Advanced Studies in Pure Mathematics 34, 2002
Minimal Surfaces, Geometric Analysis and Symplectic Geometry
pp. 255–259

A Note on the Symplectic Volume of the Moduli Space of Spatial Polygons

Tatsuru Takakura*

Abstract.

We present an altenative proof of the volume formula of the moduli space of spatial polygons, which was given by Kamiyama-Tezuka. Our method is based on the commutativity of geometric quantization and symplectic reduction, originating from a conjecture of Guillemin-Sternberg.

§1. Introduction

Consider the following space $\mathcal{M}_n$, often called the moduli space of spatial polygons;

$$\mathcal{M}_n = \{(a_1, \dots, a_n) \in (S^2)^n \,|\, a_1 + \cdots + a_n = 0\}/SO_3,$$

where $n \geq 3$ and each S^2 is the unit sphere in $\mathbb{R}^3$ with the standard SO_3-action. For simplicity, we asseme that n is odd. In this case $\mathcal{M}_n$ is a compact connected smooth manifold of dimension $2(n-3)$. The topology and geometry of this space has been studied from various points of view (see, e.g., [K-T] and references cited there). For example, it is well-known that $\mathcal{M}_n$ admits a natural symplectic (in fact, Kähler) form ω_n. Concerning symplectic properties of $\mathcal{M}_n$, Kamiyama and Tezuka [K-T] proved, among other things, the following formula[1].

2000 *Mathematics Subject Classification.* Primary 53D20; Secondary 53D50, 14L24.

*Partly supported by the Grant-in-Aid for Encouragement of Young Scientists (No. 11740048), The Ministry of Education, Science and Culture, Japan.

[1] *Added in Proof.* After this article was submitted, the author was informed that S. Martin (*Transversality theory, cobordisms, and invariants of symplectic quotients*, preprint) had also obtained this formula. His method is different from those of Kamiyama-Tezuka and us.

Theorem 1.1 ([K-T, Theorem C]). *The symplectic volume of the space $\mathcal{M}_n$ is given as follows.*

$$\mathrm{Vol}(\mathcal{M}_n) = \frac{-1}{2(n-3)!} \sum_{r=0}^{[n/2]} (-1)^r \binom{n}{r} (n-2r)^{n-3}.$$

(Strictly speaking, the volume in [K-T] is $(n-3)!$ times our $\mathrm{Vol}(\mathcal{M}_n)$. Besides, their formula is written in a slightly different form, but is indeed equivalent to ours (see [K-T, §5]).)

The aim of this paper is to give an alternative proof of the above, as well as to prove the following more general result.

Theorem 1.2. *Let $\mathcal{L}_n$ be the complex line bundle over $\mathcal{M}_n$ with $c_1(\mathcal{L}_n) = [\omega_n]$. Then for each non-negative integer k, we have*

$$\begin{aligned}\int_{\mathcal{M}_n} \mathrm{Td}(\mathcal{M}_n)\mathrm{Ch}(\mathcal{L}_n^{\otimes k}) &= \dim \left(V_k^{\otimes n}\right)^{SO_3} \\ &= -\frac{1}{2} \sum_{r=0}^{N(n,k)} (-1)^r \binom{n}{r} \binom{(n-2r)k+n-r-2}{n-3}.\end{aligned}$$

Here, V_k is the irreducible representation of SO_3 of dimension $2k+1$, $\left(V_k^{\otimes n}\right)^{SO_3}$ is the invariant subspace of $V_k^{\otimes n}$, and $N(n,k) = \left[\dfrac{kn+1}{2k+1}\right]$.

Note that a similar formula for $k=1$ is given by Kamiyama [Ka], where the results in [K-T] on the intersection pairings $\int_{\mathcal{M}_n} \alpha \cdot \beta$ $(\alpha, \beta \in H^*(\mathcal{M}_n))$ are essential. On the other hand, we do not use such information in our proof. (In fact, our approach in this paper is able to be applied to derive general intersection pairings [T].) Actually, the proof of Theorem 1.2 is a simple application of the "quantization commutes with reduction" theorem, originally due to Guillemin-Sternberg.

Acknowledgement. I am grateful to Yasuhiko Kamiyama for explaining his work and for useful discussions.

