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Local Structure of an Elliptic Fibration

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Abstract.

We classify all the projective elliptic fibrations defined over a unit polydisc whose discriminant loci are contained in a union of coordinate hyperplanes, up to the bimeromorphic equivalence relation. If the monodromies are unipotent and if general singular fibers are not of multiple type, then we can construct relative minimal models.

Contents

§ Introduction

The aim of this paper is to describe the local structure of a projective elliptic fibration over a complex analytic manifold which is a smooth morphism outside a normal crossing divisor of the base manifold. An elliptic fibration is a proper surjective morphism $f\colon X \to S$ of complex analytic varieties whose general fibers are nonsingular elliptic curves. It is not necessarily a flat morphism. We consider the case S is a unit polydisc

$$\Delta^d := \left\{(t_1, t_2, \dots, t_d) \in \mathbb{C}^d \mid |t_i| < 1 \text{ for all } i\right\}$$

and suppose further that f is smooth over $S^\star = S \setminus D$, where D is the normal crossing divisor $D := \{t_1 t_2 \cdots t_l = 0\}$ for some $1 \le l \le d$. We are interested in what kind of such elliptic fibrations exist, up to the bimeromorphic equivalence relation over S. For the purpose, it is important to understand the notion of period mappings and monodromies. A smooth fiber is an elliptic curve isomorphic to a torus $\mathbb{C}/(\mathbb{Z}\omega + \mathbb{Z})$, where the period $\omega \in \mathbb{H} := \{\omega \in \mathbb{Z} \mid \operatorname{Im}\omega > 0\}$ is determined up to the action of $\mathrm{SL}(2, \mathbb{Z})$. By considering the ambiguity, we have a period mapping (function) $\omega \colon U \to \mathbb{H}$ from the universal covering space $U \simeq \mathbb{H}^l \times \Delta^{d-l}$ of $S^\star$ into the upper half plane $\mathbb{H}$, and a monodromy representation $\rho \colon \pi_1(S^\star) \to \mathrm{SL}(2, \mathbb{Z})$ such that for $\gamma \in \pi_1(S^\star)$ and $z \in U$,

$$\omega(\gamma z) = \frac{a_\gamma \omega(z) + b_\gamma}{c_\gamma \omega(z) + d_\gamma}, \quad \text{where} \quad \rho(\gamma) = \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix}.$$

The period function and the monodromy representation define a polarized variation of Hodge structures of rank two and weight one [G] (cf. §1). Thus f induces a variation of Hodge structures $H(f)$ on $S^\star$. We classify all the variations of Hodge structures over $S^\star$ in §2 and §3. After fixing a variation of Hodge structures H, we shall classify elliptic fibrations by determining the following set $\mathcal{E}^+(S, D, H)$: Let $(f \colon X \to S, \phi)$ be a pair of a projective elliptic fibration f smooth over $S^\star$ and an isomorphism $\phi \colon H \simeq H(f)$ as variations of Hodge structures. Two such pairs $(f_1 \colon X_1 \to S, \phi_1)$ and $(f_2 \colon X_2 \to S, \phi_2)$ are called bimeromorphically equivalent over S, if there is a bimeromorphic mapping $\varphi \colon X_1 \cdots\to X_2$ over S such that the induced isomorphism $\varphi^* \colon H(f_2) \simeq H(f_1)$ satisfies $\phi_1 = \varphi^* \circ \phi_2$. The set $\mathcal{E}^+(S, D, H)$ is defined to be the set of bimeromorphic equivalence classes of all such pairs. For any variation of Hodge structures H on $S^\star$, we have a projective elliptic fibration $p \colon B(H) \to S$ with $H \simeq H(p)$ which admits a section $S \to B(H)$. This is uniquely determined up to the bimeromorphic equivalence relation over S and is called the *basic elliptic fibration* associated with H. It determines a distinguished element of $\mathcal{E}^+(S, D, H)$, and thus it is also called the *basic member*. For the study of other elements of $\mathcal{E}^+(S, D, H)$, we first consider a special case where the following two conditions are satisfied:

- The monodromy matrices $\rho(\gamma)$ are all unipotent;
- The fibration f admits a meromorphic section over a neighborhood of any point of $S \setminus Z$ for a Zariski closed subset Z of codimension greater than one.

Under the situation, our Theorems 4.3.1 and 4.3.2 state that f is bimeromorphically equivalent to a basic elliptic fibration. Further the basic elliptic fibration is a smooth elliptic fibration or a *toric model* which is constructed by the method of toroidal embedding theory ([KKMS]).

These are minimal elliptic fibrations. Next, for a general elliptic fibration $f\colon X \to S$, we have a finite ramified covering of the form

$$\tau\colon T = \Delta^l \times \Delta^{d-l} \longrightarrow S = \Delta^l \times \Delta^{d-l}$$
$$(\theta_1, \theta_2, \ldots, \theta_l, t_{l+1}, t_{l+2}, \ldots, t_d) \mapsto (\theta_1^{m_1}, \theta_2^{m_2}, \ldots, \theta_l^{m_l}, t_{l+1}, t_{l+2}, \ldots, t_d)$$

for some $m_i \geq 1$ such that $f_T\colon X \times_S T \to T$ satisfies the above two conditions. Hence f_T is bimeromorphically equivalent to $p_T\colon B(H) \times_S T \to T$. Therefore the bimeromorphic equivalence class of f is determined by a meromorphic action of the Galois group $\mathrm{Gal}(\tau)$ on p_T. The basic elliptic fibration p is a group object over $S^\star$. Hence the sheaf $\mathfrak{S}_{H/S}$ of germs of meromorphic sections of p is a sheaf of abelian groups. Since we always fix a marking of variation of Hodge structures, the action of an element of the Galois group is written as the translation by a meromorphic section of p_T. Therefore, $\mathcal{E}^+(S, D, H)$ is identified with the inductive limit of Galois cohomology groups

$$\varinjlim H^1(\mathrm{Gal}(\tau), H^0(T, \mathfrak{S}_{H_T/T})),$$

where H_T is the pullback of H on $T^\star := \tau^{-1}(S^\star)$ and the limit is taken over all such coverings τ described as above. We shall describe the limits and the actions of Galois groups on basic elliptic fibrations in §§5–7.

Background

The study of elliptic fibration was developed by Kodaira's work on elliptic surfaces, i.e., elliptic fibrations over curves, in [Kd1] and [Kd2].

In the work, first the classification of singular fibers of minimal elliptic fibrations is obtained by a calculation of intersection numbers of irreducible components. The following types of singular fibers are listed (cf. Figure 3 and Figure 4): ${}_m\mathrm{I}_a$, I_a^*, II, II*, III, III*, IV, IV*, where $a \geq 0$ and $m \geq 1$.

Next, the basic elliptic fibration is constructed from the data of period function and monodromy representation defined on a Zariski-open subset of the base curve, which were essentially called *functional* and *homological* invariants, respectively. The construction is natural over the Zariski-open subset. To obtain an extension of the basic fibration to the whole base curve, we may assume that the curve is a unit disc Δ and the Zariski-open subset is the punctured disc $\Delta^\star = \Delta \setminus \{0\}$. If the monodromy matrix is trivial, then the period mapping is single-valued and thus the smooth basic fibration is naturally extended. In the case the monodromy is unipotent of infinite order, i.e., I_a, $(a > 0)$, Kodaira made a technical construction of the basic fibration. But now it can be replaced by the method of toroidal embedding theory ([KKMS]). For other cases of monodromy matrices, a branched covering $\Delta \to \Delta$ reduces to the study of actions of the Galois group on the above basic fibrations. The quotient spaces, their desingularizations and further their (relative) minimal models are obtained by careful calculations. The result corresponds to the classification of non-multiple singular fibers. Kodaira proved that every elliptic surface without multiple fibers is a twist of the basic member by the translation by local sections. Thus the set of such fibrations is essentially identified with the cohomology group $H^1(S, \mathfrak{S})$, where S is the base curve and $\mathfrak{S}$ is the sheaf of germs of sections of the basic fibration.

For general elliptic fibrations, Kodaira showed in [Kd2] that every multiple fibers are obtained from an elliptic surface without multiple fibers by *logarithmic transformations.*

His theory contained not only local but also global properties of elliptic fibrations. This was generalized to the study of degenerations of abelian varieties, where a particular open subset of the basic elliptic fibration is considered as the *Néron model.* In the purely algebraic situation, birational equivalence classes are determined only by smooth parts, or more strictly, by generic fibers. Hence the study of multiple fibers is replaced by that of Galois cohomology groups $H^1(\mathrm{Gal}(L/K), E(L))$, where E is an elliptic curve with origin (hence is fixed a group structure) over a field K, L/K is a Galois extension, $E(L)$ is the group of L-valued points.

In the analytic situation, Kawai ([Kwi]) succeeded in generalizing the construction of basic members to the case of elliptic fibrations over surfaces, where the resulting ambient spaces were not necessarily nonsingular. Ueno ([U]) obtained their desingularizations, which however are not distinguished models in their bimeromorphic equivalence classes. To obtain a good model, we had to wait the development of the minimal model theory.

Their basic members were also determined by functional and homological invariants. Now we know that giving these invariants is equivalent to giving a polarized variation of Hodge structures of weight one and rank two (cf. [G]). This is also equivalent to giving a *Weierstrass model* [Ny4]. It was proved that every elliptic fibration admitting a section is bimeromorphically equivalent to a Weierstrass model. Before [Ny4], Miranda ([Mi]) studied the desingularizations of Weierstrass models over surfaces, where he obtained flat minimal models after changing the base surface by blow-ups.

Compared with the progress in the study of elliptic fibration admitting a global section, few results were known for general elliptic fibrations. For example, some interesting examples are found in the case multiple fibers appear. Especially, Fujimoto ([Fm]) constructed them by a generalization of logarithmic transformation. Some of them induce examples of deformations of complex manifolds under which the plurigenera are not invariant.

The minimal model theory of projective varieties (cf. [KMM]) together with its generalization to complex analytic varieties [Ny3, §4] allows us to study the minimality of elliptic fibrations. For the classification of elliptic fibrations, it is essential to determine the relative minimal models. Since Mori ([Mo]) has proved the three-dimensional flip conjecture, there exist relative minimal models for a given projective elliptic fibration over a surface. These minimal models usually have terminal singularities and are not uniquely determined in their bimeromorphic equivalence classes. However, every two bimeromorphically equivalent minimal models are connected by a sequence of flops [Kw4] and [Kl2]. We have studied elliptic fibrations over surfaces by applying the minimal model theory in [Ny5], whose Main Theorem corresponds to Theorems 4.3.1 and 4.3.2.

Previous version

The author intended to write this paper as "Elliptic fibrations over surfaces II," that is a continuation of [Ny5]. He considered the cases of non-unipotent monodromies and of multiple fibers, by taking a suitable finite Kummer covering $\Delta^2 \to \Delta^2$. The study was reduced to that of Galois actions on special basic fibrations. The classification of the actions was to be the contents of "Part II." But a few months later, the author obtained a generalization of Main Theorem of [Ny5] to the higher dimensional case. The three-dimensional flip theorem ([Mo]) was essential in the proof in [Ny5]. He found a new idea to prove it without using the flip theorem. By the progress, the classification of actions of covering groups is also extended to higher dimensional case. This is essentially reduced to calculating Galois cohomology groups. The first version [Ny7] appeared in a preprint series of Department of Mathematics, Faculty of Science, University of Tokyo in 1991.

The construction of the first version is as follows: §§1–3 are devoted to some basics on elliptic fibrations. The basic properties on period functions for smooth elliptic fibrations are explained in §1. Especially, variations of Hodge structures, basic elliptic fibrations and their torsors are discussed. In §2, monodromy representations over a product of punctured discs are studied. The types of monodromies are classified into: I_0, $\mathrm{I}_0^{(*)}$, $\mathrm{II}^{(*)}$, $\mathrm{III}^{(*)}$, $\mathrm{IV}_+^{(*)}$, $\mathrm{IV}_-^{(*)}$, $\mathrm{I}_{(+)}$, $\mathrm{I}_{(+)}^{(*)}$ (cf. Table 2). The sets of smooth elliptic fibrations over the base with a fixed variation of Hodge structures are calculated in each type. Thus all the smooth elliptic fibrations over the product of punctured discs are described. However, the calculation of Galois cohomology groups contains some errors. The canonical extension of the variation of Hodge structures to $S = \Delta^d$ is explained in §3. We have some results on locally projective or Kähler elliptic fibrations from fundamental isomorphisms Corollary 3.2.1 for direct image sheaves of canonical sheaves and from torsion free theorems for the higher direct image sheaves. Examples of non-Kähler elliptic fibrations are given. In §4, toric models are constructed, which are basic elliptic fibrations corresponding to variations of Hodge structures with non-trivial but only unipotent monodromies. These are given by the method of toroidal embedding theory. Similar constructions appeared in the study of degeneration of abelian varieties (cf. [Nk], [Nm]). The last part of §4 is devoted to proving the main results Theorems 4.3.1 and 4.3.2, which are generalizations of Main Theorem in [Ny5]. In §5, elliptic fibrations over curves are studied from a viewpoint of toric models. In §6, the case of finite monodromies is studied and possible elliptic fibrations are described as the quotient space of basic smooth fibration by an action of Galois group. In §7, the case of infinite monodromies are treated. However, the calculation of some Galois cohomology groups in the case of $\mathrm{I}^{(*)}_{(+)}$ is not clearly mentioned. It had two appendices, where elliptic fibrations over surfaces are studied by the method of minimal model theory. In Appendix A, the study of elliptic fibrations over a surface is shown to be reduced, in some sense, to that of *standard* elliptic fibrations. They are relative minimal fibrations with only equi-dimensional fibers and satisfy more conditions. In Appendix B, the good minimal model conjecture is proved for compact Kähler threefolds admitting elliptic fibrations. This is a generalization of the unpublished paper [Ny6].

Present version

The author left the first version untouched about five years. In the period, he received a paper [DG] of Dolgachev and Gross, where elliptic fibrations over surfaces are studied in the purely algebraic situation. For the use of étale cohomology theory, they looked carefully at the models obtained by Miranda ([Mi]) and calculated similar Galois cohomology groups. In our first version, the author did not understand the importance of describing the groups. In the study of the groups, he found a kind of generalization of étale cohomology theory, by which we can consider global structures of elliptic fibrations in the analytic situation. This is named the ∂-étale cohomology theory and is written in [Ny8] in 1996. The results on Galois cohomology groups in this paper are also derived from [Ny8], since the structure of $\mathcal{E}^+(S, D, H)$ is studied in more general case. Under the influence of [Ny8], the author decided to write a new version of this paper. The preparation however has been slow.

The major difference between previous and present versions is as follows: §0 is added. Here an elementary descent theory, G-linearization, and torsors are discussed. §§0–3 are still preliminary sections. In §2, we change the base space $S^\star$ to be a product of punctured discs and polydiscs, i.e., $S^\star = (\Delta^\star)^l \times \Delta^{d-l}$. We divide the case $\mathrm{I}^{(*)}_{(+)}$ into three subcases (cf. Table 3). By a similar method to [Ny8], we calculate the related group cohomologies in each type. Similarly to the previous version, all the smooth elliptic fibrations over $S^\star$ are described. In §3, we explain more on the canonical extension of a variation of Hodge structures of rank two, weight one defined on $S^\star$ to $S := \Delta^d$. In particular, we determine possible period functions. In §4, we add a discussion on a kind of generalization of torsors in §4.1, which are torsors in a sense of bimeromorphic geometry. It is important, since our basic fibrations are not group objects, but have group structures in meromorphic sense. We also give an extension Theorem 4.1.1 of smooth projective elliptic fibrations. The description of toric models in §§4.2 and 4.3 are essentially same as before, except the following two things:

- The proof of Proposition 4.2.12 is replaced. Original argument is combinatorial and the new one is an application of the theory of elliptic surfaces.
- Another proof of Corollary 4.3.3 is added, in which the toric models are not used. This is based on an argument of Viehweg in [V, 9.10].

§§5–7 are devoted to the calculation of the set $\mathcal{E}^+(S, D, H)$ and the description of any projective elliptic fibrations over $S = \Delta^d$ with discriminant locus D. In §5, we consider not only the case S is a curve but the case $l = 1$, i.e., the discriminant locus D is a smooth divisor. We have unique minimal models in this case. We treat the case H has a finite monodromy group in §6 and the remaining case in §7. The calculation is much simpler than that in the previous version. In Appendix B, B.8 is corrected.

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Notation

We use the same notation as in [Ny3], [Ny4], and [KMM], and need the following in addition.

Complex analytic space: We treat only complex analytic spaces which are Hausdorff and have countable open bases. A complex analytic variety means an irreducible and reduced complex analytic space. A complex analytic manifold means a nonsingular complex analytic variety. Every complex analytic manifolds should be connected.

Polydisc: Let Δ^d be the d-dimensional unit polydisc

$$\{(t_1, t_2, \dots, t_d) \in \mathbb{C}^d \mid |t_i| < 1 \text{ for } 1 \le i \le d\}$$

with respect to a coordinate system $(t_1, t_2, \dots, t_d)$. The coordinate hyperplane $\{t_i = 0\}$ is often denoted by D_i. We denote by $\Delta^\star$ the punctured disc $\Delta \setminus \{0\}$. Thus $(\Delta^\star)^l \times \Delta^{d-l} \simeq \Delta^d \setminus \bigcup_{i=1}^l D_i$.

Exponential mapping: We denote the function $\exp(2\pi\sqrt{-1}z)$ by $e(z)$ for $z \in \mathbb{C}$. The universal covering space of the punctured disc $\Delta^\star$ is isomorphic to the upper half plane $\mathbb{H} := \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$. The function $z \mapsto e(z)$ induces a universal covering mapping $\mathbb{H} \to \Delta^\star$.

Pullback of open subsets: Let $f\colon V \to W$ be a morphism of complex analytic spaces. For an open subset $U \subset W$, we shall denote the pullback $f^{-1}(U)$ by $V_{|U}$.

Morphisms over a fixed base space: Let $f\colon X \to S$ and $g\colon Y \to S$ be morphisms of complex analytic spaces. A morphism

$h\colon X \to Y$ is called a morphism over S, if $f = g \circ h$. A complex analytic space over S is a morphism $f\colon X \to S$ from a complex analytic space.

Duals: Dual objects are indicated by $\vee$. For example, we denote by $\mathcal{F}^\vee$ the dual $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$ for an $\mathcal{O}_X$-module $\mathcal{F}$ of a complex analytic space X.

Elimination: For a sequence of letters $a_1, a_2, \ldots, a_n$ and for $1 \leq i \leq n$, if we eliminate a_i, then we denote the remaining sequence by $a_1, a_2, \ldots, \widehat{a_i}, \ldots, a_n$.

Special complex numbers: We write $\boldsymbol{\omega} := e(2\pi\sqrt{-1}/3)$ and $\boldsymbol{i} := \sqrt{-1}$.

Fibrations and Projective morphisms: A proper surjective morphism $f\colon X \to S$ of complex analytic varieties is called a *fibration* if X and S are normal and if any fibers of f are connected. A proper morphism f is called a *projective* morphism if there exists an f-ample line bundle (invertible sheaf) on X (cf. [Ny3]). $f\colon X \to S$ is called a *locally projective morphism*, if there is an open covering $\bigcup S_\lambda = S$ such that $X_{|S_\lambda} \to S_\lambda$ is a projective morphism for any λ. Note that the composite of two projective morphisms is not always projective. This is only a locally projective morphism. The composite of two locally projective morphisms is not always locally projective.

Minimal models: A fibration $f\colon X \to S$ is said to be a *minimal* fibration (or a *minimal model*) over a point $P \in S$, if the following conditions are all satisfied:

(1) f is a (locally) projective morphism;
(2) X has only terminal singularities;
(3) X is $\mathbb{Q}$-factorial over P (cf. [Ny3, §4]);
(4) the canonical divisor (class) K_X of X is f-nef over P, i.e., the intersection numbers $(K_X \cdot C) \geq 0$ for any irreducible curves C such that $f(C) = P$.

Although, sometimes, a fibration $f\colon X \to S$ is said to be a minimal fibration even if it does not satisfy the condition (3).

Elliptic fibrations: A fibration $f\colon X \to S$ is called an *elliptic fibration* if general fibers are nonsingular elliptic curves. In this paper, we shall treat mainly the projective elliptic fibrations (cf. §3.3). If $\dim S = 1$, every elliptic fibration is a locally projective morphism. But there is an elliptic fibration over Δ^2 whose central fiber is a Hopf surface (cf. Examples 3.3.4 and 3.3.5).

Sections: Let $f\colon X \to Y$ be a proper surjective morphism between complex analytic varieties. A closed subvariety $\Sigma \subset X$ is called a *section* of f if f induces an isomorphism $\Sigma \simeq Y$. If $\Sigma \to Y$ is a bimeromorphic morphism, then Σ is called by a *meromorphic section.* Furthermore a morphism $\sigma\colon Y \to X$ such that $f \circ \sigma = \mathrm{id}_Y$ is also called a section of f.

Variation of Hodge structures: (cf. §1.1.) Since we consider elliptic fibrations, we treat variations of Hodge structures only of rank two and of weight one. Further we always assume such variation of Hodge structures admits a polarization defined over $\mathbb{Z}$.

§0. Descent theory

0.1. General situation

Let X be a complex analytic space and let $\sigma\colon G \times X \to X$ be a left action of a discrete group G. Suppose that the action is properly discontinuous. For the quotient morphism $\tau\colon X \to Y := G\backslash X$, there is a canonical morphism $G \times X \ni (g, x) \mapsto (gx, x) \in X \times_Y X$. It is an isomorphism if the action is fixed point free. For a complex analytic spaces Z, let $F(Z)$ be one of the following categories:

(1) The category of sheaves of abelian groups over Z;

(2) The category of complex analytic spaces over Z.

Then we have pullback functors $\tau^*\colon F(Y) \to F(X)$, $\sigma^*, p_2^*\colon F(X) \to F(G \times X)$, and

$$p_{31}^*, p_{32}^*, p_{21}^*\colon F(G \times X) \to F(G \times G \times X),$$

where $p_2\colon G \times X \to X$ denotes the second projection and p_{31}, p_{32}, p_{21} the morphisms $G \times G \times X \to G \times X$ defined by

$$p_{31}\colon (g, h, x) \mapsto (gh, x),\ p_{32}\colon (g, h, x) \mapsto (h, x),\ p_{21}\colon (g, h, x) \mapsto (g, hx).$$

Suppose that there is an isomorphism $\psi\colon \xi \simeq \tau^*\eta$ for objects $\xi \in F(X)$ and $\eta \in F(Y)$. Then we have a natural isomorphism

$$\phi := \phi(\eta, \psi) := p_2^*(\psi)^{-1} \circ \sigma^*(\psi)\colon \sigma^*\xi \to \sigma^*\tau^*\eta = p_2^*\tau^*\eta \to p_2^*\xi,$$

which satisfies the following cocycle condition:

$$p_{31}^*(\phi) = p_{32}^*(\phi) \circ p_{21}^*(\phi). \tag{0.1}$$

Definition 0.1.1. A pair (ξ, ϕ) consisting of an object $\xi \in F(X)$ and an isomorphism $\phi\colon \sigma^*\xi \to p_2^*\xi$ satisfying the cocycle condition (0.1), is called a *G-equivariant object* of $F(X)$. A morphism $f\colon (\xi_1, \phi_1) \to (\xi_2, \phi_2)$ is defined to be a morphism $f\colon \xi_1 \to \xi_2$ in $F(X)$ such that $\phi_1 \circ p_2^*(f) = \phi_2 \circ \sigma^*(f)$. We denote by $F(X, G)$ the category of G-equivariant objects of $F(X)$.

Let us denote by $L_g: X \to X$ the action of $g \in G$. We can identify L_g with the composite $X = \{g\} \times X \subset G \times X \xrightarrow{\sigma} X$. For an isomorphism $\phi: \sigma^*\xi \to p_2^*\xi$ in $F(X)$ and for an element $g \in G$, let ϕ_g be the morphism $\phi_g = \phi_{|\{g\}\times X}: L_g^*\xi \to \xi$. Then the cocycle condition (0.1) is equivalent to

$$\phi_{gh} = \phi_h \circ L_h^*(\phi_g) \tag{0.2}$$

for any $g, h \in G$. Thus ϕ of a G-equivariant object (ξ, ϕ) is determined by the collection $\{\phi_g\}$ satisfying (0.2).

The natural functor $\tau^*: F(Y) \to F(X)$ factors through $F(X, G) \to F(X)$. As in the usual descent theory, we have the following:

Lemma 0.1.2. *Suppose that the action of G on X is free. Then the natural functor $\tau^*: F(Y) \to F(X, G)$ gives an equivalence of categories. That is, for a G-equivariant object $(\xi, \phi) \in F(X, G)$, there exist an object $\eta \in F(Y)$ and an isomorphism $\psi: \xi \simeq \tau^*\eta$ such that $\phi = \phi(\eta, \psi)$, and furthermore, the pair (η, ψ) is uniquely determined up to the following equivalence relation: $(\eta, \psi) \sim (\eta', \psi')$ if and only if there is an isomorphism $\theta: \eta \to \eta'$ such that $\psi' = \tau^*(\theta) \circ \psi$.*

For two G-equivariant objects (ξ_1, ϕ_1) and (ξ_2, ϕ_2) of $F(X)$, the set $\mathrm{Hom}_{F(X)}(\xi_1, \xi_2)$ of morphisms admits a right action of G as follows: For $g \in G$ and a morphism $f: \xi_1 \to \xi_2$,

$$f^g := \phi_{2,g} \circ L_g^*(f) \circ \phi_{1,g}^{-1}: \xi_1 \xrightarrow{\phi_{1,g}^{-1}} L_g^*\xi_1 \xrightarrow{L_g^*(f)} L_g^*\xi_2 \xrightarrow{\phi_{2,g}} \xi_2.$$

Similarly, for a G-equivariant object (ξ, ϕ), we have a right action of G on the automorphism group $\mathrm{Aut}_{F(X)}(\xi)$.

Lemma 0.1.3. *Let (ξ, ϕ) be a G-equivariant object of $F(X)$. Then the set of isomorphism classes of G-equivariant objects of the form (ξ, ϕ') is identified with the cohomology set $H^1(G, \mathrm{Aut}_{F(X)}(\xi))$, where the action of G on $\mathrm{Aut}_{F(X)}(\xi)$ is determined by ϕ.*

Proof. Let $\phi'_g: L_g^*\xi \to \xi$ be the restriction of ϕ' to $\{g\} \times X$ and set $\rho(g) := \phi'_g \circ \phi_g^{-1} \in \mathrm{Aut}_{F(X)}(\xi)$. Then for $g, h \in G$, we have

$$\rho(gh) = \rho(h) \circ \phi_h \circ L_h^*(\rho(g)) \circ \phi_h^{-1} = \rho(h) \circ \rho(g)^h.$$

Thus $\{\rho(g)\}$ defines a cocycle in $Z^1(G, \mathrm{Aut}_{F(X)}(\xi))$. Conversely, for a cocycle $\{\rho(g)\}$, the collection $\{\phi'_g := \rho(g) \circ \phi_g\}$ defines an isomorphism ϕ' satisfying the cocycle condition (0.1). Suppose that two cocycles $\{\rho_1(g)\}$ and $\{\rho_2(g)\}$ define two isomorphisms $\phi_1, \phi_2: \sigma^*\xi \to p_2^*\xi$, respectively.

Then (ξ, ϕ_1) is isomorphic to (ξ, ϕ_2) in $F(X, G)$ if and only if $\{\rho_1(g)\}$ and $\{\rho_2(g)\}$ are cohomologous. Q.E.D.

Corollary 0.1.4. *Suppose that the action of G on X is free. Let η be an object of $F(Y)$. Then the set of isomorphism classes of $\eta' \in F(Y)$ admitting an isomorphism $\tau^*\eta \simeq \tau^*\eta'$ is identified with the cohomology set $H^1(G, \mathrm{Aut}(\tau^*\eta))$.*

0.2. G-linearization

Let X, Y, G be same as before. We shall recall the notion of G-linearization (cf. [Mu1]). For a sheaf $\mathcal{F}$ of abelian groups on X, a G-linearization is an isomorphism $\phi\colon \sigma^{-1}\mathcal{F} \to p_2^{-1}\mathcal{F}$ satisfying the cocycle condition $p_{31}^*(\phi) = p_{32}^*(\phi) \circ p_{21}^*(\phi)$. Therefore this is the case $F(Z)$ is the category of sheaves of abelian groups on Z. For two G-linearized sheaves $(\mathcal{F}_1, \phi_1)$ and $(\mathcal{F}_2, \phi_2)$, the tensor product $\mathcal{F}_1 \otimes \mathcal{F}_2$ has a G-linearization $\phi_1 \otimes \phi_2$. A G-linearization ϕ on the sheaf $\mathcal{H}om(\mathcal{F}_1, \mathcal{F}_2)$ is given by

$$\phi\colon \mathcal{H}om(\sigma^{-1}\mathcal{F}_1, \sigma^{-1}\mathcal{F}_2) \ni \alpha \mapsto \phi_2 \circ \alpha \circ \phi_1^{-1} \in \mathcal{H}om(p_2^{-1}\mathcal{F}_1, p_2^{-1}\mathcal{F}_2).$$

As in §0.1, from a G-linearization on $\mathcal{F}$, we have a right action of G on the set $H^0(X, \mathcal{F}) = \mathrm{Hom}_X(\mathbb{Z}_X, \mathcal{F})$. This is called the dual action of G in [Mu1]. Therefore, the direct image sheaf $\tau_*\mathcal{F}$ also admits the right action of G. Let $\mathcal{G}$ be the G-invariant part of $\tau_*\mathcal{F}$, i.e., $\mathcal{G} := \mathcal{H}om_{\mathbb{Z}_Y[G]}(\mathbb{Z}_Y, \tau_*\mathcal{F})$. If the action of G is free, then there is an isomorphism $\mathcal{F} \simeq \tau^{-1}\mathcal{G}$ by Lemma 0.1.2. The set of isomorphism classes of G-linearizations of $\mathcal{F}$ is identified with the cohomology set $H^1(G, \mathrm{Aut}(\mathcal{F}))$ by Lemma 0.1.3. The cohomology groups $H^p(X, \mathcal{F}) \simeq H^p(Y, \tau_*\mathcal{F})$ have also right G-module structures, since so does $\tau_*\mathcal{F}$. Here we recall the following:

Lemma 0.2.1 (Hochschild–Serre spectral sequence). *Suppose that the action of G on X is free. Let $\mathcal{G}$ be a sheaf of abelian groups on Y. Then there is a spectral sequence:*

$$E_2^{p,q} = H^p(G, H^q(X, \tau^{-1}(\mathcal{G}))) \Longrightarrow H^{p+q}(Y, \mathcal{G}).$$

In particular, if $H^i(X, \tau^{-1}(\mathcal{G})) = 0$ for any $i > 0$, then, for all p, we have an isomorphism

$$H^p(G, H^0(X, \tau^{-1}(\mathcal{G}))) \simeq H^p(Y, \mathcal{G}).$$

Next we shall consider the case $F(Z)$ is the category of sheaves of $\mathcal{O}_Z$-modules in §0.1, where $\mathcal{O}_Z$ denotes the structure sheaf. The $\mathcal{O}_X$

has a natural G-linearization which is explicitly written as follows: The isomorphisms $\phi_g\colon L_g^{-1}\mathcal{O}_X \simeq \mathcal{O}_X$ are given by

$$H^0(gU, \mathcal{O}_X) \ni f \mapsto f^g \in H^0(U, \mathcal{O}_X),$$

where $U \subset X$ is an open subset and $f^g(z) := f(gz)$ for $z \in U$. A G-linearization of an $\mathcal{O}_X$-module $\mathcal{F}$ is called $\mathcal{O}_X$-linear if the multiplication $\mathcal{O}_X \times \mathcal{F} \to \mathcal{F}$ is compatible with G-linearizations. Then we have the category of G-linearized $\mathcal{O}_X$-modules. This is identified with the category of $\mathcal{O}_Y$-modules when G acts on X freely. For a G-linearized $\mathcal{O}_X$-module $\mathcal{F}$, the set of isomorphism classes of $\mathcal{O}_X$-linear G-linearizations is identified with the cohomology set $H^1(G, \mathrm{Aut}_{\mathcal{O}_X}(\mathcal{F}))$ by Lemma 0.1.3.

Next we shall consider a special case. Let M be an abelian group. Suppose that X is connected and that there is a G-linearization ϕ on the constant sheaf $M_X := M \otimes \mathbb{Z}_X$ which is different from the trivial G-linearization induced from $M_Y := M \otimes \mathbb{Z}_Y$. The ϕ corresponds to an element of $H^1(G, \mathrm{Aut}(M_X))$ by Lemma 0.1.3. Since X is connected, we have $\mathrm{Aut}(M_X) = \mathrm{Aut}(M)$ and thus G acts trivially on $\mathrm{Aut}(M)$. Therefore, the cohomology set $H^1(G, \mathrm{Aut}(M_X))$ is identified with the set $\mathrm{Hom}_{\mathrm{anti}}(G, \mathrm{Aut}(M))$ of anti-group homomorphisms from G to $\mathrm{Aut}(M)$. Thus M has a right G-module structure, which is nothing but the right module structure of $M = H^0(X, M_X)$ induced from ϕ. If M has a right G-module structure, then there is uniquely a group homomorphism $\rho\colon G \to \mathrm{Aut}(M)$ such that $x^g = \rho(g^{-1})(x)$ for any $x \in M$. The tensor product $M \otimes_{\mathbb{Z}} \mathcal{O}_X$ has a natural G-linearization induced from $M_X \simeq \tau^{-1}M_Y$ and from the natural G-linearization of $\mathcal{O}_X$. We have another G-linearization of $M \otimes_{\mathbb{Z}} \mathcal{O}_X$ from ϕ above. Thus we have:

Lemma 0.2.2. *Suppose that X is connected and let M be an abelian group. Then the set of G-linearizations of the constant sheaf M_X is identified with the set* $\mathrm{Hom}(G, \mathrm{Aut}(M))$ *of group homomorphisms. For a homomorphism $\rho\colon G \to \mathrm{Aut}(M)$, the corresponding G-linearizations of M_X and $M \otimes_{\mathbb{Z}} \mathcal{O}_X$, respectively, are given in the following way*:

$$H^0(gU, M_X) = M \ni x \mapsto \rho(g^{-1})(x) \in M = H^0(U, M_X),$$
$$H^0(gU, M \otimes \mathcal{O}_X) \ni v \mapsto v^g \in H^0(U, M \otimes \mathcal{O}_X),$$

where $U \subset X$ is a connected open subset, $g \in G$ and for $z \in U$,

$$v^g(z) := \rho(g^{-1})v(gz).$$

Suppose further that Y is a connected analytic space and let H be a *locally constant sheaf* with fiber M, i.e., H is isomorphic to the constant sheaf M_Y locally on Y, and that $\tau\colon X \to Y$ is the universal covering space. Then G is isomorphic to the fundamental group $\pi_1(Y, y)$ for a point $y \in Y$ and acts on X freely. Thus there exist an isomorphism $\psi\colon M_X \simeq \tau^{-1}H$ and also a G-linearization $\phi = \phi(H, \psi)$ on M_X. Hence we have a group homomorphism $\rho\colon G = \pi_1(Y, y) \to \mathrm{Aut}(M)$, which is called the *monodromy representation* of H.

0.3. Torsors

Still let X, Y, G be same as in §0.1. We shall consider the case $F(Z)$ is the category of complex analytic spaces over Z. Let $f\colon W \to X$ be a morphism of complex analytic spaces. Suppose that there is an isomorphism $\phi\colon \sigma^*(W) := (G \times X) \times_X W \to p_2^*(W) := (G \times X) \times_X W$ over $G \times X$ satisfying the cocycle condition (0.1). Then the restriction of ϕ to $\{g\} \times X$ defines an isomorphism $\phi_g\colon L_g^* W \simeq W$. These ϕ_g satisfy the cocycle condition (0.2). From ϕ_g, we have the following commutative diagram:

$$\begin{array}{ccccc} W & \xrightarrow{\phi_g^{-1}} & L_g^* W & \longrightarrow & W \\ & & {\scriptstyle L_g^*(f)}\downarrow & & \downarrow{\scriptstyle f} \\ & & X & \xrightarrow[L_g]{} & X. \end{array}$$

Let $\varphi(g)\colon W \to W$ be the composite of the morphisms appearing at the top sequence in the diagram above. Then we have $\varphi(gh) = \varphi(g) \circ \varphi(h)$. Therefore G acts holomorphically on W from the left and the action is compatible with $W \to X$. Therefore we have the quotient space $V = G\backslash W$ over Y. If G acts on X freely, then so on W. Hence $W \to V$ is étale and $W \simeq V \times_Y X$, in the case.

Next we shall consider a special case. Suppose that the action of G on X is free. Let $B \to Y$ be an analytic space over Y admitting a group structure, i.e., the functor $Z \mapsto \mathrm{Hom}_Y(Z, B)$ from the category of complex analytic spaces over Y to the category of sets factors through the category of groups. Thus the set $B(X/Y) := \mathrm{Hom}_Y(X, B)$ is considered as the group of sections of $B_X := B \times_Y X \to X$. We have a right action of G on the group $B(X/Y) = \mathrm{Hom}_X(Y_X, B_X)$ by §0.1. There is an injection $B(X/Y) \ni \sigma \mapsto \mathrm{tr}(\sigma) \in \mathrm{Aut}_X(B_X)$, where $\mathrm{tr}(\sigma)$ is the left multiplication mapping

$$B_X = B \times_Y X \ni (b, x) \mapsto (\sigma(x)b, x) \in B_X.$$

This injection is a G-linear group homomorphism, i.e, $\mathrm{tr}(\sigma^g) = \mathrm{tr}(\sigma)^g$ for $g \in G$ and $\mathrm{tr}(\sigma_1\sigma_2) = \mathrm{tr}(\sigma_1) \circ \mathrm{tr}(\sigma_2)$. A cocycle $\{\sigma_g\}$ in $Z^1(G, B(X/Y))$

defines an element of $H^1(G, \mathrm{Aut}_X(B_X))$ and determines a smooth morphism $V \to Y$ from the quotient space $V := G\backslash B_X$ by the action:

$$B_X \ni (b, x) \mapsto (\sigma_g(x)b, gx) \in B_X.$$

Then $B \to Y$ acts on $V \to Y$ from right by:

$$V \times_Y B \ni ([b, x], b') \mapsto [bb', x] \in V,$$

where $[b, x]$ denotes the image of $(b, x) \in B_X = B \times_Y X$ under the quotient morphism $B_X \to V$. Furthermore we have a B_X-linear isomorphism $B_X \simeq V_X$.

Definition 0.3.1. A smooth morphism $V \to Y$ is called a *torsor* of $B \to Y$ if $B \to Y$ acts on $V \to Y$ from the right and there exist an open covering $\{Y_\lambda\}$ of Y and B-linear isomorphisms $B_{|Y_\lambda} \simeq V_{|Y_\lambda}$.

The set of isomorphism classes of torsors of $B \to Y$ whose pullbacks to X are trivialized is identified with the cohomology set $H^1(G, B(X/Y))$ by Lemma 0.1.3. The set of isomorphism classes of torsors of $B \to Y$ itself is identified with the cohomology set $H^1(Y, \mathcal{O}(B/Y))$, where $\mathcal{O}(B/Y)$ is the sheaf of germs of sections of $B \to Y$, i.e., $H^0(U, \mathcal{O}(B/Y)) = B(U/Y)$ for open subsets $U \subset Y$. Therefore we have an injection

$$H^1(G, B(X/Y)) \hookrightarrow H^1(Y, \mathcal{O}(B/Y)).$$

As an analogy of Lemma 0.2.1, we see that the injection is extended to a sequence:

$$H^1(G, B(X/Y)) \to H^1(Y, \mathcal{O}(B/Y)) \to H^1(X, \mathcal{O}(B_X/X)),$$

which is exact in the following sense: If an element of $H^1(Y, \mathcal{O}(B/Y))$ goes to the trivial element in $H^1(X, \mathcal{O}(B_X/X))$, then it comes from $H^1(G, B(X/Y))$.

We can also consider similar things in the case the action of G is not necessarily free. But for the resulting quotient space V, the induced morphism $V \to Y$ is not necessarily a smooth morphism. We can also consider the case that $B \to Y$ has only a meromorphic group structure and the group G is finite. By replacing $B(X/Y)$ by a group of meromorphic sections of $B_X \to X$, we obtain a meromorphic action of G on B_X from an element of $H^1(G, B(X/Y))$. Since G is finite, we have a meromorphic quotient V (up to the bimeromorphic equivalence relation) of B_X by the action.

§1. Smooth elliptic fibrations

1.1. Variation of Hodge structures of rank two and weight one

An elliptic curve C is isomorphic to a complex torus $\mathbb{C}/L$, where $L = L_\omega = \mathbb{Z} + \mathbb{Z}\omega$ for some $\omega \in \mathbb{H}$. Under a natural isomorphism $\pi_1(C) \simeq H_1(C, \mathbb{Z}) \simeq L$, we have the following two loops γ_1 and γ_0 of C corresponding to ω and 1 in L, respectively:

$$\gamma_1 \colon [0,1] \ni t \mapsto t\omega \in \mathbb{C}, \quad \gamma_0 \colon [0,1] \ni t \mapsto t \in \mathbb{C}.$$

For the coordinate z of $\mathbb{C}$, $\mathrm{d}z$ defines a holomorphic 1-form on C. Further $H^1(C, \mathbb{C})$ is spanned by the cohomology classes of $\mathrm{d}z$ and $\mathrm{d}\bar{z}$. The Hodge decomposition $H^1(C, \mathbb{C}) = H^{1,0} \oplus H^{0,1}$ is given by $H^{1,0} = \mathbb{C}\,\mathrm{d}z$ and $H^{0,1} = \mathbb{C}\,\mathrm{d}\bar{z}$. Let (e_1, e_0) be the dual base of $H^1(C, \mathbb{Z})$ to (γ_1, γ_0). Then $\mathrm{d}z = e_0 + \omega e_1$ in $H^1(C, \mathbb{C})$, since

$$\int_{\gamma_0} \mathrm{d}z = 1 \quad \text{and} \quad \int_{\gamma_1} \mathrm{d}z = \omega.$$

Let $\bigwedge^2 H^1(C, \mathbb{Z}) \simeq H^2(C, \mathbb{Z}) \to \mathbb{Z}$ be an isomorphism sending $e_0 \wedge e_1$ to 1. Let $Q \colon H^1(C, \mathbb{Z}) \times H^1(C, \mathbb{Z}) \to \mathbb{Z}$ be the induced skew symmetric bilinear form. Then

$$\int_C \frac{\sqrt{-1}}{2} \mathrm{d}z \wedge \mathrm{d}\bar{z} = \frac{\sqrt{-1}}{2} Q(\mathrm{d}z, \mathrm{d}\bar{z}) = \operatorname{Im} \omega.$$

Let $H_1(C, \mathbb{Z}) \to (H^{1,0})^\vee = \operatorname{Hom}(H^{1,0}, \mathbb{C}) \simeq \mathbb{C}$ be the homomorphism given by the integral

$$\gamma \mapsto \int_\gamma \mathrm{d}z.$$

We see that the induced homomorphism $H_1(C, \mathbb{C}) \to (H^{1,0})^\vee$ is dual to the injection $H^{1,0} \to H^1(C, \mathbb{C})$. Moreover we have a commutative diagram of exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^{1,0} & \longrightarrow & H^1(C, \mathbb{C}) & \longrightarrow & H^{0,1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow q & & \downarrow & & \\
0 & \longrightarrow & (H^{0,1})^\vee & \longrightarrow & H_1(C, \mathbb{C}) & \longrightarrow & (H^{1,0})^\vee & \longrightarrow & 0,
\end{array}$$

where $q \colon H^1(C, \mathbb{C}) \to H_1(C, \mathbb{C})$ is the isomorphism by Poincaré duality determined by Q, explicitly by $q(e_0) = \gamma_1$ and $q(e_1) = -\gamma_0$.

A polarized Hodge structure $H = (H, Q, F^\bullet)$ of rank two and of weight one is defined to be the following data (cf. [D], [G]):

(1) A free $\mathbb{Z}$-module H of rank two;
(2) A skew symmetric bilinear form $Q\colon H \times H \to \mathbb{Z}$ inducing an isomorphism $\bigwedge^2 H \simeq \mathbb{Z}$;
(3) A descending filtration of vector subspaces of $H_{\mathbb{C}} := H \otimes \mathbb{C}$:

$$0 = F^2(H_{\mathbb{C}}) \subset F^1(H_{\mathbb{C}}) \subset F^0(H_{\mathbb{C}}) = H_{\mathbb{C}}$$

satisfying the following conditions:
(a) $\dim F^1(H_{\mathbb{C}}) = 1$;
(b) $F^1(H_{\mathbb{C}}) \oplus \overline{F^1(H_{\mathbb{C}})} = H_{\mathbb{C}}$, where $\overline{F^1(H_{\mathbb{C}})}$ denotes the complex conjugate;
(c) $\sqrt{-1}Q(x, \overline{x}) > 0$ for any nonzero element x of $F^1(H_{\mathbb{C}})$.

