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Space of Geodesics on Zoll Three-Spheres

Kaoru Ono

Abstract.

The space of geodesics on a Zoll manifold, i.e., a Riemannian manifold all of whose geodesics are closed with the same minimal period, carries a natural symplectic structure. In this note, it is shown that the space of geodesics on a Zoll three-sphere is symplectomorphic to the product of two copies of two-spheres with the same area.

§1. Introduction

The geodesic flow of a Riemannian manifold (M, g) is a Hamiltonian system on its cotangent bundle T^*M. More precisely, we identify the tangent bundle TM and the cotangent bundle T^*M via the Riemannian metric and the Hamiltonian function for the geodesic flow is given by half of the square of the fiber norm on T^*M. Note that the unit cotangent bundle U^*M is an energy hypersurface, i.e., a level set of the Hamiltonian function. The trajectories of the geodesic flow $\{\Psi_t\}$ on the unit tangent bundle of (M, g) are characteristic curves on U^*M with respect to the standard symplectic structure on T^*M.

The space of geodesics, $\mathrm{Geod}(M, g)$, is the quotient space of U^*M by the characteristic flow, which may not be a nice space, in general. If all the geodesics are closed with the same minimal period, the quotient space becomes a manifold. We call a metric g enjoying this property a Zoll metric. In such a case, the space of geodesics is considered as the symplectic reduction of the energy hypersurface U^*M by the circle action and carries the natural symplectic structure. The cohomology class of the symplectic form is the negative of the first Chern class of the S^1-bundle $U^*M \to \mathrm{Geod}(M, g)$. A special feature of the symplectic

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manifold $\mathrm{Geod}(M, g)$ is that it contains Lagrangian spheres which sweep up the whole space. Namely, the set L_p of geodesics passing through a point $p \in M$ is the image of a fiber of U^*M at p by the projection $U^*M \to \mathrm{Geod}(M, g)$, and hence a Lagrangian sphere. It is obvious that $\mathrm{Geod}(M, g)$ is swept by these Lagrangian spheres.

It is known that the space of geodesics on an n-dimensional Zoll sphere is simply connected and has the same cohomology ring as that of the complex hyperquadric of complex dimension $n - 1$ (see [Y]). In the case of low dimensional spheres, Hajime Sato proved that the Chern classes of the space of geodesics of a Zoll 3-sphere has the same form as in the case of the standard 3-sphere in their cohomology rings, which are isomorphic, and the space of geodesics of a Zoll 4-sphere is diffeomorphic to the complex hyperquadric [S].

In this short note, we shall prove the following

Theorem. *The space of geodesics on any Zoll 3-sphere is symplectomorphic to the product of two copies of symplectic 2-spheres with the same area.*

Remark. All known examples of Zoll metrics, up to now, are deformations of the standard metric through Zoll metrics. (see [B], [K] for examples of Zoll metrics.) In such a case, the result of our Theorem follows from Moser's stability theorem [M]. Note also that Theorem implies the existence of homogeneous symplectomorphism between the cotangent bundles with the zero section removed of any Zoll three-sphere and the round three-sphere, which intertwines their geodesic flows.

§2. Preliminaries

In this section, we collect several results, which are necessary for our argument.

A symplectic manifold (X, ω) is called monotone[1], if the symplectic class $[\omega]$ and the first Chern class $c_1(X)$ are positively proportional:

$$c_1(X) = \lambda[\omega] \text{ for some } \lambda > 0.$$

For 4-dimensional monotone symplectic manifolds, we have the following

Theorem 2.1 (cf. [O-O]). *A monotone symplectic 4-manifold (X, ω) is diffeomorphic to one of Del-Pezzo surfaces. If X is minimal,*

[1] Monotonicity is usually defined to be the positive proportionality of $[\omega]$ and $c_1(M)$, as homomorphisms from $\pi_2(M)$ to $\mathbf{R}$. Our assumption here is stronger than this "spherical monotonicity", in case that M is not simply connected.

i.e., X does not contain (-1) symplectic sphere, then it must be symplectomorphic to either $\mathbf{C}P^2$ with a multiple of the Fubini-Study form or the product of two symplectic 2-spheres with the same area.

The result is obtained by combining a consequence of Taubes' theorem "SW=GW" and McDuff's theorem [MD] concerning rational or ruled symplectic 4-manifolds. In [O-O], we used Taubes' theorem and found a rational curve with non-negative intersection number. Then McDuff's theorem implies that it must be a rational or ruled surface, and we get classification up to diffeomorphism. The statement about symplectic structure follows from the uniqueness result [MD] (see also [L-MD]).

Remark 2.2. As we mentioned, the space of geodesics on a Zoll 3-sphere is a cohomology $S^2 \times S^2$. So it is sufficient to show that $\mathrm{Geod}(M, g)$ is a monotone symplectic manifold.

Remark 2.3. It is possible to prove our theorem by combining Theorem 2.1 with Sato's result mentioned above, which implies the monotonicity. Here we shall give another method of computing the Chern class.

