion free and $\mathcal{F}$ is locally free, since they are a first and a second syzygy, respectively, on a non-singular surface.

We can find an effective divisor Z which is vertical with respect to f and satisfies $c_1(L) = K_{S/B} - Z$. It is the fixed part of the linear system

$|K_{S/B}+f^*\mathcal{L}|$ for any sufficiently ample divisor $\mathcal{L}$ on B; we have $Z = Z_n$ in the notation of (A). Note also that $c_2(L)$ is nothing but the length of the isolated base points of the variable part (with natural scheme structure).

Lemma 1.4. *$\delta(\mathcal{F})$ is non-negative.*

Proof. We assume that $\delta(\mathcal{F}) < 0$ and show that this leads us to a contradiction. Let $\mathcal{G}$ and D be as in Lemma 1.3. Since F is nef, we always have $FD \geq 0$. Since $D^2 > 0$ and $F^2 = 0$, Hodge's index theorem implies that $FD > 0$. We have

$$FD = \mathrm{rk}(\mathcal{F})\deg(\mathcal{G}|_F) - \mathrm{rk}(\mathcal{G})\deg(\mathcal{F}|_F).$$

Thus $FD > 0$ means that $\mathcal{F}|_F$ has a destabilizing subsheaf $\mathcal{G}|_F$. But it is impossible by Lemma 1.2. Therefore, $\delta(\mathcal{F}) \geq 0$. Q.E.D.

We calculate $\delta(\mathcal{F})$. From the exact sequence

$$0 \to \mathcal{F} \to f^*f_*\omega_{S/B} \to L \to 0$$

we get

$$c_1(\mathcal{F}) = f^*\det(f_*\omega_{S/B}) - c_1(L)$$

and $\chi(\mathcal{F}) = \chi(f^*f_*\omega_{S/B}) - \chi(L)$. Since the Riemann-Roch theorem gives us

$$\chi(\mathcal{F}) = \frac{c_1(\mathcal{F})(c_1(\mathcal{F}) - K_S)}{2} - c_2(\mathcal{F}) + \mathrm{rk}(\mathcal{F})\chi(\mathcal{O}_S),$$

we get

$$c_2(\mathcal{F}) = c_1(L)^2 - 2(g-1)\chi_f - c_2(L).$$

Hence, using $c_1 = c_1(L) = K_{S/B} - Z$, we get

$$\begin{aligned} & \delta(\mathcal{F}) \\ = \; & 2(g-1)(c_1^2 - 2(g-1)\chi_f - c_2(L)) - (g-2)(c_1^2 - 2\chi_f c_1 F) \\ = \; & gK_{S/B}^2 - 4(g-1)\chi_f - g(2K_{S/B} - Z)Z - 2(g-1)c_2(L). \end{aligned}$$

Now, $\delta(\mathcal{F}) \geq 0$ is equivalent to

$$K_{S/B}^2 \geq \frac{4(g-1)}{g}\chi_f + (2K_{S/B} - Z)Z + \frac{2(g-1)}{g}c_2(L).$$

Here, $K_{S/B}$ is nef, $Z^2 \leq 0$ (since Z is vertical), and $c_2(L) \geq 0$. Therefore, we get the slope inequality (1.1.4) as desired.

1.3. Further remarks

(1) It is known that if $\lambda_f = 4 - 4/g$, then f is necessarily a hyperelliptic fibration (see [96], [54]). Hence the lower bound of the slope of non-hyperelliptic fibrations is bigger. Unfortunately, the precise bound is known only for small genus:

- $g = 3$: $\lambda_f \geq 3$ ([53], [23], [83], [44], [45])
- $g = 4$: $\lambda_f \geq 24/7$, and $\lambda_f \geq 7/2$ if a general fiber has two distinct g_3^1's ([25], [54])
- $g = 5$: $\lambda_f \geq 40/11$ for f trigonal[1] and $\lambda_f \geq 4$ for f non-trigonal ([54])

For trigonal fibrations, we gave in [55] a very rough bound $14(g-1)/(3g+1)$, while it goes up to $24(g-1)/(5g+1)$ when semi-stable [86]. For bielliptic fibrations of genus $g \geq 5$, Barja [12] has shown $\lambda_f \geq 4$. It would be very interesting to find the accurate lower bound of the slope of non-hyperelliptic fibrations. See §2.3 below for an attempt.

(2) For a fibration $f : S \to B$, we put $q_f = q(S) - b$. When $q_f > 0$, we call f an irregular fibration. In this case, we can construct an unramified covering of S not coming from those of B, inducing fibrations over B with slope equals the original λ_f and of an arbitrary large genus. Hence the slope inequality (formally letting $g \to \infty$) gives us:

Theorem (Xiao [96]). *If $q_f > 0$, then $K_{S/B}^2 \geq 4\chi_f$.*

It is known that the inequality is sharp when $q_f = 1$, but may not when $q_f \geq 2$. It would be interesting to have a sharper lower bound for the slope of irregular fibrations including g and q_f. An attempt along this line can be found in a recent paper by Barja and Zucconi [14] (see also [13]). For hyperelliptic fibrations, Xiao gave further interesting results in [98] and [99].

§2. Slope equality and the local invariants

2.1. Horikawa index and local signature

Fiber germ. Let Δ be a small open neighborhood of the origin $0 \in \mathbb{C}$, usually a disk around 0, and let $f_\Delta : S_\Delta \to \Delta$ be a relatively minimal fibration of genus g. We assume that the critical value of f_Δ, if exists, is 0. Let $f_{\Delta'} : S_{\Delta'} \to \Delta'$ be another such fibration. We say that f_Δ and $f_{\Delta'}$ are (analytically) equivalent if there exists a biholomorphic

[1] This may not be sharp.

map $\phi : S_\Delta|_{\Delta\cap\Delta'} \to S_{\Delta'}|_{\Delta\cap\Delta'}$ such that $f_\Delta|_{\Delta\cap\Delta'} = f'_\Delta|_{\Delta\cap\Delta'} \circ \phi$. This equivalence relation leads us to the notion of the (singular) *fiber germ* at 0. A representative $f_\Delta : S_\Delta \to \Delta$ of a fiber germ will be simply called a degeneration.

Similarly, we can introduce the notion of topological equivalence by considering an orientation preserving homeomorphism ϕ.

Slope equality. Let us consider and fix a certain property (*) for curves of genus g, and consider the set $\mathcal{C}(*)$ of all the fiber germs of genus g whose general fiber satisfies (*). A function $\mathrm{Ind} : \mathcal{C}(*) \to \mathbb{Q}$ is called a *Horikawa index* with respect to (*) if the following are satisfied for any global fibration $f : S \to B$ of genus g whose general fiber satisfies (*):

1. $\mathrm{Ind}(f^{-1}P)$ is a non-negative rational number for any $P \in B$ and is equal to zero if the fiber over P satisfies (*).
2. there exists a rational number λ, which depends only on g and (*), satisfying $4 - 4/g \leq \lambda \leq 12$ and

$$K^2_{S/B} = \lambda\chi_f + \sum_{P\in B} \mathrm{Ind}(f^{-1}P). \tag{2.1.1}$$

Suppose that there exists a global fibration $f : S \to B$ of genus g whose general fiber satisfies (*) and $\mathrm{Ind}(f^{-1}P) = 0$ holds for any $P \in B$. When invariants of such fibrations can fill (almost) all the lattice points on the line $y = \lambda x$, the equality (2.1.1) will be called a *slope equality*. Once it is established, λ gives us the slope of the line which is the lower bound of the zone of the existence of fibrations with property (*), and the Horikawa index measures how far it is from the lower bound.

The equality of type (2.1.1) was first obtained by Horikawa in [41] for genus 2 fibrations, and it was applied successfully to surfaces of general type near the Noether line [42]. This is the reason why we call Ind by his name.

Definition 2.1. Suppose that a Horikawa index with respect to (*) is given. Let $f_\Delta : S_\Delta \to \Delta$ be a degeneration which represents an element in $\mathcal{C}(*)$. We call such a degeneration a (*)-degeneration for simplicity. The central fiber $f^{-1}(0)$ is called a *critical* fiber if $\mathrm{Ind}(f^{-1}(0)) > 0$. As usual, it is called a *singular* fiber if $e_f(f^{-1}(0)) > 0$.

Local signature. Since a projective surface S is an orientable 4-dimensional differentiable manifold, the signature $\mathrm{Sign}(S)$ of the intersection form on H^2 is one of the most important topological invariants.

A function $\sigma : \mathcal{C}(*) \to \mathbb{Q}$ is called a *local signature* with respect to (*) if the following are satisfied for any global fibration $f : S \to B$ of genus g whose general fiber satisfies (*):

1. $\sigma(f^{-1}P) = 0$ if the fiber over $P \in B$ is non-singular and satisfies (*), and
2. the signature can be expressed as

$$\mathrm{Sign}(S) = \sum_{P \in B} \sigma(f^{-1}P) \tag{2.1.2}$$

Suppose that (2.1.1) is obtained with $\lambda < 12$. Since the signature is given by

$$\mathrm{Sign}(S) = \frac{1}{3}(c_1^2(S) - 2c_2(S)) = K_{S/B}^2 - 8\chi_f$$

if we put

$$\sigma(f^{-1}P) := \frac{4}{12-\lambda}\mathrm{Ind}(f^{-1}P) - \frac{8-\lambda}{12-\lambda}e_f(f^{-1}P) \tag{2.1.3}$$

then (1.1.1) and (2.1.1) give us

$$\mathrm{Sign}(S) = \sum_{P \in B} \sigma(f^{-1}P).$$

Hence, σ is a local signature with respect to (*).

Conversely, suppose that we are given a local signature and that we know all values $\sigma(f^{-1}P)$. If there exists a rational number μ with $-4 - 4/g \leq \mu \leq 4$ such that

1. the inequality

$$\mu(\sigma(f^{-1}P) + e_f(f^{-1}P)) \leq 4\sigma(f^{-1}P) \tag{2.1.4}$$

   holds for all fiber germs $f^{-1}P \in \mathcal{C}(*)$, and
2. the equality holds in (2.1.4) as long as the fiber $f^{-1}P$ satisfies (*),

then putting $\lambda = \mu + 8$, we will be able to define $\mathrm{Ind}(f^{-1}P)$ by using (2.1.3) and get an equality of type (2.1.1).

Remark. Though we naturally expect that Horikawa index and Local signature are unique if exist, the above definitions can say nothing about that. The local invariants will have desired properties if we can find a nice condition (*).

In the following two subsections, we shall describe two cases where the equality of type (2.1.1) is known.

2.2. Hyperelliptic fibrations—double coverings

The slope inequality was first proved for hyperelliptic fibrations by Persson [81] and Horikawa [42], Part V. They used the double covering method which has been one of the important tools in the surface theory.

Though our concern is in hyperelliptic fibrations, we start from a more general setting: Let $f : S \to B$ be a relatively minimal fibration of genus $g \geq 2$ and assume that a general fiber F has an involution which extends to an involution ι of the whole S. We are in such a situation when f is a hyperelliptic fibration: Since the relative canonical map is a generically finite rational map of degree two, S has an involution which restricts to the hyperelliptic involution on F.

Let $\rho : \hat{S} \to S$ be the blowing-ups at all the isolated fixed points of $\langle\iota\rangle$-action, and let $\hat{\iota}$ be the induced involution on $\hat{S}$. Since the fixed locus of $\hat{\iota}$ is of codimension 1, the quotient space $\hat{W} := \hat{S}/\langle\hat{\iota}\rangle$ is non-singular, and the quotient map $\hat{S} \to \hat{W}$ is a finite double covering. Then the branch locus $\hat{R}$ is a non-singular divisor on $\hat{W}$, and we can find a line bundle $\hat{L}$ on $\hat{W}$ satisfying $[\hat{R}] = 2\hat{L}$ and such that $\hat{S}$ is isomorphic to the double covering of $\hat{W}$ constructed in the total space of $\hat{L}$ in the usual way.

$$\begin{tikzcd} \hat{S} \arrow[rr] \arrow[d, "\rho"'] & & \hat{W} \arrow[d] \\ S \arrow[dr, "f"'] & & W \arrow[dl, "\pi"] \\ & B & \end{tikzcd}$$

Since $\hat{W}$ has a natural fibration over B, say of genus h, we can take a relatively minimal model $\pi : W \to B$. Such W is unique when $h \geq 1$. On the other hand, if $h = 0$, that is, $\hat{W}$ is a ruled surface over B, a relatively minimal model is not unique, and we can move from a model to another via elementary transformations (that is, blow up a fiber at a point and then blow down the proper transform).