The Q is called the polarization of H and $\{F^p\}$ is called the Hodge filtration. The condition (c) is called the Hodge–Riemann bilinear relation. For the elliptic curve C above, the data $(H^1(C, \mathbb{Z}), Q, F^1 = H^{1,0})$ form a polarized Hodge structure of rank two and of weight one. Conversely, any polarized Hodge structure of rank two and of weight one defines an elliptic curve inducing the same Hodge structure.

Let S be a complex analytic variety. A polarized variation of Hodge structures $H = (H, Q, \mathcal{F}^\bullet)$ of rank two and weight one over S is defined to be the following data (cf. [D], [G]):

(1) A locally constant sheaf H with fiber $\mathbb{Z}^{\oplus 2}$;
(2) A skew symmetric bilinear form $Q\colon H \times H \to \mathbb{Z}_S$ inducing an isomorphism $\bigwedge^2 H \simeq \mathbb{Z}_S$;
(3) A descending sequence of holomorphic subbundles:

$$0 = \mathcal{F}^2(\mathcal{H}) \subset \mathcal{F}^1(\mathcal{H}) \subset \mathcal{F}^0(\mathcal{H}) = \mathcal{H} := H \otimes \mathcal{O}_S,$$

where the restriction $(H_s, Q_s, \mathcal{F}^\bullet \otimes \mathbb{C}(s))$ to the fiber over any point $s \in S$ forms a polarized Hodge structure of rank two and of weight one.

Note that the Griffiths transversality condition is satisfied automatically in this case.

Example 1.1.1. Let $f\colon X \to S$ be a smooth elliptic fibration, i.e., a smooth proper surjective morphism with elliptic curves as fibers. Then $H := R^1 f_* \mathbb{Z}_X$ is a locally constant sheaf with fiber $\mathbb{Z}^{\oplus 2}$. The cup product $R^1 f_* \mathbb{Z}_X \times R^1 f_* \mathbb{Z}_X \to R^2 f_* \mathbb{Z}_X$ and the trace map $R^2 f_* \mathbb{Z}_X \simeq \mathbb{Z}_S$ define a skew symmetric bilinear form Q on H. Let

$$0 \to f^{-1}\mathcal{O}_S \to \mathcal{O}_X \stackrel{\mathrm{d}_{X/S}}{\to} \Omega^1_{X/S} \to 0$$

be the relative Poincaré exact sequence. By taking higher direct images, we have an exact sequence:

$$0 \to f_*\Omega^1_{X/S} \to R^1 f_* f^{-1}\mathcal{O}_S \simeq H \otimes \mathcal{O}_S \to R^1 f_* \mathcal{O}_X \to 0.$$

Let $\mathcal{F}^1(\mathcal{H})$ be the subbundle $f_*\Omega^1_{X/S}$ of $\mathcal{H} := H \otimes \mathcal{O}_S$. Then the conditions (*a*), (*b*), (*c*) above are satisfied on each fiber. Thus we have a variation of Hodge structures of weight one and rank two from a smooth elliptic fibration.

Let H be a variation of Hodge structures of rank two and weight one whose local constant system H is trivial. Then we can choose a base (e_0, e_1) of $H^0(S, H)$ so that $Q(e_0, e_1) = 1$. Denoting $\mathcal{L}_H := \mathcal{H}/\mathcal{F}^1(\mathcal{H})$, we have a surjection $r\colon \mathcal{O}_S^{\oplus 2} \simeq \mathcal{H} \to \mathcal{L}_H$. The sections $r(e_0)$ and $r(e_1)$ of $\mathcal{L}_H$ are nowhere vanishing. We then define a function by

$$\omega(z) := -\frac{r(e_0)}{r(e_1)}$$

for $z \in S$. The Hodge subbundle $\mathcal{F}^1(\mathcal{H})$ is generated by $\omega(z)e_1 + e_0$. Hence the Hodge–Riemann bilinear relation implies that $\operatorname{Im}\omega(z) > 0$, i.e., ω is a mapping into the upper half plane $\mathbb{H}$. Let $(e_0^\sharp, e_1^\sharp)$ be another base of $H^0(S, H)$ with $Q(e_0^\sharp, e_1^\sharp) = 1$. Then

$$(e_1, e_0) = (e_1^\sharp, e_0^\sharp)\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

for a matrix in $\mathrm{SL}(2, \mathbb{Z})$. Let $\omega^\sharp(z) = -r(e_0^\sharp)/r(e_1^\sharp)$ be the similarly defined function. Since $\omega^\sharp(z)e_1^\sharp + e_0^\sharp$ is also a generator of $\mathcal{F}^1(\mathcal{H})$, there is a nowhere vanishing holomorphic function $u(z)$ such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \omega(z) \\ 1 \end{pmatrix} = u(z)\begin{pmatrix} \omega^\sharp(z) \\ 1 \end{pmatrix}.$$

Thus $u(z) = c\omega(z) + d$ and $\omega^\sharp(z) = (a\omega(z) + b)/(c\omega(z) + d)$.

Definition 1.1.2. The $\omega(z)$ is called the period function.

Next suppose further that there is a properly discontinuous action of a discrete group Γ on S and that the variation of Hodge structures H admits a Γ-linearization. This means that the locally constant system H and Hodge filtrations $\mathcal{F}^\bullet(\mathcal{H})$ admit compatible Γ-linearizations which preserve the polarization Q. For the right Γ-module structure of $H^0(S, H)$, we have a group homomorphism $\rho\colon \Gamma \to \operatorname{Aut}(H^0(S, H))$ such that $x^\gamma = \rho(\gamma)^{-1}x$ for $x \in H^0(S, H)$. Since Q is preserved, we have a matrix

$$\rho(\gamma) = \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix}$$

in $\mathrm{SL}(2, \mathbb{Z})$ such that $(e_1^\gamma, e_0^\gamma) = (e_1, e_0)\rho(\gamma)^{-1}$. We shall write an element of $H^0(S, H) = \mathbb{Z}e_1 \oplus \mathbb{Z}e_0$ by a column vector ${}^{\mathrm{t}}(x, y)$ consisting of integers which corresponds to $xe_1 + ye_0$. Let $H^\vee$ be the dual locally constant system $\mathcal{H}om(H, \mathbb{Z}_S)$ and let $(e_0^\vee, e_1^\vee)$ be the dual base to (e_0, e_1). We identify a row vector (m, n) consisting of integers with $me_1^\vee + ne_0^\vee$ in $H^0(S, H^\vee)$. Since $(m, n) \cdot {}^{\mathrm{t}}(x, y)^\gamma = (m, n)\rho(\gamma^{-1})\, {}^{\mathrm{t}}(x, y)$, the right Γ-module structure of $H^0(S, H^\vee)$ is described by $(m, n)^\gamma = (m, n)\rho(\gamma)$. Let $q\colon H \to H^\vee$ be the isomorphism defined by $q(x)(y) = Q(x, y)$ for $x, y \in H$. Then we have $q(e_0) = e_1^\vee$, and $q(e_1) = -e_0^\vee$. More explicitly, we have

$$q\begin{pmatrix} x \\ y \end{pmatrix} = (x, y)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Thus q is compatible with Γ-linearizations, since

$${}^{\mathrm{t}}\rho(\gamma^{-1}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \rho(\gamma) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{-1}.$$

We shall also write an element of $H^0(S, \mathcal{H})$ by a column vector

$$\boldsymbol{v}(z) = \begin{pmatrix} f(z) \\ g(z) \end{pmatrix}$$

consisting of global holomorphic functions which corresponds to $f(z)e_1 + g(z)e_0$. Then the right Γ-module structure of $H^0(S, \mathcal{H})$ is given by $\boldsymbol{v}(z)^\gamma = \rho(\gamma)^{-1}\boldsymbol{v}(\gamma z)$. Since $\mathcal{F}^1(\mathcal{H})$ is generated by $\omega(z)e_1 + e_0$, for each $\gamma \in \Gamma$, we have

$$\rho(\gamma)\begin{pmatrix} \omega(z) \\ 1 \end{pmatrix} = (c_\gamma \omega(z) + d_\gamma)\begin{pmatrix} \omega(\gamma z) \\ 1 \end{pmatrix}.$$

In particular, we have

$$\omega(\gamma z) = \frac{a_\gamma \omega(z) + b_\gamma}{c_\gamma \omega(z) + d_\gamma}. \tag{1.1}$$

Now we have the following commutative diagram of exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F}^1(\mathcal{H}) & \longrightarrow & \mathcal{H} & \longrightarrow & \mathcal{L}_H & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ 0 & \longrightarrow & \mathcal{O}_S & \xrightarrow{\ i\ } & \mathcal{O}_S^{\oplus 2} & \xrightarrow{\ p\ } & \mathcal{O}_S & \longrightarrow & 0, \end{array}$$

where i and p are defined by:

$$i\colon 1 \mapsto \begin{pmatrix} \omega(z) \\ 1 \end{pmatrix} \quad \text{and} \quad p\colon \begin{pmatrix} \alpha(z) \\ \beta(z) \end{pmatrix} \mapsto \alpha(z) - \omega(z)\beta(z).$$

There is a Γ-linearization on $\mathcal{O}_S \simeq \mathcal{L}_H$. The induced right action of Γ on $H^0(S, \mathcal{O}_S)$ is described as follows: For a holomorphic function $f(z)$ on S and for $\gamma \in \Gamma$,

$$f^\gamma(z) := (c_\gamma \omega(z) + d_\gamma) f(\gamma z).$$

The homomorphism $\mathcal{H} \to \mathcal{L}_H$ is isomorphic to the dual of $\mathcal{F}^1(\mathcal{H}) \to \mathcal{H}$. Thus the composite

$$H^\vee \xrightarrow{q^{-1}} H \otimes \mathcal{O}_S \to \mathcal{L}_H$$

is induced from the following Γ-linearized homomorphism:

$$H^\vee \simeq \mathbb{Z}_S^{\oplus 2} \ni (m, n) \mapsto m\omega(z) + n \in \mathcal{O}_S.$$

Next, we consider a polarized variation of Hodge structures H of rank two and of weight one on a complex analytic variety S whose local constant system is not necessarily constant. Let $\tau\colon U \to S$ be the universal covering mapping. Then $\tau^{-1}H = (\tau^{-1}H, Q, \tau^*\mathcal{F}^\bullet(\mathcal{H}))$ is a variation of Hodge structures with a trivial locally constant system. We have an action of the fundamental group $\Gamma = \pi_1(S, s)$ for a point $s \in S$ on U and a Γ-linearization on the variation of Hodge structures $\tau^{-1}H$. Thus by the previous argument, we have a period function $\omega(z)$ for $z \in U$ and a monodromy representation $\rho\colon \Gamma \to \mathrm{SL}(2, \mathbb{Z})$ satisfying (1.1). Let $\mathcal{L}_H$ denote the quotient $\mathcal{H}/\mathcal{F}^1(\mathcal{H})$. Then the homomorphism $\tau^{-1}H \to \tau^*\mathcal{L}_H$ is isomorphic to

$$\mathbb{Z}_U^{\oplus 2} \ni (m, n) \mapsto m\omega(z) + n \in \mathcal{O}_U.$$

Here the right actions of $\gamma \in \Gamma$ on $H^0(U, \mathbb{Z}_U^{\oplus 2})$ and $H^0(U, \mathcal{O}_U)$ are given by:

$$(m, n) \mapsto (m, n)\rho(\gamma) \quad \text{and} \quad f(z) \mapsto f^\gamma(z) := (c_\gamma w(z) + d_\gamma) f(\gamma z).$$

Therefore a polarized variation of Hodge structures of rank two and of weight one is determined by a monodromy representation $\rho\colon \pi_1(S, s) \to \mathrm{SL}(2, \mathbb{Z})$ and a period function $\omega\colon U \to \mathbb{H}$ satisfying the condition (1.1).

1.2. Smooth basic elliptic fibrations

We shall define the *basic elliptic fibration* associated with the variation of Hodge structures H on S. Let $\mathbb{V}(H) := \mathbb{V}(\mathcal{L}_H) \to S$ be the holomorphic line bundle associated with the invertible sheaf $\mathcal{L}_H$. For the injection $H \hookrightarrow \mathcal{L}_H$, we have a corresponding subspace $\mathbb{L}(H) \subset \mathbb{V}(H)$ étale over S. Locally on S, $\mathbb{V}(H) \simeq S \times \mathbb{C}$ and $\mathbb{L}(H) \simeq S \times \mathbb{Z}^2$. Since $\mathbb{L}(H)$ is a discrete subgroup of $\mathbb{V}(H)$ over S, we can define the quotient $B(H) := \mathbb{V}(H)/\mathbb{L}(H)$. This is also described in the following way: Let $\omega(z)$ for $z \in U$ and $\rho(\gamma)$ for $\gamma \in \Gamma = \pi_1(S, s)$, respectively, be the period function and the monodromy representation defined as before. For $\gamma \in \Gamma$ and $(m, n) \in \mathbb{Z}^{\oplus 2}$, we define an automorphism $\Phi(\gamma, (m, n))$ of $U \times \mathbb{C}$ by:

$$U \times \mathbb{C} \ni (z, \zeta) \mapsto \left(\gamma z, \frac{\zeta + m\omega(z) + n}{c_\gamma \omega(z) + d_\gamma}\right).$$

Then for any $\gamma_1, \gamma_2 \in \Gamma$ and $(m_1, n_1), (m_2, n_2) \in \mathbb{Z}^{\oplus 2}$, we obtain

$$\Phi(\gamma_1, (m_1, n_1)) \circ \Phi(\gamma_2, (m_2, n_2)) = \Phi(\gamma_1\gamma_2, (m_3, n_3)),$$
$$\text{where} \quad (m_3, n_3) = (m_2, n_2) + (m_1, n_1)\rho(\gamma_2).$$

Thus the semi-direct product $\Gamma \ltimes \mathbb{Z}^{\oplus 2}$ acts on $U \times \mathbb{C}$ from the left. Since this action is properly discontinuous and fixed point free, we have the quotient variety $B(H)$ smooth over S. By the argument of §1.1, the quotient of $U \times \mathbb{C}$ by the subgroup $\Gamma \ltimes 0$ is isomorphic to $\mathbb{V}(H)$. Therefore, we have an elliptic fibration $p \colon B(H) \to S$ canonically from H. The zero section of $\mathbb{V}(H) \to S$ defines a section $\sigma_0 \colon S \to B(H)$. Note that $R^1 p_* \mathbb{Z}_{B(H)} \simeq H$ as variations of Hodge structures. By the construction, $p \colon B(H) \to S$ has a group structure whose zero section is σ_0.

Definition 1.2.1 (cf. [Kd1]). The elliptic fibration $p \colon B(H) \to S$ is said to be the *smooth basic elliptic fibration* associated with the polarized variation of Hodge structures H.

Let $\sigma \colon S \to B(H)$ be another section of p. Since $p \colon B(H) \to S$ has a group structure, we have the translation morphism $\mathrm{tr}(\sigma) \colon B(H) \to B(H)$ over S. Then $\mathrm{tr}(\sigma)$ preserves the variation of Hodge structures H, i.e.,

$$\mathrm{tr}(\sigma)^* \colon R^1 p_* \mathbb{Z}_{B(H)} \to R^1 p_* \mathbb{Z}_{B(H)}$$

is the identity mapping. Conversely, we have the following:

Lemma 1.2.2. *Let $\varphi \colon B(H) \to B(H)$ be an automorphism over S which induces the identity on $H = R^1 p_* \mathbb{Z}_{B(H)}$. Then $\varphi = \mathrm{tr}(\sigma)$ for a section $\sigma \colon S \to B(H)$.*

Proof. Let $\sigma\colon S \to B$ be the composite of the zero section $\sigma_0\colon S \to B$ and $\varphi\colon B \to B$. Then the composite of φ and the inverse of the translation $\mathrm{tr}(\sigma)$ also induces the identity on $R^1p_*\mathbb{Z}_B$. Thus it is enough to prove that φ is the identity morphism provided that φ preserves the zero section. We see that this should be an identity on any fiber, by a property of automorphisms of elliptic curves. Q.E.D.

Some properties on morphisms of elliptic curves are generalized to:

Lemma 1.2.3.

(1) *Let H_1 and H_2 be two variations of Hodge structures of weight one and rank two over S and let $\varphi\colon B(H_1) \to B(H_2)$ be a morphism over S. Then $\varphi = \mathrm{tr}(\sigma) \circ \phi$ for the translation morphism $\mathrm{tr}(\sigma)$ by a section $\sigma\colon S \to B(H_2)$ and a group homomorphism $\phi\colon B(H_1) \to B(H_2)$ over S.*

(2) *Let $\phi\colon B(H) \to B(H)$ be an automorphism over S preserving the zero section. Then the order of ϕ is finite and is one of $\{1, 2, 3, 4, 6\}$.*

Proposition 1.2.4 (cf. [Kd1]). *Let $f\colon X \to S$ be a smooth elliptic fibration of complex analytic varieties such that $H \simeq R^1f_*\mathbb{Z}_X$ as variations of Hodge structures. Assume that f admits a section $\sigma\colon S \to X$. Then there exists an isomorphism $h\colon X \to B(H)$ over S such that $h \circ \sigma = \sigma_0$.*

Proof. Let $\Delta_X \subset X \times_S X$ be the diagonal locus, $\Sigma := \sigma(S) \subset X$, p_1, p_2 the first and the second projections, respectively, and let $\Sigma_X := p_2^{-1}(\Sigma) \subset X \times_S X$. We consider the invertible sheaf

$$\mathcal{N} := \mathcal{O}_{X\times_S X}(\Delta_X - \Sigma_X).$$

Then for any $x \in X$, we have an isomorphism

$$\mathcal{N}_{|p_1^{-1}(x)} \simeq \mathcal{O}_{f^{-1}(f(x))}([x] - [\sigma(f(x))]),$$

where $[x]$ denotes the prime divisor supported at x on the elliptic curve $f^{-1}(f(x))$. Let c be the image of $\mathcal{N}$ under the natural homomorphism

$$H^1(X \times_S X, \mathcal{O}^\star_{X\times_S X}) \to H^0(X, R^1p_{1*}\mathcal{O}^\star_{X\times_S X}).$$

We shall also consider the following exact sequence induced from the exponential sequence on $X \times_S X$:

$$0 \to R^1p_{1*}\mathbb{Z}_{X\times_S X} \to R^1p_{1*}\mathcal{O}_{X\times_S X} \to R^1p_{1*}\mathcal{O}^\star_{X\times_S X} \to \\ \to R^2p_{1*}\mathbb{Z}_{X\times_S X} \simeq \mathbb{Z}_X.$$

We infer that $R^1p_{1*}\mathbb{Z}_{X\times_S X} \simeq f^{-1}H$, $R^1p_{1*}\mathcal{O}_{X\times_S X} \simeq f^*\mathcal{L}_H$, and that the natural inclusion $H \hookrightarrow \mathcal{L}_H$ determined by the variation of Hodge structures induces the injection $R^1p_{1*}\mathbb{Z}_{X\times_S X} \to R^1p_{1*}\mathcal{O}_{X\times_S X}$ above. Let $\mathfrak{S}$ be the cokernel of $f^{-1}H \hookrightarrow f^*\mathcal{L}_H$. Then $c \in H^0(X, \mathfrak{S})$. Since $f^*\mathcal{L}_H \to \mathfrak{S}$ is surjective, we have an open covering $\{X_\lambda\}$ of X and sections $\alpha_\lambda \in H^0(X_\lambda, f^*\mathcal{L}_H)$ such that $c_{|X_\lambda}$ is the image of α_λ. Then

$$\alpha_{\lambda|X_\lambda\cap X_\mu} - \alpha_{\mu|X_\lambda\cap X_\mu} \in H^0(X_\lambda \cap X_\mu, f^{-1}H).$$

The α_λ defines a morphism $h_\lambda\colon X_\lambda \to \mathbb{V}(H)$ over S. Further $h_\lambda(x) - h_\mu(x) \in \mathbb{L}(H)$ for $x \in X_\lambda \cap X_\mu$. Therefore we have a global morphism $h\colon X \to B(H) = \mathbb{V}(H)/\mathbb{L}(H)$ over S. By construction, h does not depend on the choices of open coverings $\{X_\lambda\}$ and sections $\{\alpha_\lambda\}$.

We shall show that $h(\Sigma)$ coincides with the zero section of $B(H) \to S$. By considering the restrictions to $\Sigma \simeq S$ of $f^{-1}H$, $f^*\mathcal{L}_H$, $\mathfrak{S}$, and $R^1p_{1*}\mathcal{O}^\star_{X\times_S X}$, we have the following commutative diagram:

$$\begin{array}{ccc} H^0(X,\mathfrak{S}) & \longrightarrow & H^0(X, R^1p_{1*}\mathcal{O}^\star_{X\times_S X}) \\ \downarrow & & \downarrow \\ H^0(S, \mathcal{L}_H/H) & \longrightarrow & H^0(S, R^1f_*\mathcal{O}^\star_X). \end{array}$$

The both horizontal homomorphisms are injective. From an isomorphism $\mathcal{N}_{|\Sigma\times_S X} \simeq \mathcal{O}_X$ and the commutative diagram

$$\begin{array}{ccc} H^1(X\times_S X, \mathcal{O}^\star_{X\times_S X}) & \longrightarrow & H^1(\Sigma\times_S X, \mathcal{O}^\star_{\Sigma\times_S X}) \\ \downarrow & & \downarrow \\ H^0(X, R^1p_{1*}\mathcal{O}^\star_{X\times_S X}) & \longrightarrow & H^0(S, R^1f_*\mathcal{O}^\star_X), \end{array}$$

we infer that the image of c in $H^0(S, \mathcal{L}_H/H)$ is zero. Thus $h(\Sigma)$ coincides with the zero section.

Finally, we shall prove that h is an isomorphism. We have only to check it on each fiber of $X \to S$. The restriction of h to a fiber $E := f^{-1}(P)$ is essentially isomorphic to:

$$E \ni x \mapsto \mathcal{O}([x] - [\sigma(P)]) \in \mathrm{Pic}^0(E).$$

Therefore this is an isomorphism. Q.E.D.

We thus obtain a one to one correspondence between the set of isomorphism classes of smooth basic elliptic fibrations and that of polarized variations of Hodge structures of rank two, weight one over S. Next, we shall relate them with Weierstrass models [MS], [Ny4]. Let

$(\mathcal{L}, \alpha, \beta)$ be a triplet consisting of an invertible sheaf $\mathcal{L}$ on S and sections $\alpha \in H^0(S, \mathcal{L}^{\otimes(-4)})$, $\beta \in H^0(S, \mathcal{L}^{\otimes(-6)})$ such that $0 \neq 4\alpha^3 + 27\beta^2 \in H^0(S, \mathcal{L}^{\otimes(-12)})$. For the $\mathbb{P}^2$-bundle $p \colon \mathbb{P} := \mathbb{P}(\mathcal{O}_S \oplus \mathcal{L}^{\otimes 2} \oplus \mathcal{L}^{\otimes 3}) \to S$, let $\mathcal{O}(1)$ be the tautological line bundle. According to the natural inclusions

$$\mathcal{O}_S \hookrightarrow \mathcal{O}_S \oplus \mathcal{L}^{\otimes 2} \oplus \mathcal{L}^{\otimes 3}, \ \mathcal{L}^{\otimes 2} \hookrightarrow \mathcal{O}_S \oplus \mathcal{L}^{\otimes 2} \oplus \mathcal{L}^{\otimes 3}, \ \mathcal{L}^{\otimes 3} \hookrightarrow \mathcal{O}_S \oplus \mathcal{L}^{\otimes 2} \oplus \mathcal{L}^{\otimes 3},$$

we have sections $Z \in H^0(\mathbb{P}, \mathcal{O}(1))$, $X \in H^0(\mathbb{P}, \mathcal{O}(1) \otimes p^*(\mathcal{L}^{\otimes(-2)}))$, and $Y \in H^0(\mathbb{P}, \mathcal{O}(1) \otimes p^*(\mathcal{L}^{\otimes(-3)}))$. Then $Y^2Z - (X^3 + \alpha XZ^2 + \beta Z^3)$ is a global section of $\mathcal{O}(3) \otimes p^*\mathcal{L}^{\otimes(-12)}$. The zero locus of the section is called the Weierstrass model and denoted by $W_S(\mathcal{L}, \alpha, \beta)$. The section $\{X = Z = 0\}$ of $\mathbb{P} \to S$ is contained in $W_S(\mathcal{L}, \alpha, \beta)$, which is called the canonical section.

Fact 1.2.5 ([MS]). Let $f \colon X \to S$ be a smooth elliptic fibration admitting a section $\sigma \colon S \to X$. Then there exist a triplet $(\mathcal{L}, \alpha, \beta)$ and an isomorphism $\mu \colon X \to W_S(\mathcal{L}, \alpha, \beta)$ over S such that $\mu \circ \sigma$ is the canonical section.

In this case, $\mathcal{L} \simeq R^1 f_* \mathcal{O}_X$ and the discriminant $4\alpha^3 + 27\beta^2$ is a nowhere vanishing section. Therefore the following three sets can be identified to each other:

- The set of isomorphism classes of variations of Hodge structures of weight one and rank two over S;
- The set of isomorphism classes of smooth basic elliptic fibrations over S;
- The set of triplets $(\mathcal{L}, \alpha, \beta)$ as above with $4\alpha^3 + 27\beta^2$ nowhere vanishing, modulo the following equivalence relation: $(\mathcal{L}, \alpha, \beta) \sim (\mathcal{L}', \alpha', \beta')$ if and only if there is a nowhere vanishing section $\varepsilon \in H^0(S, \mathcal{L}' \otimes \mathcal{L}^{\otimes(-1)})$ such that $\alpha = \varepsilon^4 \alpha'$ and $\beta = \varepsilon^6 \beta'$.

Remark 1.2.6. Let us consider the case $S = \mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ and $\omega(z) = z$ for $z \in \mathbb{H}$. Then ω defines a variation of Hodge structures and the corresponding smooth basic elliptic fibration over $\mathbb{H}$ is sometimes called the "universal" elliptic fibration. By the theory of Weierstrass' $\wp$-function, this is isomorphic to the Weierstrass model

$$W_{\mathbb{H}}(\mathcal{O}_{\mathbb{H}}, \boldsymbol{\alpha}, \boldsymbol{\beta}) = \{\, ((X : Y : Z), z) \in \mathbb{P}^2 \times \mathbb{H} \mid \\ Y^2 Z = X^3 + \boldsymbol{\alpha}(z) X Z^2 + \boldsymbol{\beta}(z) Z^3 \},$$

where $\boldsymbol{\alpha}(z) := -15 G_4(z)$, $\boldsymbol{\beta}(z) := -35 G_6(z)$, and $G_k(z)$ is the Eisenstein series

$$G_k(z) := \sum_{(0,0) \neq (m,n) \in \mathbb{Z}^{\oplus 2}} (mz + n)^{-k}$$

of weight k. The following facts are known.

(1) $4\boldsymbol{\alpha}(z)^3 + 27\boldsymbol{\beta}(z)^2 = -(1/16)\boldsymbol{\Delta}(z)$, where $\boldsymbol{\Delta}(z)$ is the cusp form of weight 12 of the modular group $\mathrm{SL}(2,\mathbb{Z})$ represented by

$$\boldsymbol{\Delta}(z) = (2\pi)^{12} q \prod_{\nu=1}^{\infty} (1 - q^{\nu})^{24}$$

for $q = \exp(2\pi\sqrt{-1}z)$.

(2) The $\mathrm{SL}(2,\mathbb{Z})$-invariant function

$$\boldsymbol{j}(z) := \frac{4\boldsymbol{\alpha}(z)^3}{4\boldsymbol{\alpha}(z)^3 + 27\boldsymbol{\beta}(z)^2}$$

is called the *elliptic modular function* and induces an isomorphism $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H} \simeq \mathbb{C}$.

(3) The function $\boldsymbol{j}(z) - 12^{-3}q^{-1}$ is a holomorphic function near $q = 0$.

Definition 1.2.7. Let H be a polarized variation of Hodge structures of rank two weight one. The *J-function* of H is defined to be $J(t) := \boldsymbol{j}(\omega(z))$, where $\tau(z) = t \in S$. The J-function of a smooth elliptic fibration $X \to S$ should be the J-function of the corresponding polarized variation of Hodge structures.

In particular, the J-function of a smooth Weierstrass model $W(\mathcal{L}, \alpha, \beta) \to S$ is given by $J(t) = 4\alpha^3/(4\alpha^3 + 27\beta^2)$.

In papers [Kd1] and [Kwi], the J-function is called the *functional invariant* and the monodromy representation of H (or that restricted to the Zariski-open subset $\{t \in S\colon J(t) \neq 0, 1\}$) is called the *homological invariant.* Here a period function ω is a multi-valued analytic function satisfying $\boldsymbol{j}(\omega) = J$ and the condition (1.1). A pair consisting of such a period function and a monodromy representation is called a *characteristic pair* in [U].

1.3. General smooth elliptic fibrations

Let $\mathfrak{S}_H$ be the sheaf of germs of sections of the smooth basic elliptic fibration $p\colon B := B(H) \to S$. Then this is a sheaf of abelian groups. From the surjection $\mathbb{V}(H) \to B(H)$, we have the following exponential exact sequence (cf. [Kd1]):

$$0 \to H \to \mathcal{L}_H \to \mathfrak{S}_H \to 0.$$

For $\eta \in H^1(S, \mathfrak{S}_H)$, we can define a torsor $B(H)^{\eta} \to S$ of $p\colon B \to S$. By a similar argument to [Kd1], we can prove the following:

Proposition 1.3.1 (cf. [Kd1, 10.1]). *Any smooth elliptic fibration $f\colon X \to S$ with an isomorphism $R^1f_*\mathbb{Z}_X \simeq H$ is isomorphic to $B(H)^\eta \to S$ for some $\eta \in H^1(S, \mathfrak{S}_H)$.*

Proof. Since f is smooth, we have an open covering $\{S_\lambda\}_{\lambda\in\Lambda}$ of S and sections $S_\lambda \to X_{|U_\lambda}$. Therefore there exist isomorphisms $\phi_\lambda\colon X_{|U_\lambda} \to B_{|U_\lambda}$ by Proposition 1.2.4. Here we may assume that the induced isomorphisms $\phi_\lambda^*\colon (R^1p_*\mathbb{Z}_B)_{|U_\lambda} \to (R^1f_*\mathbb{Z}_X)_{|U_\lambda}$ are glued to the given isomorphism $H \simeq R^1f_*\mathbb{Z}_X$. Let us consider the composites $\varphi_{\lambda,\mu} := (\phi_\lambda \circ \phi_\mu^{-1})_{|U_\lambda\cap U_\mu}$. Then $\varphi_{\lambda,\mu}$ induces the identity on $(R^1p_*\mathbb{Z}_B)_{|U_\lambda\cap U_\mu}$. Thus by Lemma 1.2.2, there exists a section $\eta_{\lambda,\mu}$ such that $\varphi_{\lambda,\mu}$ is the translation morphism $\mathrm{tr}(\eta_{\lambda,\mu})$. Since $\varphi_{\lambda,\mu}\circ\varphi_{\mu,\nu}\circ\varphi_{\nu,\lambda}$ is identical over $U_\lambda\cap U_\mu\cap U_\nu$ for $\lambda,\mu,\nu \in \Lambda$, we have $\eta_{\lambda,\mu}+\eta_{\mu,\nu}+\eta_{\nu,\lambda} = 0$ over $U_\lambda\cap U_\mu\cap U_\nu$. Therefore $f\colon X \to S$ is isomorphic to $B(H)^\eta$ for $\eta = \{\eta_{\lambda,\mu}\}_{\lambda,\mu\in\Lambda}$. Q.E.D.

We shall explain more about the cohomology class η. For a smooth elliptic fibration $f\colon X \simeq B(H)^\eta \to S$, let us consider the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R^1f_*\mathbb{Z}_X & \longrightarrow & R^1f_*\mathcal{O}_X & \longrightarrow & R^1f_*\mathcal{O}_X^\star & \longrightarrow & R^2f_*\mathbb{Z}_X \\ & & \| & & \| & & & & \\ 0 & \longrightarrow & H & \longrightarrow & \mathcal{L}_H & \longrightarrow & \mathfrak{S}_H & \longrightarrow & 0. \end{array}$$

Then we have a homomorphism $\Phi_X\colon \mathfrak{S}_H \to R^1f_*\mathcal{O}_X^\star$ such that the sequence

$$(1.2) \qquad 0 \to \mathfrak{S}_H \overset{\Phi_X}{\to} R^1f_*\mathcal{O}_X^\star \to R^2f_*\mathbb{Z}_X \simeq \mathbb{Z}_S \to 0$$

is exact. We have the following description of Φ_X: Let $\Sigma_0 := \sigma_0(S) \subset B(H)$ be the zero section and let Σ be a section of $p\colon B(H)_{|\mathcal{U}} \to \mathcal{U}$ for an open subset $\mathcal{U} \subset S$. We shall take an open covering $\{U_\lambda\}$ of S and isomorphisms $\phi_\lambda\colon X_{|U_\lambda} \to B(H)_{|U_\lambda}$ as in the proof of Proposition 1.3.1. Let $\mathcal{M}_\lambda$ be the invertible sheaf

$$\phi_\lambda^*(\mathcal{O}(\Sigma-\Sigma_0)_{|B(H)_{|U_\lambda\cap\mathcal{U}}}).$$

Since $\phi_\lambda \circ \phi_\mu^{-1}$ on $B(H)_{|U_\lambda\cap U_\mu}$ is the translation mapping by a section $\eta_{\lambda,\mu}$, there exist invertible sheaves $\mathcal{N}_{\lambda,\mu}$ on $U_\lambda\cap U_\mu\cap\mathcal{U}$ such that

$$(\mathcal{M}_\lambda)_{|U_\lambda\cap U_\mu\cap\mathcal{U}} \simeq (\mathcal{M}_\mu)_{|U_\lambda\cap U_\mu\cap\mathcal{U}} \otimes f^*\mathcal{N}_{\lambda,\mu}.$$

Therefore we have an element $\Phi_X(\Sigma) \in H^0(\mathcal{U}, R^1f_*\mathcal{O}_X^\star)$, which does not depend on the choices of open coverings $\{U_\lambda\}$ and isomorphisms

ϕ_λ. This is a description of the homomorphism $\Phi_X\colon H^0(\mathcal{U}, \mathfrak{S}_H) \to H^0(\mathcal{U}, R^1 f_* \mathcal{O}_X^\star)$. Let us consider a connecting homomorphism

$$\mathbb{Z} = H^0(S, \mathbb{Z}_S) \to H^1(S, \mathfrak{S}_H) \tag{1.3}$$

of the sequence (1.2). By the description of Φ_X, we see that the image of 1 is just η. Thus we have proved:

Lemma 1.3.2 (cf. [Ny8]). *Let $f\colon X \to S$ be a smooth elliptic fibration with H as a variation of Hodge structures. Suppose that $X \simeq B(H)^\eta$ over S. Then there exists an exact sequence* (1.2) *and the image of* 1 *under the connecting homomorphism* (1.3) *is η.*

Proposition 1.3.3 (cf. [Kd1, 11.5]). *Let $f\colon X \to S$ be a smooth elliptic fibration and let $\eta \in H^1(S, \mathfrak{S}_H)$ be the corresponding cohomology class. Then the following three conditions are equivalent*:

(1) *There is a prime divisor $D \subset X$ dominating S*;
(2) *The smooth elliptic fibration $f\colon X \to S$ is a projective morphism, i.e., there is an f-ample line bundle on X*;
(3) *η is a torsion element of $H^1(S, \mathfrak{S}_H)$.*

Proof. (1) $\Longrightarrow$ (2): The invertible sheaf $\mathcal{O}_X(D)$ is f-ample.
(2) $\Longrightarrow$ (3): By (1.2), we have the following long exact sequence:

$$0 \to H^0(S, \mathfrak{S}_H) \to H^0(S, R^1 f_* \mathcal{O}_X^\star) \to H^0(S, \mathbb{Z}) \to H^1(S, \mathfrak{S}_H).$$

An f-ample invertible sheaf defines an element of $H^0(S, R^1 f_* \mathcal{O}_X^\star)$, which is mapped to a positive integer in $\mathbb{Z} = H^0(S, \mathbb{Z})$. Thus by Lemma 1.3.2, the η is a torsion element.

(3) $\Longrightarrow$ (1): Let us assume that $m\eta = 0$ for a positive integer m. We shall consider the multiplication by m:

$$m\times\colon B(H) \ni b \mapsto mb = b + \cdots + b \in B(H).$$

Then by gluing $m\times\colon B(H)_{|S_\lambda} \to B(H)_{|S_\lambda}$, we have an étale finite morphism $\mu\colon X \simeq B(H)^\eta \to B(H)^{m\eta} \simeq B(H)$. Thus an irreducible component D of $\mu^*(\Sigma)$ dominates S. Q.E.D.

By the proof, we can take a divisor $D \subset X$ in (1) to be étale over S. However in general there is a prime divisor of X which is not étale over S.

Example 1.3.4. Let us consider the ruled surface $\Sigma_1 := \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-1)) \to \mathbb{P}^1$ and a double covering $E \to \mathbb{P}^1$ from an elliptic curve. Let V be the fiber product $\Sigma_1 \times_{\mathbb{P}^1} E$. By considering the blow-down $\Sigma_1 \to \mathbb{P}^2$

of the unique (-1)-curve, we have a morphism $h: V \to \Sigma_1 \times E \to \mathbb{P}^2 \times E$. The image $h(V) \subset \mathbb{P}^2 \times E$ dominates $\mathbb{P}^2$, but $h(V) \to \mathbb{P}^2$ is not étale.

In the case the mapping degree of $D \to S$ is one, we have:

Lemma 1.3.5. *Let $f: X \to S$ be a smooth elliptic fibration over a complex* manifold *S. If a prime divisor D of X dominates S bimeromorphically, then $D \to S$ is isomorphic.*

Proof. Suppose that $h := f_{|D}: D \to S$ is not an isomorphism. Then the support of a non-trivial fiber $h^{-1}(s)$ is an elliptic curve. On the other hand, we have a bimeromorphic morphism $\nu: M \to D$ from a manifold M such that every non-trivial fiber of $h \circ \nu: M \to S$ is a union of rational curves. This is a contradiction. Q.E.D.

1.4. Smooth elliptic fibrations whose pullbacks are basic

Let $f: X \to S$ be a smooth elliptic fibration, $\tau: U \to S$ the universal covering mapping, and let $\Gamma = \pi_1(S, s)$. Suppose that the pullback $f_U: U \times_S X \to U$ admits a global section. Let $p: B = B(H) \to S$ be the basic smooth elliptic fibration associated with the variation of Hodge structures H induced from f. By Proposition 1.3.1, f is considered to be a torsor of p and it corresponds to a cohomology class η in $H^1(S, \mathfrak{S}_H)$. Further by Lemma 0.1.3, η is contained in $H^1(\Gamma, H^0(U, \tau^{-1}\mathfrak{S}_H))$, where we consider the following edge sequence of the Hochschild–Serre spectral sequence Lemma 0.2.1:

$$0 \to H^1(\Gamma, H^0(U, \tau^{-1}\mathfrak{S}_H)) \to H^1(S, \mathfrak{S}_H) \to H^0(\Gamma, H^1(U, \tau^{-1}\mathfrak{S}_H)).$$

Looking at the exact sequence:

$$(1.4) \quad 0 \to \tau^{-1}H \simeq \mathbb{Z}_U^{\oplus 2} \to \tau^{-1}(\mathcal{L}_H) \simeq \mathcal{O}_U \to \tau^{-1}\mathfrak{S}_H \simeq \mathfrak{S}_{\tau^{-1}H} \to 0,$$

we have an isomorphism

$$H^0(U, \mathfrak{S}_{\tau^{-1}H}) \simeq H^0(U, \mathcal{O}_U)/(\mathbb{Z}\omega + \mathbb{Z}),$$

where $\omega: U \to \mathbb{H}$ is the period function, since U is simply connected. Hence an element of $H^1(\Gamma, H^0(U, \mathfrak{S}_{\tau^{-1}H}))$ is represented by a collection of global holomorphic functions $\{F_\gamma(z)\}_{\gamma \in \Gamma}$ on U satisfying the cocycle condition:

$$(1.5) \qquad F_{\gamma\delta}(z) \equiv F_\delta(z) + (c_\delta\omega(z) + d_\delta)F_\gamma(\delta z) \mod \mathbb{Z}\omega(z) + \mathbb{Z},$$

for $z \in U$ and $\gamma, \delta \in \Gamma$ (cf. §1.1). Two collections $\{F_\gamma^{(1)}(z)\}$ and $\{F_\gamma^{(2)}(z)\}$ of holomorphic functions determine the same cohomology class in $H^1(\Gamma, H^0(U, \mathfrak{S}_{\tau^{-1}H}))$ if and only if there is a global holomorphic function $H(z)$ on U such that

$$(1.6)\quad F_\gamma^{(1)}(z) - F_\gamma^{(2)}(z) \equiv H(z) - (c_\gamma\omega(z) + d_\gamma)H(\gamma z) \mod \mathbb{Z}\omega(z) + \mathbb{Z}.$$

Let $F := \{F_\gamma(z)\}$ be a collection satisfying (1.5) and let $B_U := B(\tau^{-1}H) \simeq U \times_S B(H)$. Then F defines a left action of Γ on B_U, which is described as follows: For $\gamma \in \Gamma$, let us define the following automorphism of $U \times \mathbb{C}$:

$$U \times \mathbb{C} \ni (z, \zeta) \mapsto \left(\gamma z, \frac{\zeta + F_\gamma(z)}{c_\gamma\omega(z) + d_\gamma}\right).$$

Then it induces an automorphism $\Phi_F(\gamma)$ of $B_U \simeq U \times \mathbb{C}/(\mathbb{Z}\omega + \mathbb{Z})$. Here we have $\Phi_F(\gamma) \circ \Phi_F(\delta) = \Phi_F(\gamma\delta)$ for $\gamma, \delta \in \Gamma$. Thus we have the left action by Φ_F which is compatible with the action of Γ on U. Let B^F be the quotient $\Gamma\backslash B_U$ by the action. Then we have a smooth elliptic fibration $p^F : B^F \to S$. Therefore we have:

Lemma 1.4.1. *Let $f: X \to S$ be a smooth elliptic fibration which induces the variation of Hodge structures H on S. Suppose that $U \times_S X \to U$ admits a global section for the universal covering mapping $U \to S$. Then there is a collection of global holomorphic functions $F = \{F_\gamma(z)\}_{\gamma \in \pi_1(S,s)}$ on U satisfying the condition* (1.5) *such that f is isomorphic to $p^F : B^F \to S$ over S.*

Remark. Since $B_U \simeq U \times \mathbb{C}/(\mathbb{Z}\omega + \mathbb{Z})$, we can describe B^F as the quotient of $U \times \mathbb{C}$ by an action of a suitable group. Let $\Phi_F(\gamma, m, n)$ be an automorphism of $U \times \mathbb{C}$ defined by

$$U \times \mathbb{C} \ni (z, \zeta) \mapsto \left(\gamma z, \frac{\zeta + F_\gamma(z) + m\omega(z) + n}{c_\gamma\omega(z) + d_\gamma}\right),$$

for $m, n \in \mathbb{Z}$. For $\gamma, \delta \in \Gamma$, let us define a pair $(A_{\gamma,\delta}, B_{\gamma,\delta})$ of integers by

$$A_{\gamma,\delta}\omega(z) + B_{\gamma,\delta} := F_\delta(z) - F_{\gamma\delta}(z) + (c_\delta\omega(z) + d_\delta)F_\gamma(\delta z).$$

Then we have $\Phi_F(\gamma, m_1, n_1) \circ \Phi_F(\delta, m_2, n_2) = \Phi_F(\gamma\delta, m_3, n_3)$, where

$$(m_3, n_3) = (m_2, n_2) + (m_1, n_1)\begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix} + (A_{\gamma,\delta}, B_{\gamma,\delta}).$$

Let c be the cohomology class determined by $\{(A_{\gamma,\delta}, B_{\gamma,\delta})\}$ in $H^2(\Gamma, \mathbb{Z}^{\oplus 2})$. Then we have a corresponding group $G(c)$ which is an extension of Γ by the right Γ-module $\mathbb{Z}^{\oplus 2}$. We have a left action of $G(c)$ on $U \times \mathbb{C}$ by Φ_F. The correspondence $F \mapsto c$ induces a homomorphism $H^1(\Gamma, H^0(U, \mathfrak{S}_{\tau^{-1}H})) \to H^2(\Gamma, \mathbb{Z}^{\oplus 2})$, which is derived from the exact sequence (1.4). The B^F is isomorphic to the quotient space $G(c)\backslash(U \times \mathbb{C})$.