The unit cotangent bundle U^*M carries a contact structure $\xi \subset TU^*M$, which inherits the structure of symplectic vector bundle from the canonical symplectic form on T^*M. The tangent bundle along fibers of $U^*M \to M$ is a Lagrangian subbundle, which we denote by Ver. Associated to a closed geodesic $\gamma : [0, L] \to M$, we consider the loop $\mathcal{L}_\gamma$ of Lagrangian subspaces in ξ:

$$\mathcal{L}_{\gamma(t)} = d\Psi_t(\mathrm{Ver}_{\gamma(0)}).$$

We denote by $\mu(\gamma)$ the Maslov index of the loop $\mathcal{L}_\gamma$ with respect to the vertical Lagrangian distribution Ver. The relation between the Maslov index $\mu(\gamma)$ and the Morse index of the geodesic γ is established in [D]. Here we recall the following fact.

Lemma 2.4 (cf. [B, Theorm 7.23]). *$\mu(\gamma)$ must be 4. (the Morse index $= 2$, the nullity $= 2$)*

§3. Proof of Theorem

Let g_0 be the standard metric on $M = S^3$ and g a metric, all of whose geodesics are closed with the common minimal period. By multiplying a suitable real number to g, we may assume that the unit cotangent ball bundle $B^{g_0}M$ with respect to g_0 is strictly contained in the unit cotangent ball bundle $B^g M$ with respect to g. By the assumption that characteristics on unit cotangent bundles $U^*_{g_0}M$ and $U^*_g M$ are periodic (with different periods), we can apply symplectic cutting construction [L] along boundaries to the compact symplectic manifold with boundary $B^g M \setminus Int(B^{g_0}M)$ and get a closed symplectic 6-manifold (Z, ω). Namely, we consider the orbit spaces of boundaries by the circle actions and replace the boundaries of the compact manifold by the orbit spaces. It turns out that the space obtained is a smooth manifold, if the circle actions are free, and carries a natural symplectic structure. This is the symplectic cutting construction. It is clear that $\mathrm{Geod}(M, g_0)$ and $\mathrm{Geod}(M, g)$ are symplectically embedded in Z.

For the standard metric g_0, the space of geodesics $\mathrm{Geod}(M, g_0)$ is diffeomorphic to the complex hyperquadric of complex dimension 2, i.e., the product of two copies of S^2. We denote by α_1 and β_1 the Poincaré duals of $S^2 \times \mathrm{pt}$ and $\mathrm{pt} \times S^2$, respectively.

Since the cohomology ring of $\mathrm{Geod}(M, g)$ is also isomorphic to that of $S^2 \times S^2$, we denote by α_2 and β_2 the corresponding generators of the second cohomology group so that $\alpha_2 \cup \beta_2[\mathrm{Geod}(M, g)] = 1$ and $\alpha_2^2[\mathrm{Geod}(M, g)] = \beta_2^2[\mathrm{Geod}(M, g)] = 0$.

Lemma 3.1. *The first Chern class of* $\mathrm{Geod}(M, g)$ *is* $\pm 2(\alpha_2 + \beta_2)$.

Proof. Note that the signature of $\mathrm{Geod}(M, g)$ is zero. Hence the first Pontrjagin class must be zero. Since $p_1 = c_1^2 - 2c_2$ and c_2 is identified with the Euler number, $c_1^2[\mathrm{Geod}(M, g)] = 8$. For a simply connected 4-manifold, the fact that the intersection form is of even type implies the vanishing of the second Stiefel-Whitney class w_2. Combining the fact that c_1 modulo 2 equals w_2, we have $c_1(M) = \pm 2(\alpha_2 + \beta_2)$. □

Remark 3.2. The symplectic manifold, which has the same cohomology ring as $S^2 \times S^2$, contains an embedded Lagrangian sphere, only if the symplectic class is proportional to $\alpha_2 + \beta_2$. The reason is the following. The self intersection number of a Lagrangian two-sphere must be -2 and its homology class must be $\pm(\alpha_2 - \beta_2)$. Then the Lagrangian condition implies that the integration of the symplectic form over α_2 and β_2 are the same, which implies the proportionality, although it is not neccessarily positive proportionality.

We may change, if necessary, the orientation on both factors of $S^2 \times S^2$ simultaneously and assume that the symplectic class is a positive multiple of $\alpha_2 + \beta_2$. Then, once we know that $c_1(\mathrm{Geod}(M, g)) = 2(\alpha_2 + \beta_2)$, our theorem follows from Theorem 2.1. The rest of the argument is devoted to computing the first Chern class.