We let R be the direct image of $\hat{R}$ in W as divisor. Then R is a reduced divisor with a natural line bundle L satisfying $2L = [R]$. Hence we can construct in the total space of L a normal surface S' birational to S, in the usual way. Note that, by Hurwitz formula, we have

$$2g - 2 = 2(2h - 2) + R\Gamma, \tag{2.2.1}$$

where Γ is a general fiber of π.

Since S' is a divisor in a non-singular threefold, the dualizing sheaf $\omega_{S'}$ is an invertible sheaf. More concretely, it is induced by $K_W + L$.

We can calculate the numerical invariants of $f' : S' \to B$ as follows:

$$\omega^2_{S'/B} = 2(K_{W/B} + L)^2 = 2K^2_{W/B} + 2K_{W/B}R + R^2/2$$

Since

$$\begin{aligned}\chi(\mathcal{O}_{S'}) &= \chi(\mathcal{O}_W) + \chi(\mathcal{O}_W(-L)) \\ &= 2\chi(\mathcal{O}_W) + R^2/8 + K_W R/4\end{aligned}$$

we get

$$\begin{aligned}\chi_{f'} &= 2\chi(\mathcal{O}_W) + R^2/8 + K_W R/4 - (g-1)(b-1) \\ &= 2\chi_\pi + R^2/8 + K_W R/4 - (b-1)R\Gamma/2 \\ &= 2\chi_\pi + R^2/8 + K_{W/B}R/4.\end{aligned}$$

From these, it follows

$$\omega^2_{S'/B} - 4\chi_{f'} = 2(K^2_{W/B} - 4\chi_\pi) + K_{W/B}R. \tag{2.2.2}$$

The singularities of S' can be resolved in a natural way by the *canonical resolution* (or even resolution). Let P be a singular point of R of multiplicity m_1. Let $\sigma_1 : W_1 \to W$ be the blowing-up at P, $E = \sigma_1^{-1}(P)$ the exceptional (-1)-curve. Then $R_1 = \sigma_1^* R - 2[m_1/2]E$ is a reduced divisor and $[R_1] = 2L_1$, where $L_1 = \sigma_1^* L - [m_1/2][E]$. Furthermore, the double covering of W_1 branched along R_1 (constructed in L_1) has a birational morphism onto S'. Continuing this process, we get a sequence

$$\begin{array}{ccccccc} S_n & \to & S_{n-1} & \to \cdots \to & S_1 & \to & S' \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ W_n & \to & W_{n-1} & \to \cdots \to & W_1 & \to & W \end{array} \tag{2.2.3}$$

such that the branch locus R_n of $S_n \to W_n$ is non-singular. If we choose such a sequence of minimal length, then it can be shown that S_n is isomorphic to $\hat{S}$. Let $m_1, \ldots, m_n$ denote the multiplicity sequence. Then a calculation shows that the differences of invariants are given by

$$\chi_{f'} - \chi_{f_n} = \frac{1}{2}\sum_{i=1}^{n} \left[\frac{m_i}{2}\right]\left(\left[\frac{m_i}{2}\right] - 1\right), \tag{2.2.4}$$

$$\omega^2_{S'/B} - K^2_{S_n/B} = 2\sum_{i=1}^{n}\left(\left[\frac{m_i}{2}\right] - 1\right)^2 \tag{2.2.5}$$

Therefore, letting ϵ denote the number of blowing-downs to obtain S from $\hat{S}$, we get

$$K^2_{S/B} - 4\chi_f = 2(K^2_{W/B} - 4\chi_\pi) + K_{W/B}R + \sum_{i=1}^{n}\left(\left[\frac{m_i}{2}\right] - 1\right) + \epsilon \tag{2.2.6}$$

We assume now that f is a hyperelliptic fibration. By (2.2.1), we have $R\Gamma = 2g+2$. Since we can write $R \equiv -(g+1)K_{W/B} + \alpha\Gamma$ with some integer α, it follows from (2.2.2) that

$$\omega^2_{S'/B} = \frac{4(g-1)}{g}\chi_{f'}$$

because $K^2_{W/B} = \chi_\pi = 0$. (N.B. Curiously enough, the equality always holds in the slope inequality for the singular model $f' : S' \to B$.) Then, taking the contribution of the resolution into account, we get

$$K^2_{S/B} = \frac{4(g-1)}{g}\chi_f + \frac{2}{g}\sum_i \left(\left[\frac{m_i}{2}\right] - 1\right)\left(g - \left[\frac{m_i}{2}\right]\right) + \epsilon. \tag{2.2.7}$$

We can take W so that all the m_i satisfy $m_i \leq g+2$ (e.g., [99]). It follows that all the summands in the correction term are non- negative, and we get the slope inequality. It may be clear that the correction term can be localized to give us the Horikawa index. Needless to say, the property (*) in §2.1 is "hyperelliptic" here.

Endo [31] defined topologically the local signature for hyperelliptic fibrations. In analytic case, it coincides with ours obtained as in §2.1 using the Horikawa index (see [2] and [31]).

Remark. When $h = 1$, it follows from (2.2.6) that $K^2_{S/B} \geq 4\chi_f$ as shown by Barja in [12].

2.3. Fibrations of general Clifford index —relative canonical algebra

The second method to get a slope equality is the use of the relative canonical algebra ([23], [83], [56]).

For any non-negative integer n, put $\mathcal{R}_n = f_*(\omega^{\otimes n}_{S/B})$. These are nef locally free sheaves satisfying

$$\mathrm{rk}\mathcal{R}_n = \begin{cases} 1 & \text{if } n = 0, \\ g & \text{if } n = 1, \\ (2n-1)(g-1) & \text{if } n \geq 2 \end{cases}$$

and

$$\deg \mathcal{R}_n = \begin{cases} 0 & \text{if } n = 0, \\ \chi_f & \text{if } n = 1, \\ \frac{1}{2}n(n-1)K^2_{S/B} + \chi_f & \text{if } n \geq 2. \end{cases}$$

The relative canonical algebra for f is the $\mathcal{O}_B$-algebra

$$\mathcal{R}(f) = \bigoplus_{n \geq 0} \mathcal{R}_n.$$

It is attractive for a naive reason that its Proj gives the relative canonical model.

Let X be a non-singular projective curve of genus $g \geq 2$. The Clifford index of X is defined as

$$\mathrm{Cliff}(X) = \min\{\deg L - 2\dim|L| \mid L \in \mathrm{Pic}(X),\ h^0(L) > 1,\ h^1(L) > 1\}$$

when $g \geq 4$. If $g = 2$, we put $\mathrm{Cliff}(X) = 0$. If $g = 3$, we put $\mathrm{Cliff}(X) = 0$ or 1 according to whether X is hyperelliptic or not. See [27] for further properties of the Clifford index. The Clifford index of a fibration $f : S \to B$, which we denote by $\mathrm{Cliff}(f)$, is defined as that of a general fiber F. Then we have $0 \leq \mathrm{Cliff}(f) \leq (g-1)/2$.

Let $f : S \to B$ be a fibration with $\mathrm{Cliff}(f) > 0$. The contraction with the identity in $H^0(B, End(\mathcal{R}_1))$ and the multiplication define an $\mathcal{O}_B$-linear map $d_{i,j} : \bigwedge^i \mathcal{R}_1 \otimes \mathcal{R}_j \to \bigwedge^{i-1} \mathcal{R}_1 \otimes \mathcal{R}_{j+1}$ satisfying $\mathrm{Im}(d_{i,j}) \subset \mathrm{Ker}(d_{i-1,j+1})$. Hence, as in [34], we can consider a Koszul complex

$$0 \to \wedge^{c+1}\mathcal{R}_1 \to \cdots \to \wedge^i \mathcal{R}_1 \otimes \mathcal{R}_{c+1-i} \xrightarrow{d_{i,c+1-i}} \wedge^{i-1}\mathcal{R}_1 \otimes \mathcal{R}_{c+2-i} \to$$
$$\cdots \to \mathcal{R}_1 \otimes \mathcal{R}_c \to \mathcal{R}_{c+1} \to 0$$

for $c = \mathrm{Cliff}(f)$. We put

$$\mathcal{K}_{i,j} = \mathrm{Ker}(d_{i,j})/\mathrm{Im}(d_{i+1,j-1}).$$

Then $\mathcal{K}_{i,c+1-i}$ is a torsion sheaf for $0 \leq i \leq c-2$, because the Koszul complex

$$\begin{aligned} &\textstyle\bigwedge^{i+1} H^0(K_F) \otimes H^0((c-i)K_F) \to \\ &\quad \textstyle\bigwedge^{i} H^0(K_F) \otimes H^0((c-i+1)K_F) \to \\ &\qquad \textstyle\bigwedge^{i-1} H^0(K_F) \otimes H^0((c-i+2)K_F) \end{aligned} \tag{2.3.1}$$

is exact at the middle term for any general fiber F by [34]. The differential $d_{c+1,0} : \bigwedge^{c+1} \mathcal{R}_1 \to \bigwedge^c \mathcal{R}_1 \otimes \mathcal{R}_1$ is clearly injective and the quotient is locally free. We have

$$\mathcal{K}_{c,1} \simeq \mathrm{Ker}\left(\frac{\bigwedge^c \mathcal{R}_1 \otimes \mathcal{R}_1}{\bigwedge^{c+1} \mathcal{R}_1} \xrightarrow{\bar{d}_{c,1}} \bigwedge^{c-1} \mathcal{R}_1 \otimes \mathcal{R}_2 \right)$$

where $\bar{d}_{c,1}$ denote the natural map induced from $d_{c,1}$. It is locally free being a subsheaf of a locally free sheaf on a smooth curve. We have

$$\sum_{i=0}^{c+1}(-1)^i \deg\left(\bigwedge^{c+1-i} \mathcal{R}_1 \otimes \mathcal{R}_i\right) = \sum_{i=0}^{c+1}(-1)^i \deg(\mathcal{K}_{c+1-i,i}).$$

By using formulae giving rank and degree of $\mathcal{R}_n$, we get

$$\begin{aligned} K_{S/B}^2 = & \frac{(g-1)(g+2-2c)}{g-c}\chi_f \\ & -\begin{pmatrix} g-3 \\ c-1 \end{pmatrix}^{-1} \sum_{i=0}^{c+1}(-1)^{i+1} \deg(\mathcal{K}_{c+1-i,i}). \end{aligned} \tag{2.3.2}$$

Note that, if Green's conjecture [34] on syzygies of canonical curves holds, $\mathcal{K}_{c-1,2}$ is a torsion sheaf for $c = \mathrm{Cliff}(f)$. If $\mathcal{K}_{c-1,2}$ is torsion, then the rank of $\mathcal{K}_{c,1}$ is given by

$$\mathrm{rk}(\mathcal{K}_{c,1}) = \begin{pmatrix} g-1 \\ c-1 \end{pmatrix} \frac{(g-1-c)(g-1-2c)}{c+1}. \tag{2.3.3}$$

We now assume that g is odd and $\mathrm{Cliff}(f) = c = (g-1)/2$. Then Green's conjecture is true ([40] and [88]) and we see that $\mathcal{K}_{(g-3)/2,2}$ is torsion. Furthermore, we have $\mathcal{K}_{(g-1)/2,1} = 0$ because its rank is zero by (2.3.3). Hence (2.3.2) gives us:

$$K_{S/B}^2 = \frac{6(g-1)}{g+1}\chi_f + \sum_{P\in B} \mathrm{Ind}(f^{-1}P), \tag{2.3.4}$$

where

$$\mathrm{Ind}(f^{-1}P) := \begin{pmatrix} g-3 \\ (g-3)/2 \end{pmatrix}^{-1} \sum_{i=2}^{(g+1)/2} (-1)^i \mathrm{length}(\mathcal{K}_{(g+1)/2-i,i})_P.$$

We can show ([56]) that $\mathrm{Ind}(f^{-1}P)$ is non-negative, and it is zero only if $|K_F|$ has no base points and (2.3.1) with $i = (g-3)/2$ is exact at the middle term for $F = f^{-1}P$. Therefore, we get the slope equality (2.3.4) with respect to the condition (*): $\mathrm{Cliff}(F) = (g-1)/2$.