§2. Smooth elliptic fibrations over $(\Delta^\star)^l \times \Delta^{d-l}$

2.1. Monodromy representations

Let S be a d-dimensional unit polydisc Δ^d with a coordinate system $(t_1, t_2, \dots, t_d)$, i.e.,

$$S = \{(t_1, t_2, \dots, t_d) \in \mathbb{C}^d \mid |t_i| < 1 \text{ for any } i\}$$

for a positive integer d. Let D be a divisor $\{t_1 t_2 \cdots t_l = 0\}$ for a positive integer $l \le d$, i.e., $D = \sum_{i=1}^{l} D_i$, where $D_i = \{t_i = 0\}$ is the i-th coordinate hyperplane. We denote by $S^\star$ the complement $S \setminus D$ and by $j\colon S^\star \hookrightarrow S$ the natural inclusion. Since $S^\star$ is isomorphic to $(\Delta^\star)^l \times \Delta^{d-l}$, the universal covering space U of $S^\star$ is isomorphic to $\mathbb{H}^l \times \Delta^{d-l}$, where $\mathbb{H}$ is the upper half plane $\{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$. For a coordinate system $z = (z_1, z_2, \dots, z_l, t_{l+1}, \dots, t_d)$ of U, the universal covering mapping $\boldsymbol{e}\colon U \to S^\star$ is given by:

$$\boldsymbol{e}(z) = (e(z_1), e(z_2), \dots, e(z_l), t_{l+1}, \dots, t_d),$$

where $e(z) := \exp(2\pi\sqrt{-1}z)$. For $1 \le i \le l$, let γ_i be the automorphism of U defined by:

$$(z_1, z_2, \dots, z_l, t') \mapsto (z_1, z_2, \dots, z_{i-1}, z_i + 1, z_{i+1}, \dots, z_l, t'),$$

where $t' = (t_{l+1}, t_{l+2}, \dots, t_d)$. Then the fundamental group $\pi_1 := \pi_1(S^\star)$ is a free abelian group of rank l generated by $\gamma_1, \gamma_2, \dots, \gamma_l$.

In this section, we shall consider smooth elliptic fibrations defined over $S^\star$. First of all, we shall describe all the variations of Hodge structures of rank two and weight one defined over $S^\star$. Note that the monodromy matrices are quasi-unipotent by Borel's lemma (cf. [Sc, (4.5)]). We have the following classification of quasi-unipotent matrices in $\mathrm{SL}(2, \mathbb{Z})$:

Lemma 2.1.1 (cf. [Kd1]). *A quasi-unipotent matrix in* $\mathrm{SL}(2,\mathbb{Z})$ *is conjugate exactly to one of the matrices of Table* 1 *in* $\mathrm{SL}(2,\mathbb{Z})$.

Table 1. Quasi unipotent matrices in $\mathrm{SL}(2,\mathbb{Z})$.

I_a $(a \in \mathbb{Z})$	II	III	IV
$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$
I_b^* $(b \in \mathbb{Z})$	II^*	III^*	IV^*
$\begin{pmatrix} -1 & -b \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$

The monodromy matrix $\rho(\gamma_i)$ for γ_i is said to be the *monodromy matrix around the coordinate hyperplane* D_i.

Lemma 2.1.2. *Let $g(t)$ be a holomorphic function on $t \in S^\star$ such that $e(g(t))$ is a meromorphic function on S. Then $g(t)$ is holomorphic also on S.*

Proof. There exist integers a_i for $1 \le i \le l$ and a nowhere vanishing function $u(t)$ on S such that $e(g(t)) = u(t)\prod_{i=1}^{l} t_i^{a_i}$. We have a holomorphic function $h(t)$ such that $u(t) = e(h(t))$ on S. Then for the coordinate system $(z_1, z_2, \ldots, z_l, t')$ of U, we have

$$g(t) - h(t) - \sum_{i=1}^{l} a_i z_i \in \mathbb{Z}.$$

Since this is a constant function, $a_i = 0$ for all i. Hence $g(t)$ is holomorphic on S. Q.E.D.

Lemma 2.1.3 (cf. [Kd1, 7.3]). *Let $\rho\colon \pi_1 \to \mathrm{SL}(2,\mathbb{Z})$ be the monodromy representation associated with a variation of Hodge structures of weight one rank two defined on $S^\star$. Then the conjugation by a suitable matrix in* $\mathrm{SL}(2,\mathbb{Z})$ *changes all the monodromy matrices $\rho(\gamma_i)$ to be matrices listed in Table* 1 *at the same time. If $\rho(\gamma_i)$ corresponds to the matrix of the form I_a or I_a^*, then $a \ge 0$.*

Proof. By Lemma 2.1.1, the first assertion is derived from the commutativity of $\rho(\gamma)$'s. For the rest, we may assume that $d = l = 1$ and $\rho(\gamma_1)$ is of type I_a or I_a^*. Then the period function $\omega(z)$ satisfies $\omega(z+1) = \omega(z) + a$ by (1.1). Thus the function $e(\omega(z))$ is invariant under the action of π_1. Thus there is a holomorphic function $W(t)$ on $S^\star$ such that $W(e(z)) = e(\omega(z))$. Since $|W(t)| < 1$ for any $t \in S^\star$, $W(t)$ is still holomorphic over $0 \in S$. On the other hand, the function $\omega(z) - az$ is also invariant under the action of π_1. Thus we have a holomorphic function $g(t)$ on $S^\star$ such that $g(e(z)) = \omega(z) - az$. Then $W(t) = t^a e(g(t))$. Thus $g(t)$ is also holomorphic over $0 \in S$ by Lemma 2.1.2. Therefore $a \ge 0$. Q.E.D.

We can define the types of monodromy representations $\rho\colon \pi_1 \to \mathrm{SL}(2,\mathbb{Z})$ as in Table 2. By Lemmas 2.1.1 and 2.1.3, any monodromy representation ρ is one of types as above, up to conjugation in $\mathrm{SL}(2,\mathbb{Z})$. We call the image of ρ by the *monodromy group*. If the monodromy group is not finite, then ρ is of type $\mathrm{I}_{(+)}$ or $\mathrm{I}_{(+)}^{(*)}$. In the case $\mathrm{I}_{(+)}$, we set $\boldsymbol{a} := (a_1, a_2, \ldots, a_l) \in \mathbb{Z}^{\oplus l}$, where $\rho(\gamma_i)$ is of type I_{a_i} for $1 \le i \le l$. Further we put $\alpha := \gcd(\boldsymbol{a})$. In the cases $\mathrm{I}_0^{(*)}$ and $\mathrm{I}_{(+)}^{(*)}$, let c_i be one of $\{0, 1\}$

Table 2. Type of monodromy representations.

I_0	All the $\rho(\gamma_i)$ are of type I_0.
$\mathrm{I}_0^{(*)}$	One of $\rho(\gamma_i)$ is of type I_0^*. Others are of types I_0 or I_0^*.
$\mathrm{II}^{(*)}$	One of $\rho(\gamma_i)$ is of type II or II^*. Others are of types I_0, I_0^*, II, II^*, IV or IV^*.
$\mathrm{III}^{(*)}$	One of $\rho(\gamma_i)$ is of type III or III^*. Others are of types I_0, I_0^*, III or III^*.
$\mathrm{IV}_+^{(*)}$	One of $\rho(\gamma_i)$ is of type IV or IV^*. Others are of types I_0, IV or IV^*.
$\mathrm{IV}_-^{(*)}$	One of $\rho(\gamma_i)$ is of type IV or IV^* and another $\rho(\gamma_j)$ is of type I_0^*. Others are of types I_0, I_0^*, IV or IV^*.
$\mathrm{I}_{(+)}$	Any $\rho(\gamma_i)$ is of type I_{a_i}, where one of a_i is positive.
$\mathrm{I}_{(+)}^{(*)}$	One of $\rho(\gamma_i)$ is of type $\mathrm{I}_{a_i}^*$. Others are of types I_{a_j} or $\mathrm{I}_{a_j}^*$, where one of a_i is positive.

Table 3. Subcases of $\mathrm{I}_{(+)}^{(*)}$.

$\mathrm{I}_{(+)}^{(*)}(0)$	$\boldsymbol{a}^* \equiv 0 \mod 2$
$\mathrm{I}_{(+)}^{(*)}(1)$	$\boldsymbol{a}^* \equiv \boldsymbol{c} \mod 2$
$\mathrm{I}_{(+)}^{(*)}(2)$	$\boldsymbol{a}^* \wedge \boldsymbol{c} \not\equiv 0 \mod 2$

such that $(-1)^{c_i}$ is the eigenvalue of $\rho(\gamma_i)$. We set $\boldsymbol{c} := (c_1, c_2, \ldots, c_l)$. In the case $\mathrm{I}_{(+)}^{(*)}$, we further define $a_i^* := (-1)^{c_i} a_i$, where $\rho(\gamma_i)$ is of type I_{a_i} or $\mathrm{I}_{a_i}^*$. We also set $\boldsymbol{a}^* := (a_1^*, a_2^*, \ldots, a_l^*)$. We divide the case $\mathrm{I}_{(+)}^{(*)}$ into three subcases as in Table 3.

Proposition 2.1.4. *Let $\rho\colon \pi_1 \to \mathrm{SL}(2,\mathbb{Z})$ and $\omega(z)$, respectively, be the monodromy representation and the period function associated with a variation of Hodge structures of rank two and weight one defined over $S^\star$. The following four conditions are equivalent:*

(1) *The type of the monodromy representation is either $\mathrm{I}_{(+)}$ or $\mathrm{I}_{(+)}^{(*)}$;*
(2) *There exists a holomorphic function h on S such that the period function is given by*

$$\omega(z) = \sum_{i=1}^{l} a_i z_i + h(t),$$

where one of a_i is positive;

(3) *The J-function $J(t) = \boldsymbol{j}(\omega(z))$ is not holomorphic at $\{0\}$ in S;*

(4) *The monodromy group is not a finite group.*

Proof. (1) $\Longrightarrow$ (2): For the function $g(z) := \omega(z) - \sum_{i=1}^{l} a_i z_i$, we have $g(\gamma_i z) = g(z)$ for any γ_i. Thus there is a holomorphic function $h(t)$ on $t \in S^\star$ such that $g(z) = h(t)$. As in the proof of Lemma 2.1.3, we have also a holomorphic function $W(t)$ on S such that $W(t) = \prod_{i=1}^{l} t_i^{a_i} e(h(t))$ on $S^\star$. Therefore $e(h(t))$ is meromorphic on S. Thus $h(t)$ is still holomorphic on S by Lemma 2.1.2.

(2) $\Longrightarrow$ (3): By (2), we have $e(\omega(z)) = u(t) \prod_{i=1}^{l} t_i^{a_i}$ for a nowhere vanishing function $u(t)$ on S. Thus by Remark 1.2.6, $J(t)$ is a meromorphic function with poles of order a_i on each coordinate hyperplane D_i.

(3) $\Longrightarrow$ (4): Suppose that the monodromy group is finite. Then we can take a Kummer covering

$$\tau\colon \Delta^d = \Delta^l \times \Delta^{d-l} \ni \theta = (\theta_1, \theta_2, \ldots, \theta_l, t') \mapsto (\theta_1^{m_1}, \theta_2^{m_2}, \ldots, \theta_l^{m_l}, t') \in S$$

such that the pullback of the variation of Hodge structures on $\tau^{-1}(S^\star)$ has a trivial monodromy group. Thus there exists a holomorphic function $H(\theta)$ on Δ^d such that $\omega(z) = H(\theta) = H(\theta', t')$ for $\theta' = (e(z_1/m_1), e(z_2/m_2), \ldots, e(z_l/m_l))$. Hence $J(t) = \boldsymbol{j}(\omega(z)) = \boldsymbol{j}(H(\theta))$ is holomorphic on Δ^d. Thus $J(t)$ is also holomorphic on S.

(4) $\Longrightarrow$ (1): Trivial. Q.E.D.

Corollary 2.1.5. *The J-function $J(t) = \boldsymbol{j}(\omega(z))$ induces a holomorphic map $J\colon S \to \mathbb{P}^1$. The image contains ∞ if and only if the monodromy representation is of type $\mathrm{I}_{(+)}$ or $\mathrm{I}_{(+)}^{(*)}$.*

The classification of possible period functions $\omega(z)$ is given in Corollary 3.1.6.

2.2. Classification of smooth projective elliptic fibrations over $(\Delta^\star)^l \times \Delta^{d-l}$

Let H be a variation of Hodge structures of weight one and rank two on $S^\star = (\Delta^\star)^l \times \Delta^{d-l}$. We may assume that for the monodromy representation $\rho\colon \pi_1 \to \mathrm{SL}(2, \mathbb{Z})$, every $\rho(\gamma)$ for $\gamma \in \pi_1$ are matrices listed in Table 1. The Hodge filtrations are determined by the period function $\omega(z)$ on U such that

$$\omega(\gamma z) = \frac{a_\gamma \omega(z) + b_\gamma}{c_\gamma \omega(z) + d_\gamma}, \quad \text{where} \quad \rho(\gamma) = \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix}.$$

By Proposition 1.3.1, any smooth elliptic fibration on $S^\star$ is isomorphic to $B(H)^\eta \to S^\star$ for some $\eta \in H^1(S^\star, \mathfrak{S}_H)$. Let us consider the exact sequence

$$0 \to H \to \mathcal{L}_H \to \mathfrak{S}_H \to 0.$$

Since $H^i(U, \mathbb{Z}) = H^i(U, \mathcal{O}_U) = 0$ for $i > 0$, applying Lemma 0.2.1, we have isomorphisms:

$$H^p(S^\star, H) \simeq H^p(\pi_1, \mathbb{Z}^{\oplus 2}), \quad H^p(S^\star, \mathcal{L}_H) \simeq H^p(\pi_1, H^0(U, \mathcal{O}_U)),$$
$$H^p(S^\star, \mathfrak{S}_H) \simeq H^p(\pi_1, H^0(U, \mathfrak{S}_{\boldsymbol{e}^{-1}H})),$$
$$H^1(S^\star, \mathfrak{S}_H) \simeq H^2(S^\star, H) \simeq H^2(\pi_1, \mathbb{Z}^{\oplus 2})$$

for any p. From the vanishing $H^1(U, \boldsymbol{e}^{-1}\mathfrak{S}_H) = 0$, we see that any smooth elliptic fibration over U admits a global section. Therefore by Lemma 1.4.1, for any smooth elliptic fibration $X \to S^\star$ having H as a variation of Hodge structures, there is a collection of holomorphic functions $F := \{F_\gamma(z) \mid \gamma \in \pi_1\}$ on U such that F satisfies the condition (1.5) and $X \simeq B(H)^F$ over $S^\star$.

Theorem 2.2.1 ([Ny8, (3.1)]). *The group cohomology groups $H^p(\pi_1, \mathbb{Z}^{\oplus 2})$ are calculated as in Table 4.*

Proof. Let $R := \mathbb{Z}[\pi_1] = \mathbb{Z}[\gamma_1^{\pm 1}, \gamma_2^{\pm 1}, \dots, \gamma_l^{\pm 1}]$ be the group ring for $\pi_1 \simeq \mathbb{Z}^{\oplus l}$. Then we have a standard resolution

$$\cdots \to \bigwedge^{p+1}(R^{\oplus l}) \to \bigwedge^{p}(R^{\oplus l}) \to \cdots \to R \to \mathbb{Z} \to 0$$

of the trivial π_1-module $\mathbb{Z}$, where the canonical base $e_{i_0} \wedge e_{i_1} \wedge \cdots \wedge e_{i_p} \in \bigwedge^{p+1}(R^{\oplus l})$ for $1 \le i_0 < i_1 < \cdots < i_p \le l$ is mapped to

$$\sum_{j=0}^{p} (-1)^j e_{i_0} \wedge e_{i_1} \wedge \cdots \wedge \widehat{e_{i_j}} \wedge \cdots \wedge e_{i_p}(1 - \gamma_{i_j}).$$

The group cohomology $H^p(\pi_1, \mathbb{Z}^{\oplus 2})$ for the right π_1-module $\mathbb{Z}^{\oplus 2}$ is isomorphic to the p-th cohomology of the complex

$$\cdots \to \mathrm{Hom}_R(\bigwedge^{p}(R^{\oplus l}), \mathbb{Z}^{\oplus 2}) \xrightarrow{\mathrm{d}^p} \mathrm{Hom}_R(\bigwedge^{p+1}(R^{\oplus l}), \mathbb{Z}^{\oplus 2}) \to \cdots.$$

Let I be the unit matrix. Then the d^p is described as:

$$\mathrm{d}^p(x)(e_{i_0} \wedge e_{i_1} \wedge \cdots \wedge e_{i_p}) = \sum_{j=0}^{p} (-1)^j x(e_{i_0} \wedge e_{i_1} \wedge \cdots \wedge \widehat{e_{i_j}} \wedge \cdots \wedge e_{i_p})(I - \rho(\gamma_{i_j})).$$

We denote $\boldsymbol{i} := \sqrt{-1}$ and $\boldsymbol{\omega} := \exp(2\pi\sqrt{-1}/3)$. Let A be the commutative algebra defined as follows:

$$A := \begin{cases} \mathbb{Z}, & \text{in the cases } \mathrm{I}_0,\ \mathrm{I}_0^{(*)}; \\ \mathbb{Z}[\boldsymbol{\omega}], & \text{in the cases } \mathrm{II}^{(*)},\ \mathrm{IV}_+^{(*)},\ \mathrm{IV}_-^{(*)}; \\ \mathbb{Z}[\boldsymbol{i}], & \text{in the case } \mathrm{III}^{(*)}; \\ \mathbb{Z}[\varepsilon]/(\varepsilon^2), & \text{in the cases } \mathrm{I}_{(+)},\ \mathrm{I}_{(+)}^{(*)}. \end{cases}$$

Then we can consider $\mathbb{Z}^{\oplus 2}$ as an A-module by regarding the elements $\boldsymbol{i}$, $\boldsymbol{\omega}$ and ε as:

$$\boldsymbol{i} \leftrightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{\omega} \leftrightarrow \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \quad \varepsilon \leftrightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Thus there is a natural ring homomorphism $\phi\colon R \to A$ from which the R-module structure of $\mathbb{Z}^{\oplus 2}$ is derived. More precisely, $\phi(\gamma)$ is determined according to types of the matrix $\rho(\gamma)$ as in Table 5. For all the cases except I_0, $\mathrm{I}_0^{(*)}$, we have the following isomorphism $\mathbb{Z}^{\oplus 2} \simeq A$ as

Table 4. List of cohomology groups $H^p(\pi_1, \mathbb{Z}^{\oplus 2})$.

Type	H^0	H^1	H^p $(p \geq 2)$
I_0	$\mathbb{Z}^{\oplus 2}$	$\mathbb{Z}^{\oplus 2l}$	$\mathbb{Z}^{\oplus 2\binom{l}{p}}$
$\mathrm{I}_0^{(*)}$	0	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2\binom{l-1}{p-1}}$
$\mathrm{II}^{(*)}$	0	0	0
$\mathrm{III}^{(*)}$	0	$\mathbb{Z}/2\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus\binom{l-1}{p-1}}$
$\mathrm{IV}_+^{(*)}$	0	$\mathbb{Z}/3\mathbb{Z}$	$(\mathbb{Z}/3\mathbb{Z})^{\oplus\binom{l-1}{p-1}}$
$\mathrm{IV}_-^{(*)}$	0	0	0
$\mathrm{I}_{(+)}$	$\mathbb{Z}$	$\mathbb{Z}^{\oplus l} \oplus \mathbb{Z}/\alpha\mathbb{Z}$	$\mathbb{Z}^{\oplus\binom{l}{p}} \oplus (\mathbb{Z}/\alpha\mathbb{Z})^{\oplus\binom{l-1}{p-1}}$
$\mathrm{I}_{(+)}^{(*)}(0)$	0	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2\binom{l-1}{p-1}}$
$\mathrm{I}_{(+)}^{(*)}(1)$	0	$\mathbb{Z}/4\mathbb{Z}$	$(\mathbb{Z}/4\mathbb{Z})^{\oplus\binom{l-1}{p-1}}$
$\mathrm{I}_{(+)}^{(*)}(2)$	0	$\mathbb{Z}/2\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus\binom{l-1}{p-1}}$

Table 5. Image of γ.

γ	I_0	I_0^*	II	II*	III	III*	IV	IV*	I_a	I_a^*
$\phi(\gamma)$	1	-1	$-\boldsymbol{\omega}$	$-\boldsymbol{\omega}^2$	$-\boldsymbol{i}$	$\boldsymbol{i}$	$\boldsymbol{\omega}^2$	$\boldsymbol{\omega}$	$1+a\varepsilon$	$-(1+a\varepsilon)$

A-modules:

$$\mathbb{Z}^{\oplus 2} \ni (m,n) \mapsto \begin{cases} m\boldsymbol{\omega}+n, & \text{in the cases } \mathrm{II}^{(*)}, \mathrm{IV}_+^{(*)}, \mathrm{IV}_-^{(*)}; \\ m\boldsymbol{i}+n, & \text{in the case } \mathrm{III}^{(*)}; \\ m+n\varepsilon, & \text{in the cases } \mathrm{I}_{(+)}, \mathrm{I}_{(+)}^{(*)}. \end{cases}$$

We define $b_i := 1-\phi(\gamma_i) \in A$ and $\boldsymbol{b} := (b_1, b_2, \dots, b_l) \in A^{\oplus l}$. Then $H^p(\pi_1, \mathbb{Z}^{\oplus 2})$ is isomorphic to the p-th cohomology group of the following complex:

$$0 \to M \overset{\boldsymbol{b}\wedge}{\to} M \otimes_A (A^{\oplus l}) \overset{\boldsymbol{b}\wedge}{\to} M \otimes_A \bigwedge^2 (A^{\oplus l}) \overset{\boldsymbol{b}\wedge}{\to} \cdots,$$

where $M = \mathbb{Z}^{\oplus 2}$ as an A-module. Here for $\boldsymbol{x} \in M \otimes \bigwedge^p (A^{\oplus l})$, $\boldsymbol{b} \wedge \boldsymbol{x}$ is defined as follows: Let $x_{i_1, i_2, \dots, i_p}$ be the $(i_1, i_2, \dots, i_p)$-coefficient of $\boldsymbol{x}$ for $1 \le i_1 < i_2 < \cdots < i_p \le l$. Then the $(i_0, i_1, \dots, i_p)$-coefficient of $\boldsymbol{b} \wedge \boldsymbol{x}$ for $1 \le i_0 < i_1 < \cdots < i_p \le l$ is defined by:

$$\sum_{j=0}^{p} (-1)^j x_{i_0, i_1, \dots, \widehat{i_j}, \dots, i_p} \cdot b_j.$$

We shall calculate the cohomology group $H^p = H^p(\pi_1, \mathbb{Z}^{\oplus 2})$ in each type of monodromy representations.

The case I_0: We have $A = \mathbb{Z}$ and $\boldsymbol{b} = 0$. Thus $H^p \simeq \mathbb{Z}^{\oplus 2} \otimes \bigwedge^p (\mathbb{Z}^{\oplus l})$.

The cases $\mathrm{II}^{(*)}$ and $\mathrm{IV}_-^{(*)}$: In the case $\mathrm{II}^{(*)}$, one of b_i is $1+\boldsymbol{\omega} = -\boldsymbol{\omega}^2$ or $1+\boldsymbol{\omega}^2 = -\boldsymbol{\omega}$. Since these are units of $A = \mathbb{Z}[\boldsymbol{\omega}]$, there is a matrix $P \in \mathrm{GL}(l, A)$ such that $\boldsymbol{b} = (1, 0, \dots, 0)P$. Therefore for an $\boldsymbol{x} \in \bigwedge^p (A^{\oplus l})$, $\boldsymbol{b} \wedge \boldsymbol{x} = 0$ if and only if $\boldsymbol{x} = \boldsymbol{b} \wedge \boldsymbol{y}$ for some $\boldsymbol{y}$. Hence $H^p = 0$ for any p. In the case $\mathrm{IV}_-^{(*)}$, one of b_i is $1-\boldsymbol{\omega}$ or $1-\boldsymbol{\omega}^2$ and another b_j is 2. Since $(1-\boldsymbol{\omega})-2 = \boldsymbol{\omega}^2$ and $(1-\boldsymbol{\omega}^2)-2 = \boldsymbol{\omega}$ are units in A, by the same reason as above, we have $H^p = 0$ for any p.

The case $\mathrm{I}_0^{(*)}$: We have $A = \mathbb{Z}$ and $\boldsymbol{b} = 2\boldsymbol{c}$. Then we can find a matrix $P \in \mathrm{GL}(l, \mathbb{Z})$ such that $\boldsymbol{c} = (1, 0, \dots, 0)P$. Therefore, if $\boldsymbol{x} \in \bigwedge^p (\mathbb{Z}^{\oplus l})$ satisfies $\boldsymbol{b} \wedge \boldsymbol{x} = 2\boldsymbol{c} \wedge \boldsymbol{x} = 0$, then $\boldsymbol{x} = \boldsymbol{c} \wedge \boldsymbol{y}$ for some $\boldsymbol{y} \in \bigwedge^{p-1} (\mathbb{Z}^{\oplus l})$. Suppose that the $\boldsymbol{x} = \boldsymbol{c} \wedge \boldsymbol{y}$ is written by $\boldsymbol{b} \wedge \boldsymbol{y}'$ for some

$\boldsymbol{y}' \in \bigwedge^{p-1}(\mathbb{Z}^{\oplus l})$. Then $\boldsymbol{y} - 2\boldsymbol{y}' \in \operatorname{Ker}(\boldsymbol{c}\wedge)$. This implies

$$\text{(2.1)} \qquad \boldsymbol{y} \mod 2 \in \operatorname{Im}\left(\boldsymbol{c}\wedge\colon \bigwedge^{p-2}((\mathbb{Z}/2\mathbb{Z})^{\oplus l}) \to \bigwedge^{p-1}((\mathbb{Z}/2\mathbb{Z})^{\oplus l})\right).$$

Conversely, if $\boldsymbol{y}$ satisfies the condition (2.1), then $\boldsymbol{c}\wedge\boldsymbol{y} = \boldsymbol{b}\wedge\boldsymbol{y}'$ for some $\boldsymbol{y}'$. Therefore we have

$$H^p \simeq \mathbb{Z}^{\oplus 2} \otimes \bigwedge^{p-1}((\mathbb{Z}/2\mathbb{Z})^{\oplus l}/(\mathbb{Z}/2\mathbb{Z})\boldsymbol{c}) \simeq \mathbb{Z}^{\oplus 2} \otimes \bigwedge^{p-1}((\mathbb{Z}/2\mathbb{Z})^{\oplus (l-1)}).$$

The cases $\mathrm{III}^{(*)}$, $\mathrm{IV}^{(*)}_{+}$, $\mathrm{I}^{(*)}_{(+)}(0)$, $\mathrm{I}^{(*)}_{(+)}(1)$: We have an element $\boldsymbol{u} \in A^{\oplus l}$ such that $\boldsymbol{u} = (1, 0, \dots, 0) \cdot P$ for some $P \in \mathrm{GL}(l, A)$ and $\boldsymbol{b} = \delta\boldsymbol{u}$. More explicitly, we can choose

$$\delta = \begin{cases} 1 - \boldsymbol{i}, & \text{in the case } \mathrm{III}^{(*)}; \\ 1 - \boldsymbol{\omega}, & \text{in the case } \mathrm{IV}^{(*)}_{+}; \\ 2, & \text{in the case } \mathrm{I}^{(*)}_{(+)}(0); \\ 2 - \varepsilon, & \text{in the case } \mathrm{I}^{(*)}_{(+)}(1). \end{cases}$$

Further $\boldsymbol{u} = \boldsymbol{c} - (\varepsilon/2)\boldsymbol{a}^*$ and $\boldsymbol{u} = \boldsymbol{c} - (\varepsilon/2)(\boldsymbol{a}^* - \boldsymbol{c})$ in the cases $\mathrm{I}^{(*)}_{(+)}(0)$ and $\mathrm{I}^{(*)}_{(+)}(1)$, respectively. Therefore for an $\boldsymbol{x} \in \bigwedge^{p}(A^{\oplus l})$, $\boldsymbol{b}\wedge\boldsymbol{x} = 0$ if and only if $\boldsymbol{x} = \boldsymbol{u} \wedge \boldsymbol{y}$ for some $\boldsymbol{y}$. For such $\boldsymbol{y} \in \bigwedge^{p-1}(A^{\oplus l})$, the condition: $\boldsymbol{u} \wedge \boldsymbol{y} = \boldsymbol{b} \wedge \boldsymbol{y}'$ for some $\boldsymbol{y}' \in \bigwedge^{p-1}(A^{\oplus l})$ is equivalent to: $\boldsymbol{y} \mod \delta$ is contained in the image of $\boldsymbol{u}\wedge$. Thus

$$H^p \simeq \bigwedge^{p-1}((A/\delta A)^{\oplus l}/(A/\delta A)\boldsymbol{u}) \simeq \bigwedge^{p-1}((A/\delta A)^{\oplus (l-1)}).$$

We note that

$$A/\delta A \simeq \begin{cases} \mathbb{Z}/2\mathbb{Z}, & \text{in the case } \mathrm{III}^{(*)}; \\ \mathbb{Z}/3\mathbb{Z}, & \text{in the case } \mathrm{IV}^{(*)}_{+}; \\ (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}, & \text{in the case } \mathrm{I}^{(*)}_{(+)}(0); \\ \mathbb{Z}/4\mathbb{Z}, & \text{in the case } \mathrm{I}^{(*)}_{(+)}(1). \end{cases}$$

The case $\mathrm{I}_{(+)}$: Let $\boldsymbol{u} \in \mathbb{Z}^{\oplus l}$ be the vector such that $\boldsymbol{a} = \alpha\boldsymbol{u}$. Then $\boldsymbol{u} = (1, 0, \dots, 0) \cdot P$ for some $P \in \mathrm{GL}(l, \mathbb{Z})$. We have $\boldsymbol{b} = -\alpha\varepsilon\boldsymbol{u}$. We take an element $\boldsymbol{x} = \boldsymbol{x}_0 + \varepsilon\boldsymbol{x}_1 \in \bigwedge^{p}(A^{\oplus l})$, where $\boldsymbol{x}_0, \boldsymbol{x}_1 \in \bigwedge^{p}(\mathbb{Z}^{\oplus l})$. Then $\boldsymbol{b} \wedge \boldsymbol{x} = 0$ if and only if $\boldsymbol{x}_0 = \boldsymbol{u} \wedge \boldsymbol{y}_0$ for some $\boldsymbol{y}_0 \in \bigwedge^{p-1}(\mathbb{Z}^{\oplus l})$. Furthermore for such $\boldsymbol{y}_0$ and $\boldsymbol{x}_1$, $\boldsymbol{u} \wedge \boldsymbol{y}_0 + \varepsilon\boldsymbol{x}_1 = \boldsymbol{b} \wedge \boldsymbol{v}$ for some $\boldsymbol{v} \in \bigwedge^{p-1}(A^{\oplus l})$ if and only if $\boldsymbol{u} \wedge \boldsymbol{y}_0 = 0$ and $\boldsymbol{x}_1 = \alpha \boldsymbol{u} \wedge \boldsymbol{v}_0$ for some $\boldsymbol{v}_0 \in \bigwedge^{p-1}(\mathbb{Z}^{\oplus l})$. Therefore H^p is isomorphic to

$$\begin{aligned}
&\mathrm{Im}\left(\boldsymbol{u}\wedge\colon \bigwedge^{p-1}(\mathbb{Z}^{\oplus l}) \to \bigwedge^{p}(\mathbb{Z}^{\oplus l})\right) \oplus \mathrm{Coker}\left(\alpha\boldsymbol{u}\wedge\colon \bigwedge^{p-1}(\mathbb{Z}^{\oplus l}) \to \bigwedge^{p}(\mathbb{Z}^{\oplus l})\right) \\
&\simeq \bigwedge^{p-1}(\mathbb{Z}^{\oplus l}/\mathbb{Z}\boldsymbol{u}) \oplus \bigwedge^{p}(\mathbb{Z}^{\oplus l}) \Big/ (\alpha \boldsymbol{u} \wedge \bigwedge^{p-1}(\mathbb{Z}^{\oplus l})) \\
&\simeq \bigwedge^{p-1}(\mathbb{Z}^{\oplus l}/\mathbb{Z}\boldsymbol{u}) \oplus \bigwedge^{p}(\mathbb{Z}^{\oplus l}/\mathbb{Z}\boldsymbol{u}) \oplus \left(\bigwedge^{p-1}(\mathbb{Z}^{\oplus l}/\mathbb{Z}\boldsymbol{u}) \otimes \mathbb{Z}/\alpha\mathbb{Z}\right) \\
&\simeq \mathbb{Z}^{\binom{l}{p}} \oplus (\mathbb{Z}/\alpha\mathbb{Z})^{\oplus\binom{l-1}{p-1}}.
\end{aligned}$$

The case $\mathrm{I}^{(*)}_{(+)}(2)$: We have $\boldsymbol{b} = 2\boldsymbol{c} - \varepsilon \boldsymbol{a}^*$, where $\boldsymbol{c} \wedge \boldsymbol{a}^* \not\equiv 0 \mod 2$. Let us take an element $\boldsymbol{x} = \boldsymbol{x}_0 + \varepsilon \boldsymbol{x}_1 \in \bigwedge^p(A^{\oplus l})$, where $\boldsymbol{x}_0, \boldsymbol{x}_1 \in \bigwedge^p(\mathbb{Z}^{\oplus l})$. Suppose that $\boldsymbol{b} \wedge \boldsymbol{x} = 0$. Then $\boldsymbol{c} \wedge \boldsymbol{x}_0 = 0$ and $\boldsymbol{a}^* \wedge \boldsymbol{x}_0 = 2\boldsymbol{c} \wedge \boldsymbol{x}_1$. Thus there exist $\boldsymbol{y}_0, \boldsymbol{y}_1 \in \bigwedge^{p-1}(\mathbb{Z}^{\oplus l})$ such that

$$\boldsymbol{x}_0 = \boldsymbol{c} \wedge \boldsymbol{y}_0, \quad 2\boldsymbol{x}_1 = -\boldsymbol{a}^* \wedge \boldsymbol{y}_0 + \boldsymbol{c} \wedge \boldsymbol{y}_1.$$

Since $\boldsymbol{c} \wedge \boldsymbol{a}^* \not\equiv 0 \mod 2$, we have $\boldsymbol{z}_0, \boldsymbol{z}_1, \boldsymbol{z}_2 \in \bigwedge^{p-2}(\mathbb{Z}^{\oplus l})$ and $\boldsymbol{y}_0', \boldsymbol{y}_1' \in \bigwedge^{p-1}(\mathbb{Z}^{\oplus l})$ such that

$$\boldsymbol{y}_0 = \boldsymbol{c} \wedge \boldsymbol{z}_0 + \boldsymbol{a}^* \wedge \boldsymbol{z}_1 + 2\boldsymbol{y}_0', \quad \boldsymbol{y}_1 = \boldsymbol{c} \wedge \boldsymbol{z}_2 - \boldsymbol{a}^* \wedge \boldsymbol{z}_0 + 2\boldsymbol{y}_1'.$$

Therefore we have

$$(2.2) \qquad \boldsymbol{x}_0 = \boldsymbol{c} \wedge \boldsymbol{a}^* \wedge \boldsymbol{z}_1 + 2\boldsymbol{c} \wedge \boldsymbol{y}_0', \quad \boldsymbol{x}_1 = -\boldsymbol{a}^* \wedge \boldsymbol{y}_0' + \boldsymbol{c} \wedge \boldsymbol{y}_1'.$$

Conversely, if there exist $\boldsymbol{z}_1, \boldsymbol{y}_0', \boldsymbol{y}_1'$ satisfying (2.2), then $\boldsymbol{x} = \boldsymbol{x}_0 + \varepsilon \boldsymbol{x}_1$ satisfies $\boldsymbol{b} \wedge \boldsymbol{x} = 0$. Next for such $\boldsymbol{z}_1, \boldsymbol{y}_0', \boldsymbol{y}_1'$, suppose that $\boldsymbol{x}_0 + \varepsilon \boldsymbol{x}_1 = \boldsymbol{b} \wedge (\boldsymbol{w}_0 + \varepsilon \boldsymbol{w}_1)$ for some $\boldsymbol{w}_0, \boldsymbol{w}_1 \in \bigwedge^{p-1}(\mathbb{Z}^{\oplus l})$. Then we have

$$\boldsymbol{c} \wedge \boldsymbol{a}^* \wedge \boldsymbol{z}_1 + 2\boldsymbol{c} \wedge \boldsymbol{y}_0' = 2\boldsymbol{c} \wedge \boldsymbol{w}_0, \quad -\boldsymbol{a}^* \wedge \boldsymbol{y}_0' + \boldsymbol{c} \wedge \boldsymbol{y}_1' = 2\boldsymbol{c} \wedge \boldsymbol{w}_1 - \boldsymbol{a}^* \wedge \boldsymbol{w}_0.$$

Therefore there exist $\boldsymbol{v}_0, \boldsymbol{v}_1 \in \bigwedge^{p-3}(\mathbb{Z}^{\oplus 2})$ and $\boldsymbol{z}_1', \boldsymbol{q} \in \bigwedge^{p-2}(\mathbb{Z}^{\oplus l})$ such that

$$\begin{aligned}
&\boldsymbol{z}_1 = \boldsymbol{a}^* \wedge \boldsymbol{v}_0 + \boldsymbol{c} \wedge \boldsymbol{v}_1 + 2\boldsymbol{z}_1', \quad \boldsymbol{w}_0 = \boldsymbol{y}_0' + \boldsymbol{a}^* \wedge \boldsymbol{z}_1' + \boldsymbol{c} \wedge \boldsymbol{q}, \\
&\boldsymbol{c} \wedge \boldsymbol{y}_1' = \boldsymbol{c} \wedge (\boldsymbol{a}^* \wedge \boldsymbol{q} + 2\boldsymbol{w}_1).
\end{aligned}$$

Hence we see

$$(2.3) \qquad \boldsymbol{c} \wedge \boldsymbol{a}^* \wedge \boldsymbol{z}_1 \equiv 0 \mod 2, \quad \boldsymbol{c} \wedge \boldsymbol{a}^* \wedge \boldsymbol{y}_1' \equiv 0 \mod 2.$$

Conversely, if $\boldsymbol{z}_1$ and $\boldsymbol{y}'_1$ satisfy the condition (2.3), then $\boldsymbol{x}_0 + \varepsilon\boldsymbol{x}_1 = \boldsymbol{b}\wedge(\boldsymbol{w}_0+\varepsilon\boldsymbol{w}_1)$ for some $\boldsymbol{w}_0, \boldsymbol{w}_1 \in \bigwedge^{p-1}(\mathbb{Z}^{\oplus l})$. Therefore H^p is isomorphic to

$$\operatorname{Im}\left(\boldsymbol{c}\wedge\boldsymbol{a}^*\wedge\colon \bigwedge^{p-2}(\mathbb{Z}^{\oplus l}) \to \bigwedge^{p}(\mathbb{Z}/2\mathbb{Z})^{\oplus l}\right) \oplus \operatorname{Im}\left(\boldsymbol{c}\wedge\boldsymbol{a}^*\wedge\colon \bigwedge^{p-1}(\mathbb{Z}^{\oplus l}) \to \bigwedge^{p+1}(\mathbb{Z}/2\mathbb{Z})^{\oplus l}\right)$$
$$\simeq \bigwedge^{p-2}(\mathbb{Z}/2\mathbb{Z})^{\oplus(l-2)} \oplus \bigwedge^{p-1}(\mathbb{Z}/2\mathbb{Z})^{\oplus(l-2)} \simeq (\mathbb{Z}/2\mathbb{Z})^{\oplus\binom{l-1}{p-1}}.$$

Thus we are done. Q.E.D.

In order to obtain a collection $\{F_\gamma(z)\}$ of holomorphic functions on $U = \mathbb{H}^l \times \Delta^{d-l}$ satisfying (1.5), it is enough to have a collection of holomorphic functions $F = \{F_i(z)\}_{i=1}^l$ satisfying the condition:

$$(2.4)\quad F_j(z) - (c_{\gamma_i}\omega(z) + d_{\gamma_i})F_j(\gamma_i z) \equiv F_i(z) - (c_{\gamma_j}\omega(z) + d_{\gamma_j})F_i(\gamma_j z) \mod \mathbb{Z}\omega(z) + \mathbb{Z}$$

for all $1 \le i, j \le l$. Once we have a collection F satisfying the condition (2.4), then we have a smooth elliptic fibration $B^F \to S^\star$ as the quotient of B_U by the following action of $\gamma_i \in \pi_1 \simeq \mathbb{Z}^{\oplus l}$:

$$[z, \zeta] \mapsto \left[\gamma_i z, \frac{\zeta + F_i(z)}{c_{\gamma_i}\omega(z) + d_{\gamma_i}}\right],$$

where $[z, \zeta]$ denotes the image of $(z, \zeta) \in U \times \mathbb{C}$ under the morphism $U \times \mathbb{C} \to B_U \simeq U \times \mathbb{C}/(\mathbb{Z}\omega + \mathbb{Z})$. For two collections $F = \{F_i\}_{i=1}^l$ and $F' = \{F'_i\}_{i=1}^l$, they induce same elliptic fibration if and only if there exists a holomorphic function $H(z)$ on U such that

$$(2.5)\quad F_i(z) - F'_i(z) \equiv H(z) - (c_{\gamma_i}\omega(z) + d_{\gamma_i})H(\gamma_i z) \mod \mathbb{Z}\omega(z) + \mathbb{Z}.$$

For a collection F, let $(P_{i,j}, Q_{i,j})$ for $1 \le i < j \le l$ be pairs of integers defined by

$$P_{i,j}\omega(z) + Q_{i,j} := F_i(z) - (c_{\gamma_j}\omega(z) + d_{\gamma_j})F_i(\gamma_j z) - (F_j(z) - (c_{\gamma_i}\omega(z) + d_{\gamma_i})F_j(\gamma_i z)).$$

Then $\{(P_{i,j}, Q_{i,j})\}$ defines an element $\boldsymbol{x}$ of $M \otimes_A \bigwedge^2(A^{\oplus l})$, where $M = \mathbb{Z}^{\oplus 2}$ as an A-module (cf. Theorem 2.2.1). Here $\boldsymbol{b} \wedge \boldsymbol{x} = 0$. In order to determine all the possible smooth elliptic fibrations, we have only to find collections F of holomorphic functions which cover all the representatives $\boldsymbol{x}$ of the cohomology group $H^2(\pi_1, \mathbb{Z}^{\oplus 2})$. We shall give such collections of holomorphic functions explicitly.

The case I_0: Let $m_{i,j}$ and $n_{i,j}$ be integers for $1 \le i, j \le l$ such that $m_{j,i} = -m_{i,j}$ and $n_{j,i} = -n_{i,j}$. We have

$$F_j(z) - F_j(\gamma_i z) - (F_i(z) - F_i(\gamma_j z)) = m_{i,j}\omega(z) + n_{i,j}$$

for the functions

$$F_i(z) := \frac{1}{2}\sum_{k=1}^{l}(m_{i,k}\omega(z) + n_{i,k})z_k.$$

Thus the cohomology class in $H^2(\pi_1, \mathbb{Z}^{\oplus 2})$ induced from the collection $\{F_i(z)\}$ is essentially $\{(m_{i,j}, n_{i,j})\} \in (\bigwedge^2(\mathbb{Z}^{\oplus l}))^{\oplus 2}$.

The case $\mathrm{I}_0^{(*)}$: In the proof of Theorem 2.2.1, if $\boldsymbol{x} \in \bigwedge^2(\mathbb{Z}^{\oplus l})$ satisfies $\boldsymbol{b} \wedge \boldsymbol{x} = 0$, then $\boldsymbol{x} = \boldsymbol{c} \wedge \boldsymbol{y}$ for some $\boldsymbol{y} \in \mathbb{Z}^{\oplus l}$. Let $\boldsymbol{y}_1 := (m_1, m_2, \ldots, m_l)$ and $\boldsymbol{y}_2 := (n_1, n_2, \ldots, n_l)$ be elements of $\mathbb{Z}^{\oplus l}$. Then for any $0 \le i \ne j \le l$, the (i, j) components of vectors $\boldsymbol{c} \wedge \boldsymbol{y}_1$ and $\boldsymbol{c} \wedge \boldsymbol{y}_2$ are $c_i m_j - c_j m_i$ and $c_i n_j - c_j n_i$, respectively. We have

$$\begin{aligned} F_j(z) - (-1)^{c_i}F_j(\gamma_i z) - (F_i(z) - (-1)^{c_j}F_i(\gamma_j z)) \\ = (c_i m_j - c_j m_i)\omega(z) + (c_i n_j - c_j n_i) \end{aligned}$$

for the functions

$$F_i(z) := (m_i/2)\omega(z) + (n_i/2).$$

Therefore these collections $\{F_i\}$ induce all the cohomology classes in $H^2(\pi_1, \mathbb{Z}^{\oplus 2})$. By the proof of Theorem 2.2.1, for two collections of integers (m_i, n_i) and (m_i', n_i'), the corresponding $\{F_i\}$'s determine the same cohomology class if and only if

$$m_i - m_i' \equiv k_1 c_i \mod 2 \quad \text{and} \quad n_i - n_i' \equiv k_2 c_i \mod 2$$

for some integers k_1, k_2.