§2. Preliminaries

2.1. First, we need to specify the symplectic structure ω_n. For this purpose, let us describe the symplectic manifold $(\mathcal{M}_n, \omega_n)$ as a symplectic (Kähler) quotient, or a reduced phase space due to Marsden-Weinstein. (See, e.g., [Ki], for this notion, as well as the relation to the geometric invariant theory.) Let us consider the symplectic manifold

$(\mathbb{P}^1, 2\omega_{\mathrm{FS}})^n$, that is, the product of n-copies of complex projective line with twice the Fubini-Study Kähler form. (The Fubini-Study Kähler form ω_{FS} is supposed to be normalized by $[\omega_{\mathrm{FS}}] = c_1(H) \in H^2(\mathbb{P}^1)$, where H is the hyperplane bundle on $\mathbb{P}^1$.) The diagonal action of $SO_3 = SU_2/\{\pm 1\}$ on $(\mathbb{P}^1, 2\omega_{\mathrm{FS}})^n$ is Hamiltonian. In fact, if we identify each $\mathbb{P}^1$ with a coadjoint orbit of SO_3 in $\mathfrak{so}_3^*$ (or a round 2-sphere in $\mathbb{R}^3$), then the moment map $\Phi : (\mathbb{P}^1)^n \to \mathfrak{so}_3^* \cong \mathbb{R}^3$ is given by $\Phi(a_1, \ldots, a_n) = a_1 + \cdots + a_n$. Now our symplectic manifold $(\mathcal{M}_n, \omega_n)$ is defined as the symplectic quotient $\Phi^{-1}(0)/SO_3$ with the reduced symplectic form.

Remark. In [K-T], the symplectic structure of $\mathcal{M}_n$ is defined differently. But it does coincide with ours.

Moreover, $\mathcal{M}_n$ inherits a complex structure J_n from the standard one on $(\mathbb{P}^1)^n$, so that $(\mathcal{M}_n, \omega_n, J_n)$ is a Kähler manifold (called the Kähler quotient).

Let $\mathbb{L} = H^{\otimes 2}$. It is an SO_3-equivariant holomorphic line bundle over $\mathbb{P}^1$. Then the outer tensor product $\mathbb{L}^{\boxtimes n}$ over $(\mathbb{P}^1)^n$ naturally defines a holomorphic line bundle $\mathcal{L}_n$ over $\mathcal{M}_n$ such that $c_1(\mathcal{L}_n) = [\omega_n]$. In particular, $\mathcal{M}_n$ is projective.

2.2. Now let us recall the "quantization commutes with reduction" theorem. This theorem arose from a conjecture of Guillemin-Sternberg [G-S], and has been proved (and improved) by several people and by various methods. See, e.g., [S] for a survey and references on this topic. We do not intend to state it in full generality, but restrict ourselves only to the following special situation.

For a compact complex manifold X and a holomorphic line bundle L over X, define $\chi(X, L) := \sum (-1)^i H^i(X, \mathcal{O}(L))$, as a virtual vector space. By the Hirzebruch-Riemann-Roch theorem (or Atiyah-Singer index theorem), we have $\dim \chi(X, L) = \int_X \mathrm{Td}(X)\mathrm{Ch}(L)$. If a compact group G acts holomorphically on X and L is G-equivariant, we can regard $\chi(X, L)$ as a virtual representation of G, i.e., an element of the representation ring $R(G)$ of G.

Suppose in addition that X admits an G-invariant Kähler form ω and L admits an G-invariant hermitian connection ∇ such that $c_1(\nabla) = \omega$. Then G-action on X is Hamiltonian. Let $\Phi : X \to \mathfrak{g}^*$ be the moment map. For simplicity, we assume 0 is a regular value of Φ and G acts freely on $\Phi^{-1}(0)$, so that the Kähler quotient $X_G = \Phi^{-1}(0)/G$ is a smooth Kähler manifold and we have the reduced holomorphic line bundle L_G over X_G. The theorem we need is the following.

Theorem 2.1. *Under the assumption as above, we have $\chi(X, L)^G = \chi(X_G, L_G)$, where the left hand side is the G-invariant part of the virtual representation $\chi(X, L)$ of G.*

Remarks. (1) By the argument in §2.1, we are able to apply this theorem to the case when $G = SO_3$, $(X, L) = ((\mathbb{P}^1)^n, \mathbb{L}^{\boxtimes n})$ (with an appropriate invariant hermitian connection) and $(X_G, L_G) = (\mathcal{M}_n, \mathcal{L}_n)$.

(2) The original result in [G-S] states that $H^0(X, \mathcal{O}(L))^G = H^0(X_G, \mathcal{O}(L_G))$ for the spaces of global holomorphic sections instead of χ. Since we can show that the higer cohomologies vanish for $((\mathbb{P}^1)^n, \mathbb{L}^{\boxtimes n})$ and $(\mathcal{M}_n, \mathcal{L}_n)$, this may be enough for our purpose.