We can define the local signature as described in 2.1. But, this time, it is not known whether our local signature coincides with the topological one introduced by Furuta [33].

Remarks. (1) We emphasize here that a critical fiber is not necessarily a singular fiber and vice versa. For example, let $f : S \to B$ be a Kodaira fibration of genus three. Then $K^2_{S/B} = 12\chi_f$. On the other hand, since $\mathrm{Cliff}(f) = 1$, we have

$$K^2_{S/B} = 3\chi_f + \sum_{P\in B} \mathrm{Ind}(f^{-1}P)$$

by (2.3.4). Hence

$$\sum_{P\in B} \mathrm{Ind}(f^{-1}P) = 9\chi_f.$$

Recall that any fiber of f is non-singular. It follows that f has exactly $9\chi_f$ non-singular hyperelliptic fibers (counting infinitely near ones) which are critical.

(2) Beside its importance, the structure of the relative canonical algebra is not well understood. Miles Reid conjectured that it is generated in degrees lesser than or equal to three, and related in degrees lesser than or equal to six (1-2-3 conjecture [83]). It is shown in [58] that the relative canonical algebra is generated in degrees lesser than or equal to four, and that the 1-2-3 conjecture breaks down in one case that the fiber is a multiple fiber whose canonical linear system contains a (-1) elliptic cycle as a fixed part. The annoying exception actually occurs [59].

(3) To prove the uniqueness of the local signature, Nariya Kawazumi kindly suggests us to argue as follows. Let $\mathcal{M}_g$ be the moduli space of curves of odd genus $g\ geq 3$, and let D be the locus on which the Clifford index drops, that is, D is the "k-gonal locus" in the sense of [38]. Recall that $H^2(\mathcal{M}_g, \mathbb{Q}) \simeq \mathbb{Q}$ by Harer's theorem [36] and is generated by the 1st Morita-Mumford class ([74], [71]). Let τ denote Meyer's signature cocycle [68]. Then it is known the class $-3[\tau] \in H^2(\mathcal{M}_g)$ is the 1st Morita-Mumford class [72]. What we saw above is that $[\tau]$ goes to 0 via the natural map $H^2(\mathcal{M}_g) \to H^2(\mathcal{M}_g \setminus D)$. Since it factors as

$$H^2(\mathcal{M}_g) \to H^2(\pi_1(\mathcal{M}_g \setminus D)) \hookrightarrow H^2(\mathcal{M}_g \setminus D)$$

we see that $[\tau]$ also goes to 0 in $H^2(\pi_1(\mathcal{M}_g \setminus D))$. It follows that there exists a function $\eta : \pi_1(\mathcal{M}_g\ setminus D) \to \mathbb{Q}$ such that η goes to τ. Then to show the uniqueness, we only have to show that $H^1(\mathcal{M}_g \setminus D) = 0$.

Consider the exact sequence of cohomology groups with coefficients in $\mathbb{Q}$:

$$H^1(\mathcal{M}_g) \to H^1(\mathcal{M}_g\backslash D) \to H^2(\mathcal{M}_g, \mathcal{M}_g\backslash D) \to H^2(\mathcal{M}_g) \to H^2(\mathcal{M}_g\backslash D)$$

Since we have $H^1(\mathcal{M}_g) = 0$ by [36], it is also sufficient to see that $H^2(\mathcal{M}_g, \mathcal{M}_g \setminus D) \simeq \mathbb{Q}$.

For $g = 3$, the argument works, because $\mathcal{M}_3 \setminus D$ is nothing but the moduli of smooth plane quartics.

(4) We imagined that the Clifford index gives us a good condition (*) to get a slope equality. But the above computation implies that the answer seems to be negative. For example, consider a fibration of even genus with general Clifford index $g/2 - 1$. Then (2.3.2) gives us

$$K_{S/B}^2 = \frac{8(g-1)}{g+2}\chi_f - \binom{g-3}{g/2-2} \sum_{i=1}^{g/2} (-1)^{i+1} \deg(\mathcal{K}_{g/2-i,i}).$$

However, it contains the part involving $\mathcal{K}_{g/2-1,1}$ which has not been localized yet. So, we will need some additional (or even, completely different) assumptions.

Problem. Find a nicer condition (*) giving us a slope equality.

Recently, Yoshikawa [100] defined the local signature for general Lefschetz fibrations. His generality condition (*) is the vanishing of all even theta characteristics. However, it seems hard to induce the slope equality from his local signature, since we must compute all possible values $\sigma(F)$ to get λ as in (2.1.1). We hope that his or one of the conditions given by Furuta [33] is the right answer to the above question.

§3. Topological classification of degenerations

3.1. Moduli point and topological monodromy

Here we study singular fiber germs, especially their topological nature.

Let $f : S \to \Delta$ be the normally minimal model of a degeneration of genus g, with a unique singular fiber $F = f^{-1}(0)$. That is, it is the model such that the singularities of the reduced scheme of F are at most normal crossing and any (-1) curve in the components of F meets other components of F at more than two points. Note that such a model uniquely exists among the birational equivalence class of f.

Recall that, in the study of degenerations of elliptic curves by Kodaira [51], the notion of the limit of the period and the monodromy in $\mathrm{SL}(2, \mathbb{Z})$ are essential. We introduce their candidates in our context: the moduli point and topological monodromy.

Moduli point. Let $\mathcal{M}_g$ be the moduli space of smooth curves of genus g. We can associate to our normally minimal model $f : S \to \Delta$ the moduli map $\alpha_f : \Delta^* \to \mathcal{M}_g$, where $\Delta^* = \Delta \setminus \{0\}$. Since Δ is one-dimensional, α_f extends to a morphism

$$\overline{\alpha}_f : \Delta \longrightarrow \overline{\mathcal{M}}_g$$

where $\overline{\mathcal{M}}_g$ is the moduli space of stable curves of genus g [29]. We call $\overline{\alpha}_f(0)$ the *moduli point* of f. Let $\widetilde{f} : \widetilde{S} \to \Delta$ be the stable reduction of f. (Namely, let $\Delta \to \Delta$ be the cyclic covering between disks branching only at the origin so that the desingularization of the pull back of f by this map induces a semi-stable family $S' \to \Delta$. We obtain $\widetilde{S}$ from S' by contracting chains of (-2) curves. Thus $\widetilde{S}$ has at most rational double points of type A as its singularity.) Then the stable curve which corresponds to the moduli point of f coincides with the fiber $\widetilde{f}^{-1}(0)$.

Topological monodromy. Let Γ_g be the mapping class group of genus g. Choose $t_0 \in \Delta^*$ and put $F_0 = f^{-1}(t_0)$. The monodromy action of $\pi_1(\Delta^*, t_0) \simeq \mathbb{Z}$ induces modulo isotopy an orientation-preserving homeomorphism $\phi_f : F_0 \to F_0$. Since the change of the base point t_0 corresponds to the conjugation in Γ_g, f determines an element $[\phi_f]$ of the set $\widehat{\Gamma}_g$ of all conjugacy classes of Γ_g. We call $[\phi_f]$ the *topological monodromy* of f.

Obviously, one can associate another monodromy: By choosing a suitable symplectic 1-homology basis of F_0, we can define the homomorphism $\eta : \Gamma_g \to \mathrm{Sp}(2g, \mathbb{Z})$. We call the equivalence class of $\eta(\phi_f)$ in the conjugacy class of $\mathrm{Sp}(2g, \mathbb{Z})$ the *homological monodromy* of f.

An advantage of considering the topological monodromy is that it can distinguish degenerations whose ϕ_f's are contained in the kernel of the map η, i.e. the Torelli group (see also [6]). For instance, we consider the degeneration of genus 2 whose singular fiber is a stable curve consisting of two nonsingular components with one node. Then the topological monodromy is the integral Dehn twist along a *separated curve*, i.e. a simple closed real curve on the Riemann surface whose complement is disconnected. Since this curve does not intersect the usual symplectic homology basis, the homological monodromy is trivial. (See also [76], p. 350.)

From now on, monodromy means topological monodromy, and we sometimes identify it with the representative homeomorphism.

3.2. Namikawa-Ueno's example

Contrary to the case of degenerations of elliptic curves [51], the singular fiber F (as a one-dimensional non-reduced analytic space) in general can determine neither the analytic structure nor the topological structure of the degeneration itself when $g \geq 2$. In some sense, this phenomenon is analogous to the fact that the exceptional curve of the good resolution of an isolated surface singularity does not necessarily determine the germ of the original singularity itself.

Namikawa-Ueno found the series of examples implying the above fact. Among them, the simplest ones are of type $2I_{0-0}$ in [77], p.159, and of type II^*_{1-0} in [77], p.172. Namely, there exist two degenerations $f_i : S_i \to \Delta$ $(i = 1, 2)$ of genus 2 such that

(i) $F_i = f_i^{-1}(0)$ $(i = 1, 2)$ are mutually isomorphic and each of them is written as $2C_1 + 2C_2 + C_3 + C_4$ where C_1 is a nonsingular elliptic curve, C_j $(2 \leq j \leq 4)$ are nonsingular rational curves, $C_1^2 = C_2^2 = -1$, $C_3^2 = C_4^2 = -2$ and $C_1C_2 = C_2C_3 = C_2C_4 = 1$,

(ii) $\overline{\alpha}_{f_1}(0)$ is a stable curve consisting of two nonsingular components with one node, while $\overline{\alpha}_{f_2}(0)$ is an irreducible stable curve with one node,

(iii) ϕ_{f_1} (resp. ϕ_{f_2}) is of order 2, and is the rotation of angle π around the diameter of a separated curve (resp. a non-separated curve). These homeomorphisms are not conjugate each other in Γ_2 (see also [64], §7).

Here we reconstruct these examples. The method is nothing but the "converse process of stable reduction", and is a special case of [87], Part II.

We first construct f_1. Let E be an elliptic curve. We fix a point P_0 on E. Let U be a small open disk neighborhood of P_0 in E. Let E_i $(i = 1, 2)$ be two copies of E, and we fix the identification maps $\tau_i : E \to E_i$. Put $P_i = \tau_i(P_0)$ and $U_i = \tau_i(U)$. Let z_i be the local coordinate on U_i such that $P_i = \{z_i = 0\}$ and $U_i = \{|z_i| < 1\}$. Let Δ_i $(i = 1, 2)$ be two copies of a unit disk defined by $\{t_i \in \mathbb{C} | |t_i| < 1\}$. We put

$$V_i = \{(z_i, t_i) \in U_i \times \Delta_i \mid 0 \leq |z_i| \leq |t_i|^2\}.$$

Let W be an analytic subset in a three-dimensional polydisk defined by

$$W = \{(x_1, x_2, t) \in \Delta^3 \mid x_1x_2 = t^2, \ |x_1| < 1, |x_2| < 1, |t| < 1\}.$$

Then W has an A_1-singularity at $Q = \{(x_1, x_2, t) = (0, 0, 0)\}$. We obtain an analytic surface $\widetilde{S}_1$ by patching up three pieces $E_1 \times \Delta_1 \setminus V_1$, $E_2 \times \Delta_2 \setminus V_2$ and W in the following way: Patch $W \setminus \{Q\}$ and $U_1 \times \Delta_1 \setminus V_1$

by $x_1 = z_1$, $x_2 = t_1^2/z_1$ and $t = t_1$. Patch $W \setminus \{Q\}$ and $U_2 \times \Delta_2 \setminus V_2$ by $x_2 = z_2$, $x_1 = t_2^2/z_2$ and $t = t_2$. Then we have a natural holomorphic map $\widetilde{f}_1 : \widetilde{S}_1 \to \Delta$. The general fiber of $\widetilde{f}_1$ is a smooth curve of genus 2, since it is a connected sum of two tori. The singular fiber $\widetilde{f}_1^{-1}(0)$ is a stable curve of genus 2 consisting of two smooth components with one node, and $\widetilde{S}_1$ has an A_1-singularity at Q.

Now we define three maps $E_1 \times \Delta_1 \setminus V_1 \to E_2 \times \Delta_2 \setminus V_2$, $E_2 \times \Delta_2 \setminus V_2 \to E_1 \times \Delta_1 \setminus V_1$ and $W \to W$ by $(P, t_1) \mapsto (\tau_2 \circ \tau_1^{-1}(P), -t_2)$, $(P', t_2) \mapsto (\tau_1 \circ \tau_2^{-1}(P'), -t_1)$ and $(x_1, x_2, t) \mapsto (x_2, x_1, -t)$, respectively. These maps are well-patched and define an involution $\widetilde{\tau} : \widetilde{S}_1 \to \widetilde{S}_1$.