Let γ_1 and γ_2 be orbits of the geodesic flow on $U^*_{g_0}M$ and $U^*_g M$, respectively. We can regard them as transversal loops in the projective cotangent bundle (PT^*M, ξ), by identifying $U^*_g M$ and $U^*_{g_0}M$ with projective cotangent bundle using the projection

$$T^*M \setminus O_M \to PT^*M = (T^*M \setminus O_M)/\mathbf{R}^+.$$

Since PT^*M is diffeomorphic to $S^3 \times S^2$, it is a 5-dimensional simply connected manifold. Then we can find an isotopy of smooth embeddings between loops γ_1 and γ_2. By a C^0-small perturbation of the isotopy, we get an isotopy through transversally embedded loops. Hence we obtain the following

Lemma 3.3. *Two transversal loops γ_1 and γ_2 above are transversally isotopic.*

Using this lemma, we can construct an symplectically embedded 2-sphere as follows. Let $\{\gamma_t\}$ denote the transversal isotopy between γ_1 and γ_2. We assume that it is independent of t near $t = 1, 2$, respectively. Take a family of metrics $g_t = (2-t)g_0 + (t-1)g$. Using the identification between $U^{g_t}M$ and PT^*M, we consider that γ_t is a loop in $U^{g_t}M$. Note that the unit cotangent bundles with respect to g_t are mutually disjoint. We may rescale the metric the metric small real number so that the unit cotangent ball bundle $B^{g_0}M$ is sufficiently small and the t-direction is stretched out and the map

$$F : S^1 \times [1,2] \to B^g M \setminus \mathrm{Int}(B^{g_0}M)$$

given by $(s,t) \mapsto \gamma_t(s)$ yields a symplectically embedded cylinder. It is easy to see that after symplectic cutting construction, we get an symplectically embedded sphere C in Z, which intersects $\mathrm{Geod}(M, g_0)$ and $\mathrm{Geod}(M, g)$ with intersection index 1. We denote by N and S the points in C corresponding to γ_1 and γ_2, respectively.

By the Mayer-Vietris exact sequence, we have

Lemma 3.4. *The second homology group $H_2(Z;\mathbf{Z})$ is generated by $H_2(\mathrm{Geod}(M, g_0; \mathbf{Z}))$ and C.*

We identify $\mathrm{Geod}(M, g_0)$ with $S^2 \times S^2$ and denote by A and B the generators $[S^2 \times \mathrm{pt}]$ and $[\mathrm{pt} \times S^2]$ of $H_2(S^2 \times S^2; \mathbf{Z})$.

Then the image of $H_2(\mathrm{Geod}(M, g); \mathbf{Z})$ in $H_2(Z; \mathbf{Z})$ is the submodule spanned by $A+C$ and $B+C$. Note that $\omega\cdot(A+C) = \omega\cdot(B+C) > 0$. Since Lemma 3.1 implies that $c_1(\mathrm{Geod}(M, g))[A+C] = c_1(\mathrm{Geod}(M, g))[B+C]$, it suffices to prove that $c_1(\mathrm{Geod}(M, g))[A + C] > 0$. $\mathrm{Geod}(M, g)$ is a symplectic submanifold, so that an almost complex submanifold with respect to a compatible almost complex structure and we have

$$c_1(\mathrm{Geod}(M,g)) = c_1(TZ|_{\mathrm{Geod}(M,g)}) - PD([\mathrm{Geod}(M,g)]),$$

where $PD([\mathrm{Geod}(M,g)])$ is the Poincaré dual of the fundamental class of $\mathrm{Geod}(M,g)$. Hence we have $c_1(\mathrm{Geod}(M,g))[A+C] = c_1(Z)[A+C]-1$ and $c_1(Z)[A+C] = c_1(Z)[A] + c_1(Z)[C]$. From the explicit description of a neighborhood of $\mathrm{Geod}(M,g_0)$, we have $c_1(Z)[A] = 1$.

We have the following

Lemma 3.5. $c_1(Z)[C] = 2$.

Proof. Since C is a symplectically embedded sphere, we may assume that TC is a complex subbundle of a complex vector bundle $TZ|_C$. We denote by N_C its normal bundle. Then we have

$$c_1(Z)[C] = c_1(TC)[C] + c_1(N_C)[C] = 2 + c_1(N_C)[C].$$

Note that N_C restricted to $C\setminus\{N,S\}$ is identified as the restriction of ξ and contains a Lagrangian subbundle given by the vertical distribution. Around N and S, they may have non-trivial Maslov index and may not extend over C. But by Lemma 2.4, these Maslov indices are the same. Since the bundle N_C is the quotient of $\xi|_{F(S^1\times[1,2]}$ by the circle actions along boundaries, i.e., the differential of the geodesic flows $d\Psi_t$ with respect to the Zoll metrics g_0 and g, this implies that the symplectic vector bundle N_C is trivial and has vanishing first Chern class. Therefore we have $c_1(Z)[C] = 2$. □

Combining our computation, we have

$$c_1(\mathrm{Geod}(M,g))[A+C] = c_1(Z)[A] + c_1(Z)[C] - 1 = 1 + 2 - 1 = 2 > 0,$$

which completes the proof of the Theorem.

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Department of Mathematics
Hokkaido University
Sapporo, 060-0810
Japan
ono@math.sci.hokudai.ac.jp