The cases $\mathrm{II}^{(*)}$ *and* $\mathrm{IV}_-^{(*)}$: We have $H^2(\pi_1, \mathbb{Z}^{\oplus 2}) = 0$. Hence it is enough to set $F_i(z) = 0$ for all i.

The cases $\mathrm{III}^{(*)}$, $\mathrm{IV}_+^{(*)}$, $\mathrm{I}_{(+)}^{(*)}(0)$, $\mathrm{I}_{(+)}^{(*)}(1)$: In the proof of Theorem 2.2.1, we write $\boldsymbol{b} = \delta\boldsymbol{u}$ and $\boldsymbol{u} = (1, 0, \ldots, 0) \cdot P$ for some $P \in \mathrm{GL}(l, A)$. If $\boldsymbol{x} \in \bigwedge^2(A^{\oplus l})$ satisfies $\boldsymbol{b} \wedge \boldsymbol{x} = 0$, then $\boldsymbol{x} = \boldsymbol{u} \wedge \boldsymbol{y}$ for some $\boldsymbol{y} \in A^{\oplus l}$.

For $\boldsymbol{y} = (y_1, y_2, \ldots, y_l) \in A^{\oplus l}$, there are integers m_i, n_i for $1 \le i \le l$ such that

$$y_i = \begin{cases} m_i \boldsymbol{i} + n_i, & \text{in the case III}^{(*)}; \\ m_i \boldsymbol{\omega} + n_i, & \text{in the case IV}_+^{(*)}; \\ m_i + n_i \varepsilon & \text{in the cases I}_{(+)}^{(*)}(0),\ \text{I}_{(+)}^{(*)}(1). \end{cases}$$

Looking at $\boldsymbol{u} \wedge \boldsymbol{y} = \delta^{-1}\boldsymbol{b} \wedge \boldsymbol{y}$, we define rational numbers p_i, q_i for $1 \le i \le l$ by

$$\delta^{-1} y_i = \begin{cases} p_i \boldsymbol{i} + q_i, & \text{in the case III}^{(*)}; \\ p_i \boldsymbol{\omega} + q_i & \text{in the case IV}_+^{(*)}; \\ p_i + q_i \varepsilon & \text{in the cases I}_{(+)}^{(*)}(0),\ \text{I}_{(+)}^{(*)}(1). \end{cases}$$

We set $F_i(z) := p_i \omega(z) + q_i$. Then $(P_{i,j}, Q_{i,j})$ defined by $\{F_i\}$ as above is calculated by

$$(P_{i,j}, Q_{i,j}) = (p_i, q_i)(I - \rho(\gamma_j)) - (p_j, q_j)(I - \rho(\gamma_i)).$$

Thus $\boldsymbol{x}$ induced from $(P_{i,j}, Q_{i,j})$ corresponds to $\boldsymbol{u} \wedge \boldsymbol{y}$. Hence such collections $\{F_i(z)\}$ cover all the cohomology classes in $H^2(\pi_1, \mathbb{Z}^{\oplus 2})$. We have the following expression of $F_i(z)$ by means of (m_i, n_i):

$$F_i(z) = \begin{cases} \dfrac{m_i + n_i}{2}\omega(z) + \dfrac{n_i - m_i}{2}, & \text{in the case III}^{(*)}; \\ \dfrac{m_i + n_i}{3}\omega(z) + \dfrac{2n_i - m_i}{3}, & \text{in the case IV}_+^{(*)}; \\ \dfrac{m_i}{2}\omega(z) + \dfrac{n_i}{2}, & \text{in the case I}_{(+)}^{(*)}(0); \\ \dfrac{m_i}{2}\omega(z) + \dfrac{m_i + 2n_i}{4}, & \text{in the case I}_{(+)}^{(*)}(1). \end{cases}$$

For two collections of pairs of integers $\{(m_i, n_i)\}$ and $\{(m'_i, n'_i)\}$, they define a same cohomology class if and only if there is an integer k such that

$$k\delta^{-1} b_i = \begin{cases} (m_i - m'_i)\boldsymbol{i} + (n_i - n'_i), & \text{in the case III}^{(*)}; \\ (m_i - m'_i)\boldsymbol{\omega} + (n_i - n'_i), & \text{in the case IV}_+^{(*)}; \\ (m_i - m'_i) + (n_i - n'_i)\varepsilon, & \text{in the cases I}_{(+)}^{(*)}(0),\ \text{I}_{(+)}^{(*)}(1). \end{cases}$$

The case $\text{I}_{(+)}$: Let $\omega(z) = \sum_{i=1}^{l} a_i z_i + h(t)$ be the period function. Let m_i and $n_{i,j}$ are integers for $1 \le i, j \le l$ such that $n_{j,i} = -n_{i,j}$. Let

$\alpha = \gcd(a_1, a_2, \dots, a_l)$. We set

$$F_i(z) := \frac{1}{2\alpha}(m_i\omega(z)^2 - \sum_{k=1}^{l}(m_i a_k^2 + \alpha n_{i,k})z_k).$$

Then for $1 \le i, j \le l$,

$$F_j(\gamma_i z) - F_j(z) - (F_i(\gamma_j z) - F_i(z)) = \frac{1}{\alpha}(a_i m_j - a_j m_i)\omega(z) + n_{i,j}.$$

By the proof of Theorem 2.2.1, these $\{F_i\}$ cover all the cohomology classes in $H^2(\pi_1, \mathbb{Z}^{\oplus 2})$. For two collections $\{m_i, n_{i,j}\}$ and $\{m'_i, n'_{i,j}\}$, the corresponding $\{F_i\}$'s determine same cohomology class if and only if there exists a vector $(v_1, v_2, \dots, v_l) \in \mathbb{Z}^{\oplus l}$ such that

$$a_i(m_j - m'_j) = a_j(m_i - m'_i) \quad \text{and} \quad n_{i,j} - n'_{i,j} = a_i v_j - a_j v_i.$$

The case $\mathrm{I}^{(*)}_{(+)}(2)$: By the proof of Theorem 2.2.1, if an $\boldsymbol{x} \in \bigwedge^2(A^{\oplus l})$ satisfies $\boldsymbol{b} \wedge \boldsymbol{x} = 0$, where $A = \mathbb{Z}[\varepsilon]$, $\boldsymbol{b} = 2\boldsymbol{c} - \varepsilon\boldsymbol{a}^*$, then there exist vectors $\boldsymbol{y}'_1, \boldsymbol{w}_0, \boldsymbol{w}_1 \in \mathbb{Z}^{\oplus l}$ and an integer z_1 such that

$$\boldsymbol{x} = z_1\boldsymbol{c} \wedge \boldsymbol{a}^* + \varepsilon\boldsymbol{c} \wedge \boldsymbol{y}'_1 + \boldsymbol{b} \wedge (\boldsymbol{w}_0 + \varepsilon\boldsymbol{w}_1).$$

We denote $z_1 = m$ and $\boldsymbol{y}'_1 = (n_1, n_2, \dots, n_l)$. If we set $p_i = (m/2)a_i^*$, $q_i = n_i/2$, then

$$(2c_i - \varepsilon a_i^*)(p_j + \varepsilon q_j) - (2c_j - \varepsilon a_j^*)(p_i + \varepsilon q_i) = m(c_i a_j^* - c_j a_i^*) + \varepsilon(c_i n_j - c_j n_i).$$

Let us consider the functions:

$$F_i(z) = \frac{m a_i^*}{2}\omega(z) + \frac{n_i}{2}.$$

Then $\{F_i(z)\}$ satisfies the cocycle condition (2.4) and every element in $H^2(\pi_1, \mathbb{Z}^{\oplus 2})$ is induced from such $\{F_i(z)\}$ for some $m, n_1, n_2, \dots, n_l \in \mathbb{Z}$. For two collections of integers $\{m, n_1, n_2, \dots, n_l\}$, $\{m', n'_1, n'_2, \dots, n'_l\}$, the corresponding $\{F_i(z)\}$'s determine same cohomology class if and only if $m \equiv m' \mod 2$ and

$$(n_1, n_2, \dots, n_l) \wedge \boldsymbol{c} \wedge \boldsymbol{a}^* \equiv (n'_1, n'_2, \dots, n'_l) \wedge \boldsymbol{c} \wedge \boldsymbol{a}^* \mod 2.$$

Therefore by combining with Proposition 1.3.3, we have:

Theorem 2.2.2. *Let* $f\colon X \to S^\star \simeq (\Delta^\star)^l \times \Delta^{d-l}$ *be a smooth elliptic fibration. Then* X *is isomorphic to the quotient of the total space* B_U *of the basic fibration* $B_U \to U$ *by the following action of* $\pi_1 \simeq \mathbb{Z}^{\oplus l}$:

$$B_U \ni [z, \zeta] \mapsto \left[\gamma_i z, \frac{\zeta + F_i(z)}{c_{\gamma_i}\omega(z) + d_{\gamma_i}}\right],$$

Table 6. Collections of holomorphic functions.

Type	$F_i(z)$	Condition
I_0	$\frac{1}{2}\sum_{k=1}^{l}(m_{i,k}\omega(z)+n_{i,k})z_k.$	$m_{i,j}, n_{i,j} \in \mathbb{Z}$: $m_{j,i}=-m_{i,j}$, $n_{j,i}=-n_{i,j}$
$\mathrm{I}_0^{(*)}$	$\frac{m_i}{2}\omega(z)+\frac{n_i}{2}$	$m_i, n_i \in \mathbb{Z}$
$\mathrm{II}^{(*)}$	0	
$\mathrm{III}^{(*)}$	$\frac{m_i+n_i}{2}\omega(z)+\frac{n_i-m_i}{2}$	$m_i, n_i \in \mathbb{Z}$
$\mathrm{IV}_+^{(*)}$	$\frac{m_i+n_i}{3}\omega(z)+\frac{2n_i-m_i}{3}$	$m_i, n_i \in \mathbb{Z}$
$\mathrm{IV}_-^{(*)}$	0	
$\mathrm{I}_{(+)}$	$\frac{1}{2\alpha}\left(m_i\omega(z)^2-\sum_{k=1}^{l}(m_i a_k^2+\alpha n_{i,k})z_k\right)$	$m_i, n_{i,j} \in \mathbb{Z}$: $n_{j,i}=-n_{i,j}$
$\mathrm{I}_{(+)}^{(*)}(0)$	$\frac{m_i}{2}\omega(z)+\frac{n_i}{2}$	$m_i, n_i \in \mathbb{Z}$
$\mathrm{I}_{(+)}^{(*)}(1)$	$\frac{m_i}{2}\omega(z)+\frac{m_i+2n_i}{4}$	$m_i, n_i \in \mathbb{Z}$
$\mathrm{I}_{(+)}^{(*)}(2)$	$\frac{ma_i^*}{2}\omega(z)+\frac{n_i}{2}$	$m, n_i \in \mathbb{Z}$

where $[z,\zeta] \in B_U$ *is the image of a point* $(z,\zeta) \in U \times \mathbb{C}$, *and* $\{F_i(z)\}$ *is one of the collections of holomorphic functions listed in Table* 6. *If* H *is not of type* I_0 *nor* $\mathrm{I}_{(+)}$, *then* f *is a projective morphism. If* H *is of type* I_0 *and* f *is projective, then we can take* $F_i(z)=0$ *for any* i.

§3. Canonical extensions of variations of Hodge structures

3.1. Canonical extensions

Let H be a variation of Hodge structures of weight one, rank two on $S^\star = (\Delta^\star)^l \times \Delta^{d-l}$. As in §§1 and 2, H is determined by the monodromy representation $\rho\colon \pi_1 := \pi_1(S^\star) \simeq \mathbb{Z}^{\oplus l} \to \mathrm{SL}(2,\mathbb{Z})$ and the period function $\omega(z)$. Let $\rho(\gamma) = S(\gamma)U(\gamma)$ be a decomposition of the monodromy matrix $\rho(\gamma)$ for $\gamma \in \pi_1$ such that $S(\gamma)$ is semi-simple, $U(\gamma)$ is unipotent, and $S(\gamma)U(\gamma) = U(\gamma)S(\gamma)$. If H is one of types I_0, $\mathrm{I}_0^{(*)}$, $\mathrm{I}_{(+)}$, $\mathrm{I}_{(+)}^{(*)}$, then $S(\gamma) = \pm I$ for any $\gamma \in \pi_1$, where I denotes the unit matrix. If H is of other type, then $U(\gamma) = I$ for any $\gamma \in \pi_1$. Thus all $S(\gamma)$ and $U(\gamma)$ are uniquely determined and commute to each other. The eigenvalue of

Table 7. Order of $S(\gamma_i)$.

$\rho(\gamma_i)$	I_a	I_b^*	II	II*	III	III*	IV	IV*
m_i	1	2	6	6	4	4	3	3

$S(\gamma_i)$ is contained in $\{\pm 1, \pm\boldsymbol{\omega}^{\pm 1}, \pm\boldsymbol{i}\}$. Let m_i be its order (cf. Table 7). Now we consider the *unipotent reduction* of H. This is a Kummer covering defined by:

$$\tau \colon T = \Delta^l \times \Delta^{d-l} \ni \theta = (\theta_1, \theta_2, \dots, \theta_l, t') \longmapsto (\theta_1^{m_1}, \theta_2^{m_2}, \dots, \theta_l^{m_l}, t') \in \Delta^l \times \Delta^{d-l} = S.$$

We denote $T^\star := \tau^{-1}(S^\star)$ and let $H_T := \tau^{-1}H$ be the induced variation of Hodge structures on $T^\star$. Then all the monodromy matrices of H_T are unipotent. Let N_i, R_i^u and R_i^ℓ for $1 \le i \le l$ be the matrices satisfying the following four conditions:

(1) N_i is nilpotent and $\exp(N_i) = U(\gamma_i)$;
(2) R_i^u, R_i^ℓ are semi-simple and $\exp(R_i^u) = \exp(R_i^\ell) = S(\gamma_i)$;
(3) All the eigenvalues of R_i^u are contained in $2\pi\sqrt{-1}\,(-1, 0]$;
(4) All the eigenvalues of R_i^ℓ are contained in $2\pi\sqrt{-1}\,[0, 1)$.

Then these matrices also commute to each other. Let M_i^u and M_i^ℓ be the matrices $R_i^u + N_i$ and $R_i^\ell + N_i$, respectively, for $1 \le i \le l$. Let $\boldsymbol{e} \colon U = \mathbb{H}^l \times \Delta^{d-l} \to (\Delta^\star)^l \times \Delta^{d-l} = S^\star$ be the universal covering map defined in §2.1 and let (e_0, e_1) be the basis of $H^0(U, \boldsymbol{e}^{-1}H) \simeq \mathbb{Z}^{\oplus 2}$ defined in §1.1. The e_1 and e_0, respectively, are identified with column vectors

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad e_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and they satisfy $Q(e_0, e_1) = 1$, where Q is the polarization of H. Let $\mathcal{H} = H \otimes \mathcal{O}_{S^\star}$. Then $\boldsymbol{e}^{-1}\mathcal{H} \simeq \mathcal{O}_U^{\oplus 2}$. Therefore as in §2.1, the right action of $\gamma \in \pi_1$ induced from $\mathcal{H}$ on $H^0(U, \mathcal{O}^{\oplus 2})$ is written by

$$\boldsymbol{v}^\gamma(z) := \rho(\gamma)^{-1}\boldsymbol{v}(\gamma z),$$

where we consider $\boldsymbol{v}(z) \in H^0(U, \mathcal{O}_U^{\oplus 2})$ as a column vector. The holomorphic vector $\boldsymbol{v}(z)$ is invariant under this action if and only if $\boldsymbol{v} = \boldsymbol{e}^* v$ for some $v \in H^0(S^\star, \mathcal{H})$. Therefore for holomorphic vectors

$$^u\boldsymbol{v}_1(z) := \exp(\sum_{i=1}^{l} z_i M_i^u) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad {}^u\boldsymbol{v}_0(z) := \exp(\sum_{i=1}^{l} z_i M_i^u) \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$^\ell\boldsymbol{v}_1(z) := \exp(\sum_{i=1}^{l} z_i M_i^\ell) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad {}^\ell\boldsymbol{v}_0(z) := \exp(\sum_{i=1}^{l} z_i M_i^\ell) \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

we have global holomorphic sections ${}^uv_1$, ${}^uv_0$, ${}^\ell v_1$, ${}^\ell v_0 \in \Gamma(S^\star, \mathcal{H})$ such that ${}^u\boldsymbol{v}_i = \boldsymbol{e}^*({}^uv_i)$ and ${}^\ell\boldsymbol{v}_i = \boldsymbol{e}^*({}^\ell v_i)$ for $i = 0, 1$. Thus $\mathcal{H} = \mathcal{O}_{S^\star}\, {}^uv_1 \oplus \mathcal{O}_{S^\star}\, {}^uv_0 = \mathcal{O}_{S^\star}\, {}^\ell v_1 \oplus \mathcal{O}_{S^\star}\, {}^\ell v_0$.

Definition 3.1.1 (cf. [Kl1], [Mw]). The *upper* and the *lower canonical extensions* ${}^u\mathcal{H}$ and ${}^\ell\mathcal{H}$ of $\mathcal{H}$ to S are defined to be the subsheaves $\mathcal{O}_S\, {}^uv_1 \oplus \mathcal{O}_S\, {}^uv_2$ and $\mathcal{O}_S\, {}^\ell v_1 \oplus \mathcal{O}_S\, {}^\ell v_2$ of $j_*\mathcal{H}$, respectively, where $j\colon S^\star \hookrightarrow S$ denotes the open immersion. We define the induced filtration by:

$$\mathcal{F}^p({}^u\mathcal{H}) := j_*\mathcal{F}^p(\mathcal{H}) \cap {}^u\mathcal{H}, \quad \mathcal{F}^p({}^\ell\mathcal{H}) := j_*\mathcal{F}^p(\mathcal{H}) \cap {}^\ell\mathcal{H}$$

and define a quotient sheaf $\mathcal{L}_{H/S} := {}^\ell\mathcal{H}/\mathcal{F}^1({}^\ell\mathcal{H})$.

Remark. (1). We have ${}^\ell\mathcal{H} \subset {}^u\mathcal{H}$. If the monodromy matrices $\rho(\gamma_i)$ are all unipotent, then ${}^u\mathcal{H} = {}^\ell\mathcal{H}$. Thus ${}^u(\tau^*\mathcal{H}) = {}^\ell(\tau^*\mathcal{H})$. We see that ${}^\ell\mathcal{H}$ is the Gal(τ)-invariant part of $\tau_*({}^\ell(\tau^*\mathcal{H}))$ and that ${}^\ell(\mathcal{H}^\vee) \simeq ({}^u\mathcal{H})^\vee$, where $\mathcal{F}^\vee$ denotes the dual $\mathcal{H}om(\mathcal{F}, \mathcal{O})$.

(2). Let H be a variation of Hodge structures of weight one, rank two on $M \setminus D$, where M is a complex manifold and D is a normal crossing divisor on M. Then the local canonical extensions ${}^u\mathcal{H}$ and ${}^\ell\mathcal{H}$ are patched together. Thus we can define globally the upper and the lower canonical extensions to M.

The following result is known as a part of the nilpotent orbit theorem [Sc].

Lemma 3.1.2. *$\mathcal{F}^1({}^u\mathcal{H})$ and $\mathcal{F}^1({}^\ell\mathcal{H})$ are subbundles of rank one of ${}^u\mathcal{H}$ and ${}^\ell\mathcal{H}$, respectively. In particular, $\mathcal{L}_{H/S}$ is an invertible sheaf.*

Proof. $\mathcal{F}^1({}^\ell\mathcal{H})$ is the Gal(τ)-invariant part of $\tau_*\mathcal{F}^1({}^\ell(\tau^*\mathcal{H}))$ for the unipotent reduction. If $\mathcal{F}^1({}^\ell(\tau^*\mathcal{H}))$ is a subbundle of ${}^\ell(\tau^*\mathcal{H})$, then $\mathcal{F}^1({}^\ell\mathcal{H})$ is also a subbundle of ${}^\ell\mathcal{H}$, $\mathcal{L}_{H/S}$ is an invertible sheaf, and $\mathcal{F}^1({}^u\mathcal{H})$ is an invertible sheaf dual to $\mathcal{L}_{H/S}$. Thus we may assume that the monodromy of H is unipotent. We consider a generator $\omega(z)e_1 + e_0$ of $\boldsymbol{e}^*\mathcal{F}^1(\mathcal{H})$ corresponding to

$$\begin{pmatrix} \omega(z) \\ 1 \end{pmatrix}.$$

Now we have $\omega(z) = \sum_{i=1}^{l} a_i z_i + h(t)$ for a holomorphic function $h(t)$ by Proposition 2.1.4 and

$$M_i^\ell = N_i = \begin{pmatrix} 0 & a_i \\ 0 & 0 \end{pmatrix},$$

where we consider $a_i = 0$ in the case H is of type I_0. Thus

$$\begin{pmatrix} \omega(z) \\ 1 \end{pmatrix} = \exp(\sum_{i=1}^{l} z_i M_i^\ell) \begin{pmatrix} h(t) \\ 1 \end{pmatrix}.$$

Therefore the generator is written by $h(t)\,{}^\ell v_1 + {}^\ell v_2$. Hence $\mathcal{F}^1({}^\ell\mathcal{H})$ is generated by $h(t)\,{}^\ell v_1 + {}^\ell v_2$ and is a subbundle of ${}^\ell\mathcal{H}$. Q.E.D.

Lemma 3.1.3. *There exist natural injections $j_*H \to {}^\ell\mathcal{H}$ and $j_*H \to {}^\ell\mathcal{H} \to \mathcal{L}_{H/S}$.*

Proof. We have only to check the image of $j_*H \to j_*\mathcal{H}$ is contained in ${}^\ell\mathcal{H}$, since $H \cap \mathcal{F}^1(\mathcal{H}) = 0$. The stalk $(j_*H)_0$ is the π_1-invariant part of $\Gamma(U, \boldsymbol{e}^{-1}H)$. If H is neither of types I_0 nor $\mathrm{I}_{(+)}$, then the stalk $(j_*H)_0$ is zero. Assume that H is of type $\mathrm{I}_{(+)}$. Then

$$M_i^\ell = \begin{pmatrix} 0 & a_i \\ 0 & 0 \end{pmatrix}$$

for any i. Hence the stalk $(j_*H)_0 \simeq \mathbb{Z}$ is generated by $e_1 \in H^0(U, \boldsymbol{e}^{-1}H)$ above and

$${}^\ell\boldsymbol{v}_1(z) = \exp(\sum_{i=1}^{l} z_i M_i^\ell) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Therefore the image of e_1 is contained in $({}^\ell\mathcal{H})_0$. Finally, assume that H is of type I_0. Then all $M_i^\ell = 0$. Thus ${}^\ell\boldsymbol{v}_1(z) = e_1$ and ${}^\ell\boldsymbol{v}_0(z) = e_0$. Hence images of e_1, e_0 are contained in $({}^\ell\mathcal{H})_0$. Thus $j_*H \subset {}^\ell\mathcal{H}$. Q.E.D.

For the period function $\omega(z)$, we have:

$$\begin{pmatrix} \omega(\gamma z) \\ 1 \end{pmatrix} = (c_\gamma \omega(z) + d_\gamma)^{-1} \rho(\gamma) \begin{pmatrix} \omega(z) \\ 1 \end{pmatrix}$$

for any $\gamma \in \pi_1$, from the formula (1.1). Let us consider the following holomorphic vectors

$${}^u\mathbf{V}(z) := \exp(-\sum_i z_i M_i^u) \begin{pmatrix} \omega(z) \\ 1 \end{pmatrix}, \quad {}^\ell\mathbf{V}(z) := \exp(-\sum_i z_i M_i^\ell) \begin{pmatrix} \omega(z) \\ 1 \end{pmatrix},$$

$$\boldsymbol{u}(z) := \exp(-\sum_i z_i N_i) \begin{pmatrix} \omega(z) \\ 1 \end{pmatrix}.$$

Then for $\gamma \in \pi_1$, we have:

$$\begin{aligned} {}^u\mathbf{V}(\gamma z) &= (c_\gamma \omega(z) + d_\gamma)^{-1}\, {}^u\mathbf{V}(z), \\ {}^\ell\mathbf{V}(\gamma z) &= (c_\gamma \omega(z) + d_\gamma)^{-1}\, {}^\ell\mathbf{V}(z), \\ \boldsymbol{u}(\gamma z) &= (c_\gamma \omega(z) + d_\gamma)^{-1} S(\gamma) \boldsymbol{u}(z). \end{aligned} \tag{3.1}$$

Since $S(\gamma) = I$ for $\gamma \in \pi_1(T^\star) = \bigoplus_{i=1}^{l} m_i\mathbb{Z} \subset \pi_1$ and since

$$N_i = \begin{pmatrix} 0 & a_i \\ 0 & 0 \end{pmatrix}$$

for some $a_i \geq 0$, there exists a holomorphic function $h(\theta)$ defined on $T^\star$ such that

$$\boldsymbol{u}(z) = \begin{pmatrix} h(\theta) \\ 1 \end{pmatrix}, \quad \omega(z) = \sum_{i=1}^{l} a_i z_i + h(\theta),$$

where we write $\theta = (\theta', t') \in (\Delta^\star)^l \times \Delta^{d-l} = T^\star$. By Proposition 2.1.4, $h(\theta)$ is holomorphic on T. Note that one of a_i is positive if and only if the monodromy group is not finite. For γ_i, let $\gamma_i\theta$ be the point

$$(\theta_1, \theta_2, \ldots, \theta_{i-1}, e(1/m_i)\theta_i, \theta_{i+1}, \ldots, \theta_l, t').$$

Then we can define $\gamma\theta$ also for $\gamma \in \pi_1$. By (3.1), we have:

$$\begin{pmatrix} h(\gamma\theta) \\ 1 \end{pmatrix} = (c_\gamma \omega(z) + d_\gamma)^{-1} S(\gamma) \begin{pmatrix} h(\theta) \\ 1 \end{pmatrix}. \tag{3.2}$$

Note that $c_\gamma \omega(z) + d_\gamma = c_\gamma h(\theta) + d_\gamma$. Therefore

$$S(\gamma) \begin{pmatrix} h(0) \\ 1 \end{pmatrix} = (c_\gamma h(0) + d_\gamma) \begin{pmatrix} h(0) \\ 1 \end{pmatrix}.$$

Thus, $c_\gamma h(0) + d_\gamma$ is an eigenvalue of $S(\gamma)$. If $S(\gamma) = \pm I$ for any γ, i.e., H is one of types I_0, $\mathrm{I}_0^{(*)}$, $\mathrm{I}_{(+)}$, $\mathrm{I}_{(+)}^{(*)}$, then $h(\theta)$ is a holomorphic function on $t \in S$. In the case H is one of types $\mathrm{II}^{(*)}$, $\mathrm{III}^{(*)}$, $\mathrm{IV}_+^{(*)}$, $\mathrm{IV}_-^{(*)}$, we define the matrix:

$$P := \begin{pmatrix} h(0) & \overline{h(0)} \\ 1 & 1 \end{pmatrix}.$$

Then we have

$$P^{-1} S(\gamma) P = \begin{pmatrix} (c_\gamma h(0) + d_\gamma) & 0 \\ 0 & (c_\gamma h(0) + d_\gamma)^{-1} \end{pmatrix}$$

for any $\gamma \in \pi_1$. In any case of H, we have integers $-m_i < p_i, q_i \leq 0$ and $0 \leq p'_i, q'_i < m_i$ for $1 \leq i \leq l$ satisfying the following condition (cf. Table 8):

$$c_{\gamma_i} h(0) + d_{\gamma_i} = e\left(\frac{p_i}{m_i}\right) = e\left(\frac{p'_i}{m_i}\right) = e\left(-\frac{q_i}{m_i}\right) = e\left(-\frac{q'_i}{m_i}\right).$$

Here $-p_i = q'_i$ and $-q_i = p'_i$. We define rational numbers $\delta_i := -p_i/m_i$.

Lemma 3.1.4. *Suppose that H is one of types $\mathrm{II}^{(*)}$, $\mathrm{III}^{(*)}$, $\mathrm{IV}_+^{(*)}$, $\mathrm{IV}_-^{(*)}$. Let A be the algebra defined in the proof of Theorem* 2.2.1 *and let*

Table 8. Related numbers for monodromy matrices.

$\rho(\gamma_i)$	I_a	I_b^*	II	II*	III	III*	IV	IV*
m_i	1	2	6	6	4	4	3	3
$-p_i = q_i'$	0	1	1	5	1	3	1	2
$-q_i = p_i'$	0	1	5	1	3	1	2	1
$\delta_i := -p_i/m_i$	0	1/2	1/6	5/6	1/4	3/4	1/3	2/3

w_A be the constant defined by:

$$w_A = \begin{cases} \boldsymbol{\omega}, & \text{in the case } A = \mathbb{Z}[\boldsymbol{\omega}]; \\ \boldsymbol{i}, & \text{in the case } A = \mathbb{Z}[\boldsymbol{i}]. \end{cases}$$

Then $\phi(\gamma) = c_\gamma h(0) + d_\gamma$ in $A \subset \mathbb{C}$ for any γ, where $\phi(\gamma)$ is defined in Table 5. In particular, $h(0) = w_A$. Let $\psi(\theta)$ be the holomorphic function

$$\psi(\theta) := \frac{h(\theta) - w_A}{h(\theta) - \overline{w_A}}$$

defined on T. Then it satisfies the following conditions:

(1) $\psi(0) = 0$;
(2) *For any $\theta \in T$ and $\gamma \in \pi_1$, $|\psi(\theta)| < 1$ and $\psi(\gamma\theta) = \phi(\gamma)^{-2}\psi(\theta)$;*
(3) *For any $\theta \in T$, and $\gamma \in \pi_1$,*

$$h(\theta) = \frac{w_A - \overline{w_A}\psi(\theta)}{1 - \psi(\theta)} \quad \text{and} \quad c_\gamma h(\theta) + d_\gamma = \phi(\gamma)\frac{1 - \psi(\gamma\theta)}{1 - \psi(\theta)};$$

(4) *There is a holomorphic function $\psi_0(t)$ on S such that $|\psi_0(t)| \leq 1$ for any $t \in S$ and*

$$\psi(\theta) = \psi_0(t) \prod_{i=1}^{l} \theta_i^{\langle 2\delta_i \rangle m_i}.$$

Proof. Since all $a_i = 0$ in this case, we have $\omega(z) = h(\theta)$. Thus $\operatorname{Im} h(\theta) > 0$. Since $c_\gamma h(0) + d_\gamma$ is an eigenvalue of $\rho(\gamma) = S(\gamma)$ and $\operatorname{Im} h(0) > 0$, we have $\phi(\gamma) = c_\gamma h(0) + d_\gamma$. The equality $h(0) = w_A$ is derived from Table 1. Then we have

$$P^{-1}S(\gamma)P = \begin{pmatrix} \phi(\gamma) & 0 \\ 0 & \phi(\gamma)^{-1} \end{pmatrix},$$

for any $\gamma \in \pi_1$ and for the matrix

$$P = \begin{pmatrix} h(0) & \overline{h(0)} \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} w_A & \overline{w_A} \\ 1 & 1 \end{pmatrix}.$$

Thus for the holomorphic vector

$$\begin{pmatrix} a(\theta) \\ b(\theta) \end{pmatrix} := P^{-1}\begin{pmatrix} h(\theta) \\ 1 \end{pmatrix},$$

we have

$$a(\theta) = \frac{h(\theta) - \overline{w_A}}{w_A - \overline{w_A}}, \qquad b(\theta) = \frac{-h(\theta) + w_A}{w_A - \overline{w_A}},$$
$$a(\gamma\theta) = (c_\gamma h(\theta) + d_\gamma)^{-1}\phi(\gamma)a(\theta), \quad b(\gamma\theta) = (c_\gamma h(\theta) + d_\gamma)^{-1}\phi(\gamma)^{-1}b(\theta).$$

Note that $a(\theta)$ is a nowhere vanishing function on T. Since $\psi(\theta) = -a(\theta)^{-1}b(\theta)$, we have

$$h(\theta) = \frac{w_A - \overline{w_A}\psi(\theta)}{1 - \psi(\theta)} \quad \text{and} \quad \alpha(\theta)^{-1} = 1 - \psi(\theta).$$

Thus $\psi(\theta)$ satisfies the required conditions. Q.E.D.

Corollary 3.1.5. *A variation of Hodge structures H of one of types $\mathrm{II}^{(*)}$, $\mathrm{III}^{(*)}$, $\mathrm{IV}_+^{(*)}$, $\mathrm{IV}_-^{(*)}$ on $S^\star$ is determined by a surjective group homomorphism $\phi\colon \pi_1 \twoheadrightarrow A^\star \subset \mathbb{C}^\star$ and a holomorphic function $\widetilde{\psi}(z)$ on U such that $|\widetilde{\psi}(z)| < 1$ and $\widetilde{\psi}(\gamma z) = \phi(\gamma)^{-2}\widetilde{\psi}(z)$ for any $z \in U$ and $\gamma \in \pi_1$, where A is one of subalgebras $\mathbb{Z}[\boldsymbol{\omega}]$ and $\mathbb{Z}[\boldsymbol{i}]$ of $\mathbb{C}$ and $A^\star := A \cap \mathbb{C}^\star$. Here the period function is given by*

$$\omega(z) = \frac{w_A - \overline{w_A}\widetilde{\psi}(z)}{1 - \widetilde{\psi}(z)}.$$

Corollary 3.1.6. *The period function $\omega(z)$ is written in the following form according as the type of monodromy representation:*

I_0, $\mathrm{I}_0^{(*)}$: $\quad \omega(z) = h(t),$ *where* $\operatorname{Im} h(t) > 0$;

$\mathrm{I}_{(+)}, \mathrm{I}_{(+)}^{(*)}$: $\quad \omega(z) = \sum_{i=1}^{l} a_i z_i + h(t),$ *where* $\operatorname{Im} h(t) \geq 0$;

$\mathrm{II}^{(*)}, \mathrm{IV}_+^{(*)}, \mathrm{IV}_-^{(*)}$:

$$\omega(z) = \frac{\boldsymbol{\omega} - \boldsymbol{\omega}^2\psi_0(t)\prod_{i=1}^{l}\theta_i^{\langle 2\delta_i\rangle m_i}}{1 - \psi_0(t)\prod_{i=1}^{l}\theta_i^{\langle 2\delta_i\rangle m_i}}, \qquad \textit{where } |\psi_0(t)| \leq 1;$$

$$\mathrm{III}^{(*)}: \qquad \omega(z) = \frac{\boldsymbol{i} + \boldsymbol{i}\psi_0(t)\prod_{i=1}^{l}\theta_i^{\langle 2\delta_i\rangle m_i}}{1 - \psi_0(t)\prod_{i=1}^{l}\theta_i^{\langle 2\delta_i\rangle m_i}}, \qquad \textit{where } |\psi_0(t)| \leq 1.$$

We shall describe generators of $\mathcal{F}^1({}^u\mathcal{H})$ and $\mathcal{F}^1({}^\ell\mathcal{H})$, explicitly. In the case H is one of types $\mathrm{II}^{(*)}$, $\mathrm{III}^{(*)}$, $\mathrm{IV}_+^{(*)}$, $\mathrm{IV}_-^{(*)}$, we can write

$$P^{-1}\begin{pmatrix} h(\theta) \\ 1 \end{pmatrix} = \frac{1}{1-\psi(\theta)}\begin{pmatrix} 1 \\ -\psi(\theta) \end{pmatrix},$$

where the function $\psi(\theta)$ is written by

$$\psi(\theta) = \psi_0(t)\prod_{i=1}^{l}\theta_i^{\langle 2\delta_i\rangle m_i}$$

for a holomorphic function $\psi_0(t)$ defined on S. We see that $(p_i+p_i')/m_i + \langle 2\delta_i\rangle$ is 0 or 1 for any i. Let us define holomorphic functions $A(t)$ and $B(t)$ on S by:

$$\begin{pmatrix} A(t) \\ B(t) \end{pmatrix} := P\begin{pmatrix} 1 \\ -\psi_0(t)\prod_{i=1}^{l}\theta_i^{p_i+p_i'+\langle 2\delta_i\rangle m_i} \end{pmatrix}.$$

We define also a holomorphic function $a(\theta) := (1-\psi(\theta))^{-1}$ over T. In the case H is one of types I_0, $\mathrm{I}_0^{(*)}$, $\mathrm{I}_{(+)}$, $\mathrm{I}_{(+)}^{(*)}$, we set $P := I$,

$$\begin{pmatrix} A(t) \\ B(t) \end{pmatrix} := \begin{pmatrix} h(\theta) \\ 1 \end{pmatrix},$$

and $a(\theta) := 1$. Then we see that $A(t)$ and $B(t)$ have no common zeros on S in any case.

Lemma 3.1.7. *$A(t)\,{}^u v_1 + B(t)\,{}^u v_0$ and $A(t)\,{}^\ell v_1 + B(t)\,{}^\ell v_0$ are generators of $\mathcal{F}^1({}^u\mathcal{H})$ and $\mathcal{F}^1({}^\ell\mathcal{H})$, respectively.*

Proof. We can write

$${}^u\mathbf{V}(z) = \exp\Big(-\sum_{i=1}^{l} z_i R_i^u\Big)\begin{pmatrix} h(\theta) \\ 1 \end{pmatrix}, \quad {}^\ell\mathbf{V}(z) = \exp\Big(-\sum_{i=1}^{l} z_i R_i^\ell\Big)\begin{pmatrix} h(\theta) \\ 1 \end{pmatrix}.$$

By definition, we see

$$P^{-1}R_i^u P = 2\pi\sqrt{-1}\begin{pmatrix} \dfrac{p_i}{m_i} & 0 \\ 0 & \dfrac{q_i}{m_i} \end{pmatrix}, \quad PR_i^\ell P^{-1} = 2\pi\sqrt{-1}\begin{pmatrix} \dfrac{p_i'}{m_i} & 0 \\ 0 & \dfrac{q_i'}{m_i} \end{pmatrix}.$$

Therefore we have

$$(3.3) \qquad {}^u\mathbf{V}(z) = a(\theta)\prod_{i=1}^{l}\theta_i^{-p_i}\begin{pmatrix}A(t)\\B(t)\end{pmatrix},$$

$$(3.4) \qquad {}^\ell\mathbf{V}(z) = a(\theta)\prod_{i=1}^{l}\theta_i^{-p'_i}\begin{pmatrix}A(t)\\B(t)\end{pmatrix}.$$

Let $v \in \Gamma(S, {}^u\mathcal{H})$ be a section such that $v_{|S^\star} \in \Gamma(S^\star, \mathcal{F}^1(\mathcal{H}))$. Then $v = u_1(t)\,{}^uv_1 + u_0(t)\,{}^uv_0$ for some holomorphic functions $u_1(t), u_0(t)$ on S and $\boldsymbol{e}^*(v) = \widetilde{\varphi}(z)(e_0 + \omega(z)e_1)$ for a holomorphic function $\widetilde{\varphi}(z)$ on U. Now $\boldsymbol{e}^*(v)$ corresponds to the vector

$$\exp(\sum_{i=1}^{l} z_i M_i^u)\begin{pmatrix}u_1(t)\\u_0(t)\end{pmatrix}.$$

Hence

$$\begin{pmatrix}u_1(t)\\u_0(t)\end{pmatrix} = \widetilde{\varphi}(z)^u\mathbf{V}(z) = \widetilde{\varphi}(z)a(\theta)\prod_{i=1}^{l}\theta_i^{-p_i}\begin{pmatrix}A(t)\\B(t)\end{pmatrix}$$

by (3.3). Thus $\widetilde{\varphi}(z)a(\theta)\prod_{i=1}^{l}\theta_i^{-p_i}$ is a holomorphic function $\varphi(t)$ on S. Therefore $v = \varphi(t)(A(t)\,{}^uv_1 + B(t)\,{}^uv_0)$. Thus $A(t)\,{}^uv_1 + B(t)\,{}^uv_0$ is a generator of $\mathcal{F}^1({}^u\mathcal{H})$. Similarly we can prove $A(t)\,{}^\ell v_1 + B(t)\,{}^\ell v_0$ is a generator of $\mathcal{F}^1({}^\ell\mathcal{H})$ by using (3.4). Q.E.D.

Let $\boldsymbol{\Delta}(z)$ be the cusp form of weight 12 (cf. Remark 1.2.6). Let the section $\xi \in H^0(S^\star, \mathcal{F}^1(\mathcal{H})^{\otimes 12})$ correspond to $\boldsymbol{\Delta}(\omega(z))(\omega(z)e_1 + e_0)^{\otimes 12}$.

Corollary 3.1.8 (cf. [U]). *The ξ extends to a holomorphic section of $\mathcal{F}^1({}^u\mathcal{H})^{\otimes 12}$ over S. The effective divisor* div(ξ) *is written by* $\sum_{i=1}^{l}(a_i + 12\delta_i)D_i$.

Proof. By the argument above, $\omega(z) = \sum_{i=1}^{l} a_i z_i + h(\theta)$ for non-negative integers a_i and a holomorphic function $h(\theta)$ on T. Note that if $a_i > 0$ then monodromy matrix $\rho(\gamma_i)$ is of type I_{a_i} or $\mathrm{I}^*_{a_i}$. By Remark 1.2.6, the function $\boldsymbol{\Delta}(\omega(z))$ is written as $u(\theta)\prod_{i=1}^{l}\theta_i^{m_i a_i}$ for a nowhere vanishing function $u(\theta)$ on T. On the other hand, by (3.3), we have

$$\omega(z)e_1 + e_0 = a(\theta)(\prod_{i=1}^{l}\theta_i^{-p_i})(A(t)\,{}^uv_1 + B(t)\,{}^uv_0),$$

where $A(t)\,{}^uv_1 + B(t)\,{}^uv_0$ is a generator of $\mathcal{F}^1({}^u\mathcal{H})$ by Lemma 3.1.7. By computing the vanishing order of $\prod_i \theta_i^{-12p_i}\prod_{i=1}^{l}\theta_i^{m_i a_i}$ along each coordinate hyperplane $\{\theta_i = 0\}$, we are done. Q.E.D.

3.2. Torsion free theorems

Let $f\colon Y \to M$ be an elliptic fibration (not necessarily projective) between complex manifolds such that f is smooth outside a normal crossing divisor $D = \bigcup D_i$ on M. Then we have a variation of Hodge structures $H := (R^1 f_* \mathbb{Z}_Y)_{|M \setminus D}$. Let ${}^u\mathcal{H}$ and ${}^\ell\mathcal{H}$ be the upper and the lower canonical extensions, respectively of $\mathcal{H} = H \otimes \mathcal{O}_{M\setminus D}$ to M defined in Definition 3.1.1 (cf. [Kl1], [Mw]). Also we denote by $\mathcal{F}^p({}^u\mathcal{H})$ and $\mathcal{F}^p({}^\ell\mathcal{H})$ the induced p-th filtrations (cf. Definition 3.1.1). As a corollary of Corollary 3.1.8, we have:

Corollary 3.2.1 ([Kw2]). *Let $J\colon M \to \mathbb{P}^1$ be the J-function associated with H. Then there is an isomorphism*

$$\mathcal{F}^1({}^u\mathcal{H})^{\otimes 12} \simeq J^*\mathcal{O}(1) \otimes \mathcal{O}_M(\sum 12\delta_i D_i),$$

where the rational numbers δ_i are determined by the types of the monodromy matrices around D_i as in Table 8.

Proof. By Corollary 3.1.8, $\xi = \boldsymbol{\Delta}(\omega(z))(\omega(z)e_1 + e_0)^{\otimes 12}$ is a section of $H^0(M, \mathcal{F}^1({}^u\mathcal{H})^{\otimes 12})$ such that $\operatorname{div}(\xi) = \sum_i (a_i + 12\delta_i)D_i$. Here if $a_i = a > 0$, then the monodromy matrix around D_i is of type I_a or I_a^*. Thus we have the isomorphism $J^*\mathcal{O}(1) \simeq \mathcal{O}(\sum a_i D_i)$ by Proposition 2.1.4, which implies the expected isomorphism. Q.E.D.

The following theorem was proved by [Kl1] (cf. [Ny2]) for algebraic case and by [Mw] for projective morphisms. On the other hand, Saito independently proved this by using his theory of Hodge modules in [Sa1]. He also had a generalization to the case of Kähler morphisms (cf. [Sa2], [Sa3]). Takegoshi ([Ta]) also gives another proof for Kähler morphisms by an L^2-method.

Theorem 3.2.2. *Let $\pi\colon X \to W$ be a projective surjective morphism from a complex analytic manifold X onto a complex analytic variety W. Then the higher direct images $R^i\pi_*\omega_X$ are torsion free for $i \geq 0$. Moreover the following properties hold*:

(1) *Assume that W is nonsingular and π is smooth outside a normal crossing divisor D of W. Let $d = \dim X - \dim W$ and let ${}^u\mathcal{H}^{d+i}$ be the upper canonical extension of the variation of Hodge structures $(R^{d+i}\pi_*\mathbb{Z}_X)_{|W\setminus D}$ for any $i \geq 0$. Then we have*

$$R^i\pi_*\omega_{X/W} \simeq \mathcal{F}^d({}^u\mathcal{H}^{d+i}),$$

where $\mathcal{F}^d$ denotes the induced d-th filter from the Hodge filtration;

(2) *Assume that there is a projective morphism* $f\colon W \to V$ *to a complex analytic variety* V. *Then for an* f*-ample invertible sheaf* $\mathcal{A}$ *of* W *and for integers* $p > 0$ *and* $i \geq 0$, *we have*

$$R^p f_*(\mathcal{A} \otimes R^i \pi_* \omega_X) = 0.$$

We shall consider the above theorem in the case of (not necessarily projective) elliptic fibrations.