Let $f_1^\sharp : S_1^\sharp := \widetilde{S}_1/\langle \widetilde{\tau} \rangle \longrightarrow \Delta$ be the quotient of $\widetilde{f}_1$ by $\widetilde{\tau}$ where the base disk is the quotient space of the original disk by the involution $t \to -t$. Then it is easy to see that the resolution space S_1 of $S_1^\sharp$ induces the desired degeneration $f_1 : S_1 \to \Delta$.

Next we construct f_2. Let E' be another elliptic curve so that there exists a free action $\rho : E' \to E'$ of order 2 such that the quotient of E' by ρ coincides with E. We fix two points Q_i $(i = 1, 2)$ on E' which satisfy $\rho(Q_1) = Q_2$ so that Q_i goes to P_0 by the quotient map $E' \to E$. We fix an open disk neighborhood U_i' of Q_i which satisfy $\rho(U_1') = U_2'$ and $U_1' \cap U_2' = \emptyset$. We define the closed region V_i' of $U_i' \times \Delta$ similarly as in the previous example. We obtain an analytic surface $\widetilde{S}_2$ by patching two pieces $(E' \times \Delta) \setminus (V_1' \cup V_2')$ and W along the locus $(U_1' \times \Delta \setminus V_1') \coprod (U_2' \times \Delta \setminus V_2')$ and $W \setminus \{Q\}$ similarly. We obtain the family $\widetilde{f}_2 : \widetilde{S}_2 \to \Delta$ where $\widetilde{f}_2^{-1}(0)$ is an irreducible stable curve of genus 2 with one node. $\widetilde{S}_2$ has a natural involution so that the resolution space of the quotient space of $\widetilde{S}_2$ by this involution induces the desired degeneration.

3.3. Matsumoto-Montesinos' theorem and classification

In order to study the monodromy of degenerations, we first review the theorem of Thurston [92] about the classification of elements in Γ_g.

Let Σ be a Riemann surface of genus g with k disk boundaries with negative Euler number, i.e. $g + k - 2 > 0$. Let $\phi : \Sigma \to \Sigma$ be a homeomorphism. (Homeomorphism is always assumed to be orientation-preserving.) We call ϕ is *reducible* by an *admissible system of cut curves* Γ if;

(a) Γ is a disjoint union of simple closed curves on Σ,
(b) any connected component of the complement of Γ has negative Euler number,
(c) ϕ preserves Γ as a set.

Thurston's theorem *Any homeomorphism ϕ of Σ is isotopic to a diffeomorphism ϕ' of Σ such that, either*

(i) *ϕ' is periodic, i.e. there exists a positive integer n such that $(\phi')^n$ is the identity map, or*

(ii) *ϕ' is pseudo-Anosov, i.e. there exists a real number $\lambda > 1$ and a pair of transverse measured foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ such that $\phi'(\mathcal{F}^s) = 1/\lambda\mathcal{F}^s$ and $\phi'(\mathcal{F}^u) = \lambda\mathcal{F}^u$, or*

(iii) *ϕ' is reducible by a certain admissible system Γ of cut curves. Γ has a ϕ'-invariant annular neighborhood $\mathcal{A}(\Gamma)$ so that the complement $\Sigma \setminus \mathcal{A}(\Gamma)$ has a decomposition $\coprod_j \Sigma^{(j)}$ ($\Sigma^{(j)}$ may be disconnected) which satisfies;*
(iiia) *ϕ' preserves each component $\Sigma^{(j)}$,*
(iiib) *each component map $\phi'|_{\Sigma^{(j)}} : \Sigma^{(j)} \to \Sigma^{(j)}$ satisfies (i) or (ii).*

If all the component maps of ϕ' satisfy (i), we call ϕ *pseudo-periodic.*

Back to our situation, we consider (local 1-parameter) degenerations of curves of genus $g \geq 2$. It is known that the monodromy of degeneration is pseudo-periodic (cf. [26], [46]). Moreover this map is of negative twist, i.e. has a property of one-sided chirality ([85], [30]). More precisely, the power of this map which stabilize any curve of the admissible system induces on its annular neighborhood an integral Dehn twist of right-hand direction.

Let $\mathcal{S}_g$ be the set of topological equivalence classes of degenerations in 2.1, and let $\widehat{\mathcal{P}}_g^-$ be the subset of $\widehat{\Gamma}_g$ whose elements consist of the conjugacy classes of pseudo-periodic maps of negative twist. By taking the monodromy, we have a natural map

$$\rho : \mathcal{S}_g \longrightarrow \widehat{\mathcal{P}}_g^-.$$

Then:

Matsumoto-Montesinos' theorem *The map ρ is bijective.*

We summarize the key points of their proof [64]. For any $\phi \in \widehat{\mathcal{P}}_g^-$, they construct the degeneration whose monodromy coincides with ϕ in the following way. Among the isotopy class of ϕ, they choose an extremal element which is in some sense "the nearest" to a locally holomorphic map, and is called the superstandard form of ϕ. According to the nature of the superstandard form, they construct a certain topological covering map

$$\pi_\phi : \Sigma_g \longrightarrow V_\phi$$

form a Riemann surface Σ_g of genus g. V_ϕ is the topological underlying space of a Riemann surface with nodes, and a multiplicity is attached to each component of it. The map π_ϕ is generically finite-to-one, but some simple closed curves collapse to points. They call V_ϕ the *generalized quotient space* of ϕ, and call π_ϕ the generalized quotient map of ϕ.

They construct a fiber bundle of Riemann surfaces $f_\epsilon : \mathcal{V}_\epsilon \to S^1_\epsilon = \{t \in \mathbb{C} \mid |t| = \epsilon\}$ such that the monodromy of f_ϵ coincides with the superstandard form of ϕ and the "diameter of the vanishing cycle" tend to 0 if $\epsilon \mapsto 0$. They put the generalized quotient space V_ϕ over the origin, and associate the fiber space

$$f : S = \coprod_{0<\epsilon<1} \mathcal{V}_\epsilon \coprod V_\phi \longrightarrow \Delta = \coprod_{0<\epsilon<1} S^1_\epsilon \coprod \{0\}.$$

f naturally admit a complex structure and is the desired degeneration. Especially V_ϕ is nothing but the singular fiber of f (see also [18], [94]).

We discuss more closely around this theorem. First we assume that ϕ is a periodic map of order n. Let P be a point in Σ_g. There is a positive integer $\alpha(P)$ such that the points $P, \phi(P), \cdots, \phi^{\alpha(P)-1}(P)$ are mutually distinct and $\phi^{\alpha(P)}(P) = P$. We have $\alpha(P) = n$ for a generic point P. While if $\alpha(P) < n$, we call P a *multiple point* of ϕ.

Let C be an oriented simple closed curve on Σ_g. (We write it as $\vec{C}$ when we want to emphasize its orientation.) Let $m = m(C)$ be the smallest positive integer such that $\phi^m(\vec{C}) = \vec{C}$, i.e. $\phi^m(C) = C$ as a set and ϕ^m preserves the orientation of C. The restriction $\phi^m|_C$ is a periodic map of order, say, $\lambda \geq 1$. Note that $n = m\lambda$. Let Q be any point on C, and suppose that the images of Q under the iteration of ϕ^m are ordered as $(Q, \phi^{m\sigma}(Q), \phi^{2m\sigma}(Q), \ldots, \phi^{(\lambda-1)m\sigma}(Q))$ when we view in the direction of $\vec{C}$, where σ is an integer with $0 \leq \sigma \leq \lambda - 1$ and g.c.d$(\sigma, \lambda) = 1$. Let δ be the integer which satisfies

$$\sigma\delta \equiv 1 \pmod{\lambda}, \quad 0 \leq \delta \leq \lambda - 1.$$

Then the action of ϕ^m on $\vec{C}$ is topologically equivalent to the rotation of angle $2\pi\delta/\lambda$ with a suitable parameterization of $\vec{C}$ as an oriented circle. Nielsen [78] called the triple (m, λ, σ) the *valency* of $\vec{C}$ with respect to ϕ. The *valency of a multiple point* P is defined to be the valency of the boundary curve ∂D_P, oriented from the outside of an invariant disk neighborhood D_P of P.

Nielsen's theorem ([78], §11) says that the conjugacy class of a periodic map is completely determined by the order n and the *total valency*, i.e. the set of valencies of multiple points.

In order to determine the total valency, the following method is useful (cf. [9], §1). By Kerckhoff's theorem [50], we may assume that ϕ is an analytic automorphism under a certain complex structure. Let $\Pi : \Sigma_g \to \Sigma_{g'}$ be the quotient n-fold cyclic covering of ϕ, where g' is the genus of the base. We denote by $\lambda_1, \dots, \lambda_l$ the ramification indices of Π and let $(n/\lambda_i, \lambda_i, \sigma_i)$ $(1 \le i \le l)$ be the valencies of ϕ of the ramification points. By combining the following facts, we can determine the total valency:

(i) (Hurwitz formula) $2(g-1)/n = 2(g'-1) + \Sigma_{i=1}^{l}(1 - 1/\lambda_i)$.
(ii) (Nielsen's integral condition [78], (4.6)) $\Sigma_{i=1}^{l}\sigma_i/\lambda_i$ is an integer.
(iii) (Wiman [95]) $n \le 4g+2$.
(iv) (Harvey [39]) Set $M = \text{l.c.m}(\lambda_1, \dots, \lambda_l)$. Then;
(iv-a) $\text{l.c.m}(\lambda_1, \dots, \widehat{\lambda_i}, \dots, \lambda_l) = M$ for all i, where $\widehat{\lambda_i}$ means the omission of λ_i.
(iv-b) M divides n, and if $g' = 0$, $n = M$.
(iv-c) $l \ne 1$, and $l \ge 3$ if $g' = 0$.
(iv-d) If $2|M$, the number of $\lambda_1, \dots, \lambda_l$ which are divisible by the maximal power of 2 dividing M is even.

Example 3.1. The classification of non-identical conjugacy classes of periodic maps of genus 1 and 2 is as follows. The result is classical for $g = 1$. We apply the above criterion for $g = 2$. For simplicity, if we have the total valency $(n/\lambda_i, \lambda_i, \sigma_i)$ $(1 \le i \le l)$, we symbolically write it as $\sigma_1/\lambda_1 + \ldots + \sigma_l/\lambda_l$. (We do not specify g' when $g' = 0$.)

(i) $g = 1$:

1. $n = 6; 1/6 + 1/3 + 1/2.\ 5/6 + 2/3 + 1/2.$
2. $n = 4; 1/4 + 1/4 + 1/2.\ 3/4 + 3/4 + 1/2.$
3. $n = 3; 1/3 + 1/3 + 1/3.\ 2/3 + 2/3 + 2/3.$
4. $n = 2; 1/2 + 1/2 + 1/2 + 1/2.$
5. $g' = 1$, n is arbitrary and Π is an unramified covering.

(ii) $g = 2$:

1. $n = 10; 1/10+2/5+1/2.\ 3/10+1/5+1/2.\ 7/10+4/5+1/2.\ 9/10+3/5+1/2.$
2. $n = 8; 1/8 + 3/8 + 1/2.\ 5/8 + 7/8 + 1/2.$
3. $n = 6; 1/6 + 1/6 + 2/3.\ 5/6 + 5/6 + 1/3.\ 1/3 + 2/3 + 1/2 + 1/2.$
4. $n = 5; 1/5 + 1/5 + 3/5.\ 1/5 + 2/5 + 2/5.\ 2/5 + 4/5 + 4/5.\ 3/5 + 3/5 + 4/5.$
5. $n = 4, 1/4 + 3/4 + 1/2 + 1/2.$
6. $n = 3, 1/3 + 1/3 + 2/3 + 2/3.$
7. $n = 2, 1/2 + 1/2 + 1/2 + 1/2 + 1/2 + 1/2.$
8. $g' = 1$, $n = 2$ and $1/2 + 1/2.$

According to [64], we associate to the valency (m, λ, σ) a sequence of integers $a_0 > a_1 > \cdots > a_l = 1$ such that

$$a_0 = \lambda, a_1 = \sigma, a_{j+1} + a_{j-1} \equiv 0 \pmod{a_j} \quad (0 \le j \le l-1).$$

We set $m_j = ma_j$, and call $m_1, \cdots, m_l$ the *multiplicity sequence* of the valency. Then the "tail part" of the generalized quotient space is the open disk with multiplicity m_0 followed by a tree of $\mathbb{P}^1$'s with multiplicity $m_1, \cdots, m_l$ meeting transversally (see [65], p.72, Figure 1).