(3) If we replace L (resp. L_G) to $L^{\otimes k}$ (resp. $L_G^{\otimes k}$) for a non-negative integer k, the same formula holds. It is obvious when $k \geq 1$. See [M-S] for the case $k = 0$.

§3. Proofs

Theorem 1.1 follows from Theorem 1.2, since

$$\begin{aligned}\mathrm{Vol}(\mathcal{M}_n) &= \int_{\mathcal{M}_n} \exp(\omega_n) = \int_{\mathcal{M}_n} \mathrm{Ch}(\mathcal{L}_n) \\ &= \frac{1}{k^{n-3}} \int_{\mathcal{M}_n} \mathrm{Ch}(\mathcal{L}_n^{\otimes k}) = \lim_{k \to \infty} \frac{1}{k^{n-3}} \int_{\mathcal{M}_n} \mathrm{Td}(\mathcal{M}_n)\mathrm{Ch}(\mathcal{L}_n^{\otimes k}).\end{aligned}$$

In order to prove Theorem 1.2, note that, for each $k (\geq 0)$,

$$\begin{aligned}\int_{\mathcal{M}_n} \mathrm{Td}(\mathcal{M}_n)\mathrm{Ch}(\mathcal{L}_n^{\otimes k}) = \chi(\mathcal{M}_n, \mathcal{L}_n^{\otimes k}) &= \chi((\mathbb{P}^1)^n, (\mathbb{L}^{\otimes k})^{\boxtimes n})^{SO_3} \\ &= \left(\chi(\mathbb{P}^1, \mathbb{L}^{\otimes k})^{\otimes n}\right)^{SO_3} \\ &= \left(V_k^{\otimes n}\right)^{SO_3}.\end{aligned}$$

Indeed, the first equality is a direct consequence of Theorem 2.1, the second one is due to the multiplicative property for χ, and the third one is an elementary fact about the representations of SO_3 (which is a typical example of the Borel-Weil theorem).

Now, the invariant part of the representation is computed by the integration of its character. Namely,

$$\begin{aligned}\dim \left(V_k^{\otimes n}\right)^{SO_3} &= \frac{1}{\pi}\int_0^{2\pi}\left(\frac{\sin(2k+1)\theta}{\sin\theta}\right)^n \sin^2\theta d\theta \\ &= -\frac{1}{2}\mathrm{Res}_{z=0}\,\frac{1}{z}\left(\frac{z^{2k+1}-z^{-(2k+1)}}{z-z^{-1}}\right)^n (z-z^{-1})^2 \\ &= -\frac{1}{2}\mathrm{Res}_{z=0}\,\frac{1}{z}z^{-2(kn+1)}(z^{2(2k+1)}-1)^n(z^2-1)^{-(n-2)}.\end{aligned}$$

By considering the Laurent expansion, we obtain Theorem 1.2.

Remark. When n is even(≥ 4), the space $\mathcal{M}_n$ has singularities. Nevertheless, Theorem 1.1 holds also in this case (as proved in [K-T]). So does Theorem 1.2, after modifying the definition of $\chi(\mathcal{M}_n, \mathcal{L}_n^{\otimes k})$. These follow from a generalization of Theorem 2.1 to singular quotients (see [M-S]).

References

[G-S] V. Guillemin and S. Sternberg, *Geometric quantization and multiplicities of group representations*, Invent. Math., **67** (1982), 515–538.

[K-T] Y. Kamiyama and M. Tezuka, *Symplectic volume of the moduli space of spatial polygons*, J. Math. Kyoto Univ., **39** (1999), 557–575.

[Ka] Y. Kamiyama, *Sheaf cohomology of the moduli space of spatial polygons and lattice points*, Tokyo J. Math., **24** (2001), 205–209.

[Ki] F. C. Kirwan, *Cohomology of quotients in symplectic and algebraic geometry*, Mathematical Notes, vol. 31, Princeton Univ. Press, Princeton, NJ, 1984.

[M-S] E. Meinrenken and R. Sjamaar, *Singular reduction and quantization*, Topology, **38** (1999), 699–762.

[S] R. Sjmaar, *Symplectic reduction and Riemann-Roch formulas for multiplicities*, Bull. Amer. Math. Soc., **33** (1996), 327–338.

[T] T. Takakura, *Intesection theory on symplectic quotients of products of spheres*, Internat. J. Math., **12** (2001), 97–111.

Department of Mathematics
Chuo University
1-13-27 Kasuga, Bunkyo-ku
Tokyo 112-8551
Japan
takakura@math.chuo-u.ac.jp