Theorem 3.2.3. *Let* $f\colon Y \to W$ *be an elliptic fibration, which is not necessarily projective, between complex manifolds. Suppose that* f *is smooth outside a normal crossing divisor on* W. *Then there exist the following isomorphisms*:

$$R^i f_* \omega_{Y/W} \simeq \begin{cases} \mathcal{F}^1({}^u\mathcal{H}), & i = 0; \\ \mathcal{O}_S, & i = 1; \\ 0, & i > 1, \end{cases} \quad R^i f_* \mathcal{O}_Y \simeq \begin{cases} \mathcal{O}_S, & i = 0; \\ Gr^0_{\mathcal{F}}({}^\ell\mathcal{H}), & i = 1; \\ 0, & i > 1. \end{cases}$$

Proof. If f is a locally projective morphism, then these are isomorphic by Theorem 3.2.2. If $\dim W = 1$, then f is a flat morphism. Thus f is a locally projective morphism by Claim 3.2.4 below. Thus even in the case $\dim W > 1$, the double duals of $R^i f_* \omega_{Y/W}$ and $R^i f_* \mathcal{O}_Y$, respectively, are isomorphic to the right hand side of the corresponding formula. Hence we have only to check the formula locally on W. We may assume that the monodromy matrices are unipotent by taking the unipotent reduction. By the flattening theorem, we have a proper bimeromorphic morphism $\mu\colon M \to W$ from a nonsingular manifold M such that the fiber product $Y \times_W M \to M$ induces a flat morphism $g\colon Z \to M$ from the main component Z of $Y \times_W M$. We may assume that g is smooth outside a normal crossing divisor D of M.

Claim 3.2.4. *g is locally a projective morphism.*

Proof. Let us consider the following exact sequence induced by g_* from an exponential sequence:

$$R^1 g_* \mathcal{O}_Z \to R^1 g_* \mathcal{O}_Z^\star \to R^2 g_* \mathbb{Z}_Z \to R^2 g_* \mathcal{O}_Z.$$

Now we have $R^2 g_* \mathcal{O}_Z = 0$. Note that the stalk $(R^2 g_* \mathbb{Z}_Z)_P$ is isomorphic to $H^2(g^{-1}(P), \mathbb{Z})$ for $P \in M$. Thus we have an invertible sheaf $\mathcal{L}$ on an open neighborhood of $g^{-1}(P)$ such that the intersection numbers $\mathcal{L} \cdot C$ are positive for any irreducible components of $C \subset g^{-1}(P)$. Hence $\mathcal{L}$ is g-ample over an open neighborhood of $\{P\}$. Q.E.D.

Proof of Theorem 3.2.3 *continued.* There is an elliptic fibration $\pi\colon X \to M$ from a complex manifold X such that π is bimeromorphically equivalent to g over M and that π and g are isomorphic to each other over $M \setminus D$. By Claim 3.2.4, g is bimeromorphically equivalent to a projective morphism locally over M. Since the canonical extension of $(R^1\pi_*\mathbb{Z}_X)_{|M\setminus D} \otimes \mathcal{O}_{M\setminus D}$ and the induced filtrations are pullbacks of the corresponding sheaves on W, we have:

$$R^i\pi_*\omega_{X/M} \simeq \begin{cases} \mu^*(\mathcal{F}^1(\mathcal{H})), & \text{if } i = 0; \\ \mathcal{O}_M, & \text{if } i = 1; \\ 0, & \text{otherwise,} \end{cases}$$

by Theorem 3.2.2. Since $R^i\mu_*\omega_M = 0$ for $i > 0$, we have:

$$R^i f_*\omega_{Y/W} \simeq R^i(\mu \circ \pi)_*\omega_{X/W} \simeq \begin{cases} \mathcal{F}^1(\mathcal{H}), & \text{if } i = 0; \\ \mathcal{O}_W, & \text{if } i = 1; \\ 0, & \text{otherwise.} \end{cases}$$

By duality, we also have the isomorphisms for $R^i f_*\mathcal{O}_Y$. Q.E.D.

Remark. By the same argument, we can show such isomorphisms exist in the case the general fibers of $Y \to S$ are curves. But if it is not an elliptic fibration, then $Y \to S$ is bimeromorphically equivalent to a projective morphism. Therefore this is already proved by Theorem 3.2.2.

Corollary 3.2.5 (cf. [U, 6.1], [Kw2, 20]). *Let $f\colon Y \to W$ be an elliptic fibration between complex manifolds such that f is smooth outside a normal crossing divisor $D = \cup D_i$. Then we have:*

$$(f_*\omega_{Y/W})^{\otimes 12} \simeq J^*\mathcal{O}(1) \otimes \mathcal{O}(\textstyle\sum 12\delta_i D_i).$$

Now we shall prove the following:

Theorem 3.2.6 (Torsion free theorem). *Let $\pi\colon X \to W$ be an elliptic fibration from a complex manifold X onto a complex analytic variety W. Then the higher direct image sheaves $R^i\pi_*\omega_X$ are torsion free. Further if there exist a projective morphism $f\colon W \to V$ onto a complex analytic variety V and an f-ample invertible sheaf $\mathcal{A}$ on W, then*

$$R^p f_*(\mathcal{A} \otimes R^i\pi_*\omega_X) = 0$$

for any $p > 0$ and $i \geq 0$.

Proof. First we shall show that $R^1\pi_*\omega_X$ is torsion free and that $R^i\pi_*\omega_X = 0$ for $i \geq 2$. We may assume that there exist a complex manifold M and morphisms $g\colon X \to M$ and $\mu\colon M \to W$ such that g is smooth outside a normal crossing divisor $D = \bigcup D_i$ on M, μ is a bimeromorphic morphism, and that $\pi = \mu \circ g$. Thus by Theorem 3.2.3 and Corollary 3.2.5, we have

$$(g_*\omega_X)^{\otimes 12} \simeq \omega_M^{\otimes 12} \otimes J^*\mathcal{O}(1) \otimes \mathcal{O}(\sum 12\delta_i D_i),$$

$$R^i g_*\omega_X \simeq \begin{cases} \omega_M, & \text{for } i = 1; \\ 0, & \text{for } i > 1. \end{cases}$$

Note that $g_*\omega_{X/M} - \sum \delta_i D_i$ is μ-nef. By applying the vanishing theorem [Ny3, 3.6, 3.7], we have

$$R^k\mu_*(g_*\omega_X) = 0 \quad \text{and} \quad R^k\mu_*\omega_M = 0$$

for $k > 0$. Therefore by the Leray spectral sequence, we have

$$R^i\pi_*\omega_X \simeq \begin{cases} \mu_*(g_*\omega_X), & \text{for } i = 0; \\ \mu_*(\omega_M), & \text{for } i = 1; \\ 0, & \text{for } i > 1. \end{cases}$$

Thus $R^i\pi_*\omega_X$ are torsion free. Next we shall prove the vanishing:

$$R^p f_*(\mathcal{A} \otimes R^i\pi_*\omega_X) = 0.$$

By the argument above, we have only to consider the cases $i = 0, 1$. Since $\mu^*\mathcal{A}$ is $(f \circ \mu)$-nef-big, the $g_*\omega_{X/M} - \sum_i \delta_i D_i + \mu^*\mathcal{A}$ is also $(f \circ \mu)$-nef-big. Thus by [Ny3, 3.7], we have

$$R^p(f \circ \mu)_*(\mu^*\mathcal{A} \otimes \omega_M) = 0 \quad \text{and} \quad R^p(f \circ \mu)_*(\mu^*\mathcal{A} \otimes \pi_*\omega_X) = 0$$

for $p > 0$. Thus by the argument above, we have the desired vanishing. Q.E.D.

3.3. Projective morphisms

We have the following criterion for a given proper surjective morphism to be locally projective.

Proposition 3.3.1. *Let $\pi\colon X \to V$ be a proper surjective morphism from a complex analytic manifold X onto a complex analytic variety V. Suppose that the stalk $(R^2\pi_*\mathcal{O}_X)_P = 0$ for a point $P \in V$. Then π is projective over $\{P\}$ if and only if there is an open neighborhood U of P in V such that $\pi^{-1}(U)$ admits a Kähler metric.*

Proof. It is enough to prove that π is projective over $\{P\}$ under the assumption: X is Kähler. Let us consider the following exact sequence induced by π_* from the exponential sequence:

$$R^1\pi_*\mathcal{O}_X \to R^1\pi_*\mathcal{O}_X^\star \to R^2\pi_*\mathbb{Z}_X \to R^2\pi_*\mathcal{O}_X = 0.$$

We note that the stalk $(R^2\pi_*\mathbb{Z}_X)_P$ is isomorphic to $H^2(\pi^{-1}(P), \mathbb{Z})$. Let ω be a Kähler form on $\pi^{-1}(U)$ for an open neighborhood U of $\{P\}$. Then its cohomology class $[\omega]$ should be an element of $H^2(\pi^{-1}(U), \mathbb{R})$. Let us denote by the same $[\omega]$ the image of $[\omega]$ under the map $H^2(\pi^{-1}(U), \mathbb{R}) \to H^2(\pi^{-1}(P), \mathbb{R})$. We define the *Kähler cone* $\mathrm{KC}(X/V;\ P)$ over $\{P\}$ to be the subset of $H^2(\pi^{-1}(P), \mathbb{R})$ consisting of all the $[\omega]$ for Kähler forms ω defined on some neighborhoods of $\pi^{-1}(P)$.

Claim 3.3.2. *$\mathrm{KC}(X/V;\ P)$ is an open subset of $H^2(\pi^{-1}(P), \mathbb{R})$.*

Proof. Note that $(R^1\pi_*\mathcal{O}_X^\star)_P \to H^2(\pi^{-1}(P), \mathbb{Z})$ is surjective. Thus for any element $\tau \in H^2(\pi^{-1}(P), \mathbb{R})$, we have a d-closed real $(1,1)$-form η on a neighborhood of $\pi^{-1}(P)$ such that its cohomology class $[\eta]$ is τ. Let ω be a Kähler form and let η_i for $1 \leq i \leq n$ be d-closed real $(1,1)$-forms on a neighborhood of $\pi^{-1}(P)$ such that $\{[\eta_i]\}$ is a basis of $H^2(\pi^{-1}(P), \mathbb{R})$. Since $\pi^{-1}(P)$ is a compact subset, there exists a positive number ε such that if x_i are real numbers with $|x_i| < \varepsilon$, then $\omega + \sum_{1\leq i\leq n} x_i\eta_i$ is also a Kähler form on a neighborhood of $\pi^{-1}(P)$. Thus the Kähler cone is open. Q.E.D.

Proof of Proposition 3.3.1 *continued.* By the above claim, we obtain an invertible sheaf $\mathcal{L}$ on a neighborhood of $\pi^{-1}(P)$ which has a positive Hermitian metric. Thus $\mathcal{L}$ is π-ample. Therefore π is a projective morphism over $\{P\}$. Q.E.D.

As a consequence of Claim 3.2.3 and Proposition 3.3.1, we have:

Theorem 3.3.3. *Let $f\colon Y \to M$ be an elliptic fibration from a complex Kähler manifold Y onto a complex manifold M such that f is smooth outside a normal crossing divisor on M. Then f is a locally projective morphism.*

In the case M is a nonsingular curve, any elliptic fibration $Y \to M$ is a locally projective morphism. But in the case $\dim M \geq 2$, there exist non-projective elliptic fibrations.

Example 3.3.4. Let Δ^2 be the two-dimensional unit disc with a coordinate system (t_1, t_2), $\mu\colon S \to \Delta^2$ the blowing-up at $0 = (0,0) \in \Delta^2$, and let D be the exceptional divisor on S. Then S is covered by open subsets U_0 and U_1 such that

(1) $U_0 = \{(x_0, y_0) \in \mathbb{C}^2 \mid |x_0| < 1, |x_0y_0| < 1\}$, $\mu^*(t_1) = x_0$ and $\mu^*(t_2) = x_0y_0$ on U_0,
(2) $U_1 = \{(x_1, y_1) \in \mathbb{C}^2 \mid |x_1| < 1, |x_1y_1| < 1\}$, $\mu^*(t_1) = x_1y_1$ and $\mu^*(t_2) = x_1$ on U_1,

(3) $U_{0,1} := \{ (x_0, y_0) \in U_0 \mid y_0 \neq 0\}$ and $U_{1,0} := \{ (x_1, y_1) \in U_1 \mid y_1 \neq 0\}$ are isomorphic to each other by

$$\begin{cases} x_1 &= x_0 y_0, \\ y_1 &= y_0^{-1}, \end{cases} \quad \text{and} \quad \begin{cases} x_0 &= x_1 y_1, \\ y_0 &= y_1^{-1}. \end{cases}$$

We take an elliptic curve E_ρ that is the quotient manifold of $\mathbb{C}^\star$ by the action:

$$\mathbb{C}^\star \ni u \mapsto u\rho$$

for $\rho \in \mathbb{C}^\star$ with $|\rho| < 1$. Let us consider the following isomorphism:

$$U_{0,1} \times \mathbb{C}^\star \ni ((x_0, y_0),\, u) \mapsto ((x_0 y_0, y_0^{-1}),\, u y_0) \in U_{1,0} \times \mathbb{C}^\star .$$

This induces the isomorphism $U_{0,1} \times E \simeq U_{1,0} \times E$, by which we can patch $U_0 \times E$ and $U_1 \times E$. Thus we obtain a smooth elliptic fibration $f\colon X \to S$. Note that $f^{-1}(D)$ is isomorphic to the Hopf surface H_ρ, which is defined to be the quotient manifold of $\mathbb{C}^2 \setminus \{(0,0)\}$ by the action

$$\mathbb{C}^2 \setminus \{(0,0)\} \ni (z_1, z_2) \mapsto (\rho z_1, \rho z_2) .$$

Thus f is a locally projective morphism but not a projective morphism. Further the composite $\mu \circ f\colon X \to \Delta^2$ is an elliptic fibration smooth outside $\{0\}$ and the central fiber $f^{-1}\mu^{-1}(0)$ is isomorphic to the Hopf surface.

Similar constructions to this example are found in [Kt], [Ts]. We have the following generalization:

Example 3.3.5. Let us consider the following three-dimensional complex manifold:

$$M := \left\{(x, y, z_1, z_2) \in \Delta^2 \times (\mathbb{C}^2 \setminus \{(0,0)\}) \,|\, x z_2 = y z_1\right\} .$$

Here we consider the following three actions:

$$(x, y, z_1, z_2) \mapsto (\mu x, y, \mu z_1, z_2), \quad (x, \mu y, z_1, \mu z_2), \quad (x, y, \rho z_1, \rho z_2),$$

where $\mu := e(1/m)$ for a positive integer m and $\rho \in \mathbb{C}^\star$ satisfies $|\rho| < 1$. Therefore $(\mathbb{Z}/m\mathbb{Z}) \times (\mathbb{Z}/m\mathbb{Z}) \times \mathbb{Z}$ acts on M properly discontinuously and freely. Thus we have the quotient manifold X with an elliptic fibration $g\colon X \to \Delta^2$ defined by $(x, y, z_1, z_2) \mapsto (t_1, t_2) = (x^m, y^m)$. Further we have an elliptic fibration $f\colon X \to S \subset \Delta^2 \times \mathbb{P}^1$ by $(x, y, z_1, z_2) \mapsto (x^m, y^m, (z_1^m : z_2^m))$, where $\nu\colon S \to \Delta^2$ is the blowing-up at $0 = (0,0) \in \Delta^2$. Here $g = \nu \circ f$. The f is smooth outside $D = \nu^{-1}(0)$ and $f^*(D) = m f^{-1}(D)$, where the central fiber $f^{-1}(D) = g^{-1}(0)$ is the Hopf surface

H_{ρ^m} (cf. Example 3.3.4) and $f^{-1}(D) \to D$ is the induced smooth elliptic fibration. Therefore we have the following canonical bundle formula:

$$K_X \sim f^*(K_S) + (m-1)f^{-1}(D) \sim g^*(K_{\Delta^2}) + (2m-1)f^{-1}(D).$$

Note that if $m = 1$, then this f is nothing but the same f as Example 3.3.4. The multi-valued map:

$$(\Delta^2)^\circ := \Delta^2 \setminus \{(0,0)\} \ni (t_1, t_2) \mapsto (t_1^{1/m}, t_2^{1/m}, t_1^{1/m}, t_2^{1/m}) \in M$$

defines a holomorphic section. Thus by Proposition 1.2.4, $X \times_{\Delta^2} (\Delta^2)^\circ \simeq E_\rho \times (\Delta^2)^\circ$, since $(\Delta^2)^\circ$ is simply connected.

§4. Toric models

4.1. Basic elliptic fibrations

An *elliptic fibration* is defined to be a proper surjective morphism of complex analytic varieties whose general fibers are elliptic curves. An elliptic fibration is said to be *basic* if there is a meromorphic section. Let S be a d-dimensional complex manifold, D a normal crossing divisor, and let $S^\star := S \setminus D$. If an elliptic fibration over S is smooth over $S^\star$, then any meromorphic section is holomorphic over $S^\star$ by Lemma 1.3.5. Let H be a variation of Hodge structures on $S^\star$, $p^\star\colon B(H)^\star \to S^\star$ the associated smooth basic elliptic fibration, and let $\sigma_0^\star\colon S^\star \to B(H)^\star$ be the zero section. Then by [Ny4, (2.5)], there exists a *minimal* triplet $(\mathcal{L}_{H/S}, \alpha, \beta)$ on S such that the Weierstrass model $W := W(\mathcal{L}_{H/S}, \alpha, \beta) \to S$ is an extension of $p^\star$ to S and the canonical section is an extension of $\sigma_0^\star$. The W has only rational singularities and the invertible sheaf $\mathcal{L}_{H/S}$ is isomorphic to (cf. Theorem 3.2.2):

$$\mathrm{Gr}^0_{\mathcal{F}}({}^\ell\mathcal{H}) = {}^\ell\mathcal{H}/\mathcal{F}^1({}^\ell\mathcal{H}).$$

Let $\nu^\star\colon H \to H$ be an automorphism as a variation of Hodge structures over $S^\star$. Then $\nu^\star$ is of finite order, which is one of $\{1, 2, 3, 4, 6\}$. By the uniqueness of the extension W [Ny4, (2.5)], we have an automorphism $\nu\colon W \to W$ over S inducing $\nu^\star$ over $S^\star$. The automorphism ν is defined by

$$W \ni (X : Y : Z) \mapsto (\varepsilon^2 X : Y : \varepsilon^3 Z)$$

for a primitive m-th root ε of 1, where m is the order of $\nu^\star$. By taking an equivariant resolution of singularities, we have an extension $p\colon B(H) \to S$ of $p^\star$ satisfying the following conditions:

(1) $B(H)$ is nonsingular;

(2) $B(H) \to S$ admits a section $\sigma_0 \colon S \to B(H)$ which is an extension of $\sigma_0^\star$;
(3) For any such automorphism $\nu \colon W \to W$ as above, the induced bimeromorphic automorphism $\nu \colon B(H) \cdots\to B(H)$ is holomorphic.

The section $\sigma_0 \colon S \to B(H)$ is also called the zero section. By the existence of such extensions and by Proposition 1.3.3, we have the following:

Theorem 4.1.1 (cf. [Ny8]). *Let $f^\star \colon X^\star \to S^\star$ be a smooth projective elliptic fibration over a complex manifold $S^\star$. Suppose that $S^\star$ is isomorphic to a Zariski-open subset of another complex manifold S. Then $f^\star$ extends to a projective elliptic fibration over S.*

Proof. A prime divisor $R^\star$ of $X^\star$ is finite étale over $S^\star$ by Proposition 1.3.3. Let $S'^\star \to S^\star$ be the Galois closure of $R^\star \to S^\star$. Then $S'^\star$ is realized as a Zariski-open subset of a complex manifold S' and the finite étale morphism $S'^\star \to S^\star$ extends to a generically finite proper morphism $S' \to S$, by a theorem of Grauert–Remmert [GR]. Here we may assume that the Galois group G of $S'^\star \to S^\star$ acts holomorphically on S'. The pullback $X^\star \times_{S^\star} S'^\star \to S'^\star$ admits a section. Thus by the previous argument, we can extend the smooth basic elliptic fibration to a basic elliptic fibration $B(H') \to S'$, where the action of the Galois group G on $X^\star \times_{S^\star} S'^\star$ induces a holomorphic action on $B(H')$. Hence we have only to take the quotient. Q.E.D.

Let $\sigma \colon S \cdots\to B(H)$ be a meromorphic section. We denote by Σ and Σ_0 the images of σ and σ_0, respectively. Let us consider the diagonal $\Delta_{B(H)} \subset B(H) \times_S B(H)$ and take a bimeromorphic morphism $\mu \colon Z \to B(H) \times_S B(H)$ from a complex manifold Z onto the main component of $B(H) \times_S B(H)$ which is isomorphic over $B(H)^\star \times_{S^\star} B(H)^\star$. Let Δ' be the proper transform of $\Delta_{B(H)}$ in Z and let $p_1, p_2 \colon Z \to B(H)$ be the first and the second projections, respectively. We consider an invertible sheaf

$$\mathcal{N} := \mathcal{O}_Z(\Delta' - p_2^*(\Sigma_0) + p_2^*(\Sigma)).$$

Then for $b \in B(H)^\star = p^{-1}(S^\star)$, we have an isomorphism:

$$\mathcal{N}_{|p_1^{-1}(b)} \simeq \mathcal{O}_{p^{-1}(p(b))}([b] - [\sigma_0(p(b))] + [\sigma(p(b))]),$$

which is an invertible sheaf of degree one on the elliptic curve $p^{-1}(p(b))$. By replacing Z by a further blowing up, we have an effective divisor $E \subset Z$ such that $p_1^* p_{1*}\mathcal{N} \simeq \mathcal{N} \otimes \mathcal{O}(-E)$. An irreducible component E_0 of E dominates $B(H)$ bimeromorphically, and the other components do not dominate $B(H)$. Let $\mathrm{tr}(\sigma) \colon B(H) \cdots\to B(H)$ be the meromorphic mapping over S associated with the graph $\mu(E_0) \subset B(H) \times_S B(H)$. Then the restriction of $\mathrm{tr}(\sigma)$ to $p^{-1}(S^\star) = B(H)^\star$ is nothing but the translation morphism by the section σ. We call $\mathrm{tr}(\sigma)\colon B(H) \cdots\to B(H)$ by the *translation mapping* by a meromorphic section σ. By the same argument, we see that $p\colon B(H) \to S$ has a meromorphic group structure, i.e., there exist a multiplication mapping $B(H) \times_S B(H) \cdots\to B(H)$ over S and an inverse $B(H) \to B(H)$ which are extensions of the same objects for $p^\star\colon B(H)^\star \to S^\star$. We also have the following generalization of Lemma 1.2.2:

Lemma 4.1.2. *Let $\varphi\colon B(H) \cdots\to B(H)$ be a bimeromorphic mapping over S inducing the identity homomorphism on $(R^1 p_* \mathbb{Z}_{B(H)})_{|S^\star}$. Then there exists a meromorphic section $\sigma\colon S \cdots\to B(H)$ such that $\varphi = \mathrm{tr}(\sigma)$.*

In particular, every bimeromorphic automorphism $\varphi\colon B(H) \cdots\to B(H)$ over S is expressed as the composite of a translation mapping and an automorphism ν of finite order explained as before. By [Ny4, (2.1)], we have:

Lemma 4.1.3. *Let $f\colon X \to S$ be an elliptic fibration smooth over $S^\star$ which induces an isomorphism $H \simeq (R^1 f_* \mathbb{Z}_X)_{|S^\star}$ as variations of Hodge structures. Suppose that f admits a meromorphic section $\sigma\colon S \cdots\to X$. Then there exists a bimeromorphic mapping $h\colon X \cdots\to B(H)$ such that $h \circ \sigma$ is the zero section σ_0.*

Let $\mathcal{U} \subset S$ be an open subset. The set of meromorphic sections $\{\sigma\colon \mathcal{U} \cdots\to B(H)_{|\mathcal{U}}\}$ forms a subgroup of $H^0(\mathcal{U} \cap S^\star, \mathfrak{S}_H)$. From the subgroups, we define a subsheaf $\mathfrak{S}_{H/S}$ of $j_* \mathfrak{S}_H$, where j denotes the inclusion $S^\star \hookrightarrow S$. We call $\mathfrak{S}_{H/S}$ by the sheaf of germs of meromorphic sections of $B(H) \to S$. Obviously, it does not depend on the choice of $B(H)$. Let $f\colon X \to S$ be an elliptic fibration smooth over $S^\star$ and let $\phi\colon H \simeq (R^1 f_* \mathbb{Z}_X)_{|S^\star}$ be an isomorphism as variations of Hodge structures. Suppose that f admits a meromorphic section locally on S, i.e., there exist an open covering $S = \bigcup_{\lambda \in \Lambda} S_\lambda$ and meromorphic sections $S_\lambda \cdots\to X_{|S_\lambda}$ for any λ. Then we have bimeromorphic mappings $\varphi_\lambda\colon X_{|S_\lambda} \cdots\to B(H)_{|S_\lambda}$ over S_λ such that these φ_λ^* induce the given isomorphism $\phi\colon H \simeq (R^1 f_* \mathbb{Z}_X)_{|S^\star}$. Then for $\lambda, \mu \in \Lambda$, the transition mapping $\varphi_\lambda \circ \varphi_\mu^{-1}\colon B(H)_{|S_\lambda \cap S_\mu} \cdots\to B(H)_{|S_\lambda \cap S_\mu}$ is the translation mapping by a meromorphic section $\eta_{\lambda,\mu}$ of $B(H) \to S$ over $S_\lambda \cap S_\mu$. Therefore we have $\eta_{\lambda,\mu} + \eta_{\mu,\nu} + \eta_{\nu,\lambda} = 0$ for $\lambda, \mu, \nu \in \Lambda$. This collection $\{\eta_{\lambda,\mu}\}$ defines a cohomology class in $H^1(S, \mathfrak{S}_{H/S})$, which is independent of the choices of an open covering $\{S_\lambda\}$ and bimeromorphic mappings $\{\varphi_\lambda\}$. Let us denote the cohomology class by $\eta(X/S, \phi)$. Let $f'\colon X' \to S$ be another elliptic fibration smooth over $S^\star$ and let $\phi'\colon H \simeq (R^1 f'_*\mathbb{Z}_{X'})_{|S^\star}$ be an isomorphism. Assume that f' also admits a meromorphic section locally over S and that $\eta(X/S,\phi) = \eta(X'/S,\phi')$. Then there exists a bimeromorphic mapping $\psi\colon X \cdots\to X'$ over S such that $\phi = \psi^* \circ \phi'$. Therefore these $\eta(X/S,\phi)$ define a natural equivalence relation for such pairs $(X/S,\phi)$. Let $(f\colon X \to S,\phi)$ be a pair as above and suppose that X is nonsingular. Then we have the following exact sequence by Theorem 3.2.3:

$$0 \to R^1 f_*\mathbb{Z}_X \to R^1 f_*\mathcal{O}_X \to R^1 f_*\mathcal{O}_X^\star \to R^2 f_*\mathbb{Z}_X \to 0.$$

The natural homomorphism $R^1 f_*\mathbb{Z}_X \to j_*H$ induces a commutative diagram (cf. Lemma 3.1.3)

$$\begin{array}{ccc} R^1 f_*\mathbb{Z}_X & \longrightarrow & R^1 f_*\mathcal{O}_X \\ \downarrow & & \| \\ j_*H & \longrightarrow & \mathcal{L}_{H/S}. \end{array}$$

Let $\mathcal{V}_X$ be the kernel of the homomorphism

$$R^1 f_*\mathcal{O}_X^\star \to j_*((R^1 f_*\mathcal{O}_X^\star)_{|S^\star}).$$

For a meromorphic section $\sigma\colon S \cdots\to B(H)$, we can attach an invertible sheaf $\mathcal{O}_{B(H)}(\Sigma - \Sigma_0)$, where $\Sigma = \sigma(S)$ and $\Sigma_0 = \sigma_0(S)$. By considering $\varphi_\lambda^*\mathcal{O}_{B(H)}(\Sigma - \Sigma_0)$, we have an element of $H^0(S, R^1 f_*\mathcal{O}_X^\star/\mathcal{V}_X)$, since the transition mappings $\varphi_\lambda \circ \varphi_\mu^{-1}$ are translations. Therefore as in §1.3, we have an injective homomorphism

$$\Phi_X\colon \mathfrak{S}_{H/S} \to R^1 f_*\mathcal{O}_X^\star/\mathcal{V}_X,$$

which extends to the exact sequence (cf. (1.2)):

$$0 \to \mathfrak{S}_{H/S} \stackrel{\Phi_X}{\to} R^1 f_*\mathcal{O}_X^\star/\mathcal{V}_X \to \mathbb{Z}_S \to 0. \tag{4.1}$$

By the similar argument to the proof of Lemma 1.3.2, we have:

Lemma 4.1.4 (cf. [Ny8]). *The cohomology class $\eta(X/S,\phi)$ is the image of* 1 *under the connecting homomorphism*

$$\mathbb{Z} = H^0(S,\mathbb{Z}_S) \to H^1(S,\mathfrak{S}_{H/S})$$

derived from (4.1).

Proposition 4.1.5.

(1) *Let* $f\colon X \to S$ *be an elliptic fibration smooth over* $S^\star$ *admitting a meromorphic section locally over* S *and let* $\phi\colon H \simeq (R^1 f_* \mathbb{Z}_X)_{|S^\star}$ *be an isomorphism. Then the cohomology class* $\eta(X/S, \phi) \in H^1(S, \mathfrak{S}_{H/S})$ *is a torsion element if and only if* f *is bimeromorphically equivalent to a projective morphism over* S.

(2) *For a torsion element* $\eta \in H^1(S, \mathfrak{S}_{H/S})$, *there exist a pair* $(X/S, \phi)$ *as above such that* $\eta = \eta(X/S, \phi)$.

Proof. (1). Suppose that f is bimeromorphically equivalent to a projective morphism and that X is nonsingular. Then we have an invertible sheaf $\mathcal{M}$ on X such that $\deg \mathcal{M}_{|f^{-1}(s)} > 0$ for a general fiber $f^{-1}(s)$. At the exact sequence (4.1), $\mathcal{M} \in H^1(X, \mathcal{O}_X^\star)$ induces a positive integer in $\mathbb{Z} = \Gamma(S, \mathbb{Z}_S)$. Hence by Lemma 4.1.4, $\eta(X/S, \phi)$ is a torsion element. Next conversely suppose $m\eta(X/S, \phi) = 0$ for a positive integer m. Then by the same argument as in the proof of Proposition 1.3.3, we have a generically finite meromorphic mapping $X \cdots\to B(H)$ over S. Therefore $f\colon X \to S$ is bimeromorphically equivalent to a projective morphism.

(2). Suppose that $m\eta = 0$ for a positive integer m. Let $\{S_\lambda\}$ be a locally finite open covering of S and $\{\eta_{\lambda,\mu}\}$ be a cocycle of meromorphic sections of $B(H)$ representing η. We may assume that there exist meromorphic sections ξ_λ over S_λ such that $m\eta_{\lambda,\mu} = \xi_\mu - \xi_\lambda$ over $S_\lambda \cap S_\mu$. As in the previous argument we have multiplication mappings

$$\psi_\lambda\colon B(H)_{|S_\lambda} \overset{m\times}{\cdots\to} B(H)_{|S_\lambda}.$$

Then $\mathrm{tr}(\xi_\mu) \circ \mathrm{tr}(\xi_\lambda)^{-1} \circ \psi_\mu = \psi_\lambda \circ \mathrm{tr}(\eta_{\lambda,\mu})$. The meromorphic sections $\eta_{\lambda,\mu}$, ξ_λ are holomorphic over $S^\star$. Therefore we have the patching $X^\star := \bigcup_\lambda B^\star(H)_{|S^\lambda}$ by $\{\eta_{\lambda,\mu}\}$ and a finite étale morphism $\psi^\star\colon X^\star \to B(H)^\star$ over $S^\star$. By construction, the elliptic fibration $f^\star\colon X^\star \to B(H)^\star \to S^\star$ induces an isomorphism $\phi\colon H \simeq R^1 f^\star_* \mathbb{Z}_{X^\star}$ as variations of Hodge structures. By a theorem of Grauert–Remmert ([GR]), there exist a generically finite morphism $\psi\colon X \to B(H)$ extending $\psi^\star\colon X^\star \to B^\star(H)$. Then the composite $f\colon X \to B(H) \to S$ is an elliptic fibration bimeromorphically equivalent to a projective morphism. By the uniqueness of the extension of finite morphisms, we have bimeromorphic mappings $\varphi_\lambda\colon X_{|S_\lambda} \cdots\to B(H)_{|S_\lambda}$ such that $\psi_{|S_\lambda} = \psi_\lambda \circ \varphi_\lambda$ and $\varphi_\lambda = \mathrm{tr}(\eta_{\lambda,\mu}) \circ \varphi_\mu$ for λ, μ. Therefore $\eta(X/S, \phi) = \eta$. Q.E.D.

Next we consider another situation. Let $f\colon X \to S$ be an elliptic fibration smooth over $S^\star$ and let $\phi\colon H \simeq (R^1 f_* \mathbb{Z}_X)_{|S^\star}$ be an isomorphism of variations of Hodge structures. Suppose that there is a finite Galois covering $\tau \colon T \to S$ with the Galois group G such that T is a complex manifold, $\tau^{-1}(D) = D_T$ is a normal crossing divisor, τ is étale over $S^\star$, and that the fiber product $X_T := T \times_S X \to T$ admits a meromorphic section. Let H_T be the the variation of Hodge structures $\tau^{-1}H$ on $T^\star := T \setminus \tau^{-1}(D)$ and let $B(H_T) \to T$ be a similar basic elliptic fibration. Then $X_T \to T$ is bimeromorphically equivalent to $B(H_T) \to T$. Then from f, we have a cohomology class in $H^1(G, H^0(T, \mathfrak{S}_{H_T/T}))$ by the same argument as in §0.3 and §1.4. Conversely, let us take an element $\eta \in H^1(G, H^0(T, \mathfrak{S}_{H_T/T}))$. Then by an argument in §0.3, η induces a left meromorphic action of G on $B(H_T)$. This is described as follows: Let $\{\eta_g\}_{g \in G}$ be a cocycle of meromorphic sections of $B(H_T)$ representing η. Let $G \times B(H_T) \cdots\to B(H_T)$ be the natural meromorphic action of G which defines $B(H)$ as the quotient. Let $\phi_g \colon B(H_T) = \{g\} \times B(H_T) \cdots\to B(H_T)$ be induced bimeromorphic automorphisms. Then the new action of G on $B(H_T)$ is defined by $\phi'_g := \phi_g \circ \mathrm{tr}(\eta_g)$. Since G is a finite group, we can consider the quotient X of $B(H_T)$ by G. Then we obtained an elliptic fibration. Let $\mathcal{E}(S, D, H, T)$ be the set of bimeromorphic equivalence classes of pairs $(f \colon X \to S, \phi)$ consisting of an elliptic fibration $f \colon X \to S$ smooth over $S^\star$ and an isomorphism $\phi \colon H \simeq (R^1 f_* \mathbb{Z}_X)_{|S^\star}$ such that $X \times_S T \to T$ admits a meromorphic section. Here two pairs $(f_1 \colon X_1 \to S, \phi_1)$ and $(f_2 \colon X_2 \to S, \phi_2)$ are called to be bimeromorphically equivalent if there is a bimeromorphic mapping $\varphi \colon X_1 \cdots\to X_2$ over S such that $\phi_1 = \varphi^* \circ \phi_2$. Then we have:

Lemma 4.1.6. *There is a one to one correspondence between* $H^1(G, H^0(T, \mathfrak{S}_{H_T/T}))$ *and* $\mathcal{E}(S, D, H, T)$.

4.2. Construction of toric models

We fix positive integers $1 \le l \le d$ and nonnegative integers a_i for $1 \le i \le l$, where we assume $a := \sum_{i=1}^{l} a_i > 0$.

Definition 4.2.1. A map $\sigma \colon \mathbb{Z} \to \{1, 2, \ldots, l\}$ is called a *sign function* with respect to $(a_1, a_2, \ldots, a_l)$ if the following two conditions are satisfied:

(1) $\sigma(m + a) = \sigma(m)$ for $m \in \mathbb{Z}$;
(2) $a_i = \sharp\{0 \le j < a \,|\, \sigma(j) = i\}$ for any i.

For a sign function σ and for given integers b_i for $1 \le i \le l$, there exist maps $I_i \colon \mathbb{Z} \to \mathbb{Z}$ with $I_i(0) = b_i$ such that for $m \in \mathbb{Z}$,

$$I_i(m+1) = \begin{cases} I_i(m) + 1, & \text{if } \sigma(m) = i; \\ I_i(m), & \text{otherwise.} \end{cases}$$

Definition 4.2.2. We call the map I_i by the *index function* at i with respect to the sign function σ and the initial value b_i.

For any integer k, let $\mathcal{C}_k \subset \mathbb{R}^{d+1}$ be the rational polyhedral cone

$$\left\{ (u_1, \ldots, u_d, y) \in \mathbb{R}^{d+1} \,\middle|\, u_i \geq 0,\ \sum_{i=1}^{l} I_i(k) u_i \leq y \leq \sum_{i=1}^{l} I_i(k+1) u_i \right\}.$$

Then the semigroup

$$\mathcal{C}_k^\vee \cap \mathbb{Z}^{d+1} := \left\{ (n_1, \ldots, n_{d+1}) \,\middle|\, \sum_{i=1}^{d} n_i u_i + n_{d+1} y \geq 0 \text{ for } (u_i, y) \in \mathcal{C}_k \right\}$$

is finitely generated. Let R_k be the associated semigroup ring over $\mathbb{C}$. It is easy to show the following:

Lemma 4.2.3. *Let $\mathbb{C}[t_1, t_2, \ldots, t_d, s]$ be the polynomial ring of $(d+1)$-variables. Then R_k is isomorphic to a $\mathbb{C}$-subalgebra of $\mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_d^{\pm 1}, s^{\pm 1}]$ generated by monomials*

$$t_1, t_2, \ldots, t_d, \quad s \prod_{i=1}^{l} t_i^{-I_i(k)}, \quad s^{-1} \prod_{i=1}^{l} t_i^{I_i(k+1)}.$$

By the theory of torus embeddings, $\operatorname{Spec} R_k$ are patched together and form a nonsingular scheme $\mathcal{M}_{\sigma,(b_i)}$ locally of finite type over $\operatorname{Spec} \mathbb{C}[t_1, t_2, \ldots, t_d]$. We note that $\mathcal{M}_{\sigma,(0)_{i=1}^l}$ is isomorphic to $\mathcal{M}_{\sigma,(b_i)}$ by the morphism:

$$(t_1, t_2, \ldots, t_d, s) \mapsto \left(t_1, t_2, \ldots, t_d, s \prod_{i=1}^{l} t_i^{b_i} \right).$$

Thus we denote $\mathcal{M}_{\sigma,(b_i)}$ simply by $\mathcal{M}_\sigma$. Let Δ^d be the unit polydisc defined in

$$(\operatorname{Spec} \mathbb{C}[t_1, t_2, \ldots, t_d])^{\mathrm{an}} \simeq \mathbb{C}^d.$$

Let $(\Delta^d)^\star := (\Delta^\star)^l \times \Delta^{d-l}$ and let $(\Delta^d)^\circ$ be the complement of the following subset in Δ^d:

$$\bigcup_{1 \leq i < j \leq l} \{t_i = t_j = 0\}.$$

Then $(\Delta^d)^\circ = \Delta^d \setminus \operatorname{Sing} D$ and $(\Delta^d)^\star = \Delta^d \setminus \operatorname{Supp} D$, where $D = \sum_{i=1}^{l} D_i = \{t_1 t_2 \cdots t_l = 0\}$. We shall consider the analytic space

$$\mathcal{X}_\sigma := (\mathcal{M}_\sigma)^{\mathrm{an}} \times_{(\operatorname{Spec} \mathbb{C}[t_1, t_2, \ldots, t_d])^{\mathrm{an}}} \Delta^d$$

and the projection $\pi\colon \mathcal{X}_\sigma \to \Delta^d$. We define $\mathcal{X}_\sigma^\circ := \mathcal{X}_\sigma \times_{\Delta^d} (\Delta^d)^\circ$ and $\mathcal{X}_\sigma^\star := \mathcal{X}_\sigma \times_{\Delta^d} (\Delta^d)^\star$. Since $\mathcal{X}_\sigma^\star \simeq (\Delta^d)^\star \times \mathbb{C}^\star$ does not depend on the choice of σ, we write $\mathcal{X}^\star := \mathcal{X}_\sigma^\star$. Here the variable s above is considered to be a coordinate of $\mathbb{C}^\star$. Let us define

$$\mathcal{X}_\sigma^{(k)} := (\operatorname{Spec} R_k)^{\mathrm{an}} \times_{(\operatorname{Spec}\mathbb{C}[t_1,t_2,\ldots,t_d])^{\mathrm{an}}} \Delta^d.$$

Lemma 4.2.4. *$\mathcal{X}_\sigma$ is simply connected.*

Proof. By construction, $\mathcal{X}_\sigma$ contains $\mathcal{X}_\sigma^{(k)}$ as a Zariski-open subset, which is isomorphic to $\{(u,v) \in \mathbb{C}^2 \mid |uv| < 1\} \times \Delta^{d-1}$. This is simply connected. Thus $\mathcal{X}_\sigma$ is also simply connected. Q.E.D.

Lemma 4.2.5. *For any sign functions σ, $\mathcal{X}_\sigma^\circ$ are isomorphic to each other.*

Proof. For any $\mathbf{J} = (j_1, j_2, \ldots, j_l) \in \mathbb{Z}^{\oplus l}$ and for $1 \le i \le l$, we consider the algebra

$${}_iR_{\mathbf{J}} := \mathbb{C}\left[t_1, t_2, \ldots, t_d, s\prod_{k=1}^{l} t_k^{-j_k}, s^{-1}t_i\prod_{k=1}^{l} t_k^{j_k}\right].$$

It is enough to show that $\mathcal{X}_\sigma^\circ$ contains $(\operatorname{Spec}\, {}_iR_{\mathbf{J}})^{\mathrm{an}} \times_{(\operatorname{Spec}\mathbb{C}[t])^{\mathrm{an}}} (\Delta^d)^\circ$ as an open subset naturally, where $t = (t_1, t_2, \ldots, t_d)$. There is an integer m such that $\sigma(m) = i$ and $I_i(m) = j_i$. Then two algebras ${}_iR_{\mathbf{J}}$ and R_m are isomorphic to each other up to the localization by $\prod_{k\neq i} t_k$. Next we fix $i' \neq i$ and compare two algebras ${}_iR_{\mathbf{J}}$ and ${}_{i'}R_{\mathbf{J}}$. Then there is an open immersion $\operatorname{Spec}({}_iR_{\mathbf{J}}[t_i^{-1}]) \to \operatorname{Spec}({}_{i'}R_{\mathbf{J}}[t_i^{-1}])$. Hence by combining with the previous argument, we have an open immersion

$$\operatorname{Spec}\left({}_iR_{\mathbf{J}}\left[\prod\nolimits_{k\neq i'} t_k^{-1}\right]\right) \to \operatorname{Spec}\left(R_{m'}\left[\prod\nolimits_{k\neq i'} t_k^{-1}\right]\right)$$

for some m'. Thus we are done. Q.E.D.

In what follows, we also denote $\mathcal{X}^\circ = \mathcal{X}_\sigma^\circ$. Note that $\mathcal{X}^\circ$ is also simply connected, since $\operatorname{codim}(\mathcal{X}_\sigma \setminus \mathcal{X}^\circ) \ge 2$. Now we shall take a period function $\omega(z)$ on $U = \mathbb{H}^l \times \Delta^{d-l}$ of the form (cf. Proposition 2.1.4):

$$\omega(z) = \sum_{i=1}^{l} a_i z_i + h(t),$$

where $h(t)$ is a holomorphic function on Δ^d such that $\operatorname{Im} h(t) \ge 0$. Then we have a variation of Hodge structures H of type $\mathrm{I}_{(+)}$ on $(\Delta^d)^\star$ whose monodromy matrix around the coordinate hyperplane D_i is of type I_{a_i}.