In the case of periodic maps, we can construct the generalized quotient space V_ϕ by patching all the tail parts of the total valency to the "body part" $\Sigma_{g'}$.

By this method, we can easily construct all the degenerations of genus 1 and 2 with periodic monodromy ¿from the datum in Example 3.1. For instance, if $g = 1, n = 6$ and the total valency is $5/6 + 2/3 + 1/2$, then the multiplicity sequences are $(6, 5, 4, 3, 2, 1)$, $(6, 4, 2)$ and $(6, 3)$. We connect the three trees to a rational component of multiplicity 6. This induces the degeneration of type II^* in [51]. The case $g = 1, n = 6$ and $1/6 + 1/3 + 1/2$ corresponds to the normally minimal model of type II in [51]. The case $g = 2, n = 8$ and $5/8 + 7/8 + 1/2$ corresponds to type VII^* in [76], p.340, etc. The moduli points of these degenerations are smooth curves. (However we cannot determine the limit of the period matrix by this method, while it is possible by [76] for $g = 2$.)

Next we assume that ϕ is a reducible pseudo-periodic map. We identify ϕ with its representative ϕ' in Thurston's theorem. Let Γ_i be a simple closed curve of the admissible system Γ of ϕ. Let β_i the smallest positive integer such that $\phi^{\beta_i}(\vec{\Gamma}_i) = \vec{\Gamma}_i$, and let L_i be the smallest positive integer such that ϕ^{L_i} is an integral Dehn twist (say e_i-times). According to [78], we define the *screw number* of Γ_i by $s(\Gamma_i) = e_i\beta_i/\ L_i$. If β_i is even and $\phi^{\beta_i/2}(\vec{\Gamma}_i) = -\vec{\Gamma}_i$ (i.e. $\phi^{\beta_i/2}$ preserves Γ_i as a set but reverses its orientation), we call Γ_i *amphidrome*. Otherwise, we call it *non-amphidrome*.

Let B_k be a connected component of $\Sigma_g \setminus \Gamma$. Let b_k be the smallest positive integer such that ϕ^{b_k} stabilizes B_k. We call the periodic map $\phi^{b_k}|_{B_k}$ the *stabilized component map* of ϕ for B_k. Let $G(\Gamma)$ be the *extended partition graph* ([64]), i.e. the vertex v_{B_k} of $G(\Gamma)$ corresponds to a component B_k and the edge e_{Γ_j} corresponds to a curve Γ_j , and e_{Γ_j} is connected to v_{B_k} if Γ_j is a boundary of B_k. We apriori give an orientation on e_{Γ_j} in a suitable way. Note that, if Γ_j is amphidrome, then the action of $\phi^{\beta_i/2}$ to e_{Γ_j} changes its orientation.

Nielsen and Matsumoto-Montesinos' theorem ([79], [64]) says that the conjugacy class of a pseudo-periodic map is completely determined by

(i) the set of stabilized component maps,
(ii) the set of screw numbers,
(iii) the action of the map to the extended partition graph.

The generalized quotient space V_ϕ is constructed as follows: We first construct the "parts" of V_ϕ by the data (i), and then combine these parts by the datum (ii) and (iii).

We explain this process explicitly by the following simple examples (compare with [77], pp.158–169): Let Γ be a separated simple closed curve on a Riemann surface Σ_2 of genus 2. We write

$$\Sigma_2 = \Sigma^{(1)} \cup \mathcal{A}(\Gamma) \cup \Sigma^{(2)}$$

where $\Sigma^{(i)}$ $(i = 1, 2)$ is a torus with one disk boundary $\partial\Sigma^{(i)}$ and $\mathcal{A}(\Gamma)$ is an annulus around Γ. The graph $G(\Gamma)$ consists of two vertices $v_{\Sigma^{(1)}}$ and $v_{\Sigma^{(2)}}$ which are combined by an oriented edge e_Γ. The action of ϕ to $G(\Gamma)$ is either (a) the identity, or (b) of order 2 and is generated by the permutation $(v_{\Sigma^{(1)}}, v_{\Sigma^{(2)}})$ reversing the orientation of e_Γ.

Assume (a). We have a closed torus $\overline{\Sigma^{(i)}}$ by patching a disk U on $\Sigma^{(i)}$ along the boundary and have a periodic homeomorphism $\overline{\phi}^{(i)} : \overline{\Sigma^{(i)}} \longrightarrow \overline{\Sigma^{(i)}}$ which is an extension of the component map $\phi|_{\Sigma^{(i)}}$ $(i = 1, 2)$ so that $\overline{\phi}^{(i)}|_U$ has a unique multiple point. The total valency of $\overline{\phi}^{(i)}$ is one of the table in Example 3.1. For example, we assume that the total valency of $\overline{\phi}^{(1)}$ (resp. $\overline{\phi}^{(2)}$) is $5/6 + 2/3 + 1/2$ (resp. $2/3 + 2/3 + 2/3$). Then the valency of the curve $\partial\Sigma^{(1)} = \partial U$ (resp. $\partial\Sigma^{(2)} = \partial U$) is automatically $5/6$ (resp. $2/3$). The two parts of V_ϕ are degenerate elliptic curves of type II^* and of type IV^*. We patch them as follows: Put

$$K = -5/6 - 2/3 - s(\Gamma).$$

Note that the screw number $s(\Gamma)$ is non-positive since ϕ is of negative twist. Then K is an integer which satisfies $K \geq -1$ ([64]). If $K > 0$, we have the sequence $(6, 5, 4, 3, 2, 1, 1, \cdots, 1, 1, 2, 3)$ by adding the the multiplicity sequences of $5/6$ and $2/3$ to $(K - 1)$ times of 1. If $K = 0$, the sequence is $(6, 5, 4, 3, 2, 1, 2, 3)$. If $K = -1$, we shorten the original two sequences and get $(6, 5, 4, 3, 2, 3)$. We obtain V_ϕ by combining the above two parts along the tree of smooth rational curves which have the above multiplicity sequence. Therefore we have degenerations of type II^*-IV^*-m and of type II^*-IV^*-α in [77], p.164, 165.

Assume (b). Then $(\phi^2)_{\Sigma^{(i)}}$ $(i = 1, 2)$ are the stabilized component maps. The total valencies of these maps coincide with each other. Suppose it is $3/4 + 3/4 + 1/2$, for instance. Then the valency of each boundary curve is automatically $3/4$. Since Γ is amphidrome, we use the "tail of type D", i.e. whose dual graph is a Dynkin diagram of type D (see [65], p.73, Figure 2). We obtain V_ϕ from the double multiple of the degenerate elliptic fiber of type III^* by adding the tail of type D. We consequently get the degeneration of type $2III^*$-m in [77], p.168. The other cases are similar.

Lemma 3.2. *Matsumoto-Montesinos' theorem induces the stable reduction theorem.*

Proof. (cf. [9], §4) Let ϕ_f be the monodromy of a given degeneration $f : S \to \Delta$. Since ϕ_f is pseudo-periodic, there exists a positive integer N such that ϕ_f^N is an integral Dehn twist along the annular neighborhood of the admissible system Γ_ϕ of cut curves of ϕ. Since all the component maps of ϕ_f^N are trivial, the generalized quotient space of ϕ_f^N is a semi-stable curve. Let $\widetilde{f} : \widetilde{S} \to \Delta$ be the desingularization of the pull back of f by the cyclic covering $\Delta \to \Delta$ given by $t \mapsto t^N$. Since the monodromy of $\widetilde{f}$ coincides with ϕ_f^N, $\widetilde{f}$ is a semi-stable family. Therefore we obtain the stable family by contracting chains of (-2) curves on $\widetilde{S}$. Q.E.D.

The proof especially says that the topological type of the stable curve which is the moduli point of f can be obtained by shrinking Γ_ϕ to ordinary double points. In other words, the dual graph of the stable curve of the moduli point coincides with $G(\Gamma)$.

We can classify degenerations of curves of genus g topologically, according to the following steps:

Step 1: Classify admissible systems Γ of cut curves on Σ_g, i.e. classify stable curves of genus g.

Step 2: Classify cyclic actions on $G(\Gamma)$, i.e. classify cyclic actions on the dual graph of the stable curve.

Step 3: Classify periodic maps of Riemann surfaces of genus $\leq g$ with disk boundaries, i.e. classify cyclic analytic automorphisms of pointed curves of genus $\leq g$.

Step 4: Classify pseudo-periodic maps, i.e. classify cyclic automorphisms of stable curves.

However, we must spend a lot of time in our life for the explicit calculations for large g. In the case of $g = 3$, we have:

Proposition 3.3 ([9]). *We can explicitly classify the singular fibers, the monodromies and the open strata which contain the moduli points (i.e. the topological structure of the stable curves corresponding to the moduli points) at the same time for* $g = 3$.

For the proof, we apply the above four steps to each stable curve of genus 3. We have 42 types of such curves in total.

Remarks. (1) If we restrict the map $\rho : \mathcal{S}_g \longrightarrow \widehat{\mathcal{P}}_g^-$ to a certain class of degenerations, what happens ? For instance, let $\mathcal{SH}_g$ be the set of topological equivalence classes of hyperelliptic degenerations (i.e. degenerations whose general fibers are hyperelliptic curves). Let $\mathcal{H}_g$ be the conjugacy classes of the hyperelliptic mapping class group of genus g, i.e. consisting of elements of Γ_g which commute with the hyperelliptic involution. Then we have a natural map

$$\rho' : \mathcal{SH}_g \longrightarrow \mathcal{H}_g \cap \widehat{\mathcal{P}}_g^-.$$

Endo's question asks whether ρ' is bijective. Since Ishizaka [48] has classified hyperelliptic degenerations of genus 3, we can expect an answer at least in this case.

(2) Historically Nielsen [79] first studied pseudo-periodic maps under the name "surface transformation class of algebraically finite type". From the reconstruction of Thurston's theorem by Bers [17], function theorists usually call a non-periodic pseudo-periodic map (resp. periodic map) a map of parabolic type (resp. map of elliptic type).

Contrary to the local monodromy case, if we consider the *global* monodromy of a 1-parameter degeneration, there really appear various classes in the above classification of Γ_g ([47], [84]). The systematic study in this field seems to be untouched and will be interesting.

§4. Morsification of degenerations

4.1. Definition of Morsifications

(A) Classical Morsification. Let Δ, Δ' be sufficiently small open disks. Let $f\colon S \longrightarrow \Delta$ be a degeneration of curves of genus g with a unique singular fiber $F = f^{-1}(0)$. Assume that there exist a smooth threefold M and a holomorphic map $\mathbf{f}\colon M \longrightarrow \Delta \times \Delta'$ such that the restriction $\mathbf{f}_u\colon M_u \longrightarrow \Delta \times \{u\}$ of $\mathbf{f}$ over $u \in \Delta$ satisfies the following:

(i) $\mathbf{f}_0$ coincides with f,
(ii) $\mathbf{f}_u$ $(u \neq 0)$ has $l \geq 1$ singular fiber germs $F_{u,1}, \dots, F_{u,l}$. l is independent on u.

Then we call $\mathbf{f}$ a *splitting family* of the germ F, and we symbolically write it as

$$F \longrightarrow F_{u,1} + \cdots + F_{u,l}.$$

Moreover, if $l \geq 2$, we call $\mathbf{f}$ a *proper* splitting family of F. If $l = 1$, we call $\mathbf{f}$ an *equisingular deformation* of F. If F has no proper splitting families even after equisingular deformations, then we call F an *atomic fiber*. We say that the germ F is *morsified* to the set of germs $\mathcal{S}_F = \{F^{(1)}, \dots, F^{(k)}\}$ if:

(a) each $F^{(i)}$ $(1 \leq i \leq k)$ is an atomic fiber,
(b) F is decomposed into the members of $\mathcal{S}_F$ via several splitting families.