We have $e(\omega(z)) = e(h(t)) \prod_{i=1}^{l} t_i^{a_i}$. Let us consider an automorphism ϑ of $\mathcal{X}^\star$ defined by:

$$\vartheta \colon \mathcal{X}^\star \simeq (\Delta^d)^\star \times \mathbb{C}^\star \ni (t, s) \mapsto \left(t,\, s \cdot e(h(t)) \prod_{i=1}^{l} t_i^{a_i} \right).$$

This induces also a holomorphic automorphism of $\mathcal{X}_\sigma$. In fact, we have isomorphisms $\mathcal{X}_\sigma^{(k)} \simeq \mathcal{X}_\sigma^{(k+a)}$ by ϑ. From the inequality $|e(\omega(z))| < 1$ and from a similar argument to [Nk], we have:

Lemma 4.2.6. *The action of ϑ on $\mathcal{X}_\sigma$ is properly discontinuous and fixed point free.*

Therefore we can define the quotient manifold X_σ, which has a structure of an elliptic fibration $p \colon X_\sigma \to \Delta^d$. Here we usually assume that all the initial values $b_i = 0$. By composing the morphism

$$\Delta^d \ni t \mapsto (t, 1) \in \mathcal{X}_\sigma$$

with the quotient morphism $q \colon \mathcal{X}_\sigma \to X_\sigma$, we have a section $\sigma_0 \colon \Delta^d \to X_\sigma$.

Definition 4.2.7. We call $p \colon X_\sigma \to \Delta^d$ by the *toric model* of type σ and the section $\sigma_0 \colon \Delta^d \to X_\sigma$ by the *zero section* of p.

Lemma 4.2.8. *We have the following properties on the toric model:*

(1) *The period function of p is of the form $\omega(z) = \sum_{i=1}^{d} a_i z_i + h(t)$ on $\mathbb{H}^l \times \Delta^{d-l}$;*
(2) *The monodromy matrix $\rho(\gamma_i)$ is of type I_{a_i};*
(3) *The fiber $p^{-1}(0)$ is isomorphic to a cycle of rational curves, the number of whose components is $a = \sum_{i=1}^{d} a_i$. In particular, p is a flat morphism;*
(4) *The canonical bundle of X_σ is trivial and hence $p \colon X_\sigma \to \Delta^d$ is a minimal elliptic fibration.*

Proof. It is enough to prove (4). Let us consider the meromorphic $(d+1)$-form

$$dt_1 \wedge dt_2 \wedge \cdots \wedge dt_d \wedge \frac{ds}{s}$$

on $\mathcal{X}_\sigma$. It is easily checked that this is holomorphic and is a nowhere vanishing section of the canonical bundle of $\mathcal{X}_\sigma$. Further this is invariant under ϑ. Thus this induces a nowhere vanishing section of the canonical bundle of X_σ. Q.E.D.

Theorem 4.2.9 (minimal model). *Let $f\colon Y \to \Delta^d$ be a minimal projective elliptic fibration which is bimeromorphically equivalent to $p\colon X_\sigma \to \Delta^d$. Then there exist a sign function σ' and a bimeromorphic morphism $X_{\sigma'} \to Y$ over Δ^d such that $X_{\sigma'}$ is a $\mathbb{Q}$-factorialization of Y.*

We divide the proof into the following 4 steps.

Step 1. Since $f\colon Y \to \Delta^d$ and $p\colon X_\sigma \to \Delta^d$ are minimal models, the bimeromorphic mapping $Y \cdots\to X_\sigma$ is an isomorphism in codimension one. Let A be a general irreducible divisor on Y which is f-ample. Then its proper transform Γ in X_σ is also an irreducible divisor. Suppose that Γ is p-nef. Then Γ is p-semi-ample by [Ny3, 4.8, 4.10]. Since Y and X_σ are isomorphic in codimension one, we have a bimeromorphic morphism $X_\sigma \to Y$ over Δ^d sending Γ to A. Therefore we are done in the case Γ is p-nef.

Next assume that Γ is not p-nef. It is enough to find a suitable sign function σ' such that the proper transform of A in $X_{\sigma'}$ is relatively nef over Δ^d.

Step 2. Now the positive integer $a = \sum_{i=1}^{l} a_i$ satisfies $a \geq 2$. Otherwise, the central fiber $p^{-1}(0)$ is irreducible, so Γ is p-nef. For $k \in \mathbb{Z}$, let $\widetilde{C}_k$ be the irreducible component of the central fiber $\pi^{-1}(0) \subset \mathcal{X}_\sigma$ which intersects $\mathcal{X}_\sigma^{(k)} \cap \mathcal{X}_\sigma^{(k+1)}$, $\widetilde{E}_k$ the component of $\pi^{-1}(D_{\sigma(k)})$ containing $\widetilde{C}_k$, and let $\widetilde{F}_k$ be the component of $\pi^{-1}(D_{\sigma(k+1)})$ containing $\widetilde{C}_k$. Also for $\kappa \in \mathbb{Z}/a\mathbb{Z}$, let C_κ be the image of $\widetilde{C}_k$ under the quotient morphism $\mathcal{X}_\sigma \to X_\sigma$, where $k \mod a = \kappa$. Further let E_κ and F_κ be the images of $\widetilde{E}_k$ and $\widetilde{F}_k$, respectively. The following lemma is easily shown:

Lemma 4.2.10.

(1) *If $\sigma(\kappa) = \sigma(\kappa+1)$, then $E_\kappa = F_\kappa \simeq \mathbb{P}^1 \times D_{\sigma(\kappa)}$ over $D_{\sigma(\kappa)}$ and C_κ is the central fiber of $E_\kappa \to D_{\sigma(\kappa)}$. In particular, the normal bundle $N_{C_\kappa/\mathcal{X}_\sigma}$ is isomorphic to $\mathcal{O}(-2) \oplus \mathcal{O}^{\oplus(d-1)}$.*

(2) *If $\sigma(\kappa) \neq \sigma(\kappa+1)$, then the complete intersection $E_\kappa \cap F_\kappa$ is isomorphic to $\mathbb{P}^1 \times \left(D_{\sigma(\kappa)} \cap D_{\sigma(\kappa+1)}\right)$ over $D_{\sigma(\kappa)} \cap D_{\sigma(\kappa+1)}$, where C_κ is the central fiber of $E_\kappa \cap F_\kappa \to D_{\sigma(\kappa)} \cap D_{\sigma(\kappa+1)}$.*

(3) *In the case $\sigma(\kappa) \neq \sigma(\kappa+1)$, the normal bundle of $E_\kappa \cap F_\kappa$ in $\mathcal{X}_\sigma$ is isomorphic to $p_1^*\mathcal{O}(-1)^{\oplus 2}$, where p_1 is the first projection*

$$E_\kappa \cap F_\kappa \simeq \mathbb{P}^1 \times \left(D_{\sigma(\kappa)} \cap D_{\sigma(\kappa+1)}\right) \to \mathbb{P}^1 \,.$$

In particular, the normal bundle $N_{C_\kappa/\mathcal{X}_\sigma}$ is isomorphic to $\mathcal{O}(-1)^{\oplus 2} \oplus \mathcal{O}^{\oplus(d-2)}$.

Step 3. Since Γ is not p-nef, there exists a curve $C_\kappa \subset p^{-1}(0)$ such that $\Gamma \cdot C_\kappa < 0$. If $\sigma(\kappa) = \sigma(\kappa+1)$, then $\Gamma \cdot \gamma < 0$ for any fiber γ of

$E_\kappa \to D_{\sigma(\kappa)}$. Thus $E_\kappa \subset \Gamma$. This is impossible, since Γ is irreducible and is dominating Δ^d. Therefore $\sigma(\kappa) \neq \sigma(\kappa+1)$. Let $X'_\sigma \to X_\sigma$ be the blowing-up along $E_\kappa \cap F_\kappa$. By Lemma 4.2.10, we can blow-down X'_σ along the other ruling of the exceptional divisor which is isomorphic to

$$\mathbb{P}^1 \times \mathbb{P}^1 \times \left(D_{\sigma(\kappa)} \cap D_{\sigma(\kappa+1)}\right).$$

Thus we obtain another manifold X''_σ. (cf. Figure 1). By considering this process on $\mathcal{X}_\sigma$ and by applying the torus embedding theory, we see that $X''_\sigma \simeq X_{\sigma'}$ for a sign function σ' determined by

$$\sigma'(j) := \begin{cases} \sigma(\kappa+1), & \text{if } j = \kappa; \\ \sigma(\kappa), & \text{if } j = \kappa+1; \\ \sigma(j), & \text{otherwise.} \end{cases}$$

Let C'_κ be the fiber over $0 \in \Delta^d$ of the image of the exceptional divisor, C'_j the proper transform of C_j for $\kappa \neq j \in \mathbb{Z}/a\mathbb{Z}$ and let Γ' be the proper transform of Γ. Then we have:

Lemma 4.2.11.

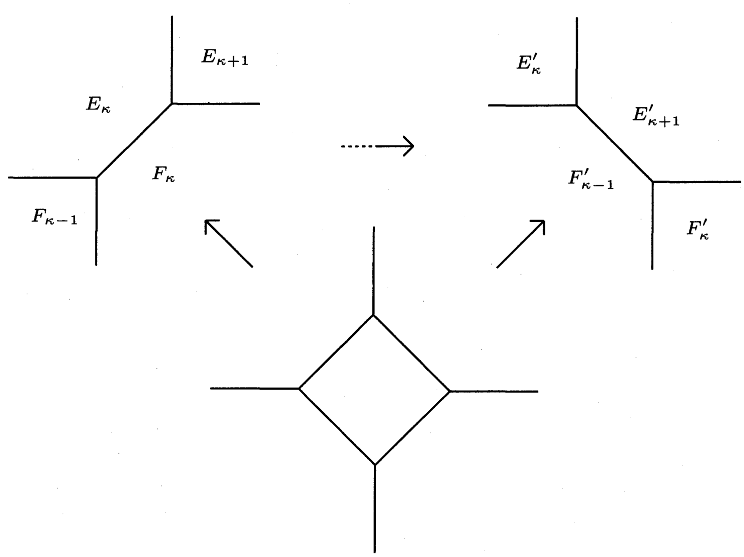

Figure 1. Flop.

(1) *If* $a > 2$, *then*

$$\Gamma' \cdot C'_j = \begin{cases} -\Gamma \cdot C_\kappa, & \text{if } j = \kappa; \\ \Gamma \cdot C_j + \Gamma \cdot C_\kappa, & \text{if } j = \kappa - 1 \text{ or } j = \kappa + 1; \\ \Gamma \cdot C_j, & \text{otherwise.} \end{cases}$$

(2) *If* $a = 2$, *then*

$$\Gamma' \cdot C'_j = \begin{cases} -\Gamma \cdot C_\kappa, & \text{if } j = \kappa; \\ \Gamma \cdot C_j + 2\Gamma \cdot C_\kappa, & \text{if } j = \kappa + 1. \end{cases}$$

Step 4. Let δ be the covering degree of $\Gamma \to \Delta^d$ and let us consider the following set of mappings:

$$S_\delta^{(a)} := \left\{ \phi \colon \mathbb{Z}/a\mathbb{Z} \to \mathbb{Z} \,\middle|\, \sum_{x \in \mathbb{Z}/a\mathbb{Z}} \phi(x) = \delta \right\}.$$

An element $\phi \in S_\delta^{(a)}$ is called *nef* if $\phi(x) \geq 0$ for all $x \in \mathbb{Z}/a\mathbb{Z}$. For $y \in \mathbb{Z}/a\mathbb{Z}$ and $\phi \in S_\delta^{(a)}$ with $\phi(y) < 0$, we define the *flop* $F_y(\phi) \in S_\delta^{(a)}$ of ϕ at y as follows:

(1) (cf. Figure 2) If $a > 2$, then

$$F_y(\phi)(x) := \begin{cases} -\phi(x), & \text{if } x = y; \\ \phi(x) + \phi(y), & \text{if } x = y - 1 \text{ or } x = y + 1; \\ \phi(x), & \text{otherwise.} \end{cases}$$

(2) If $a = 2$, then

$$F_y(\phi)(x) := \begin{cases} -\phi(x), & \text{if } x = y; \\ \phi(x) + 2\phi(y), & \text{if } x = y + 1. \end{cases}$$

By the previous argument, Theorem 4.2.9 is deduced from the following:

Proposition 4.2.12 (Termination). *Any* $\phi \in S_\delta^{(a)}$ *turns to be nef after a finite number of flops.*

A lot of proofs of the proposition seem to be known. The following one is an application of the theory of elliptic surfaces.

Proof. Let $p \colon X \to \Delta$ be the toric model over a one-dimensional disc Δ, determined by a positive integer $a > 1$ and a period function $\omega(z) = az$. Then X is minimal over Δ and the central fiber $p^*(0)$

is a union of smooth rational curves, the number of whose irreducible components is a. We can write:

$$p^*(0) = \sum_{i \in \mathbb{Z}/a\mathbb{Z}} C_i.$$

For $\phi \in S_\delta^{(a)}$, let L be a Cartier divisor on X with intersection numbers $L \cdot C_i = \phi(i)$ for all i. Note that such L exists since there exist divisors Γ_i for $i \in \mathbb{Z}/a\mathbb{Z}$ such that $\Gamma_i \cdot C_j = \delta_{i,j}$. Suppose that $\phi(y) < 0$ for some $y \in \mathbb{Z}/a\mathbb{Z}$. Let us consider the divisor $L' = L + \phi(y)C_y$. Then we have $L' \cdot C_j = \phi'(j)$ for $\phi' = F_y(\phi)$. We here look at the exact sequence:

$$0 \to \mathcal{O}(L') \to \mathcal{O}(L) \to \mathcal{O}(L) \otimes \mathcal{O}_B \to 0,$$

where B is the effective divisor $-\phi(y)C_y$. We have $\chi(B, \mathcal{O}(L) \otimes \mathcal{O}_B) = 0$. Thus $h^0(B, \mathcal{O}(L) \otimes \mathcal{O}_B) = h^1(B, \mathcal{O}(L) \otimes \mathcal{O}_B)$. Let l and l' be the lengths of the skyscraper sheaves $R^1 p_* \mathcal{O}(L)$ and $R^1 p_* \mathcal{O}(L')$, respectively. If $p_* \mathcal{O}(L') \simeq p_* \mathcal{O}(L)$, then $l = l'$. If $p_* \mathcal{O}(L') \hookrightarrow p_* \mathcal{O}(L)$ is not an isomorphism, then $l' < l$. In the former case, we have

$$\mathrm{Im}(p^* p_* \mathcal{O}(L) \to \mathcal{O}(L)) \subset \mathcal{O}(L') \subset \mathcal{O}(L).$$

Thus after a finite number of flops, we come to the second situation. However the length is a nonnegative integer. Therefore we can not perform flops infinitely. Q.E.D.

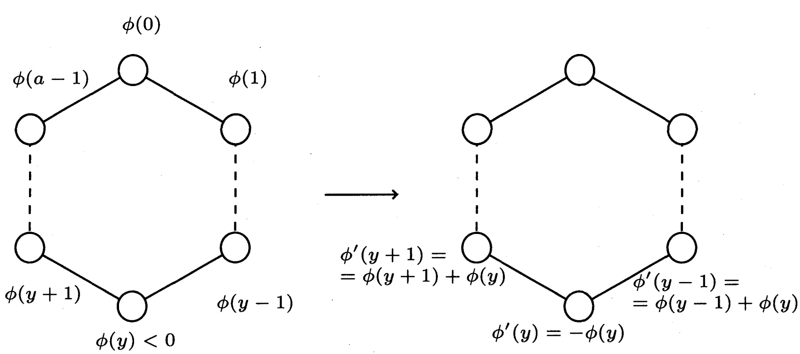

Figure 2. Rule of flops in the case $a > 2$.

Let H be the variation of Hodge structures on $(\Delta^d)^\star$ induced from the period function $\omega(z) = \sum_{i=1}^{l} a_i z_i + h(t)$ and monodromy matrices $\rho(\gamma_i)$ of type I_{a_i}. Then the group of meromorphic sections of the toric model $p\colon X_\sigma \to \Delta^d$ is isomorphic to $H^0(\Delta^d, \mathfrak{S}_{H/\Delta^d})$. We shall study the sheaf $\mathfrak{S}_{H/\Delta^d}$.

Lemma 4.2.13. *Let $\Delta^d \cdots\to X_\sigma$ be a meromorphic section. Then there exist a sign function σ' such that the induced mapping $\Delta^d \cdots\to X_{\sigma'}$ is holomorphic.*

Proof. Let $\Gamma \subset X_\sigma$ be the image of the meromorphic section. Suppose that $\Gamma \to \Delta^d$ is not an isomorphism. Then $\Gamma \cdot C < 0$ for an irreducible curve C contained in a fiber of $\Gamma \to \Delta^d$. Thus Γ is not relatively nef over Δ^d. By the same argument as in the proof of Proposition 4.2.12, we can find an expected sign function. Q.E.D.

We have the quotient morphism $\mathcal{X}_\sigma \to X_\sigma$ by the action of ϑ. Since this is the universal covering mapping of X_σ, every meromorphic section of $X_\sigma \to \Delta^d$ has a lift to $\mathcal{X}_\sigma$. Note that this is holomorphic over $(\Delta^d)^\circ$.

Lemma 4.2.14. *The sheaf of germs of meromorphic sections of $\pi\colon \mathcal{X}_\sigma \to \Delta^d$ is isomorphic to the sheaf $\mathcal{O}_{\Delta^d}(*D^+)^\star$ of germs of meromorphic functions whose zeros and poles are contained in the divisor $D^+ := \{\prod_{a_i>0} t_i = 0\}$.*

Proof. A holomorphic section of $\mathcal{X}_\sigma \to \Delta^d$ for some σ is given by $s = v(t)\prod_{i=1}^{l} t_i^{I_i(k)}$ for some holomorphic function $v(t)$ on Δ^d and an integer k, where I_i denotes the index function at i with respect to σ. By Lemma 4.2.13, we see that every meromorphic sections of π is written by

$$s = u(t)\prod_{i=1}^{l} t_i^{m_i}$$

for a nowhere vanishing function $u(t)$ on Δ^d and integers m_i, for $1 \le i \le l$ with $a_i > 0$. Q.E.D.

Then we have a natural surjective homomorphism $\mathcal{O}_{\Delta^d}(*D^+)^\star \twoheadrightarrow \mathfrak{S}_{H/\Delta^d}$. The kernel is isomorphic to $\mathbb{Z}$ whose generator corresponds to the meromorphic function $e(h(t))\prod_{i=1}^{l} t_i^{a_i}$. Therefore there exist the following two exact sequences:

$$0 \to \mathbb{Z}_{\Delta^d} \to \mathcal{O}_{\Delta^d}(*D^+)^\star \to \mathfrak{S}_{H/\Delta^d} \to 0; \tag{4.2}$$

$$0 \to \mathcal{O}_{\Delta^d}^\star \to \mathcal{O}_{\Delta^d}(*D^+)^\star \to \bigoplus_{a_i>0} \mathbb{Z}_{D_i} \to 0. \tag{4.3}$$

4.3. Smooth model and toric model theorems

Theorem 4.3.1 (Smooth model theorem). *Let $f\colon Y \to \Delta^d$ be a projective elliptic fibration such that*

(1) *f is a smooth morphism over $(\Delta^d)^\star$,*
(2) *the monodromy representation is of type I_0,*
(3) *$f^{-1}(P)$ has a reduced component for a general point P of each D_i.*

Then f is bimeromorphically equivalent to the smooth basic elliptic fibration $p\colon B(H) \to \Delta^d$, where H is the variation of Hodge structures induced from f.

Proof. We have a Zariski-open subset V of Δ^d such that Y is flat over V, $\operatorname{codim}(\Delta^d \setminus V) \geq 2$, and that $f_{|V}\colon Y_{|V} \to V$ admits a meromorphic section locally over V. Thus we obtain a cohomology class $\eta = \eta(Y_{|V}/V, \phi) \in H^1(V, \mathfrak{S}_{H/\Delta^d})$, where $\phi\colon H \simeq (R^1 f_* \mathbb{Z}_Y)_{|V}$ is an isomorphism. Note that η is a torsion element by Proposition 4.1.5. There is an exact sequence:

$$0 \to H \simeq \mathbb{Z}_{\Delta^d}^{\oplus 2} \to \mathcal{O}_{\Delta^d} \to \mathfrak{S}_{H/\Delta^d} \to 0.$$

Hence we have an isomorphism:

$$H^1(V, \mathcal{O}_V) \simeq H^1(V, \mathfrak{S}_{H/\Delta^d}),$$

since $H^i(V, \mathbb{Z}) = 0$ for $i = 1, 2$. Thus the torsion element η must be zero. This means that $Y_{|V}$ is bimeromorphically equivalent to $B(H)_{|V}$ over V. Since $B(H)\backslash p^{-1}(V)$ has codimension greater than one, the meromorphic mapping to $Y_{|V}$ extends to a meromorphic mapping $B(H) \cdots\!\to Y$ over Δ^d. Hence f is bimeromorphically equivalent to p. Q.E.D.

Theorem 4.3.2 (Toric model theorem). *Let $f\colon Y \to \Delta^d$ be a projective elliptic fibration such that*

(1) *f is a smooth morphism over $(\Delta^d)^\star$,*
(2) *the monodromy matrix $\rho(\gamma_i)$ around the coordinate hyperplane D_i is of type I_{a_i},*
(3) *one of $\{a_i\}$ is not zero,*
(4) *$f^{-1}(P)$ has a reduced component for a general point P of each D_i.*

Then f is bimeromorphically equivalent to a toric model $p\colon X_\sigma \to \Delta^d$.

Proof. Since the monodromy representation of f is of type $\mathrm{I}_{(+)}$, by Proposition 2.1.4, we may assume that the period function of f over $(\Delta^d)^\star$ is written as:

$$\omega(z) = \sum_{i=1}^{l} a_i z_i + h(t)$$

for a holomorphic function $h(t)$ on Δ^d. Let $p\colon X_\sigma \to \Delta^d$ be a toric model constructed from the variation of Hodge structures H. We have a Zariski-open subset $V \subset (\Delta^d)^\circ$ such that $f\colon Y \to \Delta^d$ is flat over V, $\operatorname{codim}((\Delta^d)^\circ \setminus V) \geq 2$ and that $f_{|V}\colon Y_{|V} \to V$ admits a meromorphic section locally over V. Therefore, we have a cohomology class $\eta(Y_{|V}/V, \phi) \in H^1(V, \mathfrak{S}_{H/\Delta^d})$, where $\phi\colon H \simeq (R^1 f_* \mathbb{Z}_Y)_{|Y^\star}$ is an isomorphism. By Proposition 4.1.5, $\eta(Y_{|V}/V, \phi)$ is a torsion element of $H^1(V, \mathfrak{S}_{H/\Delta^d})$. We have an isomorphism $H^1(V, \mathcal{O}_{\Delta^d}(*D^+)^\star) \simeq H^1(V, \mathfrak{S}_{H/\Delta^d})$ from (4.2). From (4.3), we have a surjection

$$H^0\left(V, \mathcal{O}_{\Delta^d}(*D^+)^\star\right) \twoheadrightarrow H^0\Big(V, \bigoplus_{a_i>0} \mathbb{Z}_{D_i}\Big) \simeq \bigoplus_{a_i>0} \mathbb{Z}$$

and an exact sequence

$$0 \to H^1(V, \mathcal{O}^\star_{\Delta^d}) \to H^1\left(V, \mathcal{O}_{\Delta^d}(*D^+)^\star\right) \to$$
$$\to H^1\Big(V, \bigoplus_{a_i>0} \mathbb{Z}_{D_i}\Big) \simeq \bigoplus_{a_i>0} H^1(D_i \cap V, \mathbb{Z}).$$

Note that $H^1(D_i \cap V, \mathbb{Z})$ are torsion free abelian groups and the exponential sequence on Δ^d induces the isomorphism $H^1(V, \mathcal{O}_{\Delta^d}) \simeq H^1(V, \mathcal{O}^\star_{\Delta^d})$. Hence $H^1(V, \mathcal{O}^\star_{\Delta^d})$ has a structure of $\mathbb{C}$-vector space. Therefore $H^1(V, \mathfrak{S}_{H/\Delta^d})$ is torsion free. Thus $\eta(Y_{|V}/V, \phi) = 0$, which means that $Y_{|V} \to V$ is bimeromorphically equivalent to the toric model $p\colon X_\sigma \to \Delta^d$ over V. Hence we have a bimeromorphic mapping $(X_\sigma)_{|V} \cdots\to Y_{|V}$ over V. Since $\operatorname{codim}(X_\sigma \setminus p^{-1}((\Delta^d)^\circ)) \geq 2$, $\operatorname{codim}(p^{-1}((\Delta^d)^\circ) \setminus p^{-1}(V)) \geq 2$, and since $f\colon Y \to \Delta^d$ is a projective morphism, the meromorphic mapping extends to a bimeromorphic mapping $X_\sigma \cdots\to Y$ over Δ^d.

Q.E.D.

By taking a unipotent reduction and a further Kummer coverings $\Delta^d \to \Delta^d$, we have the following:

Corollary 4.3.3. *Let $f\colon Y \to \Delta^d$ be a projective elliptic fibration smooth over $(\Delta^d)^\star$. Then there is a finite branched covering $T \to \Delta^d$ étale over $(\Delta^d)^\star$ such that $Y \times_{\Delta^d} T \to T$ admits a meromorphic section.*

We shall give another proof of Corollary 4.3.3, which is based on an argument of Viehweg ([V, 9.10]).

Proof. Let $A \subset Y$ be a prime divisor dominating Δ^d. By taking a normalization of A, we have a generically finite surjective morphism $V \to \Delta^d$ such that $Y \times_{\Delta^d} V \to V$ admits a meromorphic section. Let $V \to T \to \Delta^d$ be the Stein factorization. Then $\tau\colon T \to \Delta^d$ is a finite morphism. By taking its Galois closure, we may suppose that τ

is a Galois covering with a Galois group G. Let us consider a basic elliptic fibration $p: B(H) \to \Delta^d$ associated with the variation of Hodge structures H induced from f over $(\Delta^d)^\star$ and let B_T be the normalization of the main component of $B(H) \times_{\Delta^d} T$. We fix a bimeromorphic mapping $\varphi: Y \times_{\Delta^d} T \cdots\to B_T$ over T which keeps the isomorphism of variations of Hodge structures determined by f and p. Let $\mathfrak{S}_{H_T/T}$ be the sheaf of germs of meromorphic sections of $B_T \to T$. Then by the argument preceeding to Lemma 4.1.6, we have a cohomology class $\eta \in H^1(G, H^0(T, \mathfrak{S}_{H_T/T}))$. For a cocycle $\{\eta_g\}_{g\in G}$ representing η, we have a meromorphic mapping

$$\phi'(g) := \phi(g) \circ \mathrm{tr}(\eta_g) \colon B_T \cdots\to B_T \to B_T,$$

where $\phi(g)$ is induced from

$$\mathrm{id} \times_{\Delta^d} g \colon B(H) \times_{\Delta^d} T \to B(H) \times_{\Delta^d} T.$$

We can take $\{\eta_g\}$ to satisfy $\phi'(g) = \varphi \circ \phi_Y(g) \circ \varphi^{-1}$, where

$$\phi_Y(g) := (\mathrm{id}_Y \times_{\Delta^d} g) \colon Y \times_{\Delta^d} T \to Y \times_{\Delta^d} T.$$

Let n be the order of η. Then η comes from $H^1(G, H^0(T, K_n))$, where

$$K_n := \mathrm{Ker}(\mathfrak{S}_{H_T/T} \overset{n\times}{\to} \mathfrak{S}_{H_T/T}).$$

Thus we have a cocycle $\{\eta_g^0\}$ of $H^0(T, K_n)$ and a meromorphic section $\sigma \in H^0(T, \mathfrak{S}_{H/T})$ such that

$$\eta_g = \eta_g^0 + \sigma - \sigma^g.$$

By replacing φ by $\mathrm{tr}(\sigma) \circ \varphi$, we may assume that $\eta_g = \eta_g^0 \in H^0(T, K_n)$. For a prime divisor Γ of Δ^d with $\Gamma \cap (\Delta^d)^\star \neq \emptyset$, let Γ' be the unique irreducible component of $p^{-1}(\Gamma)$ dominating Γ. Let $R(\Gamma)$ be the ramification group for Γ, that is the subgroup of G consisting of all the elements $g \in G$ satisfying the following condition: for any prime divisor Γ_i of T dominating Γ, $g: T \to T$ induces the identity on Γ_i. If $g \in R(\Gamma)$, then $\phi'(g)$ also induces the identity on every prime divisor Γ_i' on B_T dominating Γ'. Therefore η_g coincides with the zero section at least over $\Gamma \cap (\Delta^d)^\star$. Since K_n is a local constant system with fiber $(\mathbb{Z}/n\mathbb{Z})^{\oplus 2}$ over $\tau^{-1}((\Delta^d)^\star)$, every $\eta_g = 0$. Let R be the subgroup of G generated by all such ramification subgroups $R(\Gamma)$. Then it is a normal subgroup and $\phi(g) = \phi'(g)$ for any $g \in R$. Therefore the quotient $R\backslash Y \to R\backslash T$ still admits a meromorphic section. Hence if we take such a Galois covering $\tau: T \to S$ with the degree of τ being minimal, then R must be trivial. This means that τ is unramified over $(\Delta^d)^\star$. Thus we are done. Q.E.D.

§5. Elliptic fibrations with smooth discriminant loci

Let S, D, $j\colon S^\star \hookrightarrow S$, $e\colon U \to S^\star$ be the same objects as in §2. A variation of Hodge structures H on $S^\star$ is determined by a monodromy representation $\rho\colon \pi_1 := \pi_1(S^\star) \to \mathrm{SL}(2, \mathbb{Z})$ and a period mapping $\omega\colon U \to \mathbb{H}$. They are described in Table 2 and Corollary 3.1.6.

Definition. A finite ramified covering $\tau\colon T \to S$ is called a *U-covering* if the following conditions are satisfied:

(1) $T \simeq \Delta^d = \Delta^l \times \Delta^{d-l}$ and τ is given by

$$\theta = (\theta_1, \theta_2, \ldots, \theta_l, t') \mapsto (\theta_1^{m_1}, \theta_2^{m_2}, \ldots, \theta_l^{m_l}, t') \in S,$$

for some positive integers $m_1, m_2, \ldots, m_l$;

(2) The induced variation of Hodge structures $H_T := \tau^{-1}H$ defined on $T^\star := \tau^{-1}(S^\star)$ has only unipotent monodromies, i.e., it is of type I_0 or type $\mathrm{I}_{(+)}$.

Let $\tau\colon T \to S$ be a U-covering. Then the period mapping of H is written by

$$\omega(z) = \sum_{i=1}^{l} a_i z_i + h(\theta)$$

for nonnegative integers a_i and holomorphic function $h(\theta)$ defined over T. If the monodromy group of H is finite, then all $a_i = 0$. Otherwise, $h(\theta)$ is the pullback of a holomorphic function on S. Under this situation, we shall classify projective elliptic fibrations $f\colon Y \to S$ smooth over $S^\star$ such that the induced variations of Hodge structures are isomorphic to the given H. More precisely, we shall describe the set $\mathcal{E}^+(S, D, H)$ defined as follows: Let $(f\colon Y \to S, \phi)$ be a pair of a projective elliptic fibration $f\colon Y \to S$ smooth over $S^\star$ and an isomorphism $\phi\colon H \simeq (R^1 f_* \mathbb{Z})_{|S^\star}$ as variations of Hodge structures. Two such pairs $(f_1\colon Y_1 \to S, \phi_1)$ and $(f_2\colon Y_2 \to S, \phi_2)$ are called bimeromorphically equivalent over S if there is a bimeromorphic mapping $\varphi\colon Y_1 \cdots\to Y_2$ over S such that $\phi_1 = \varphi^* \circ \phi_2$. We define $\mathcal{E}^+(S, D, H)$ to be the set of all the equivalence classes by this relation. Let $(f\colon Y \to S, \phi)$ be an element in $\mathcal{E}^+(S, D, H)$. Then by Corollary 4.3.3, we can find a U-covering $\tau\colon T \to S$ satisfying the following condition:

(5.1) $\qquad Y \times_S T \to T$ admits a meromorphic section.

Therefore, we have

$$\mathcal{E}^+(S, D, H) = \varinjlim_{T \to S} \mathcal{E}(S, D, H, T),$$

where the inductive limit is taken over all the U-coverings $\tau\colon T \to S$. Note that the set $\mathcal{E}(S, D, H, T)$ is identified with the cohomology group $H^1(\mathrm{Gal}(\tau), H^0(T, \mathfrak{S}_{H_T/T}))$ by Lemma 4.1.6. We see further that if a U-covering $\tau\colon T \to S$ satisfies the condition (5.1) for $Y \to S$, then $Y \times_S T \to T$ is bimeromorphically equivalent to a smooth morphism or a toric model by Theorems 4.3.1 and 4.3.2. For a cohomology class in $H^1(\mathrm{Gal}(\tau), H^0(T, \mathfrak{S}_{H_T/T}))$, we have a meromorphic action of the Galois group $\mathrm{Gal}(\tau)$ on the smooth or the toric model. We shall describe all such actions. In this section, we treat the case $l = 1$. We can further construct minimal models of the quotient varieties. For $l \geq 2$, we treat the case H has only finite monodromies in §6 and the rest case in §7.

5.1. Finite monodromy case

Assume that $l = 1$. We denote $a = a_1$ and $m = m_1$ for a U-covering $\tau\colon T \to S$. Thus τ is defined by $(\theta_1, t') \mapsto (\theta_1^m, t')$. Suppose that the order of the monodromy matrix $\rho(\gamma_1)$ is finite. Then $a = 0$ and $j_{T*}H_T$ is a constant sheaf for the immersion $j_T\colon T^\star \hookrightarrow T$. We denote the constant sheaf by the same symbol H_T. The exact sequence

$$0 \to H_T \to \mathcal{L}_{H_T} \to \mathfrak{S}_{H_T} \to 0$$

defined over $T^\star$ extends to:

$$0 \to \mathbb{Z}_T^{\oplus 2} \to \mathcal{O}_T \to \mathfrak{S}_{H_T/T} \to 0.$$

Every sheaves appearing in both sequences are canonically G-linearized, where $G = \mathrm{Gal}(\tau) \simeq \mathbb{Z}/m\mathbb{Z}$. Therefore the right action of G on $\mathbb{Z}^{\oplus 2} = H^0(T, \mathbb{Z}_T^{\oplus 2})$ is induced from the right multiplication of $\rho(\gamma_1)$, and that on $H^0(T, \mathcal{O}_T)$ is described by:

$$f(\theta) \mapsto (c_{\gamma_1} h(\theta) + d_{\gamma_1}) f(\gamma_1 \theta),$$

where $\omega(z) = h(\theta)$ is the period function. The right action on $H^0(T, \mathfrak{S}_{H_T/T})$ is induced from two actions above. We have an exact sequence:

$$H^1(G, H^0(T, \mathcal{O}_T)) \to H^1(G, H^0(T, \mathfrak{S}_{H_T/T})) \to \\ \to H^2(G, \mathbb{Z}^{\oplus 2}) \to H^2(G, H^0(T, \mathcal{O}_T)).$$

Here we note that $H^p(G, H^0(T, \mathcal{O}_T)) = 0$ for $p > 0$, since $H^0(T, \mathcal{O}_T)$ is a $\mathbb{C}$-vector space. Thus

$$H^1(G, H^0(T, \mathfrak{S}_{H_T/T})) \simeq H^2(G, \mathbb{Z}^{\oplus 2}).$$

Let π_1' be the fundamental group of $T^\star$. Then $\pi_1/\pi_1' \simeq G$. We have the following Hochschild–Serre spectral sequence:

$$E_2^{p,q} = H^p(G, H^q(\pi_1', \mathbb{Z}^{\oplus 2})) \Longrightarrow E^{p+q} = H^{p+q}(\pi_1, \mathbb{Z}^{\oplus 2}).$$

Suppose that H is of type I_0. Then $E^1 = \mathbb{Z}^{\oplus 2} \to E_2^{0,1} = \mathbb{Z}^{\oplus 2}$ is the multiplication map by m. Since $H^2(\pi_1, \mathbb{Z}^{\oplus 2}) = 0$, we have

$$H^2(G, \mathbb{Z}^{\oplus 2}) \simeq (\mathbb{Z}/m\mathbb{Z})^{\oplus 2}.$$

Suppose that H is not of type I_0. Then $E_2^{0,1} = H^0(G, \mathbb{Z}^{\oplus 2}) = 0$. Since $H^2(\pi_1, \mathbb{Z}^{\oplus 2}) = 0$, we have

$$H^1(G, H^0(T, \mathfrak{S}_{H_T/T})) \simeq H^2(G, \mathbb{Z}) = E_2^{2,0} = 0.$$

This means that every elliptic fibration $Y \to T$ appearing in $\mathcal{E}^+(S, D, H)$ is a basic fibration in this case. Therefore we have:

Theorem 5.1.1. *For $S = \Delta^d$, $D = \{t_1 = 0\}$ and for a variation of Hodge structures H on $S^\star = S \setminus D$ such that the order of the monodromy matrix is finite, we have the following identification:*

$$\mathcal{E}^+(S, D, H) = \begin{cases} (\mathbb{Q}/\mathbb{Z})^{\oplus 2}, & H \text{ is of type } \mathrm{I}_0, \\ 0, & \text{otherwise.} \end{cases}$$

5.2. Infinite monodromy case

Assume that the order of the monodromy matrix $\rho(\gamma_1)$ is infinite, i.e., $\rho(\gamma_1)$ is of type I_a or I_a^* for a positive integer a. The period function $\omega(z)$ is written by $\omega(z) = az_1 + h(t)$, where $h(t)$ is a holomorphic function on S. Let $\tau\colon T \to S$ be a U-covering determined by $\theta = (\theta_1, t') \mapsto (\theta_1^m, t')$. Then from the exact sequences (4.2), (4.3), we have exact sequences of right G-modules:

$$(5.2) \qquad 0 \to \mathbb{Z} \to H^0(T, \mathcal{O}_T(*D_T)^\star) \to H^0(T, \mathfrak{S}_{H_T/T}) \to 0,$$

$$(5.3) \qquad 0 \to H^0(T, \mathcal{O}_T^\star) \to H^0(T, \mathcal{O}_T(*D_T)^\star) \to H^0(D_T, \mathbb{Z}) \simeq \mathbb{Z} \to 0,$$

where $D_T = \tau^{-1}(D) = \{\theta_1 = 0\}$ and $1 \in \mathbb{Z}$ is mapped to the function $e(h(t))t_1^a$. Suppose first that $\rho(\gamma_1)$ is of type I_a. Then its action on $\mathbb{Z}$ is trivial and that on $H^0(T, \mathcal{O}_T(*D_T)^\star)$ is written as:

$$H^0(T, \mathcal{O}_T(*D_T)^\star) \ni v(\theta) = v(\theta_1, t') \mapsto v\left(e(1/m)\theta_1, t'\right).$$

Thus the action on the group $H^0(T, \mathcal{O}_T(*D_T)^\star) \simeq H^0(T, \mathcal{O}_T^\star) \oplus \mathbb{Z}$ is expressed by:

$$(u(\theta), n) = (u(\theta_1, t'), n) \mapsto (u(e(1/m)\theta_1, t')e(n/m), n).$$

Let $[u(\theta), n]$ be the image of $(u(\theta), n)$ under the homomorphism $H^0(T, \mathcal{O}_T(*D_T)^\star) \to H^0(T, \mathfrak{S}_{H_T/T})$. Since $G \simeq \mathbb{Z}/m\mathbb{Z}$, the cohomology group $H^1(G, H^0(T, \mathfrak{S}_{H_T/T}))$ is isomorphic to Z^1/B^1, where

$$Z^1 = \left\{ \xi \in H^0(T, \mathfrak{S}_{H_T/T}) \,\middle|\, \sum_{i=0}^{m-1} \xi^{\gamma_1^i} = 0 \right\},$$
$$B^1 = \left\{ \xi = \eta - \eta^{\gamma_1} \,\middle|\, \eta \in H^0(T, \mathfrak{S}_{H_T/T}) \right\}.$$

For an element $\xi = [u(\theta), n]$, this is contained in Z^1 if and only if there is an integer k such that

$$\left(e\Big(\frac{n(m-1)}{2}\Big) \prod_{i=0}^{m-1} u\left(e(i/m)\theta_1, t'\right), nm \right) = (e(kh(t)), kma)\,.$$

Therefore $n = ka$ and

$$\prod_{i=0}^{m-1} u\left(e(i/m)\theta_1, t'\right) = e\Big(kh(t) - \frac{n(m-1)}{2}\Big).$$

Hence there exist a nowhere vanishing function $u_1(\theta)$ and a positive integer n_1 such that

$$u(\theta) = e\Big(\frac{n_1 + kh(t)}{m} - \frac{ka(m-1)}{2m}\Big) u_1(\theta) u_1\left(e(1/m)\theta_1, t'\right)^{-1}.$$

If $\xi = [u(\theta), n]$ is contained in B^1, then $n = k'ma$ and

$$u(\theta) = u_2(\theta) u_2\left(e(1/m)\theta_1, t'\right)^{-1} e\left(-n'/m + k'h(t)\right),$$

for integers k', n' and a nowhere vanishing function $u_2(\theta)$. Thus $k = mk'$ and $n_1 + n' \equiv k'am(m-1)/2 \mod m$. Hence $H^1(G, H^0(T, \mathfrak{S}_{H_T/T})) \simeq \mathbb{Z}/m\mathbb{Z}$ and its generator is written by

$$\xi = \xi_m = \left[e\Big(\frac{h(t)}{m} - \frac{a(m-1)}{2m}\Big), a \right].$$

Let $T' \simeq \Delta^d \to T$ be a finite ramified covering branched only over D_T defined by $\theta' = (\theta'_1, t') \mapsto (\theta_1'^{m'}, t')$. Then $\mathrm{Gal}(T'/T) \simeq \mathbb{Z}/m'\mathbb{Z}$ and $\mathrm{Gal}(T'/S) \simeq \mathbb{Z}/mm'\mathbb{Z}$. The image of ξ_m under the homomorphism

$$H^1(\mathrm{Gal}(T/S), H^0(T, \mathfrak{S}_{H_T/T})) \to H^1(\mathrm{Gal}(T'/S), H^0(T', \mathfrak{S}_{H_{T'}/T'}))$$

is $m'\xi_{mm'}$. Therefore

$$\varinjlim_{T/S} H^1(\mathrm{Gal}(T/S), H^0(T, \mathfrak{S}_{H_T/T})) \simeq \mathbb{Q}/\mathbb{Z}.$$

Next suppose that $\rho(\gamma_1)$ is of type I_a^*. At the exact sequence (5.2), its action on $\mathbb{Z}$ is the multiplication of -1 and that on $H^0(T, \mathcal{O}_T(*D_T)^\star)$ is written by:

$$H^0(T, \mathcal{O}_T(*D_T))^\star \ni v(\theta_1, t') \mapsto v\left(e(1/m)\theta_1, t'\right)^{-1}.$$

Thus the action on the group $H^0(T, \mathcal{O}_T(*D_T)^\star) \simeq H^0(T, \mathcal{O}_T^\star) \oplus \mathbb{Z}$ is expressed by:

$$(u(\theta), n) = (u(\theta_1, t'), n) \mapsto \left(u\left(e(1/m)\theta_1, t'\right)^{-1} e(-n/m), -n\right).$$

Let $[u(\theta), n]$ be the image of $(u(\theta), n)$ under the homomorphism $H^0(T, \mathcal{O}_T(*D_T)^\star) \to H^0(T, \mathfrak{S}_{H_T/T})$. Since $G \simeq \mathbb{Z}/m\mathbb{Z}$, the cohomology group $H^1(G, H^0(T, \mathfrak{S}_{H_T/T}))$ is isomorphic to Z^1/B^1, where

$$Z^1 = \left\{\xi \in H^0(T, \mathfrak{S}_{H_T/T}) \,\middle|\, \sum_{i=0}^{m-1} \xi^{\gamma_1^i} = 0\right\},$$
$$B^1 = \left\{\xi = \eta - \eta^{\gamma_1} \,\middle|\, \eta \in H^0(T, \mathfrak{S}_{H_T/T})\right\}.$$

For an element $\xi = [u(\theta), n]$, this is contained in Z^1 if and only if

$$\left((-1)^n \prod_{i=0}^{m-1} u\left(e(i/m)\theta_1, t'\right)^{(-1)^i}, 0\right) = (1, 0).$$

By taking $\theta_1 = 0$, we see that n is even. Since $H^1(G, H^0(T, \mathcal{O}_T)) = 0$ and $H^2(G, \mathbb{Z}) = 0$, we have $H^1(G, H^0(T, \mathcal{O}_T^\star)) = 0$. Therefore there exist a nowhere vanishing function $u_1(\tau)$ such that

$$u(\theta) = u_1(\theta) u_1\left(e(1/m)\theta_1, t'\right).$$

Let $v(\theta) := e(-n/(4m))u_1(\theta)$. Then $\xi = [u(\theta), n] = \eta - \eta^{\gamma_1}$ for $\eta = [v(\theta), n/2]$. Hence ξ is contained in B^1. Therefore $H^1(G, H^0(T, \mathfrak{S}_{H_T/T})) = 0$. Thus we have:

Theorem 5.2.1. *For $S = \Delta^d$, $D = \{t_1 = 0\}$ and for a variation of Hodge structures H on $S^\star = S \setminus D$ such that the order of the monodromy matrix is infinite, we have the following identification:*

$$\mathcal{E}^+(S, D, H) = \begin{cases} \mathbb{Q}/\mathbb{Z}, & \text{in the case } \mathrm{I}_a; \\ 0, & \text{in the case } \mathrm{I}_a^*. \end{cases}$$

5.3. Minimal models

Suppose that $\rho(\gamma_1)$ is of type I_0. Let $(p/m, q/m)$ be a pair of rational numbers, where p, q, m are positive integers and $\gcd(m, p, q) = 1$. Giving such a pair modulo $\mathbb{Z}^{\oplus 2}$ is equivalent to giving an element of $(\mathbb{Q}/\mathbb{Z})^{\oplus 2}$ whose order is m. Let $\tau\colon T \to S$ be the cyclic covering defined as before with mapping degree m. The smooth basic fibration $B(H_T) \to T$ is described as the quotient space of $T \times \mathbb{C}$ by the following action of $(n_1, n_2) \in \mathbb{Z}^{\oplus 2}$:

$$T \times \mathbb{C} \ni (\theta, \zeta) \mapsto (\theta, \zeta + n_1 h(t) + n_2),$$

where $h(t) = \omega(z)$ is the period function. From $(p/m, q/m)$, we have the following action of the generator 1 of $G \simeq \mathbb{Z}/m\mathbb{Z}$ on $B(H_T)$:

$$B(H_T) \ni [(\theta_1, t'), \zeta] \mapsto \left[(e(1/m)\theta_1, t'), \zeta + \frac{p}{m}h(t) + \frac{q}{m}\right].$$

An elliptic fibration $f\colon Y \to S$ smooth over $S^\star$ having H as its variation of Hodge structures is bimeromorphically equivalent to the quotient space by the action above for some $(p/m, q/m)$. Note that the action is free. Therefore the quotient space X is nonsingular. Let D_X be the support of the divisor π^*D, where $\pi\colon X \to S$ is the induced elliptic fibration. Then $\pi^*D = mD_X$ and thus the singular fibers of π are elliptic curves $\mathbb{C}/(\mathbb{Z}h(0,t') + \mathbb{Z} + \mathbb{Z}((p/m)h(0,t') + (q/m)))$ with multiplicity m. We have the canonical bundle formula:

$$\omega_X \simeq \pi^*(\omega_S) \otimes \mathcal{O}_X((m-1)D_X).$$

In particular, $\pi\colon X \to S$ is the unique minimal model in the bimeromorphic equivalence class over S.

Theorem 5.3.1. *Let $f\colon Y \to S = \Delta^d$ be a projective elliptic fibration smooth outside $D = \{t_1 = 0\}$. Suppose that the induced variation of Hodge structures is of type I_0. Then f has a unique minimal model $\pi\colon X \to S$ such that X is nonsingular, π is a flat morphism, and*

$$\omega_X \simeq \pi^*\omega_S \otimes \mathcal{O}_X((m-1)D_X)$$

for some positive integer m, where $D_X \to D$ is a smooth elliptic fibration and $\pi^*D = mD_X$.

In the case $d = 1$, the singular fiber is called of type ${}_m\mathrm{I}_0$ in Kodaira ([Kd1]) (cf. Figure 3).

Next suppose that $\rho(\gamma_1)$ is of type I_a for $a > 0$. Let p and m be coprime positive integers, $T \to S$ the finite cyclic covering

$$T \ni \theta = (\theta_1, t') \mapsto (\theta_1^m, t') \in S$$

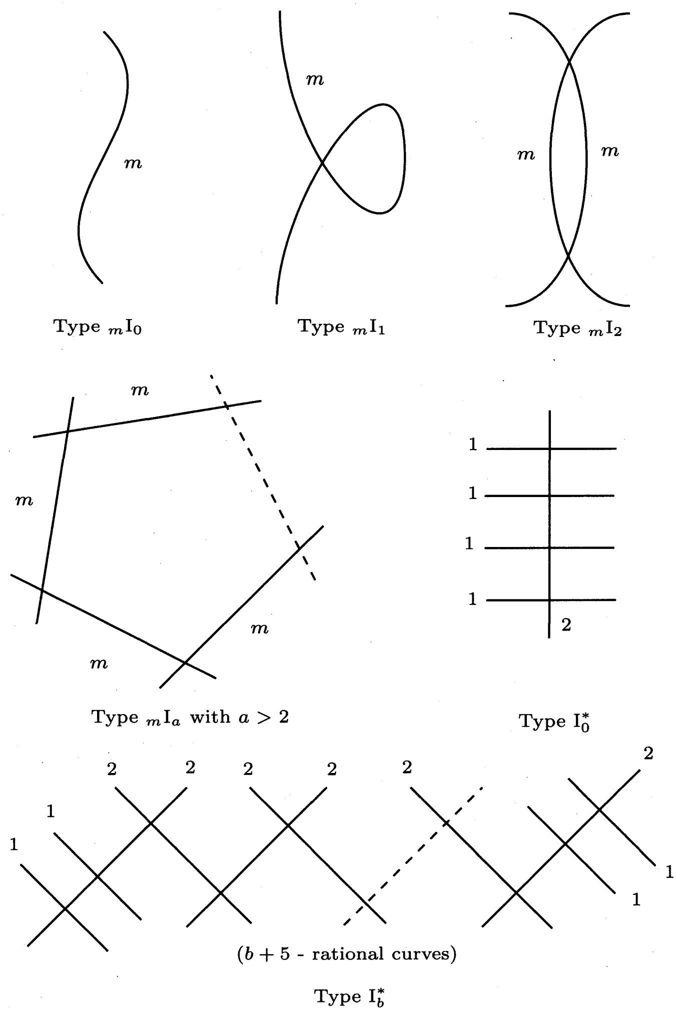

Figure 3. Singular fibers of Types $_m\mathrm{I}_a$ and I_b^*.

defined as before, and let $X \longrightarrow T$ be the toric model associated with the period function $\omega(z) = az_1 + h(t)$. The X is the quotient space of $\mathcal{X} = \bigcup_{k\in\mathbb{Z}} \mathcal{X}^{(k)}$ by the action: $\mathcal{X} \ni s \mapsto s\theta_1^{ma}e(h(t))$, where s is a coordinate of $\mathbb{C}^\star$ under the isomorphism $T^\star \times_T \mathcal{X} \simeq T^\star \times \mathbb{C}^\star$. We now

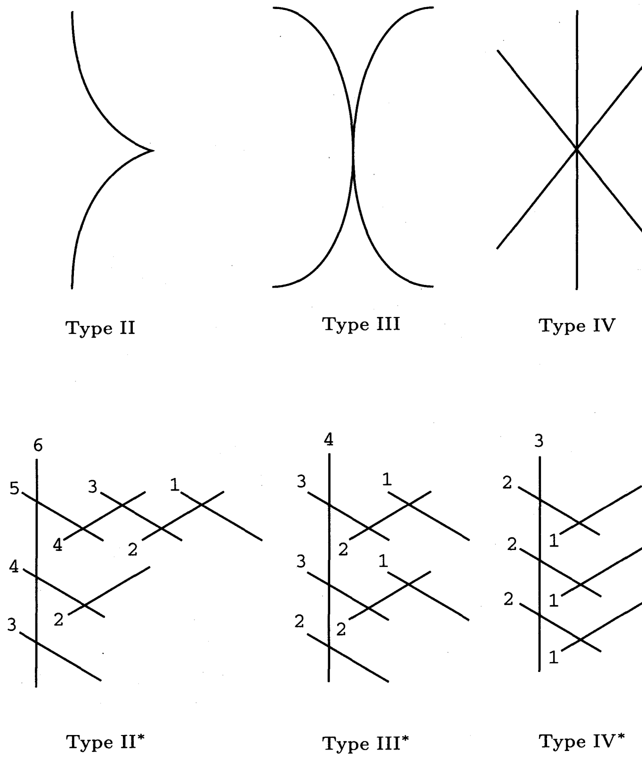

Figure 4. Singular fibers of Types II, III, IV, II*, III* and IV*.

have the following action of $\rho(\gamma_1)$ on X corresponding to $p/m \in \mathbb{Q}/\mathbb{Z}$:

$$X \ni [\theta, s] = [(\theta_1, t'), s]$$
$$\longmapsto \left[(e(1/m)\theta_1, t'), s \cdot e\left(\frac{p}{m} \left(h(t) - \frac{(m-1)a}{2} \right) \right) \theta_1^{pa} \right],$$

where $[\theta, s]$ denotes the image of the point $(\theta, s) \in T^\star \times \mathbb{C}^\star$. Note that the action is holomorphic and free on the whole space X. Let $\pi\colon Z \to S$ be the elliptic fibration obtained as the quotient by the action. Let D_Z be the support of the divisor $\pi^* D$. Then each fiber of $D_Z \to D$ is a cycle of rational curves, the number of whose irreducible components is a, and $\pi^*D = mD_Z$. The canonical bundle ω_Z is isomorphic to $\pi^*\omega_S \otimes \mathcal{O}((m-1)D_Z)$. In particular, $\pi\colon Z \to S$ is the unique minimal model of $f\colon Y \to S$. Thus we have:

Theorem 5.3.2. *Let $f\colon Y \to S = \Delta^d$ be a projective elliptic fibration which is smooth outside $D = \{t_1 = 0\}$. Suppose that the induced variation of Hodge structures is of type I_a for $a > 0$. Then f has a unique minimal model $\pi\colon Z \to S$ such that Z is nonsingular, π is a flat morphism, and*

$$\omega_Z \simeq \pi^*\omega_S \otimes \mathcal{O}_Z((m-1)D_Z)$$

for some positive integer m, where each fiber of $D_Z \to D$ is a cycle of rational curves, the number of whose irreducible components is a, and $\pi^*D = mD_Z$.

In the case $d = 1$, the singular fiber is called of type ${}_m\mathrm{I}_a$ in Kodaira ([Kd1]) (cf. Figure 3).

Finally, we shall consider the rest cases, i.e., $\rho(\gamma_1)$ is one of types I_0^*, II, II*, III, III*, IV, IV* and I_a^* for $a > 0$. We have only to give a minimal model for a basic elliptic fibration $B(H) \to S$. There are two methods. First one is starting from the description of the action of G on the smooth basic elliptic fibration or the toric model over T. After resolving the singularities of quotient spaces, we take successive contractions of the exceptional curves of the first kind. This is done in [Kd1] in the case $d = 1$. The same argument works even in the case $d > 1$, since the singularities are of similar types. In the second method, we use Weierstrass models. We may assume that $B(H) \simeq W_S(\mathcal{O}_S, \alpha, \beta)$ for some functions α, β such that $4\alpha^3 + 27\beta^2$ vanishes only over $D = \{t_1 = 0\}$. Since $\rho(\gamma_1)$ is one of such types, we see that $\alpha = \beta = 0$ on D. The singular locus of the Weierstrass model $\{Y^2Z = X^3 + \alpha(t)XZ^2 + \beta(t)Z^3\}$ is the locus $\{Y = t_1 = 0\}$. This singularity is locally isomorphic to $F \times \mathbb{C}^{d-1}$, where F is a surface singularity and is a germ of a rational double point. Therefore by taking standard resolution of singularities $Z \to W_S(\mathcal{O}_S, \alpha, \beta)$, we have a minimal elliptic fibration $\pi\colon Z \to S$. Therefore we have:

Theorem 5.3.3. *Let $f\colon Y \to S = \Delta^d$ be a projective elliptic fibration which is smooth outside $D = \{t_1 = 0\}$. Suppose that the induced variation of Hodge structures is not of type I_a $(a \geq 0)$. Then f admits a meromorphic section and has a unique minimal model $\pi\colon Z \to S$ such that Z is nonsingular, π is a flat morphism, and $\omega_Z \simeq \pi^*\omega_S$, where each fiber of $\pi^*D \to D$ is isomorphic to the singular fiber of the same type obtained in Kodaira* ([Kd1]) (cf. Figure 3 and Figure 4).

Since the minimal models are unique, we have the following result from Theorems 5.3.1, 5.3.2 and 5.3.3:

Corollary 5.3.4. *Let $f\colon Y \to S$ be an elliptic fibration over a complex manifold S. Assume the following conditions are satisfied:*

(1) *f is smooth outside a nonsingular divisor $D \subset S$;*
(2) *For any point $P \in S$, there is an open neighborhood $\mathcal{U}$ such that $Y_{|\mathcal{U}} \to \mathcal{U}$ is bimeromorphically equivalent to a projective morphism.*

Then there is a minimal elliptic fibration $\pi\colon X \to S$ for f such that X is nonsingular and π is flat.

Let $f\colon Y \to S$ be an elliptic fibration smooth outside a normal crossing divisor D. Let $C \subset S$ be a general smooth curve intersecting an irreducible component D_i transversely at one general point P. Over an open neighborhood $\mathcal{U}$ of P, we have the unique minimal elliptic fibration $Z_{\mathcal{U}} \to \mathcal{U}$ from $Y_{|\mathcal{U}} \to \mathcal{U}$, by above theorems. Then the singular fiber over P of the minimal elliptic surface obtained from the fiber product $Y \times_{\Delta^d} C \to C$ is isomorphic to that of $Z_{\mathcal{U}} \to \mathcal{U}$.

Definition 5.3.5. The *singular fiber type* of f over the divisor D_i is defined to be the type of the fiber over $P = C \cap D_i$ of the minimal elliptic surface obtained from the fiber product $Y \times_{\Delta^d} C \to C$ for a general curve C.

§6. Finite monodromy case

6.1. Cohomology groups

Let $S = \Delta^d$, $D = \{t_1 t_2 \cdots t_l = 0\} = \sum_{i=1}^{l} D_i$, $S^\star = S \setminus D$ be the same objects as in §2 and let H be a polarized variation of Hodge structures of rank two and weight one defined on $S^\star$. In this section, we treat the case the monodromy group $\mathrm{Im}(\pi_1 \to \mathrm{SL}(2,\mathbb{Z}))$ is a finite group. Then by §5, the singular fiber type over the coordinate hyperplane D_i is one of ${}_m\mathrm{I}_0$, I_0^*, II, II*, III, III*, IV and IV*.

Let $\tau\colon T \to S$ be a U-covering. Then the pullback $\tau^{-1}H$ extends trivially to a constant system $\mathbb{Z}_T^{\oplus 2}$ together with Hodge filtrations. The period function $\omega(z)$ of H is written by $\omega(z) = h(\theta)$ for a holomorphic function $h(\theta)$ on T. Let H_T be the variation of Hodge structures on T. We have an exact sequences:

$$0 \to H_T \simeq \mathbb{Z}_T^{\oplus 2} \to \mathcal{L}_{H_T} \simeq \mathcal{O}_T \to \mathfrak{S}_{H_T} \to 0.$$

Since there is a factorization $\pi_1 \to G = \mathrm{Gal}(\tau) \to \mathrm{SL}(2,\mathbb{Z})$ of the monodromy representation, the sheaves $\mathbb{Z}_T^{\oplus 2}$, $\mathcal{O}_T$, and $\mathfrak{S}_{H_T}$ are G-linearized.

By taking global sections, we have:

$$0 \to \mathbb{Z}^{\oplus 2} \to H^0(T, \mathcal{O}_T) \to H^0(T, \mathfrak{S}_{H_T}) \to 0,$$

where the right G-module structures of $\mathbb{Z}^{\oplus 2}$ and $H^0(T, \mathcal{O}_T)$ are given by:

$$\mathbb{Z}^{\oplus 2} \ni (m,n) \mapsto (m,n)\rho(\gamma) \text{ and } H^0(T, \mathcal{O}_T) \ni f(\theta) \mapsto (c_\gamma h(\theta) + d_\gamma) f(\gamma\theta),$$

respectively, for $\gamma \in \pi_1$. Since $H^0(T, \mathcal{O}_T)$ is a $\mathbb{C}$-vector space, we have $H^i(G, H^0(T, \mathcal{O}_T)) = 0$ for $i > 0$. Therefore

$$H^1(G, H^0(T, \mathfrak{S}_{H_T})) \simeq H^2(G, \mathbb{Z}^{\oplus 2}).$$

First of all, let us consider the case H is of type I_0. Then $\mathbb{Z}^{\oplus 2}$ is a trivial π_1-module. We have the following commutative diagram of exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}^{\oplus 2} & \longrightarrow & \mathbb{Q}^{\oplus 2} & \longrightarrow & (\mathbb{Q}/\mathbb{Z})^{\oplus 2} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z}^{\oplus 2} & \longrightarrow & H^0(T, \mathcal{O}_T) & \longrightarrow & H^0(T, \mathfrak{S}_{H_T}) & \longrightarrow & 0.
\end{array}$$

Here the homomorphism $\mathbb{Q}^{\oplus 2} \to H^0(T, \mathcal{O}_T)$ is given by $(q_1, q_2) \mapsto q_1 h(\theta) + q_2$. Therefore we have also an isomorphism:

$$H^1(G, (\mathbb{Q}/\mathbb{Z})^{\oplus 2}) \simeq H^1(G, H^0(T, \mathfrak{S}_{H_T})).$$

Hence the set $\mathcal{E}^+(S, D, H)$ is identified with

$$\varinjlim_{\pi_1 \twoheadrightarrow G} H^1(G, (\mathbb{Q}/\mathbb{Z})^{\oplus 2}),$$

where the limit is taken over all the finite quotient groups G of π_1. By considering the isomorphism $H^1(G, (\mathbb{Q}/\mathbb{Z})^{\oplus 2}) \simeq \mathrm{Hom}(G, (\mathbb{Q}/\mathbb{Z})^{\oplus 2})$, we have

$$\mathcal{E}^+(S, D, H) = \mathrm{Hom}(\pi_1, (\mathbb{Q}/\mathbb{Z})^{\oplus 2}) = (\mathbb{Q}/\mathbb{Z})^{\oplus 2l}.$$

Next let us consider the case H is not of type I_0. For the U-covering $T \to S$, let π_1' be the fundamental group of $T^\star$. Then $G = \pi_1/\pi_1'$ and we have the Hochschild–Serre spectral sequence:

$$E_2^{p,q} = H^p(G, H^q(\pi_1', \mathbb{Z}^{\oplus 2})) \Longrightarrow H^{p+q}(\pi_1, \mathbb{Z}^{\oplus 2}).$$

Since $\mathbb{Z}^{\oplus 2}$ is a trivial π_1'-module, we have

$$H^q(\pi_1', \mathbb{Z}^{\oplus 2}) \simeq \begin{cases} \mathbb{Z}^{\oplus 2}, & q = 0; \\ \mathrm{Hom}(\bigwedge^q \pi_1', \mathbb{Z}^{\oplus 2}), & q > 0. \end{cases}$$

We know $H^0(\pi_1, \mathbb{Z}^{\oplus 2}) = 0$ by Theorem 2.2.1. Hence $E_2^{0,1} = E_2^{0,2} = 0$. Therefore we have an injection

$$E_2^{2,0} = H^2(G, \mathbb{Z}^{\oplus 2}) \hookrightarrow H^2(\pi_1, \mathbb{Z}^{\oplus 2}).$$

We shall show it is also surjective for some U-covering $T \to S$. The cohomology group $H^2(\pi_1, \mathbb{Z}^{\oplus 2})$ parameterizes all the smooth elliptic fibrations over $S^\star$ having the same variation of Hodge structures by §2.2. Since this is a finite group by Theorem 2.2.1, they are all projective morphisms. Therefore by Theorem 4.1.1, we can extend them to projective morphisms over S. Hence

$$H^2(\pi_1, \mathbb{Z}^{\oplus 2}) = \bigcup H^2(G, \mathbb{Z}^{\oplus 2}),$$

where G is taken to be the Galois group of a U-covering. Therefore

$$\mathcal{E}^+(S, D, H) = H^2(G, \mathbb{Z}^{\oplus 2}) = H^2(\pi_1, \mathbb{Z}^{\oplus 2})$$

for some U-covering $T \to S$. As a result, we have:

Theorem 6.1.1.

$$\mathcal{E}^+(S, D, H) = \begin{cases} (\mathbb{Q}/\mathbb{Z})^{\oplus 2l}, & H \textit{ is of type } \mathrm{I}_0; \\ (\mathbb{Z}/2\mathbb{Z})^{\oplus 2(l-1)}, & H \textit{ is of type } \mathrm{I}_0^{(*)}; \\ 0, & H \textit{ is of types } \mathrm{II}^{(*)} \textit{ or } \mathrm{IV}_-^{(*)}; \\ (\mathbb{Z}/2\mathbb{Z})^{\oplus (l-1)}, & H \textit{ is of type } \mathrm{III}^{(*)}; \\ (\mathbb{Z}/3\mathbb{Z})^{\oplus (l-1)}, & H \textit{ is of type } \mathrm{IV}_+^{(*)}. \end{cases}$$

6.2. Construction

Case I_0. A variation of Hodge structures H of this type is defined only by a single-valued holomorphic period function $\omega(t) \colon S \to \mathbb{H}$. The basic smooth elliptic fibration $B(H) \to S$ associated with H is constructed as the quotient of $S \times \mathbb{C}$ by the following action of $(m, n) \in \mathbb{Z}^{\oplus 2}$:

$$(t, \zeta) \mapsto (t, \zeta + m\omega(t) + n).$$

Let (p_i, q_i) be elements of $(\mathbb{Q}/\mathbb{Z})^{\oplus 2}$ for $1 \le i \le l$. Let m_i be the order of (p_i, q_i) in $(\mathbb{Q}/\mathbb{Z})^{\oplus 2}$ and let $\tau \colon T = \Delta^d \to S$ be the Kummer covering defined by:

$$T \ni \theta = (\theta_1, \theta_2, \ldots, \theta_l, t') \mapsto (\theta_1^{m_1}, \theta_2^{m_2}, \ldots, \theta_l^{m_l}, t') \in S.$$

The action of the Galois group $G = \mathrm{Gal}(\tau) \simeq \bigoplus_{i=1}^{l} \mathbb{Z}/m_i\mathbb{Z}$ on $T \times_S B(H)$ is as follows:

$$[\theta, \zeta] \mapsto [\gamma_i\theta, \zeta + p_i\omega(t) + q_i],$$

where $[\theta, \zeta]$ denotes the image of $(\theta, \zeta) \in T \times \mathbb{C}$ in $T \times_S B(H)$. Then the quotient $G\backslash(T \times_S B(H)) \to G\backslash T \simeq S$ is an elliptic fibration corresponding to the element $\{(p_i, q_i)\}_{i=1}^{l} \in \mathcal{E}^+(S, D, H)$.

Theorem 6.2.1. *If H is of type I_0, then there exists a minimal elliptic fibration for any element of $\mathcal{E}^+(S, D, H)$.*

Proof. We consider the fixed points for the action of $\gamma \in G$ on $T \times_S B(H)$. Then we see that if

$$\gamma = \gamma_1^{k_1}\gamma_2^{k_2}\cdots\gamma_l^{k_l}$$

has a fixed point, then any points over the locus $\{\theta = \gamma\theta\}$ are fixed and

$$\sum_{i=1}^{l} k_i(p_i, q_i) = 0 \quad \text{in} \quad (\mathbb{Q}/\mathbb{Z})^{\oplus 2}. \tag{6.1}$$

Let $G_0 \subset G$ be the subgroup consisting of any γ satisfying (6.1). Then $G_0\backslash(T \times_S B(H)) \simeq (G_0\backslash T) \times_S B(H)$. Note that the singularities of $G_0\backslash T$ are described by means of a torus embedding theory. By [R], there is a toroidal partial resolution of singularities $V \to G_0\backslash T$ such that V has only terminal singularities and the canonical divisor K_V is relatively nef over S. It is constructed by a decomposition of the cone associated with $G_0\backslash T$. Since G/G_0 preserves the decomposition, we have an action of G/G_0 on V. Thus we have an action of the same group on $V \times_S B(H)$ which is bimeromorphically equivalent to that on $(G_0\backslash T) \times_S B(H)$. We see the action on $V \times_S B(H)$ is free. Thus the quotient space $X := (G/G_0)\backslash(V \times_S B(H))$ has only terminal singularities and the canonical divisor K_X is relatively nef over S. Hence we obtain a minimal model. Q.E.D.

Example 6.2.2. Let $l = d = 2$ and take $(1/2, 1/2), (1/3, 1/3) \in (\mathbb{Q}/\mathbb{Z})^{\oplus 2}$. Then $m_1 = 2$ and $m_2 = 3$. For integers k_1, k_2, $k_1(1/2, 1/2) + k_2(1/3, 1/3) = 0$ if and only if $k_1 \equiv 0 \mod 2$ and $k_2 \equiv 0 \mod 3$. Thus the action of the Galois group G on $T \times_S B(H)$ is free. Hence the quotient space $X = G\backslash(T \times_S B(H))$ is nonsingular and the elliptic fibration $f\colon X \to S = \Delta^2$ is a flat morphism. The fiber over a point of $D_1 \setminus \{(0, 0)\}$ (resp. $D_2 \setminus \{(0, 0)\}$) is a multiple fiber with multiplicity 2 (resp. 3). The central fiber is also a non-reduced curve with multiplicity 6, whose support is a nonsingular elliptic curve. We have:

$$K_X \sim f^*K_S + D_1' + 2D_2',$$

where D_i' is the irreducible component of $f^*(D_i)$ for $i = 1, 2$.

Example 6.2.3. Let $l = d = 2$ and take $(1/2, 1/2), (1/4, 3/4) \in (\mathbb{Q}/\mathbb{Z})^{\oplus 2}$. Then $m_1 = 2$ and $m_2 = 4$. For integers k_1, k_2, $k_1(1/2, 1/2) + k_2(1/4, 3/4) = 0$ if and only if k_2 is even and $k_1 + k_2/2$ is also even. Thus G_0 is generated by $\gamma_1\gamma_2^2$, which is of order 2. The action of G_0 on $T = \Delta^2$ is written by:

$$(\theta_1, \theta_2) \mapsto (-\theta_1, -\theta_2).$$

Therefore $G_0\backslash T$ has an ordinary double point as the singularity. Let $V \to G_0\backslash T$ be the minimal desingularization. Then $G/G_0 \simeq \mathbb{Z}/4\mathbb{Z}$ acts on V and the quotient space $W := (\mathbb{Z}/4\mathbb{Z})\backslash V$ is obtained by the blowing up of $S = \Delta^2$ along the ideal (t_1^2, t_2). The W has one exceptional curve $C \simeq \mathbb{P}^1$ and proper transforms D_1', D_2' of coordinate lines $D_i = \{t_i = 0\}$. The intersection $D_1' \cap C$ is one point and it is an ordinary double point. The minimal elliptic fibration $X \to W$ is smooth outside $D_1' \sqcup D_2'$, and singular fiber type over D_1' is ${}_2\mathrm{I}_0$ and that over D_2' is ${}_4\mathrm{I}_0$.

Other Cases. Let $\tau\colon T = \Delta^d \to S$ be a U-covering. The natural extension of $\tau^{-1}H$ on $T^\star$ to T is denoted by H_T. Let $\omega(z) = h(\theta)\colon T \to \mathbb{H}$ be the period function and let $B(H_T) \to T$ be the associated smooth basic elliptic fibration. The $B(H_T)$ is isomorphic to the quotient space of $T \times \mathbb{C}$ by the following action of $(m, n) \in \mathbb{Z}^{\oplus 2}$:

$$(\theta, \zeta) \mapsto (\theta, \zeta + mh(\theta) + n).$$

Let us consider functions $F_i(z)$ listed in the Table 6. These are holomorphic function over T, since the period function $\omega(z) = h(\theta)$ is so. Similarly to Theorem 2.2.2, if we take the U-covering $\tau\colon T \to S$ in a suitable way, then we can define an action of the Galois group $G = \mathrm{Gal}(\tau)$ on $B(H_T)$ by:

$$[\theta, \zeta] \mapsto \left[\gamma_i\theta, \frac{\zeta + F_i(z)}{c_{\gamma_i}h(\theta) + d_{\gamma_i}}\right].$$

The quotient by the action induces an elliptic fibration $X \to S$, which is of course the extension of the corresponding smooth elliptic fibration over $S^\star$. Since all the cohomology classes of $H^2(\pi_1, \mathbb{Z}^{\oplus 2})$ are represented by the functions $F_i(z)$, we have all the elliptic fibrations corresponding to elements of $\mathcal{E}^+(S, D, H)$. The possible singular fiber types over coordinate hyperplanes are listed in Table 9.

§7. Infinite monodromy case

Let H be a variation of Hodge structures of type $\mathrm{I}_{(+)}$ or $\mathrm{I}_{(+)}^{(*)}$ on $S^\star$. We use the same notation as in §2. The period function $\omega(z)$ is written by

Table 9. Possible singular fiber types.

Type of monodromy	Singular fiber types ($0 \leq a, m \in \mathbb{Z}$)
I_0	${}_m\mathrm{I}_0$
I_0^*	I_0, ${}_2\mathrm{I}_0$, I_0^*
$\mathrm{II}^{(*)}$	I_0, I_0^*, II, II^*, IV, IV^*
$\mathrm{III}^{(*)}$	I_0, ${}_2\mathrm{I}_0$, I_0^*, III, III^*
$\mathrm{IV}_+^{(*)}$	I_0, ${}_3\mathrm{I}_0$, IV
$\mathrm{IV}_-^{(*)}$	I_0, I_0^*, IV, IV^*
$\mathrm{I}_{(+)}$	${}_m\mathrm{I}_a$
$\mathrm{I}_{(+)}^{(*)}(0)$	I_a, ${}_2\mathrm{I}_a$, I_a^*
$\mathrm{I}_{(+)}^{(*)}(1)$	I_a, ${}_2\mathrm{I}_a$, ${}_4\mathrm{I}_a$, I_a^*
$\mathrm{I}_{(+)}^{(*)}(2)$	I_a, ${}_2\mathrm{I}_a$, I_a^*

$$\omega(z) = \sum_{i=1}^{l} a_i z_i + h(t)$$

for a holomorphic function $h(t)$ on S, where $a_i \geq 0$ and one of a_i is positive. Let $\tau\colon T \simeq \Delta^d \to S$ be a U-covering determined by $\tau^* t_i = \theta_i^{m_i}$ for $1 \leq i \leq l$. Then the monodromy matrix around the coordinate hyperplane $D_{T,i} = \{\theta_i = 0\}$ is of type $\mathrm{I}_{m_i a_i}$. Let G be the Galois group $\mathrm{Gal}(\tau) \simeq \bigoplus_{i=1}^{l} \mathbb{Z}/m_i\mathbb{Z}$ and let π' be the kernel of $\pi_1 \twoheadrightarrow G$, which is the fundamental group of $T^\star = \tau^{-1}(S^\star)$. We shall calculate the cohomology group $H^1(G, H^0(T, \mathfrak{S}_{H_T/T}))$. Let D_T^+ be the divisor $\{\prod_{a_i>0} \theta_i = 0\}$. Then as in (4.2) and (4.3), we have the following two exact sequences:

$$0 \to \mathbb{Z}_T \to \mathcal{O}_T(*D_T^+)^\star \to \mathfrak{S}_{H_T/T} \to 0; \tag{7.1}$$

$$0 \to \mathcal{O}_T^\star \to \mathcal{O}_T(*D_T^+)^\star \to \bigoplus_{a_i>0} \mathbb{Z}_{D_{T,i}} \to 0. \tag{7.2}$$

Note that the $\bigoplus_{a_i>0} \mathbb{Z}_{D_{T,i}}$ is considered to be a submodule of $R^1 j_{T*}\mathbb{Z}_{T^\star} \simeq \bigoplus_{i=1}^{l} \mathbb{Z}_{D_{T,i}}$, where j_T is an immersion $T^\star \hookrightarrow T$. By taking global sections, we have the following two exact sequences of G-modules:

$$0 \to \mathbb{Z} \to H^0(T, \mathcal{O}_T(*D_T^+)^\star) \to H^0(T, \mathfrak{S}_{H_T/T}) \to 0; \tag{7.3}$$

(7.4)

$$0 \to H^0(T, \mathcal{O}_T^\star) \to H^0(T, \mathcal{O}_T(*D_T^+)^\star) \to H^0(T, \bigoplus_{a_i>0} \mathbb{Z}_{D_{T,i}}) =: L_T^+ \to 0.$$

Here L_T^+ is considered to be a submodule of $H^0(T, R^1 j_{T*}\mathbb{Z}_{T^\star}) \simeq \mathrm{Hom}(\pi', \mathbb{Z})$. Since we fix generators γ_i of π_1, we have a natural isomorphism $\mathrm{Hom}(\pi_1, \mathbb{Z}) \simeq \mathbb{Z}^{\oplus l}$. By using this isomorphism, we identify L_T^+ with $\bigoplus_{a_i>0}(1/m_i)\mathbb{Z}$, i.e., we shall write an element of L_T^+ by $(q_1, q_2, \dots, q_l) \in \mathbb{Q}^{\oplus l}$, where $q_j = 0$ for $a_j = 0$ and $m_j q_j \in \mathbb{Z}$ for all j. By the sequence (7.4), we see that $H^0(T, \mathcal{O}_T(*D_T^+)^\star)$ is isomorphic to the direct sum $H^0(T, \mathcal{O}_T^\star) \oplus L_T^+$ as an abelian group. The isomorphism is described by:

$$(u(\theta), (q_i)) \mapsto u(\theta) \prod_{a_i>0} \theta_i^{m_i q_i}.$$

Therefore the induced action of $\gamma_j \in \pi_1$ (more precisely, the image of γ_j in G) on the direct sum is written by:

$$(u(\theta), (q_i)) \mapsto \left((u(\gamma_j\theta)e(q_j))^{(-1)^{c_j}}, (-1)^{c_j}(q_i) \right).$$

By (7.1) and (7.2), we have a homomorphism $\mathcal{O}_T^\star \to \mathfrak{S}_{H_T/T}$, from which the following exact sequence of G-modules is derived:

$$(7.5) \qquad 0 \to H^0(T, \mathcal{O}_T^\star) \to H^0(T, \mathfrak{S}_{H_T/T}) \to L_T^+/\mathbb{Z}\boldsymbol{a} \to 0,$$

where $\boldsymbol{a} := (a_1, a_2, \dots, a_l) \in \mathbb{Z}^{\oplus l} = \mathrm{Hom}(\pi_1, \mathbb{Z})$.

7.1. Case $\mathrm{I}_{(+)}$

Suppose that H is of type $\mathrm{I}_{(+)}$. Then every $c_i = 0$. Thus $\mathbb{Z}$ in the sequence (7.3) and L_T^+ are trivial G-modules. For $(u(\theta), \boldsymbol{q}) \in H^0(T, \mathcal{O}_T^\star) \oplus L_T^+$, let $[u(\theta), \boldsymbol{q}]$ be the image under the homomorphism

$$H^0(T, \mathcal{O}_T(*D_T^+)^\star) \to H^0(T, \mathfrak{S}_{H_T/T}),$$

where $\boldsymbol{q} = (q_1, q_2, \dots, q_l) \in L_T^+$. Since G is isomorphic to $\bigoplus_{i=1}^l \mathbb{Z}/m_i\mathbb{Z}$, the cohomology group $H^1(G, H^0(T, \mathfrak{S}_{H_T/T}))$ is isomorphic to Z^1/B^1, where Z^1 and B^1 is defined by:

$$Z^1 := \left\{ (\xi_i)_{i=1}^l \in H^0(T, \mathfrak{S}_{H_T/T})^{\oplus l} \;\middle|\; \sum_{r=0}^{m_i-1} \xi_i^{\gamma_i^r} = 0,\, \xi_i - \xi_i^{\gamma_j} = \xi_j - \xi_j^{\gamma_i} \right\};$$

$$B^1 := \left\{ (\eta - \eta^{\gamma_i})_{i=1}^l \;\middle|\; \eta \in H^0(T, \mathfrak{S}_{H_T/T}) \right\}.$$

Lemma 7.1.1. *For $\xi_i = [u_i(\theta), \boldsymbol{q}^i]$, the collection $(\xi_i)_{i=1}^l$ is contained in Z^1 if and only if there exist integers n_i, rational numbers λ_i for $1 \le i \le l$, and a nowhere vanishing function $v(\theta)$ defined on T satisfying the following conditions for any $1 \le i, j \le l$:*

(1) $m_i\lambda_i \in \mathbb{Z}$;
(2) $(n_i/m_i)a_j \equiv (n_j/m_j)a_i \mod \mathbb{Z}$;
(3) $\boldsymbol{q}^i = (n_i/m_i)\boldsymbol{a}$;
(4)

$$u_i(\theta) = e\left(\lambda_i - \frac{(m_i-1)n_ia_i}{2m_i} + \frac{h(t)n_i}{m_i}\right)v(\theta)v(\gamma_i\theta)^{-1}.$$

Proof. We see the condition is equivalent to the following condition by simple calculation: there exist integers n_i for $1 \le i \le l$ such that

(1)

$$\prod_{r=0}^{m_i-1} u_i(\gamma_i^r\theta) = e\left(-\frac{q_i^i m_i(m_i-1)}{2} + n_ih(t)\right),$$

(2) $m_i\boldsymbol{q}^i = n_i\boldsymbol{a}$,
(3) $u_i(\theta)u_i(\gamma_j\theta)^{-1}e(-q_j^i) = u_j(\theta)u_j(\gamma_i\theta)^{-1}e(-q_i^j)$,

for any i, j, where $\boldsymbol{q}^i = (q_1^i, q_2^i, \dots, q_l^i) \in L_T^+$. By taking $\theta = 0$ in (3), we see $q_j^i - q_i^j \in \mathbb{Z}$. Further by (2), we have $q_j^i = (n_i/m_i)a_j$ for any i, j. Therefore $(n_i/m_i)a_j \equiv (n_j/m_j)a_i \mod \mathbb{Z}$ for any i, j. Let us define

$$v_i(\theta) := u_i(\theta)e\left(\frac{(m_i-1)n_ia_i}{2m_i} - \frac{h(t)n_i}{m_i}\right).$$

Then we have $\prod_{r=0}^{m_i-1} v_i(\gamma_i^r\theta) = 1$ and $v_i(\theta)v_i(\gamma_j\theta)^{-1} = v_j(\theta)v_j(\gamma_i\theta)^{-1}$ for i, j. Thus $\{v_i(\theta)\}$ defines an element of $H^1(G, H^0(T, \mathcal{O}_T^\star))$. Thus we have rational numbers λ_i for $1 \le i \le l$ with $m_i\lambda_i \in \mathbb{Z}$ and a nowhere vanishing function $v(\theta)$ on T such that

$$v_i(\theta) = e(\lambda_i)v(\theta)v(\gamma_i\theta)^{-1}. \qquad \text{Q.E.D.}$$

By considering the condition that the collection (ξ_i) is contained in B^1, we have:

Corollary 7.1.2. *The cohomology group $H^1(G, H^0(T, \mathfrak{S}_{H_T/T}))$ is isomorphic to*

$$\bigoplus_{a_i=0} m_i^{-1}\mathbb{Z}/\mathbb{Z}$$

$$\oplus \left\{(n_i/m_i) \in \bigoplus_{i=1}^{l} m_i^{-1}\mathbb{Z}/\mathbb{Z} \,\middle|\, (n_i/m_i)a_j \equiv (n_j/m_j)a_i \mod \mathbb{Z}\right\}.$$

Since our description of $L_T^+ \subset \mathrm{Hom}(\pi_1, \mathbb{Q})$ is compatible with any further U-coverings $T' \to T \to S$, we have:

Theorem 7.1.3. *Suppose that H is of type $\mathrm{I}_{(+)}$ and the monodromy matrix around the coordinate hyperplane D_i is of type I_{a_i}. Then the set $\mathcal{E}^+(S, D, H)$ is identified with the group*

$$\bigoplus_{a_i=0} \mathbb{Q}/\mathbb{Z} \oplus \left\{ (p_i) \in \bigoplus_{i=1}^{l} \mathbb{Q}/\mathbb{Z} \,\middle|\, p_i a_j \equiv p_j a_i \mod \mathbb{Z} \text{ for any } i, j \right\}.$$

Let k be the number of indices $1 \le i \le l$ with $a_i > 0$ and let $\alpha := \gcd a_i$. Then the group is isomorphic to:

$$(\mathbb{Q}/\mathbb{Z})^{\oplus(l-k+1)} \oplus (\mathbb{Z}/\alpha\mathbb{Z})^{\oplus(l-1)}.$$

Next, we shall construct the elliptic fibration associated with an element of $\mathcal{E}^+(S, D, H)$. By Theorem 7.1.3, every element of $\mathcal{E}^+(S, D, H)$ is determined by l-pairs of rational numbers (p_i, q_i) for $1 \le i \le l$ such that $q_i = 0$ for $a_i > 0$ and that $p_i a_j \equiv p_j a_i \mod \mathbb{Z}$ for any $1 \le i, j \le l$. Let m_i be the order of (p_i, q_i) in $(\mathbb{Q}/\mathbb{Z})^{\oplus 2}$ and let $\tau \colon T = \Delta^d \to S$ be the U-covering with $\tau^* t_i = \theta_i^{m_i}$ for $1 \le i \le l$. Let $X_\sigma \to T$ be the toric model associated with the variation of Hodge structures $\tau^{-1}H$ on $T^\star$ and with a suitable sign function σ. For the universal covering space $\mathcal{X}_\sigma$ of X_σ, $\mathcal{X}^\circ$ and $\mathcal{X}^\star$ are the open subsets $(\mathcal{X}_\sigma)_{|S^\circ}$ and $(\mathcal{X}_\sigma)_{|S^\star}$, respectively. We know that $\mathcal{X}^\star \simeq T^\star \times \mathbb{C}^\star$. For $1 \le i \le l$, we have the following isomorphism of $\mathcal{X}^\star$:

$$(\theta, s) \mapsto \left(\gamma_i \theta, s \cdot e\left(q_i - \frac{p_i a_i (m_i - 1)}{2} + p_i h(t) \right) \left(\prod_{j=1}^{l} \theta_j^{m_j a_j} \right)^{p_i} \right).$$

This extends to an isomorphism of $\mathcal{X}^\circ$. Since it is compatible with the action of

$$\vartheta \colon (\theta, s) \mapsto \left(\theta, s \cdot e(h(t)) \prod_{i=1}^{l} \theta_i^{m_i a_i} \right),$$

we have a meromorphic action of $\mathrm{Gal}(\tau)$ on X_σ. If we choose a sign function σ with respect to $(m_1 a_1/n, m_2 a_2/n, \ldots, m_l a_l/n)$ for $n = \gcd(m_1 a_1, m_2 a_2, \ldots, m_l a_l)$, then the action is holomorphic. By taking the quotient, we have an expected elliptic fibration.

7.2. Case $\mathrm{I}_{(+)}^{(*)}$

Next suppose that H is of type $\mathrm{I}_{(+)}^{(*)}$. Then γ_i acts on $\mathbb{Z}$ in the sequence (7.3) and on L_T^+ as the multiplication of $(-1)^{c_i}$. By the isomorphism as an abelian group

$$H^0(T, \mathcal{O}_T(*D_T^+)^\star) \simeq H^0(T, \mathcal{O}_T^\star) \oplus L_T^+,$$

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathbb{Z} & \longrightarrow & \mathbb{Z}^{\oplus 2} & \longrightarrow & \mathbb{Z} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H^0(T, \mathcal{O}_T) & \longrightarrow & H^0(T, \mathcal{O}_T) \oplus L_T^+ & \longrightarrow & L_T^+ & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H^0(T, \mathcal{O}_T^\star) & \longrightarrow & H^0(T, \mathfrak{S}_{H_T/T}) & \longrightarrow & L_T^+/\mathbb{Z}\boldsymbol{a} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Figure 5. (cf. Lemma 7.2.1.)

we have a G-module structure on the direct sum. We can define a compatible π_1-module structure on $H^0(T, \mathcal{O}_T) \oplus L_T^+$, where the γ_j acts as

$$(f(\theta), (q_i)) \mapsto (-1)^{c_j}(f(\gamma_j \theta) + q_j, (q_i)).$$

Then we have:

Lemma 7.2.1. *We have an exact sequence of π_1-modules:*

$$0 \to \mathbb{Z}^{\oplus 2} \to H^0(T, \mathcal{O}_T) \oplus L_T^+ \to H^0(T, \mathfrak{S}_{H_T/T}) \to 0, \tag{7.6}$$

where $\mathbb{Z}^{\oplus 2}$ is the π_1-module associated with the monodromy representation $\pi_1 \longrightarrow \mathrm{SL}(2, \mathbb{Z})$ and the first homomorphism is given by:

$$\mathbb{Z}^{\oplus 2} \ni (m, n) \mapsto (mh(t) + n, m\boldsymbol{a}) \in H^0(T, \mathcal{O}_T) \oplus L_T^+.$$

Further there is the commutative diagram Figure 5, *where the left vertical sequence is induced from the exponential sequence of T, and the right vertical sequence is induced from $1 \mapsto \boldsymbol{a}$.*

Thus we have a long exact sequence:

$$\begin{aligned}
0 \to H^0(\pi_1', \mathbb{Z}^{\oplus 2}) \to & H^0(\pi_1', H^0(T, \mathcal{O}_T) \oplus L_T^+) \to H^0(T, \mathfrak{S}_{H_T/T}) \to \\
\to H^1(\pi_1', \mathbb{Z}^{\oplus 2}) \to & H^1(\pi_1', H^0(T, \mathcal{O}_T) \oplus L_T^+) \to \cdots
\end{aligned}$$

$$
\begin{array}{ccccc}
H^0(G, L_T^+/\mathbb{Z}\boldsymbol{a}) & \longrightarrow & H^1(G, H^0(T, \mathcal{O}_T^\star)) & \longrightarrow & \\
\downarrow & & \downarrow & & \\
H^0(G, H^1(\pi_1', \mathbb{Z}^{\oplus 2})) & \longrightarrow & H^2(G, H^0(\pi_1', \mathbb{Z}^{\oplus 2})) & \longrightarrow &
\end{array}
$$

$$
\begin{array}{cccccc}
\longrightarrow & H^1(G, H^0(T, \mathfrak{S}_{H_T/T})) & \longrightarrow & H^1(G, L_T^+/\mathbb{Z}\boldsymbol{a}) & \longrightarrow \\
& \downarrow & & \downarrow & \\
\longrightarrow & F^1(H^2(\pi_1, \mathbb{Z}^{\oplus 2})) & \longrightarrow & H^1(G, H^1(\pi_1', \mathbb{Z}^{\oplus 2})) & \longrightarrow
\end{array}
$$

$$
\begin{array}{cc}
\longrightarrow & H^2(G, H^0(T, \mathcal{O}_T^\star)) \\
& \downarrow \\
\longrightarrow & H^3(G, H^0(\pi_1', \mathbb{Z}^{\oplus 2})).
\end{array}
$$

Figure 6. (cf. Lemma 7.2.3.)