Namely, if F is not atomic, F has a splitting family $F \longrightarrow F_{u,1} + \cdots + F_{u,l}$. If one of the elements of $\{F_{u,1}, \dots, F_{u,l}\}$ does not belong to $\mathcal{S}_F$, say $F_{u,i_0} \notin \mathcal{S}_F$, then F_{u,i_0} has a splitting family $F_{u,i_0} \longrightarrow F'_{v,1} + \cdots + F'_{v,l'}$ (It is possible that F_{u,i_0} has an equisingular deformation $F_{u,i_0} \longrightarrow \widetilde{F_{u,i_0}}$ and then has a proper splitting family $\widetilde{F_{u,i_0}} \longrightarrow F'_{v,1} + \cdots + F'_{v,l'}$). If F'_{v,j_0} $(1 \leq j_0 \leq l')$ does not belong to $\mathcal{S}_F$, then F'_{v,j_0} has a splitting family. Continue this process. After finite steps, all the terminated fiber germs belong to $\mathcal{S}_F$.

If F is morsified to $\mathcal{S}_F$ via only one splitting family, we say that F is *directly morsified* to $\mathcal{S}_F$. The set of atomic fibers $\mathcal{S} = \{\widetilde{F}^{(1)}, \dots, \widetilde{F}^{(\widetilde{l})}\}$ is called the *complete system of atomic fibers of genus* g if any fiber germ of genus g is morsified to a subset of $\mathcal{S}$.

Now, Xiao Gang's morsification conjecture [83] can be stated as:

Morsification Conjecture. Any atomic fiber is either a fiber germ with only one Morse critical point or a multiple of a smooth curve.

(B) Algebraic Morsification. We consider a certain condition (*) for smooth curves as in §2.1 with respect to which the Horikawa index is well-defined. Let $f\colon S \longrightarrow \Delta$ be a (*)-degeneration with the unique critical or singular fiber $F = f^{-1}(0)$. Assume that there exist a smooth threefold M and a holomorphic map $\mathbf{f}\colon M \longrightarrow \Delta \times \Delta'$ such that

(i) $\mathbf{f}_0$ coincides with f,
(ii) $\mathbf{f}_u$ $(u \neq 0)$ is a (*)-degeneration and has $l \geq 1$ critical or singular fibers $F_{u,1}, \dots, F_{u,l}$, where l is independent on u,
(iii) (conservation of index) $\quad \mathrm{Ind}(F) = \sum_{i=1}^{l} \mathrm{Ind}(F_{u,i})$.

Then we call **f** a (*)-splitting family of F. In a similar way, we define (*)-atomic fiber, complete system of (*)-atomic fibers and so on.

Question. Does the condition (iii) follow from (i) and (ii) ?

(C) Topological Morsification. Let $\overline{f}\colon \overline{S} \longrightarrow \overline{\Delta}$ be a holomorphic fibration between a complex surface $\overline{S}$ with boundary and a closed disk $\overline{\Delta} = \{t \in \mathbf{C} \mid |t| \le \varepsilon\}$, i.e., $\overline{f}$ is the restriction of a certain holomorphic fibration $f^{(0)}\colon S^{(0)} \longrightarrow \Delta^{(0)} = \{t \in \mathbf{C} \mid |t| < \varepsilon + \varepsilon'\}$ to $\overline{\Delta} \subset \Delta^{(0)}$. Assume that $F = f^{-1}(0)$ is the unique singular fiber.

Suppose that there exists another holomorphic fibration $\overline{f'}\colon \overline{S'} \longrightarrow \overline{\Delta}$ with singular fibers $F_1', \ldots, F_l'$ which satisfies the following: There exists an orientation-preserving diffeomorphism $\mathcal{H}\colon \overline{S} \longrightarrow \overline{S'}$ which commutes with $\overline{f}$ and $\overline{f'}$ on the boundary, i.e., $(\overline{f'}|_{\partial\overline{S'}}) \circ \mathcal{H}|_{\partial\overline{S}} = \overline{F}|_{\partial S}$. Then we say that F splits differentiably into F_1', $ldots, F_l'$ ([63], Definition 3.4) and write $F \stackrel{C^\infty}{\longrightarrow} F_1' + \cdots + F_l'$. Based on this notion, we can similarly define C^∞-atomic fibers, C^∞-complete system of atomic fibers and so on.

Lemma 4.1 ([3], §1). *A classical splitting $F \longrightarrow F_{u,1} + \cdots + F_{u,l}$ naturally induces a C^∞-splitting $F \stackrel{C^\infty}{\longrightarrow} F_{u,1} + \cdots + F_{u,l}$ whenever $|u|$ is sufficiently small.*

The proof is an analogue of the C^∞ local triviality of the deformation space of complex manifolds without boundary.

Historical remark. As to the classical Morsification, Moishezon [70] studied elliptic fibrations and proved that any Kodaira singular fiber is directly morsified to either:

(a) an irreducible stable curve with one node, or
(b) a multiple of a smooth elliptic curve.

For the study of elliptic fibrations over multi-dimensional base via minimal model theory, see Nakayama [75].

As to the algebraic morsification, a systematic study has been done only for hyperelliptic degenerations. For $g = 2$, Horikawa [43] proved that, modulo fibers with Ind $= 0$, any critical fiber is directly morsified to a several number of a unique atomic fiber, i. e., the stable curve with two components with one node. We remark that this can be also shown by deforming the canonical algebra, since it is a complete intersection [67]. We will discuss further developements for hyperelliptic fibrations in §4.2. For non-hyperelliptic degenerations of genus three, we showed in Appendix of [57] that, modulo fibers with Ind $= 0$, any 2-connected critical fiber is morsified to (smooth) hyperelliptic fibers of Horikawa index one, by deforming the canonical algebra. Similar calculations may be possible by using results in [67]. But, the following conjecture still remains open:

Conjecture ([83]). *The complete system of atomic fibers of non-hyperelliptic degenerations of genus three consists of*

(a) *an irreducible stable curve with one node,* Ind $= 0$,
(b) *a smooth hyperelliptic curve,* Ind $= 1$,
(c) *a stable curve with two components with one node,* Ind $= 2$,
(d) *a double multiple of a smooth curve of genus two,* Ind $= 3$.

As to the topological morsification, topologists consider its analogue for a wider class of fibrations, that is, "locally analytic fibrations" (cf. [63]). For instance, a good torus fibration ([62]) is such a generalized object of an elliptic fibration, and Matsumoto [62] and Ue [93] studied the Morsification problem for good torus fibrations.

4.2. Hyperelliptic splitting families

Here, according to [2] and [3], we construct some hyperelliptic splitting families and determine the complete system of hyperelliptic atomic fibers of genus three.

We set the condition (*) in §2.1 to be hyperelliptic. Let $f\colon S \to \Delta$ be a hyperelliptic degeneration of genus g. Then a (*)-critical fiber of f is necessarily a singular fiber. In fact, hyperelliptic curves do not specialize to a non- hyperelliptic curve via a deformation by the closedness of hyperelliptic locus in $\mathcal{M}_g$. Moreover, the notion of algebraic splitting family is almost equivalent to the notion of classical splitting family by the following:

Lemma 4.2 ([3], §1). *If a hyperelliptic singular fiber germ F splits as $F \longrightarrow F_{u,1} + \cdots + F_{u,l}$, then*

$$\mathrm{Ind}(F) = \sum_{i=1}^{l} \mathrm{Ind}(F_{u,1}).$$

Proof. By (2.1.3) with $\lambda = 4(g-1)/g$, we have

$$\mathrm{Ind}(F) = \frac{2g+1}{g}\sigma(F) + \frac{g+1}{g}e_f(F).$$

On the other hand, Terasoma [89] shows that

$$\sigma(F) = \sigma_{top}(F),$$

where $\sigma_{top}(F)$ is the topological local signature of Endo citeEndo. It follows from Lemma 4.1 that

$$\sigma_{top}(F) = \sum_{i=1}^{l} \sigma_{top}(F_{u,i}), \quad e_f(F) = \sum_{i=1}^{l} e_f(F_{u,i}).$$

Therefore, we obtain the assertion. Q.E.D.

Let $f\colon S \to \Delta$ be a hyperelliptic degeneration of genus g with the unique singular fiber $F = f^{-1}(0)$. As we explained in §2.2, f is determined by the branch divisor R on $W = \mathbb{P}^1 \times \Delta$. Let $\pi\colon W \to \Delta$ be the natural projection. If a point $P \in R$ is a singular point of R or R has a component which is tangential to the fiber of π at P, we call P a *bad point* of R. Since f is smooth over $\Delta \setminus \{0\}$ by the assumption, bad points of R are on $\Gamma_0 = \pi^{-1}(0)$.

Now one of our main methods constructing splitting families of f is to find a reduced divisor $\mathbf{R}$ on $\mathbf{W} = \mathbb{P}^1 \times \Delta \times \Delta'$ such that the restriction $\mathbf{R}_u = \mathbf{R}|_{\mathbb{P}^1 \times \Delta \times \{u\}}$ over $u \in \Delta'$ satisfies the following:

(a) $\mathbf{R}_0$ coincides with the original R,

(b) The bad points of $\mathbf{R}_u$ $(u \neq 0)$ are on several fibers of the projection

$$\pi_u \colon \mathbb{P}^1 \times \Delta \times \{u\} \longrightarrow \Delta \times \{u\},$$

(c) Let $\phi\colon M' \to \mathbf{W}$ be the singular double cover branched along $\mathbf{R}$ in the total space of the square root bundle of $[\mathbf{R}]$. The natural map $f'\colon M' \to \Delta^{\ prime}$ induces a deformation of normal surface singularities. Then f' has a simultaneous resolution (after a base change if necessary), which gives us a splitting family.

We describe a sufficient condition for (c). Let $m_1, \ldots, m_n$ be the multiplicity sequence of the even resolution (2.2.3) of R. We re-order the sequence $\{[m_i/2]\}$ according to the usual ordering $\geq$ to get a number sequence $\{h_i\}_{i=1}^n$, $h_1 \geq \cdots geqh_n$. We put

$$HM(R) := \{h_i\}_{i=1}^n.$$

and call it the *half multiplicity sequence* of R. The half multiplicity sequence $HM(\mathbf{R}_u)$ of $\mathbf{R}_u$ is defined in the same way. If we have

(c′) $\qquad HM(R) = HM(\mathbf{R}_u)$

for any $u \in \Delta'$, then the summation of the geometric genus of singularities on the fiber of f' is preserved via the deformation by the formula (2.2.4). Therefore the condition (c$'$) implies the condition (c) by Laufer's theorem [60].

In order to find $\mathbf{R}$ satisfying (a), (b) and (c$'$), we use the "perturbation method on the way of its resolution process". We explain it by using the following example. (See also Example 3.6 and Figure 2 in [2].)

Let (x, t) be the system of coordinates of W, where x is the inhomogeneous coordinate of $\mathbb{P}^1$. Let R be the divisor on W defined by

$$(x^3 + t^{6n})(x-1)(x-2)(x-3) = 0.$$

Then R has the unique bad point $Q = \{x = t = 0\}$, and we have $\Gamma_0 R = 6$. The $2n$-th branch divisor R_{2n} by the composition of blow-ups $\sigma_{2n} \circ \cdots \sigma_1 \colon W_{2n} \longrightarrow W$ with their centers are infinitely near to Q is nonsingular. We have $HM(R) = \{2, \ldots, 2, 1, \ldots, 1\}$ (1,2 appears n-times). The double cover $\widetilde{S}$ of W_{2n} branched along R_{2n} is nonsingular and the natural fibration $\widetilde{f} \colon \widetilde{S} \longrightarrow \Delta$ has n (-1)-curves on $\widetilde{f}^{-1}(0)$. By contracting them, we get a semi-stable degeneration of genus two whose singular fiber $F = f^{-1}(0)$ consists of two smooth elliptic curves and n (-2)-curves. We have $\mathrm{Ind}(F) = n$ by (2.2.7). Horikawa [41] called F a singular fiber *of type* rmI_n.

Now let $R_{2k} \subset W_{2k}$ $(1 \le k \le n-1)$ be the $2k$-th branch divisor on the way of its resolution process. R_{2k} has an unique singular point Q_{2k} on the exceptional curve E_{2k} by σ_{2k}, and the equation of R_{2k} at Q_{2k} is written as $(x')^3 + t^{6(n-k)} = 0$ with respect to the natural local coordinates on a neighborhood U around $Q_{2k} \in W_{2k}$.

We can define the divisor $\widetilde{\mathbf{R}}$ on $W_{2k} \times \Delta'$ such that the equation of $\widetilde{\mathbf{R}}$ in $U \times \Delta'$, with coordinates (x', t, u), is written as

$$(x')^3 + (t-u)^{6(n-k)} = 0.$$

We resolve the singularities on the singular double cover branched along $\widetilde{\mathbf{R}}$ simultaneously with respect to the base Δ' by the relative version of the canonical resolution. Geometrically, this process is essentially the same as the following:

Let $\mathbf{R}$ be the image of $\widetilde{\mathbf{R}}$ by the contraction map $W_{2k} \times \Delta' \longrightarrow W \times \Delta'$. Then the restriction $\mathbf{R}_u$ of $\mathbf{R}$ to $W \times \{u\}$ $(u \neq 0)$ has two bad points Q' and Q'' on $\pi^{-1}(0)$ and $\pi^{-1}(u)$, respectively. The typical equation of $\mathbf{R}$ is

$$(x^3 + t^{6k})(x^3 + (t-u)^{6(n-k)})(x-1)(x-2)(x-3) = 0.$$

We have

$$HM(\mathbf{R}_u) = \{2, \dots, 2, 1, \dots, 1\} = HM(R).$$

Note that the contributions to $HM(\mathbf{R}_u)$ $(u \neq 0)$ come from both Q' and Q'', while the contributions to $HM(R)$ come from Q only. Therefore, $\mathbf{R}$ satisfies the desired conditions (a), (b) and (c$'$). The simultaneous resolution space induces the splitting family of type I_n into type I_k and type I_{n-k}.

Of course, we can define $\mathbf{R}$ directly by the above equation in this special case. However, this argument, called a *fission* in [2], is useful for constructing more complicated hyperelliptic splitting families.

Remark. The idea of the perturbation method on the way of its resolution process is originally used by A'Campo [1] for studying the Morsification of plane curve singularities. It is also applied to study the signature of the Milnor fiber of a type of normal surface singularities [8].

Assume that two reduced (possibly reducible) curves C_1 and C_2 on a nonsingular surface meet at a point Q. For a positive integer n, we say that C_1 is *n-tangential* to C_2 at Q if the following condition holds: Blow-up n-times successively at infinitely near points of Q. Then there exist a locally analytic component C_1' of C_1 at Q and a locally analytic component C_2' of C_2 at Q so that the proper transforms of C_1' and C_2' by the composition of these blow-ups still meet each other. Namely, C_1' contacts C_2' at Q of order at least n. We say that C_1 is n-tangential to C_2 if there exists a point Q such that C_1 is n-tangential to C_2 at Q. Let (W, R) be a representative of the branch curve of a germ F. If R contains Γ_0, then we set $R_{hor} := R - \Gamma_0$, and otherwise we set $R_{hor} := R$. We call R_{hor} the horizontal part of R.

We call a representative (W, R) *trivial* if one of the following two conditions is satisfied:

(i) R_{hor} intersects Γ_0 at (just) one point Q_1 such that R_{hor} is not tangential to Γ_0 at Q_1, or

(ii) R_{hor} intersects Γ_0 at (just) two points Q_1 and Q_2 such that R_{hor} is smooth at Q_1 meeting transversally to Γ_0, and R_{hor} is not tangential to Γ_0 at Q_2.

Otherwise, we call (W, R) a *non-trivial* representative. A trivial representaive (W, R) of F becomes a non-trivial representative after a finite number of elementary transformations.

With this notation, we introduce some special classes of singular fiber germs.

(i) Type 0_0: R is smooth on W and meets Γ_0 transversally except at one point P where the order of contact is two. Then the corresponding singular fiber F is an irreducible stable curve with one node. If a representative of the branch curve (W, R) has the above property, we call the fiber germ F a germ of type 0_0.

(ii) Class I: R does not contain Γ_0 and meets Γ_0 transversally except at one point P which is an ordinary singularity of R of multiplicity $2g'+2$, where g' is an integer with $1 \leq g' \leq [(g-1)/2]$. Then F is a stable curve of two components of genera g' and $g-g'-1$ with two nodes. If a representative of the branch curve (W, R) has the above property, we say that the fiber germ F belongs to class I. In fact, this class contains $[(g-1)/2]$ types.

(iii) Class II: R contains Γ_0. We can produce a minimal succession of blow-ups (2.2.3) such that any singularity of R_n is ordinary. Denote by $\mathbf{E}_j$ $(1 \leq j \leq n)$ the total transform of Γ_0 by $\sigma_j \circ \sigma_{j-1} \circ \cdots \circ \sigma_1$. Then the following conditions (a) through (e) hold:

(a) R_n contains $\mathbf{E}_n$,
(b) Any singularity of R_n has even multiplicity,
(c) For any $0 \leq i \leq n$, any singularity of R_i with even multiplicity is ordinary,
(d) If R is not 1-tangential to Γ_0 at P, then the following condition hold: Blow-up at P. Then $R_1 - \mathbf{E}_1$ is 3-tangential to $\mathbf{E}_1$,
(e) If R has an ordinary double point, then R_{hor} is 3-tangential to Γ_0.

We say that a germ F belongs to class II if at least one nontrivial representative (W, R) of the branch curve satisfies $R \supset \Gamma_0$ and moreover any nontrivial representative (W, R) of the branch curve with $R \supset \Gamma_0$ has the above conditions (a) through (e).

Proposition 4.3 ([2]). *Any hyperelliptic singular fiber germ F splits into the germs of members of type 0_0, class I and class II via several proper splitting families.*

Moreover, if $\mathrm{Ind}(F) = 0$, *then F splits into several germs of type* 0_0.

We use several variations of the method of the fission to prove it. We do not know whether class II fibers have further proper hyperelliptic splitting families or not. However, if we admit equisingular deformations, some of them really split as follows.

From now, we consider the case $g = 3$. We have the following list of fibers of type 0_0, class I and class II (class II has three types II(i), II(ii) and II(iii)):

Type 0_0: F is an irreducible stable curve with one node.

Type I: F is a stable curve with two smooth elliptic components with two nodes.

Type II(i): $R = R_{hor} + \Gamma_0$ has two bad points Q_i $(i = 1, 2)$. Q_1 (resp. Q_2) is an ordinary singularity of multiplicity four (resp. six). F is a stable curve consisting of a smooth elliptic component and a smooth genus two component meeting at a point.

Type II(ii): $R = R_{hor} + \Gamma_0$ has two bad points Q_i $(i = 1, 2)$. R_{hor} has two smooth local components at Q_i such that each of them contacts to Γ_0 to the second order. We can write F as $F = 2C_1 + 2C_2 + 2C_3$, where C_1 and C_3 are smooth elliptic curves and C_2 is a smooth rational curve with $C_1C_2 = C_2C_3 = 1$, $C_1^2 = C_3^2 = -1$ and $C_2^2 = -2$.

Type II(iii): $R = R_{hor} + \Gamma_0$ has a unique bad point Q. R_{hor} has four smooth local components at Q such that each of them contacts to Γ_0 to the second order. F is a double multiple of a smooth curve of genus two.

The Horikawa indices and the Euler contributions of these germs are:

	0_0	I	II (i)	II (ii)	II (iii)
Ind	0	2/3	5/3	10/3	8/3
e_f	1	2	1	4	2

If there were a proper splitting family of a germ in these classes, the possible case would be the following (a) or (b) by the conservation of invariants:

(a) $\mathrm{II(ii)} \longrightarrow 0_0 + 0_0 + \mathrm{II(i)} + \mathrm{II(i)}$
(b) $\mathrm{II(ii)} \longrightarrow \mathrm{I} + \mathrm{II(iii)}$

We can show that the case (a) is impossible by a certain monodromy argument, while (b) actually occurs.

Lemma 4.4. *There exists a type* II(ii) *fiber germ F such that F splits into type* I *and type* II(iii).

We have two proofs in [3], one of which is as follows. We first construct a hyperelliptic stable family $\widetilde{\mathbf{f}} : \widetilde{M} \longrightarrow \Delta \times \Delta'$ of genus three (where $\widetilde{M}$ has singularities) such that $\widetilde{\mathbf{f}}_u : \widetilde{M}_u \longrightarrow \Delta \times \{u\}$ satisfies the following:

(a) $\mathbf{f}_0$ has a unique singular fiber $\mathbf{f}_0^{-1}(0)$ of type I so that two nodes of $\mathbf{f}_0^{-1}(0)$ are A_1-singularities on $\widetilde{M}$,
(b) $\mathbf{f}_u$ $(u \neq 0)$ has two singular fibers $\mathbf{f}_u^{-1}(u)$ and $\mathbf{f}_u^{-1}(-u)$ of type I so that the nodes of them are nonsingular points on $\widetilde{M}$,

(c) there exists an involution $\iota\colon \widetilde{M} \longrightarrow \widetilde{M}$ fiber-wise with respect to the base Δ' so that τ sends $\mathbf{f}_u^{-1}(u)$ isomorphically onto $\mathbf{f}_u^{-1}(-u)$ and τ acts on $\mathbf{f}_u^{-1}(0)$ as a fixed point free automorphism.

Then the resolution space of the quotient space $\widetilde{M}/\langle\iota\rangle$ induces a desired splitting family. Since we can connect any type II(ii) fiber germ and the fiber germ F in Lemma 4.4 via equisingular deformations, we get:

Proposition 4.5 ([3]). *The complete system of hyperelliptic atomic fibers of genus three are of type* 0_0, I, II(i) *and* II(iii).

4.3. Questions and remarks

Morsification of fiber germs may be a new mathematical area involving algebraic geometry, low-dimensional topology, Teichmüller theory and so on. Many natural and fundamental questions are not settled, or, even, are not under consideration. Here we pick up some of them and give comments.

1. Direct Morsification. If a germ F has a splitting $F \longrightarrow F_{u,1}+ F_{u,2} + \cdots$ and each $F_{u,i}$ $(i = 1, 2, \dots)$ has a splitting $F_{u,i} \longrightarrow F'_{u,i,1} + F'_{u,i,2} +$ *cdots*. Then is there a family which realizes $F \longrightarrow F'_{u,1,1} + F'_{u,1,2} + \cdots F'_{u,2,1} + F'_{u,2,2} + \cdots$ directly ? If one has a complete system of atomic fibers in a certain category, then does a germ F in the category have a direct Morsification to atomic fibers ?

2. General construction of splitting families. As to the classical Morsification, Takamura began his pioneering work [87]. The main method of [87], Parts II and III, is as follows: He first reconstructs Matsumoto-Montesinos' families and shows that we can choose a "linear degeneration" as a representative of the topological equivalence class of a given germ F. The linear degeneration is constructed as a hypersurface of a certain explicit ambient threefold $X \supset S \xrightarrow{f} \Delta$. X is the "plumbing space" of the normal bundles of components of F, and is a generalization of Hirzebruch-Jung string in some sense. Terasoma's theorem [90] says that if two degenerations f_1 and f_2 are topologically equivalent, then we can connect f_1 and f_2 via several equisingular deformations. Therefore, the linear degeneration is also a representative of the equisingular deformation class of F.

Now he constructs explicitly relative deformations $\{f_u\colon X_u \supset S_u \longrightarrow \Delta\}_{u\in\Delta'}$ of f so that f_u $(u \neq 0)$ has several singular fibers. If we had $X_u \simeq X$, then the deformation would behave as an infinitesimal displacement in X. But X really has a jumping deformation similarly as in the case of rational normal scrolls ([37]), and the constructions of $\{f_u\}_{u\in\Delta'}$ become delicate.

Takamura announces that he can obtain the complete systems of classical atomic fibers of genus $g \le 5$ by this method.

As to the algebraic Morsification, our knowledge is very poor except for hyperelliptic degenerations. We expect that pioneers will come up soon. Another problem to be considered is the Morsification problem for the base locus of the canonical linear system of singular fiber germs. We imagine that $\mathrm{Bs}|K_F|$ of an atomic fiber F is simple and K_F is ample. The last assertion means that (-2)-curves will disappear along Morsification, while it is impossible under deformations of " global" surfaces.

3. Versal family. Can we describe a versal family of a given fiber germ F ? Moreover, can we prove the existence of the Morsification via the versal family ?

This question comes from an analogy of the well-known Morse theory on hypersurface singularities (e.g., [5]). For the readers' convenience, we review the argument.