Lemma 7.2.2.

(1) $H^0(\pi_1', H^0(T, \mathcal{O}_T) \oplus L_T^+) \simeq H^0(T, \mathcal{O}_T)$.

(2) *The image of* $H^0(T, \mathfrak{S}_{H_T/T}) \to H^1(\pi_1', \mathbb{Z}^{\oplus 2})$ *is isomorphic to* $L_T^+/\mathbb{Z}\boldsymbol{a}$.

Proof. Let $(f(\theta), (q_i))$ be a π_1'-invariant element of $H^0(T, \mathcal{O}_T) \oplus L_T^+$. Then $m_i q_i = 0$ for any i. Hence $q_i = 0$. Conversely, $(f(\theta), 0)$ is π_1'-invariant. Hence (1) is derived. Furthermore, we see that the injection

$$H^0(\pi_1', \mathbb{Z}^{\oplus 2}) \to H^0(\pi_1', H^0(T, \mathcal{O}_T) \oplus L_T^+)$$

is isomorphic to $\mathbb{Z} \to H^0(T, \mathcal{O}_T)$, which sends 1 to 1. Thus we have (2), by Lemma 7.2.1. Q.E.D.

By considering Hochschild–Serre's spectral sequence, we have:

Lemma 7.2.3. *The commutative diagram* Figure 6 *of exact sequences exists, in which the top sequence is a part of a long exact sequence induced from* (7.5) *and the bottom sequence is a part of the edge sequence of Hochschild–Serre's spectral sequence for* $\mathbb{Z}^{\oplus 2}$. *The* $F^i(H^2(\pi_1, \mathbb{Z}^{\oplus 2}))$ *is the filtration induced from the spectral sequence.*

Corollary 7.2.4. *The homomorphism*

$$H^1(G, H^0(T, \mathfrak{S}_{H_T/T})) \to H^2(\pi_1, \mathbb{Z}^{\oplus 2})$$

is an injection.

Proof. Let us consider the commutative diagram in Figure 6. The cokernel of $L_T^+/\mathbb{Z}\boldsymbol{a} \to H^1(\pi_1', \mathbb{Z}^{\oplus 2})$ is a torsion free group, where γ_i acts as the multiplication of $(-1)^{c_i}$. Thus the G-invariant part of the cokernel is zero. Therefore the first vertical homomorphism is an isomorphism and the fourth one is injective. The second homomorphism is also an isomorphism, since $H^i(G, H^0(T, \mathcal{O}_T)) = 0$ for $i > 0$. Therefore the homomorphism in question is injective. Q.E.D.

Theorem 7.2.5. *Suppose that H is of type $\mathrm{I}_{(+)}^{(*)}$. Then the set $\mathcal{E}^+(S, D, H)$ is identified with the group $H^2(\pi_1, \mathbb{Z}^{\oplus 2})$.*

Proof. We know $\mathcal{E}^+(S, D, H) = \varinjlim_{T \to S} H^1(G, H^0(T, \mathfrak{S}_{H_T/T}))$. Thus we have an injection $\mathcal{E}^+(S, D, H) \hookrightarrow H^2(\pi_1, \mathbb{Z}^{\oplus 2})$ by Corollary 7.2.4. On the other hand, $\mathcal{E}^+(S^\star, \emptyset, H)$ is identified with $H^2(\pi_1, \mathbb{Z}^{\oplus 2})$ by Theorem 2.2.2. Since this is a torsion group, every smooth elliptic fibration over $S^\star$ having H as a variation of Hodge structures extends to a projective elliptic fibration over S by Theorem 4.1.1. Thus the mapping $\mathcal{E}^+(S, D, H) \to H^2(\pi_1, \mathbb{Z}^{\oplus 2})$ is bijective. Q.E.D.

Next we shall construct the elliptic fibration associated with an element of $\mathcal{E}^+(S, D, H)$. Let H be a variation of Hodge structures of type $\mathrm{I}_{(+)}^{(*)}$. Let $\tau\colon T = \Delta^d \to S$ be a U-covering with $\tau^* t_i = \theta_i^{m_i}$ for sufficiently large m_i, e.g., $m_i = 4$. Let $X_\sigma \to T$ be a toric model associated with the same variation of Hodge structures as $\tau^{-1}H$. As in the previous case, let us consider $\mathcal{X}^\circ$ and $\mathcal{X}^\star$. Then $\mathcal{X}^\star \simeq T^\star \times \mathbb{C}^\star$. According to the types $\mathrm{I}_{(+)}^{(*)}(0)$, $\mathrm{I}_{(+)}^{(*)}(1)$ and $\mathrm{I}_{(+)}^{(*)}(2)$, let $F_i(z)$ be the function listed in Table 6. Since $\omega(z) = \sum a_i z_i + h(t)$, $e(F_i(z))$ is the multiple of unit holomorphic functions on S and monomials of θ_i for $1 \le i \le l$. Hence by the mapping:

$$\mathcal{X}^\star \ni (\theta, s) \mapsto \left(\gamma_i \theta, (s \cdot e(F_i(z)))^{(-1)^{c_i}}\right),$$

we have a holomorphic action of the Galois group $G = \mathrm{Gal}(\tau)$ on the $\mathcal{X}^\circ$ and a meromorphic action on the toric model X_σ. Although it is not necessarily a holomorphic action, by taking its 'quotient', we have an expected elliptic fibration. The possible singular fiber types over coordinate hyperplanes are listed in Table 9.

§Appendix A. Standard elliptic fibrations over surfaces

We shall study elliptic fibrations over normal surfaces. If the base surface is nonsingular and the fibration is smooth outside a normal crossing divisor, then the local bimeromorphic structures are classified in §6 and §7. But here we do not use these results but the flip theorem [Mo] and the flop theorem [Kw4] (cf. [Kl2]) for threefolds. We shall prove the following:

Theorem A.1. *Let $\pi\colon X \to S$ be a locally projective elliptic fibration over a normal complex analytic surface S. Then there exist a* standard *elliptic fibration $f\colon Y \to T$ and a bimeromorphic morphism $\mu\colon T \to S$ such that π and $\mu \circ f$ are bimeromorphically equivalent and K_Y is $\mu \circ f$-semi-ample.*

A *standard* elliptic fibration is defined as follows:

Definition A.2. Let $f\colon Y \to T$ be an elliptic fibration over a normal surface T. If the following conditions are satisfied, then f is said to be a *standard* elliptic fibration:

(1) Y has only terminal singularities;
(2) Y has only $\mathbb{Q}$-*factorial singularities*, i.e., for each point $y \in Y$ and for any Weil divisor D defined on a neighborhood of y, mD is Cartier at y for a positive integer m;
(3) f is a locally projective morphism;
(4) f is an equi-dimensional morphism, i.e., every fiber of f is one-dimensional;
(5) There exists an effective $\mathbb{Q}$-divisor Δ on T such that (T, Δ) is log-terminal and $K_Y \sim_{\mathbb{Q}} f^*(K_T + \Delta)$.

Remark A.3.

(1) $\mu \circ f$ may not to be a locally projective morphism.
(2) If T is nonsingular, a standard elliptic fibration $f\colon Y \to T$ is a flat morphism.
(3) For the definition of log-terminal pair, see [KMM], or [Ny3].

Therefore, the classification of threefolds admitting elliptic fibrations is reduced to that of standard elliptic fibrations.

For the proof, we recall the following semi-ampleness theorem. This was originally proved by [Kw3, 6.1] in the case S is a point. It is generalized to the algebraic case in [Ny1, 5] (cf. [KMM, 6-1-1]) and to the case X is a variety in class $\mathcal{C}$ and S is a point in [Ny3, 5.5]. Further [Ny3, 5.8] treats in a case of degenerations. But these proofs are essentially same and depend on the torsion free Theorem 3.2.2. Thus we have:

Theorem A.4 (Semi-ampleness theorem). *Let $\pi\colon X \to S$ be a projective surjective morphism from a normal complex variety X onto a complex variety S, Δ an effective $\mathbb{Q}$-divisor of X and H a $\mathbb{Q}$-divisor of X. Then H is π-semi-ample if the following conditions are satisfied:*

(1) *(X, Δ) is log-terminal;*
(2) *H and $H - (K_X + \Delta)$ are π-nef;*
(3) *$\nu(H - (K_X + \Delta)_{|X_s}) = \kappa(H - (K_X + \Delta)_{|X_s})$ for a general fiber X_s;*
(4) *$\kappa(aH - (K_X + \Delta)_{|X_s}) \geq 0$ and $\nu(aH - (K_X + \Delta)_{|X_s}) = \nu(H - (K_X + \Delta)_{|X_s})$ for some $a > 1$ on a general fiber X_s.*

Here $\nu(D)$ denotes the numerical D-dimension (cf. [KMM, 6-1-1]).

Proposition A.5. *Let $\pi\colon X \to S$ be a locally projective elliptic fibration over a surface S. Then there exist a locally projective elliptic fibration $g\colon Z \to R$, a bimeromorphic morphism $\nu\colon R \to S$, and an effective $\mathbb{Q}$-divisor Λ on R satisfying the following conditions:*

(1) *π and $\nu \circ g$ are bimeromorphically equivalent;*
(2) *Z has only terminal singularities;*
(3) *Z is $\mathbb{Q}$-factorial over any point of S;*
(4) *(R, Λ) is log-terminal;*
(5) *$K_Z \sim_{\mathbb{Q}} g^*(K_R + \Lambda)$;*
(6) *$K_R + \Lambda$ is ν-ample.*

Proof. Since π is locally projective, for each point $s \in S$ there is an open neighborhood $\mathcal{U}_s$ such that $\pi^{-1}(\mathcal{U}_s) \to \mathcal{U}_s$ is a projective morphism. Thus by applying minimal model theorem [Mo], [Ny3, §4] to $(\mathcal{U}_s, s)$, we have an elliptic fibration $h_s\colon \mathcal{Z}_s \to \mathcal{U}'_s$ such that

- $\mathcal{U}'_s \subset \mathcal{U}_s$ is also an open neighborhood of s,
- $\mathcal{Z}_s$ has only terminal singularities,
- $\mathcal{Z}_s$ is $\mathbb{Q}$-factorial over s,
- h_s is a projective morphism, bimeromorphic to π over $\mathcal{U}'_s$,
- $K_{\mathcal{Z}_s}$ is h_s-nef.

The $\mathcal{Z}_s$ is not uniquely determined in general, but by [Kw4], it is determined up to a sequence of flops. Thus except a discrete set of points of S, $\mathcal{Z}_s$ is uniquely determined. Therefore we can patch these $\mathcal{Z}_s$ and get a locally projective elliptic fibration $h\colon Z \to S$ such that

- Z has only terminal singularities,
- Z is $\mathbb{Q}$-factorial over any point of S,
- h is a locally projective morphism, bimeromorphic to π,
- K_Z is h-nef.

By Theorem A.4, we see that K_Z is h-semi-ample. Therefore there exist a bimeromorphic morphism $\nu\colon R \to S$, a $\mathbb{Q}$-Cartier divisor L on R, and an elliptic fibration $g\colon Z \to R$ such that $h = \nu \circ g$, $K_Z \sim_{\mathbb{Q}} g^*L$, and L is ν-ample. By [Ny4, 0.4], we have an effective $\mathbb{Q}$-divisor Λ on R such that (R, Λ) is log-terminal and $K_Z \sim_{\mathbb{Q}} g^*(K_R + \Lambda)$. Q.E.D.

Proposition A.6. *Let $\pi\colon X \to S$, $g\colon Z \to R$, and $\nu\colon R \to S$ be as in Proposition* A.5. *Then there exist an equi-dimensional elliptic fibration $g'\colon Z' \to T$ and a bimeromorphic morphism $\delta\colon T \to R$ satisfying the following conditions*:

(1) *$\delta \circ g'$ and g are bimeromorphically equivalent*;
(2) *$\mu := \nu \circ \delta$ and $\mu \circ g'$ are locally projective morphisms*;
(3) *Z' has only terminal singularities and is $\mathbb{Q}$-factorial over any point of S*;
(4) *$K_{Z'}$ is $\mathbb{Q}$-linearly equivalent to the pullback of $K_R + \Lambda$.*

Proof. We may assume that g is not equi-dimensional. In general, g is equi-dimensional over a neighborhood of $\nu^{-1}(s)$ for $s \in S$ except a discrete set of points. Thus we can consider locally on S. Let us take such exceptional point $P \in S$ and look at the vector spaces $N^1(Z/S; P)$, $N^1(Z/R; \nu^{-1}(P))$, $N_1(Z/R; \nu^{-1}(P))$ (cf. [Ny3, §4]), etc.

Step 1. By the assumption, there is a prime divisor E on Z such that $g(E)$ is a point and $\nu \circ g(E) = P$. Then we can take an effective divisor D on Z such that $D + kE$ is the pullback of an effective Cartier divisor on R for some integer $k > 0$ and D does not contain E. We consider the minimal model program for the log-terminal pair $(Z, \varepsilon D)$ for $0 < \varepsilon \ll 1$ in $N_1(Z/R; \nu^{-1}(P))$. Note that K_Z is $\mathbb{Q}$-linearly equivalent to the pullback of a $\mathbb{Q}$-divisor of R. If $-E$ is not g-nef, then there exist an extremal ray and its contraction morphism over R. Since extremal curves are contained in D, E can not to be contracted. By continuing such contractions and flops over $(R, \nu^{-1}(P))$, we have an elliptic fibration $q_1\colon V_1 \to R$ such that

(1) q_1 is bimeromorphically equivalent to g,
(2) V_1 has only canonical singularities,
(3) $K_{V_1} \sim_{\mathbb{Q}} q_1^*(K_R + \Lambda)$,
(4) $-E'$ is q_1-nef,

where E' is the strict transform of E in V_1. By Theorem A.4, $-E'$ is q_1-semi-ample. Therefore there exist an elliptic fibration $V_1 \to R_1$ and a bimeromorphic morphism $\delta_1\colon R_1 \to R$ such that q_1 is the composition of these morphisms and $-E'$ is the pullback of a δ_1-ample $\mathbb{Q}$-divisor on R_1. Thus δ_1 is not an isomorphism. Here we note that δ_1 and q_1 are projective morphisms over a neighborhood of $\nu^{-1}(P)$. Hence $\rho(R_1/R; \nu^{-1}(P)) > 0$. Since V_1 has only canonical singularities, we can take a crepant morphism $Z_1 \to V_1$ such that Z_1 has only terminal singularities, is $\mathbb{Q}$-factorial over P, and is projective over a neighborhood of $\nu^{-1}(P)$. Hence Z_1 and Z are isomorphic to each other in codimension one and $\rho(Z/R;\nu^{-1}(P)) = \rho(Z_1/R;\nu^{-1}(P))$.

Step 2. Further assume that the induced morphism $f_1\colon Z_1 \to R_1$ is not equi-dimensional over $\delta_1^{-1}\nu^{-1}(P)$. Then by the same argument in *Step* 1, we have morphisms $f_2\colon Z_2 \to R_2$ and $\delta_2\colon R_2 \to R_1$ such that Z_2 has only terminal singularities, is $\mathbb{Q}$-factorial and projective over P, and $\rho(R_2/R;\nu^{-1}(P)) > \rho(R_1/R;\nu^{-1}(P))$. Therefore

$$\rho(Z/R;\nu^{-1}(P)) = \rho(Z_2/R;\nu^{-1}(P)) > \\ > \rho(R_2/R;\nu^{-1}(P)) > \rho(R_1/R;\nu^{-1}(P)).$$

If f_2 is not equi-dimensional, we can continue this process. After a finite number of steps, f_m should be equi-dimensional, since $\rho(R_i/R;\nu^{-1}(P))$ are bounded. Thus we obtain the desired $Z' := Z_m$ and $T := R_m$ over P. Q.E.D.

Remark A.7. For the equi-dimensional morphism $Z' \to T$, T is uniquely determined. Because if $Z'' \to T'$ satisfies the same conditions, then Z'' and Z are isomorphic in codimension one. Thus for every prime divisor Γ on T, its proper transform in T' must be a prime divisor. Thus $T' \simeq T$. Note that $Z' \to S$ is a locally projective morphism.

Definition A.8. The morphism $Z' \to T$ in Proposition A.6 is said to be an *equi-dimensional model* of $\pi\colon X \to S$.

Lemma A.9. *Let $f\colon Y \to T$ be a minimal elliptic fibration over a surface T such that Y is $\mathbb{Q}$-factorial over any point of T and f is equi-dimensional. Then Y has only $\mathbb{Q}$-factorial singularities.*

Proof. Let $\nu\colon Y' \to Y$ be a bimeromorphic morphism whose exceptional locus is a union of discrete curves. Then Y' has only terminal singularities and ν is crepant, i.e., $K_{Y'} \sim_{\mathbb{Q}} \nu^* K_Y$. Since $f\circ\nu\colon Y' \to T$ is also equi-dimensional, $f\circ\nu$ is a locally projective morphism by the same argument as in Claim 3.2.4. Since Y is $\mathbb{Q}$-factorial over any point of T, ν must be an isomorphism. Thus by the existence of $\mathbb{Q}$-factorialization [Kw4], we are done. Q.E.D.

Proof of Theorem A.1. Let $f\colon Y \to T$ be a minimal model of an equi-dimensional model $g'\colon Z' \to T$ such that Y is $\mathbb{Q}$-factorial over any point of T. Since Y and Z' are having only terminal singularities, they are isomorphic in codimension one. Thus by flops, we can take Y to be a partial resolution of Z'. Therefore f is also equi-dimensional. Thus by Lemma A.9, f is a standard elliptic fibration. Q.E.D.

§Appendix B. Minimal models for elliptic threefolds

Minimal model theory is not yet developed for compact Kähler manifolds. But we have the following theorem in [Ny6]:

Theorem B.1. *Let X be a compact Kähler threefold of algebraic dimension two. Then X is uniruled or there exists a good minimal model of X.*

Here we say that X is *uniruled* if there exists a dominant meromorphic mapping $Y \times \mathbb{P}^1 \cdots\to X$ such that $\dim Y = \dim X - 1$. A *good minimal model* of X is defined to be a complex normal variety V satisfying the following conditions:

(1) V is bimeromorphically equivalent to X;
(2) V has only terminal singularities;
(3) The canonical divisor K_V is semi-ample.

We shall generalize to the following:

Theorem B.2. *Let $\pi: X \to B$ be a proper surjective morphism from a complex Kähler threefold X onto a complex variety B. Suppose that there exists an elliptic fibration $f: X \to S$ and a proper surjective morphism $g: S \to B$ such that $\pi = g \circ f$. Then the general fiber of $\pi: X \to B$ is uniruled or X admits a relative good minimal model over B.*

Here, a *relative good minimal model* over B is defined to be a proper surjective morphism $V \to B$ such that V has only terminal singularities and the canonical divisor K_V is relatively semi-ample over B.

We note the following lemma which is derived from Theorem 3.2.2 and from the similar argument of [Ny3, 3.12]:

Lemma B.3. *Let $f: X \to Y$ and $g: Y \to Z$ be projective morphisms of complex varieties where X is nonsingular. Let D be a $\mathbb{Q}$-divisor on X whose fractional part $\langle D \rangle$ is supported in a normal crossing divisor. Assume that there exists a g-nef-big $\mathbb{Q}$-Cartier divisor L such that $D \sim_{\mathbb{Q}} f^*(L)$. Then*

$$R^p g_*(R^i f_* \mathcal{O}_X(K_X + \ulcorner D \urcorner)) = 0$$

for $i \geq 0$ and $p > 0$, where $\ulcorner D \urcorner$ denotes the round-up *of D.*

In the case of elliptic fibrations, we have the following generalization:

Proposition B.4. *Let $f: X \to Y$ be an elliptic fibration from a complex manifold X onto a complex variety Y, $g: Y \to Z$ a projective morphism onto a complex variety Z, and let D be a $\mathbb{Q}$-divisor on X*

whose fractional part $\langle D \rangle$ is supported in a normal crossing divisor. Assume that there exists a g-nef-big $\mathbb{Q}$-Cartier divisor L such that $D \sim_{\mathbb{Q}} f^*L$. Then $R^i f_* \mathcal{O}_X(K_X + \ulcorner D \urcorner)$ is torsion free and $R^p g_*(R^i f_* \mathcal{O}_X(K_X + \ulcorner D \urcorner)) = 0$ for $i \geq 0$ and $p > 0$.

Proof. Since the statement is local on Z, we may assume that Z is a Stein space. By Lemma B.3, we may assume further that f is not bimeromorphically equivalent to a projective morphism. As in [Ny3, 3.12], we may assume that there exist a bimeromorphic morphism $\nu\colon Y' \to Y$ such that

(1) Y' is nonsingular,
(2) $g \circ \nu\colon Y' \to Z$ is a projective morphism,
(3) there is an elliptic fibration $f'\colon X \to Y'$ with $\nu \circ f' = f$,
(4) f' is smooth outside a normal crossing divisor on Y',
(5) there is an effective $\mathbb{Q}$-divisor Δ on Y' with $\nu^*L - \delta\Delta$ being $g \circ \nu$-ample for $0 < \delta \ll 1$,
(6) $\mathrm{Supp}\langle D \rangle \cup \mathrm{Supp}\, f'^*(\Delta)$ is a normal crossing divisor.

Since $\ulcorner D - \delta f'^*(\Delta) \urcorner = \ulcorner D \urcorner$, by Leray's spectral sequence, we can reduce to the situation such that $Y = Y'$ and L is g-ample. Then by the proof of [Ny3, 3.9], we may assume further that there exists a commutative diagram:

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\tilde{f}} & \widetilde{Y} \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\lambda} \\ X & \xrightarrow{f} & Y, \end{array}$$

where

(1) $\widetilde{X}$ and $\widetilde{Y}$ are nonsingular,
(2) φ is generically finite, λ is projective, and $\tilde{f}$ is an elliptic fibration,
(3) $\mathrm{Supp}\, \varphi^*\langle D \rangle$ is a normal crossing divisor and $\varphi^*(D)$ is a Cartier divisor,
(4) $\mathcal{O}_X(K_X + \ulcorner D \urcorner)$ is a direct summand of $\varphi_* \mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}} + \varphi^*(D))$.

Therefore by replacing X and Y by $\widetilde{X}$ and $\widetilde{Y}$, respectively, we can reduce to the case where the following conditions are satisfied:

(1) Y is nonsingular;
(2) $f\colon X \to Y$ is smooth outside a normal crossing divisor on Y;
(3) L is a g-ample Cartier divisor;
(4) D is a Cartier divisor with $D \sim f^*(L)$.

Then by Theorem 3.2.3, $\mathcal{F}^i := R^i f_* \mathcal{O}_X(K_X)$ are locally free sheaves. Thus $\mathcal{F}^i = 0$ for $i \geq 2$, $\mathcal{F}^1 \simeq \mathcal{O}_Y(K_Y)$ and $(\mathcal{F}^0)^{\otimes 12} \simeq \mathcal{O}_Y(12(K_Y + \Delta))$

for some effective $\mathbb{Q}$-divisor Δ whose support is a normal crossing divisor and whose *round-down* $\llcorner \Delta \lrcorner = 0$. Since

$$\mathcal{F}^0 - \Delta + L - K_Y$$

is g-ample, we are done by applying [Ny3, 3.5]. Q.E.D.

Corollary B.5. *Let $f\colon X \to Y$ be an elliptic fibration, $g\colon Y \to Z$ a projective morphism for normal complex varieties X, Y and Z. Then $R^2(g \circ f)_* \mathcal{O}_X = 0$ if the following conditions are satisfied:*

(1) *$(X, 0)$ is log-terminal;*
(2) *There is a $\mathbb{Q}$-divisor L on Y such that $-K_X \sim_{\mathbb{Q}} f^*(L)$;*
(3) *L is g-ample.*

Proof. Let $\mu\colon M \to X$ be a modification such that μ-exceptional locus is a normal crossing divisor $\bigcup E_i$. Then we have $K_M \sim_{\mathbb{Q}} \mu^*(K_X) + \sum_i a_i E_i$ for $a_i > -1$. Then for $D = \sum_i a_i E_i - K_M$, we have

$$R^p g_*(R^i(f \circ \mu)_* \mathcal{O}_M(K_M + \ulcorner D \urcorner)) = 0$$

for $p > 0$ by Proposition B.4. Since $R^i \mu_* \mathcal{O}_M(K_M + \ulcorner D \urcorner) = 0$ for $i > 0$ (cf. [Ny3, 3.6]) and $\mu_* \mathcal{O}_M(K_M + \ulcorner D \urcorner) \simeq \mathcal{O}_X$, we have $R^p g_* R^i f_* \mathcal{O}_X \simeq 0$ for $p > 0$. Since $R^i f_* \mathcal{O}_X = 0$ for $i > 1$ by Proposition B.4, $R^2(g \circ f)_* \mathcal{O}_X = 0$. Q.E.D.

Proposition B.6 (cf. [Ft1]). *Let T be a normal compact complex surface in class $\mathcal{C}$. Suppose that there is an effective $\mathbb{Q}$-divisor Δ on T such that (T, Δ) is log-terminal and $(K_T + \Delta) \cdot C \geq 0$ for any irreducible curve C on T. Then $K_T + \Delta$ is semi-ample.*

Proof. This is proved by [Ft1] in the case $a(T) = 2$. We thus assume that $a(T) < 2$. Therefore $p_g(T) > 0$ by Proposition 3.3.1. There exists an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on T which is $\mathbb{Q}$-linearly equivalent to $K_T + \Delta$. Hence $(K_T + \Delta)^2 \geq 0$. Since $a(T) < 2$, we have $(K_T + \Delta)^2 = 0$.

Step 1. Reduction to the case T is nonsingular.
Let $\mu\colon M \to T$ be the minimal resolution of singularities of T. Then we have

$$K_M \sim_{\mathbb{Q}} \mu^*(K_T + \Delta) + \sum a_i E_i$$

for $a_i > -1$, where E_i is a μ-exceptional curve or the proper transform of a component of $\operatorname{Supp} \Delta$. Since μ is minimal, $a_i \leq 0$ for all i. Let $\Delta' := -\sum_i a_i E_i$. Then $K_M + \Delta' \sim_{\mathbb{Q}} \mu^*(K_T + \Delta)$. By definition, (M, Δ') is log-terminal. Thus we may assume that T is nonsingular.

Step 2. Reduction to the case T is relatively minimal.

Let $\nu\colon T \to T_1$ be the contraction of a (-1)-curve E on T and let $\Delta_1 := \nu_*(\Delta)$. Here (-1)-curve means an exceptional curve of the first kind. Then we have $K_T + \Delta \sim_{\mathbb{Q}} \rho^*(K_{T_1} + \Delta_1) - bE$ for some $b \geq 0$, since $(K_T + \Delta) \cdot E \geq 0$. Thus $(K_{T_1} + \Delta_1) \cdot \Gamma \geq 0$ for any irreducible curve Γ on T_1 and

$$0 = (K_{T_1} + \Delta_1)^2 \geq (K_T + \Delta)^2 = 0 .$$

Hence $b = 0$. Therefore by continuing the contractions of (-1)-curves, we may assume that T is a relatively minimal model.

Step 3. Case $a(T) = 0$.
Assume that $a(T) = 0$. Then by the classification of surfaces, T is a complex torus or a K3 surface. If T is a complex torus, then $\Delta = 0$, since T has no curves. Therefore $K_T + \Delta \sim_{\mathbb{Q}} 0$. Assume that T is a K3 surface. Then by the Riemann–Roch formula,

$$h^0(m\Delta) + h^0(-m\Delta) \geq 2,$$

for any m with $m\Delta$ is Cartier. Since $a(T) = 0$, we have also $\Delta = 0$. Thus $K_T + \Delta \sim_{\mathbb{Q}} 0$.

Step 4. Case $a(T) = 1$.
Assume that $a(T) = 1$. Then there exist a minimal elliptic fibration $f\colon T \to C$ over a smooth curve C. By the canonical bundle formula, we see that $K_T \sim_{\mathbb{Q}} f^*(K_C + B)$ for an effective $\mathbb{Q}$-divisor B on C with $\llcorner B \lrcorner = 0$. Since $a(T) = 1$, no curves Γ of T dominate C. Therefore every component of Δ is contained in fibers of f. Now $K_T + \Delta$ is f-nef. Thus Δ is also f-nef. Therefore there is another effective $\mathbb{Q}$-divisor B' on C such that $\Delta \sim_{\mathbb{Q}} f^*(B')$. Therefore $K_T + \Delta \sim_{\mathbb{Q}} f^*(K_C + B + B')$. Hence $K_T + \Delta$ is semi-ample. Q.E.D.

Lemma B.7. *Let $f\colon X \to M$ be a fibration between complex manifolds whose general fiber is $\mathbb{P}^1$. Suppose that there exist two prime divisors $D_1 \neq D_2$ on X and a Cartier divisor E on D_1 such that*

(1) *D_1 and D_2 dominate M bimeromorphically,*
(2) *$\mathcal{O}_{D_1}(D_1) \simeq \mathcal{O}_{D_1}(E)$.*

Then f is bimeromorphically equivalent to the first projection $M \times \mathbb{P}^1 \to M$.

Proof. By the generically surjective homomorphism $f^*f_*\mathcal{O}_X(D_1) \to \mathcal{O}_X(D_1)$, we may assume that X is isomorphic to $\mathbb{P}_M(\mathcal{E})$ for a locally free sheaf $\mathcal{E}$ of rank two and that D_1 and D_2 are sections of f. Then there exist two exact sequences:

$$0 \to \mathcal{O}_M \to \mathcal{E} \to \mathcal{L} \to 0, \tag{B.1}$$

$$\text{(B.2)} \qquad 0 \to \mathcal{M} \to \mathcal{E} \to \mathcal{L} \otimes \mathcal{M}^{-1} \to 0,$$

where $\mathcal{L}$ and $\mathcal{M}$ are invertible sheaves on M. Here we consider that the section D_1 corresponds to the exact sequence (B.1). Then $\mathcal{O}_{D_1}(D_2)$ is isomorphic to $\mathcal{L} \otimes \mathcal{M}^{-1}$. As in the elementary transformations, we blow-up along $D_1 \cap D_2$ and contract the proper transform of $f^{-1}(f(D_1 \cap D_2))$. Then we can make $D_1 \cap D_2 = \emptyset$. Therefore we may assume that $\mathcal{E} \simeq \mathcal{O} \oplus \mathcal{L}$. By the assumption, there is a Cartier divisor L on M such that $\mathcal{L} \simeq \mathcal{O}_M(L)$. Therefore X is bimeromorphic to $M \times \mathbb{P}^1$. Q.E.D.

Proof of Theorem B.2. By taking the Stein factorization, we may assume that $\pi\colon X \to B$ is a fibration.

Step 1. Case $\pi\colon X \to B$ is a locally projective morphism.
If $\dim B = 0$, then X is projective and theorem is true by [Mo], [KMM], and [Kw3]. If $\dim B \geq 1$ and if the general fiber of π is not uniruled, then for any point $b \in B$, we have an open neighborhood $\mathcal{U}_b \subset B$ and a relative good minimal model $Z_b \to \mathcal{U}_b$ which is bimeromorphically equivalent to $\pi^{-1}(\mathcal{U}_b) \to \mathcal{U}_b$ by [Ny3, §4], [Mo], and A.4. Further except a discrete set of points $\{b_i\}$, $Z_b \to \mathcal{U}_b$ is the unique minimal model of $\pi^{-1}(\mathcal{U}_b) \to \mathcal{U}_b$. Thus we can glue these $Z_b \to \mathcal{U}_b$ and obtain a relative good minimal model $Z \to B$ of π.

In what follows, we assume that π is not a locally projective morphism.

Step 2. Inductive step.
We may assume that X and S are nonsingular and there exists a normal crossing divisor $D = \bigcup D_i$ of S such that f is smooth outside D. Then by Theorem 3.3.3, f is a locally projective morphism. Thus by applying Proposition A.5, we have an elliptic fibration $h\colon Y \to T$ between normal varieties and an effective $\mathbb{Q}$-divisor Δ_T on T such that

(1) there is a bimeromorphic morphism $\mu\colon T \to S$,
(2) $h\colon Y \to T$ is bimeromorphically equivalent to $f\colon X \to S$,
(3) Y has only terminal singularities,
(4) Y is $\mathbb{Q}$-factorial over any points of S,
(5) (T, Δ_T) is log-terminal,
(6) $K_Y \sim_{\mathbb{Q}} h^*(K_T + \Delta_T)$.

Suppose that $(K_T + \Delta_T) \cdot C < 0$ for an irreducible curve C contained in a fiber of $T \to B$ satisfying $C^2 < 0$. Then we have a contraction $\delta\colon T \to T'$ of C. Since $(K_T + \Delta_T) \cdot C < 0$, $-(K_T + \Delta_T)$ is δ-ample. Thus for $\Delta_{T'} := \delta_* \Delta_T$, $(T', \Delta_{T'})$ is also log-terminal and

$$\delta_* \mathcal{O}_T(\llcorner m(K_T + \Delta_T) \lrcorner) \simeq \mathcal{O}_{T'}(\llcorner m(K_{T'} + \Delta_{T'}) \lrcorner)$$

for any $m \geq 0$. By applying Corollary B.5 to $Y \to T \to T'$, we have $R^2(\delta \circ h)_* \mathcal{O}_Y = 0$. Therefore by Proposition 3.3.1, $\delta \circ h$ is bimeromorphic to a locally projective morphism. Hence by Proposition A.5, there is a minimal model $h'\colon Y' \to T'$ such that h' is bimeromorphically equivalent to $\delta \circ h$ and $K_{Y'}$ is h'-semi-ample. Since

$$h'_*\mathcal{O}_{Y'}(mK_{Y'}) \simeq \mathcal{O}_{T'}(m(K_{T'} + \Delta_{T'}))$$

for infinitely many m, we see that $K_{Y'} \sim_{\mathbb{Q}} h'^*(K_{T'}+\Delta_{T'})$. By continuing this process and by Theorem A.1, we may assume that the following conditions are satisfied:

(1) $h\colon Y \to T$ is bimeromorphically equivalent to $f\colon X \to S$;
(2) h is a standard elliptic fibration;
(3) $K_Y \sim_{\mathbb{Q}} h^*(K_T + \Delta_T)$, where (T, Δ_T) is log-terminal;
(4) There is no irreducible curve C on T such that $C^2 < 0$, $(K_T + \Delta_T) \cdot C < 0$, and that C is contained in a fiber of $q\colon T \to B$.

Step 3. Case $1 \leq \dim B \leq 2$.
In this case, we have $R^2 g_*\mathcal{O}_S = 0$. Thus $g\colon S \to B$ is a locally projective morphism by Proposition 3.3.1. Therefore $q\colon T \to S$ is also a locally projective morphism. Suppose that the genus $p_g(F) = 0$ for the general fibers F of $\pi\colon X \to B$. Then $\dim B = 1$, otherwise, the general fibers of π are elliptic curves. Hence the general fibers are Kähler surfaces with $p_g = 0$, so we have $R^i\pi_*\mathcal{O}_X = 0$ for $i = 1$, 2 by [St]. Using Proposition 3.3.1, we see that π is a locally projective morphism. This is a contradiction. Therefore $\pi_*\omega_X \neq 0$. Hence for any point $P \in B$, $K_T + \Delta_T$ is $\mathbb{Q}$-linearly equivalent to an effective $\mathbb{Q}$-divisor over P. Thus $K_T+\Delta_T$ is q-nef. By Theorem A.4, $K_T+\Delta_T$ is q-semi-ample. Therefore $Y \to B$ is a good minimal model in this case.

Step 4. Case $\dim B = 0$ and T is a projective surface. (cf. [Ny6])
If $K_T + \Delta_T$ is nef on T, then $K_T + \Delta_T$ is semi-ample by [Ft1]. Thus Y is a good minimal model. Next assume that $K_T + \Delta_T$ is not nef. Then by *Step* 3 and the cone theorem for (T, Δ_T), there exists a contraction morphism $\sigma\colon T \to C$ such that $\dim C < 2$. Then by Corollary B.5, we see that $R^2(\sigma \circ h)_*\mathcal{O}_Y = 0$. Therefore $\sigma \circ h$ is bimeromorphically equivalent to a locally projective morphism by Proposition 3.3.1. Thus C is a smooth curve. Let F be a general fiber of $\sigma \circ h$. Then F is a ruled surface such that $-K_F$ is semi-ample and $K_F^2 = 0$. Suppose that the irregularity $q(F) = 0$. Then we have $R^1(\sigma \circ h)_*\mathcal{O}_Y = 0$ by [St]. Thus $H^2(Y, \mathcal{O}_Y) = 0$, so Y is Moishezon by Proposition 3.3.1. This is a contradiction. Hence $q(F) = 1$ and F is a minimal ruled surface over an elliptic curve E. Thus by applying the relative minimal model theory to $\sigma \circ h\colon Y \to C$, we have a meromorphic map $\zeta\colon Y \cdots\to N$ over C, where N is a normal nonprojective surface and $N \to C$ is an elliptic fibration. For the general fiber F, ζ induces the projection $F \to E$.

Let $\widetilde{Y} \to Y$ be a modification such that $\widetilde{Y} \to N$ is a morphism and let H_1 and H_2 be general ample divisors on T. Then $D'_1 := h^*(H_1)$ and $D'_2 := h^*(H_2)$ dominate N by ζ. Thus we can take a finite covering $N' \to N$, a modification Y' of the fiber product $\widetilde{Y} \times_N N'$ and prime divisors D_1 and D_2 such that $Y' \to N'$ and D_1, D_2 satisfy the condition of Lemma B.7. Therefore Y is dominated by $N' \times \mathbb{P}^1$.

Step 5. Case $\dim B = 0$ and T is not projective.
In this case $K_T + \Delta_T$ is semi-ample by Proposition B.6. Thus we are done. Q.E.D.

Corollary B.8. *Let X be a compact Kähler threefold admitting an elliptic fibration. Then X is uniruled or there is a good minimal model of X. In each case, there exist a normal compact complex surface T, an effective $\mathbb{Q}$-divisor Δ_T, and a standard elliptic fibration $h\colon Y \to T$ such that (T, Δ_T) is log-terminal, Y is bimeromorphically equivalent to X, $K_Y \sim_{\mathbb{Q}} h^*(K_T + \Delta_T)$. If X is uniruled, then T must be projective, so the algebraic dimension $a(X) \geq 2$. If X is not uniruled, then we can take T so that $K_T + \Delta_T$ is semi-ample. If $a(X) \leq 1$ and $\kappa(X) = 0$, then there is a finite covering $\widetilde{Y} \to Y$ such that*

(1) *the covering is étale outside the non-Gorenstein locus of Y,*
(2) *$\widetilde{Y}$ is a three-dimensional complex torus or the product of an elliptic curve and a $K3$ surface.*

Proof. We have only to prove the last statement. First assume that $a(X) \leq 1$, $\kappa(X) = 0$, and $p_g(X) = 1$. Then $K_Y \sim 0$. Since T is not ruled, we see that $\Delta_T = 0$, T has only rational double points as singularities, and $K_T \sim_{\mathbb{Q}} 0$. The inequality $a(T) \leq a(X) \leq 1$ implies that T is a two-dimensional complex torus or its minimal desingularization is a K3 surface. Therefore, the elliptic fibration $h\colon Y \to T$ is smooth outside the singular locus of T by Theorem 4.3.1. For a singular point $P \in T$, there exist an open neighborhood $\mathcal{U} \subset T$ and a finite Galois covering $\mathcal{V} \to \mathcal{U}$ from a nonsingular surface étale outside P such that the normalization $\mathcal{Y}$ of $Y \times_T \mathcal{V}$ induces a smooth elliptic fibration $\mathcal{Y} \to \mathcal{V}$. Here $\mathcal{Y} \to Y$ is an étale morphism since Y has only Gorenstein terminal singularities (cf. [Kw4, 5.1]). In particular, the fiber of $h\colon Y \to T$ over P is an elliptic curve. Now we have isomorphisms $R^1h_*\mathcal{O}_Y \simeq R^1h_*\omega_Y \simeq \omega_T \simeq \mathcal{O}_T$. Since X is compact and Kähler, the natural homomorphism $H^1(Y, \mathcal{O}_Y) \to H^0(T, R^1h_*\mathcal{O}_T)$ is surjective. We infer that $q(Y) = q(T) + 1$. Thus $q(Y) = 1$ or $q(Y) = 3$ according as T is bimeromorphic to a K3 surface or T is a complex torus. Let $Y \to A$ be the Albanese mapping, which is a fiber space by [Kw1]. If $q(Y) = 3$, then Y is isomorphic to a complex torus and $h\colon Y \to T$ is a fiber bundle. Suppose that $q(Y) = 1$. Then the induced morphism

$Y \to A \times T$ is surjective, since general fibers of h dominate the elliptic curve A. In particular, every smooth fibers of h are isomorphic to each other. Let $T' \subset Y$ be a general fiber of $Y \to A$. Then T' is nonsingular and dominates T. Hence T' is a complex torus of dimension two or a K3 surface. Further $T' \to T$ is a finite morphism étale outside the singular locus of T, since every fibers of $Y \to T$ are elliptic curves and since possible exceptional curves for $T' \to T$ should be rational. We infer that the normalization $\widetilde{Y}$ of the fiber product $Y \times_T T'$ is isomorphic to the product of T' and a fiber. Since $\widetilde{Y} \to Y$ is étale outside the singular locus of Y and since Y has only Gorenstein terminal singularities, Y is nonsingular and $\widetilde{Y} \to Y$ is an étale covering.

Next, we treat the general case $\kappa(X) = 0$ and $a(X) \leq 1$. Since $K_Y \sim_{\mathbb{Q}} 0$, there is a finite covering $Y' \to Y$ such that Y' has only Gorenstein terminal singularities, the covering is étale outside the non-Gorenstein locus of Y, and $K_{Y'} \sim 0$. Let $Y' \to T' \to T$ be the Stein factorization. Then $Y' \to T'$ is also an equi-dimensional elliptic fibration. Thus Y' admits a finite étale covering $\widetilde{Y} \to Y'$ from a complex torus or the product of an elliptic curve and a K3 surface. Q.E.D.

Finally, we note that the good minimal model conjecture for non-Kähler threefolds is not true in general. For example, we have the following:

Proposition B.9. *There exists a compact complex threefold X with $\kappa(X) = 2$ such that K_Y is not semi-ample for any normal variety Y with only terminal singularities bimeromorphically equivalent to X.*

Proof. Let T be a nonsingular minimal projective surface of general type and let $\mu\colon S \to T$ be the blowing-up at a point $P \in T$. Then by Example 3.3.5, we have an elliptic fibration $f\colon X \to S$ smooth outside $D := \mu^{-1}(P)$ such that $f^*(D) = mf^{-1}(D)$ for some positive integer m, where $f^{-1}(D)$ is isomorphic to a Hopf surface. Then by the canonical bundle formula, we see that

$$K_X \sim f^*(K_S) + (m-1)f^{-1}(D) \sim f^*\mu^*(K_T) + (2m-1)f^{-1}(D).$$

Therefore $H^0(X, nK_X) \simeq H^0(T, nK_T)$ for any $n \geq 0$. Suppose that there exists a normal complex threefold Y with only terminal singularities such that it is bimeromorphically equivalent to X and K_Y is semi-ample. Then we have a projective bimeromorphic morphism $\lambda\colon Z \to Y$ and a bimeromorphic morphism $\nu\colon Z \to X$ from a complex manifold Z. By construction, we see that $\lambda^*(K_Y) \sim_{\mathbb{Q}} \nu^* f^* \mu^*(K_T)$. Therefore the proper transform of the Hopf surface $f^{-1}(D)$ must be a λ-exceptional divisor of Z. Since λ is a projective morphism, this is a contradiction. Q.E.D.

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