We consider germs of n-dimensional isolated hypersurface singularities. Among them, a singularity with Milnor number $\mu = 1$ is called an A_1-singularity. A typical equation is $x_0^2 + x_1^2 + \cdots + x_n^2 = 0$. An A_1-singularity is sometimes called a *Morse singularity*, because:

Any isolated hypersurface singularity $(V, \mathbf{0})$ with Milnor number $\mu(V, \mathbf{0}) = \mu$ is directly morsified to μ A_1-singularities.

Indeed, let $x = (x_0, x_1, \dots, x_n)$ be the system of coordinates on a domain $U \subset \mathbb{C}^{n+1}$ containing the origin $\mathbf{0}$. Let $f(x_0, x_1, \dots, x_n) = 0$ be the equation in U of the germ $(V, \mathbf{0})$. The Kuranishi space of $(V, \mathbf{0})$ is unobstructed and coincides with

$$\mathrm{Ext}^1(\Omega_V^1, \mathcal{O}_V) \simeq J_f = \mathbb{C}\{x_0, x_1, \dots, x_n\}/I,$$

where $I = (f, \partial f/\partial x_0, \dots, \partial f/\partial x_n)$. Since $\dim_{\mathbb{C}} J_f = \mu(V, \mathbf{0}) = \mu$, we identify J_f with $\mathbb{C}^\mu$. Let

$$\widetilde{F} = (\widetilde{f}, \widetilde{\pi})\colon U \times \mathbb{C}^\mu \longrightarrow \mathbb{C} \times \mathbb{C}$$

be the versal family or the universal unfolding of f ([91]). Namely, if $u = (u_1, \dots, u_\mu)$ is the system of coordinates of $mathbbC^\mu$ and $\widetilde{f} = \widetilde{f}(x, u)$, then $\{\partial\widetilde{f}/\partial u_j|_{u=0}\}_{j=1}^{l}$ generates J_f. If we fix $u \in \mathbb{C}^\mu$, then the map $\widetilde{f}(x, u)\colon U \times \{u\} \to \mathbb{C}\ times\{u\}$ is a $\mu : 1$ finite map for generic u. We define the bifurcation locus $\mathrm{Bif}(\widetilde{f})$ of $\widetilde{f}$ as the subset of $\mathbb{C}^\mu$ consisting of the elements u such that the map $\widetilde{f}(x, u)$ is not $\mu : 1$. By [61], $\mathrm{Bif}(\widetilde{f})$

is a proper analytic subset of $\mathbb{C}^\mu$. Let Δ be a local curve on $\mathbb{C}^\mu$ passing through the origin $\mathbf{0}$ such that $\Delta \bigcap \mathrm{Bif}(\widetilde{f}) = \{\mathbf{0}\}$. We restrict the versal family $\widetilde{F}$ on Δ. Then we have the splitting family which includes the direct Morsification of $(V, \mathbf{0})$.

4. Maximal molecular fiber. If we have a proper splitting $F' \longrightarrow F'_{u,1} + \cdots + F_{u,l}^{prime}$, we say that F'_{u,i_0} $(1 \leq i_0 \leq l)$ is a *proper deformation* of F'. If a germ F cannot be a proper deformation of any germ F' of the same genus, we call F a maximal molecular fiber.

Can we prove the existence of maximal molecular fibers and moreover classify them ?

As an analogy in singularity theory, we consider deformations of a germ (V, P) of a rational double point. Let $G(V, P)$ be the Dynkin diagram of (V, P), that is, the dual graph of the exceptional divisor on its minimal resolution. Then it is well-known that $G(V, P)$ is one of types A_n, D_n, E_6, E_7 and E_8, and that (V, P) is a deformation of another rational double point (V', P') if and only if $G(V, P)$ is a sub-Dynkin diagram of $G(V', P')$ ([4], [21]). Therefore E_8-singularity is the only "maximal molecular singularity" in the category of rational double points.

If we determine the complete systems of atomic fibers of genus g, then we may ask for the classification of maximal molecular fibers of genus g. Especially, the versal family of a maximal molecular fiber seems to be interesting.

5. Monodromy realizable relation. Let $\mathbf{f}' \colon M' \to \Delta'' \times \Delta'''$ be a proper splitting family of genus g of a germ F, and let $\mathcal{D}$ be the discriminant locus of $\mathbf{f}'$. We choose a closed polydisk $\overline{\Delta} \times \overline{\Delta'} \subset \Delta'' \times \Delta'''$ which satisfies $(\partial\overline{\Delta} \times \overline{\Delta'}) \cap \mathcal{D} = \emptyset$, where $\partial\overline{\Delta}$ is the boundary circle of $\overline{\Delta}$. Let $\mathbf{f} \colon \overline{M} \longrightarrow \overline{\Delta} \times \overline{\Delta'}$ be the restriction of $\mathbf{f}'$ over $\overline{\Delta} \times \overline{\Delta'}$. We fix a continuous section $\lambda \colon \overline{\Delta'} \longrightarrow \partial\overline{\Delta} \times \overline{\Delta'}$ with $pr_2 \circ \lambda = id_{\overline{\Delta'}}$. We identify the circle $\partial\overline{\Delta} \times \{u\}$ $(u \in \overline{\Delta'})$ with the oriented simple closed curve $\vec{\partial}_u$ which starts from $\lambda(u)$, goes in counter-clockwise direction and ends with $\lambda(u)$.

We fix $u_0 \in \overline{\Delta'} \backslash \{0\}$. We identify the image $\lambda([0, u_0])$ with the oriented path $\vec{\lambda}$ on $\overline{\Delta} \times \overline{\Delta'}$ which starts from $\lambda(0)$ and ends with $\lambda(u)$. Then two paths $\vec{\partial}_{u_0}$ and $\vec{\lambda} \circ \vec{\partial}_0 \circ \vec{\lambda}^{-1}$ are mutually homotopic on $(\overline{\Delta} \times \overline{\Delta'}) \setminus \mathcal{D}$.

Let $\mathbf{f}_{u_0}^{-1}(d_1), \cdots, \mathbf{f}_{u_0}^{-1}(d_l)$ be the set of all singular fibers of $\mathbf{f}_{u_0} \colon M_{u_0} \to \overline{\Delta} \times \{u_0\}$, i.e. $\mathcal{D} \cap (\overline{\Delta} \times \{u_0\}) = \{d_1, \cdots, d_l\}$. Let $\overline{U}_j$ $(1 \leq j \leq l)$ be a closed connected simple region on $\overline{\Delta} \times \{u_0\}$ which satisfies

(i) $\overline{U}_j \cap \mathcal{D} = \{d_j\}$,
(ii) the boundary $\partial\overline{U}_j$ does not contain d_j,
(iii) $\lambda(u_0) \in \partial\overline{U}_j$.

We identify $\partial\overline{U}_j$ with the oriented simple closed curve $\vec{\gamma}_j$ which starts from $\lambda(u_0)$, goes in counter-clockwise direction and ends with $\lambda(u_0)$. We change the order of $\vec{\gamma}_1, \cdots, \vec{\gamma}_l$ if necessary, then two paths $\vec{\partial}_{u_0}$ and $\vec{\gamma}_l \circ \cdots \circ \vec{\gamma}_1$ are mutually homotopic on $\overline{\Delta} \times \{u_0\} \setminus (\mathcal{D} \cap (\overline{\Delta} \times \{u_0\}))$.

Let $\phi_j \colon \mathbf{f}_{u_0}^{-1}(\lambda(u_0)) \longrightarrow \mathbf{f}_{u_0}^{-1}(\lambda(u_0))$ $(1 \leq j \leq l)$ be the monodromy diffeomorphism of the closed Riemann surface $\mathbf{f}_{u_0}^{-1}(\lambda(u_0))$ of genus g according to the path $\vec{\gamma}_j$. Similarly, let $\phi_0 \colon \mathbf{f}_0^{-1}(\lambda(0)) \longrightarrow \mathbf{f}_0^{-1}(\lambda(0))$ be the diffeomorphism according to $\vec{\partial}_0$, and let $\psi_\lambda \colon \mathbf{f}_{u_0}^{-1}(\lambda(0)) \longrightarrow \mathbf{f}_{u_0}^{-1}(\lambda(u_0))$ be the diffeomorphism according to $\vec{\lambda}$. If we identify these diffeomorphisms with their isotopy classes, we have a relation

$$(**) \qquad \psi_\lambda \circ \phi_0 \circ \psi_\lambda^{-1} = \phi_l \circ \cdots \circ \phi_1$$

in Γ_g. Note that the elements $\phi_0, \phi_1, \cdots, \phi_l$ are contained in the set of pseudo-periodic maps of negative twist $\mathcal{P}_g^-$. We call (**) the *monodromy relation* of $\mathbf{f}$.

Conversely, let $a_0, a_1, \cdots, a_l$ be elements of $\mathcal{P}_g^-$ and b an element of Γ_g which satisfy

$$(***) \qquad b a_0 b^{-1} = a_l \cdots a_1$$

in Γ_g. If this relation coincides with the monodromy relation of a certain splitting family of a certain germ F, we call (***) a *monodromy realizable relation.* Our question is:

Can we characterize intrinsically monodromy realizable relations ?

For example, the Birman-Hilden relation ([19])

$$\iota = \tau_1 \cdots \tau_{2g-2} \tau_{2g-1}^2 \tau_{2g-2} \cdots \tau_1$$

is a monodromy realizable relation, where ι is the hyperelliptic involution and τ_i $(1 \leq i \leq 2g-1)$ is a right-hand full Dehn twist along the i-th curve of the Birman-Hilden base. For the proof, see Matsumoto [63], Example A, Ito [49], or [2], Examples 3.12 and 3.14. (Note that b is an element of the hyperelliptic mapping class group by the construction of [63] and [2]. Hence $b\iota b^{-1} = \iota$.)

By taking the square of both sides of the relation, we get

$$id = \tau_1 \cdots \tau_{2g-1}^2 \cdots \tau_1^2 \cdots \tau_{2g-1}^2 \cdots \tau_1.$$

We claim that this is not a monodromy realizable relation. Indeed, the degeneration with trivial topological monodromy is a topologically trivial degeneration, i.e. the central fiber is a smooth curve. The trivial degeneration has no proper splitting family by the closedness of the discriminant locus.

6. Moduli point and monodromy of a two-parameter degeneration. quad A splitting family $\mathbf{f}\colon M \to \Delta \times \Delta'$ is nothing but a two-parameter degeneration of curves. As an analogy of the case of one-parameter degenerations discussed in §3, we should consider the moduli point and the monodromy of $\mathbf{f}$.

Note that the discriminant locus $\mathcal{D}$ of $\mathbf{f}$ has a bad singularity at the origin $\mathbf{0}$ in general. Therefore the moduli map

$$\alpha_{\mathbf{f}}\colon (\Delta \times \Delta') \setminus \mathcal{D} \longrightarrow \mathcal{M}_g$$

is not necessarily extended to a *morphism* from $\Delta \times \Delta'$ to $\overline{\mathcal{M}}_g$. More precisely, if we fix $u \in \Delta'$, then the one-parameter degeneration $\mathbf{f}_u\colon M_u \longrightarrow \Delta \times \{u\}$ has the extension of the moduli map $\overline{\alpha}_{\mathbf{f}_u} : \Delta \times \{u\} \longrightarrow \overline{\mathcal{M}}_g$ as a morphism. If we define the extension $\overline{\alpha}_{\mathbf{f}}\colon \Delta \times \Delta' \longrightarrow \overline{\mathcal{M}}_g$ of $\alpha_{\mathbf{f}}$ by $\overline{\alpha}_{\mathbf{f}}(t,u) := \overline{\alpha}_{\mathbf{f}_u}(t)$, then the map fails to be continuous at $\mathbf{0}$ in general. We have many examples such that the maps $\overline{\alpha}_{\mathbf{f}}$ "jump" at $\mathbf{0}$. For instance, the family of Example 3.12 in [2] has such a property. Therefore, the moduli point is not well-defined in the usual sense.

Is the space $\overline{\mathcal{M}}_g$ too "narrow" to consider this type of problems ?

We can also consider the topological monodromy

$$\pi_1(\Delta \times \Delta' \setminus \mathcal{D}) \longrightarrow \widehat{\Gamma}_g$$

in a similar way. However, the Matsumoto-Montesinos type arguments seem to be unknown.

7. Global Morsification. We should consider the Morsification problem for global pencils $f\colon S \longrightarrow B$ discussed in §§1, 2. We guess that this may be the original motivation of Xiao-Reid and Horikawa. However, almost nothing seems known.

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