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The Topology of Toric HyperKähler Manifolds

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Abstract.

The topology of hyperKähler quotients of quaternionic vector spaces by tori is studied. We discuss the relation between their topology and a combinatorial property of some polyhedral complexes. As its simple application we compute their Chern classes.

§1. Introduction

The topology of symplectic quotients has been intensively studied in the last two decades. Especially, Kirwan's theory enables us to compute the Betti numbers of symplectic quotients [9], and thanks to the theory of Jeffrey and Kirwan [8] we can investigate their cohomology rings. On the other hand, various classes of hyperKähler quotients were introduced and studied in detail by many authors, but their topology has not yet been studied well. Recently, in this regard Bielawski and Dancer studied hyperKähler quotients of quaternionic vector spaces $\mathbf{H}^N$ by subtori of T^N, which they call toric hyperKähler manifolds [2].

Being influenced by their work, we intend to study the topology of toric hyperKähler manifolds. It should be remarked that every toric hyperKähler manifold, if we deform its hyperKähler structure appropriately, contains a union of projective toric manifolds as its deformation retract. Because of this fact we call it the core of the toric hyperKähler manifold. Generally speaking, the topology of projective toric manifolds is well-known [4]. However, since they intersect in a complicated way, it is not easy to study the topology of the core. Concerning this, in [10] we determined their cohomology rings.

In this note we also study the topology of toric hyperKähler manifolds. The structure of the core is described by a polyhedral complex

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associated to it. We discuss the relation between the topology of toric hyperKähler manifolds and a combinatorial property of the associated polyhedral complex. As its simple application we compute the total Chern class of toric hyperKähler manifolds.

In Section 2 we define toric hyperKähler manifolds and describe their cohomology rings, which is proved in [10]. The relation of the topology of toric hyperKähler manifolds and their associated polyhedral complexes is studied in Section 3. In Section 4 we compute their Chern classes.

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§2. Toric hyperKähler manifolds

In this section we define toric hyperKähler manifolds and describe their cohomology rings.

First, let us recall the hyperKähler structure on the quaternionic vector space $\mathbf{H}^N$. Let $\{1, I_1, I_2, I_3\}$ be the standard basis of $\mathbf{H}$. On $\mathbf{H}^N$ we define three complex structures by the multiplication of I_1, I_2, I_3 from the left, respectively. We denote these complex structures also by I_1, I_2, I_3. The real torus $T^N = \{\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbf{C}^N \| \alpha_i| = 1\}$ acts on $\mathbf{H}^N$ from the right diagonally, and preserves its hyperKähler structure. If we identify $\xi \in \mathbf{H}^N$ with $(z, w) \in \mathbf{C}^N \times \mathbf{C}^N$ by $\xi = z + wI_2$, then the action is given by

$$(z, w)\alpha = (z\alpha, w\alpha^{-1}).$$

Let K be a subtorus of T^N with Lie algebra $k \subset t^N$. Then we have the torus $T^n = T^N/K$ with Lie algebra $t^n = t^N/k$. Moreover, we have the following exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & k & \xrightarrow{\iota} & t^N & \xrightarrow{\pi} & t^n & \longrightarrow & 0, \\ 0 & \longleftarrow & k^* & \xleftarrow{\iota^*} & (t^N)^* & \xleftarrow{\pi^*} & (t^n)^* & \longleftarrow & 0. \end{array}$$

Since the action of K on $\mathbf{H}^N$ preserves its hyperKähler structure, we obtain the hyperKähler moment map

$$\mu_K = (\mu_{K,1}, \mu_{K,2}, \mu_{K,3})\colon \mathbf{H}^N \to k^* \otimes \mathbf{R}^3,$$

which is given by

$$\mu_{K,1}(z,w) = \pi \sum_{i=1}^{N} (|z_i|^2 - |w_i|^2)\iota^* u_i,$$

$$(\mu_{K,2} + \sqrt{-1}\mu_{K,3})(z,w) = -2\pi\sqrt{-1} \sum_{i=1}^{N} z_i w_i \iota^* u_i,$$

where $\{u_1, \ldots, u_N\} \subset (t^N)^*$ is the dual basis of the standard basis $\{X_1, \ldots, X_N\} \subset t^N$. Now we define toric hyperKähler manifolds.

Definition. If $\nu \in k^* \otimes \mathbf{R}^3$ is a regular value of the hyperKähler moment map μ_K and if the action of K on $\mu_K^{-1}(\nu)$ is free, we call the hyperKähler quotient

$$X(\nu) = \mu_K^{-1}(\nu)/K$$

a toric hyperKähler manifold.

Note that $X(\nu)$ is a $4n$ dimensional hyperKähler manifold. We denote its hyperKähler structure by $(g_\nu, I_{\nu,1}, I_{\nu,2}, I_{\nu,3})$. The torus $T^n = T^N/K$ acts on $X(\nu)$, preserving its hyperKähler structure. This action gives the hyperKähler moment map

$$\mu_{T^n} = (\mu_{T^n,1}, \mu_{T^n,2}, \mu_{T^n,3}) \colon X(\nu) \to (t^n)^* \otimes \mathbf{R}^3.$$

The terminology 'a toric hyperKähler manifold' is due to Bielawski and Dancer [2]. One of their results is the following:

Fact 2.1. *The diffeomorphism type of a toric hyperKähler manifold $X(\nu)$ is independent of the choice of ν.*

In [10], for each $h \in (t^N_{\mathbf{Z}})^* = \sum_{i=1}^N \mathbf{Z}u_i$, we constructed a holomorphic line budle L_h on $X(\nu)$ with respect to the complex structure $I_{\nu,1}$. The equation $z_i = 0$ defines a divisor D_{u_i} on $X(\nu)$, and we showed that the holomorphic line bundle defined by the divisor D_{u_i} is L_{u_i}. Moreover, we showed that the dual line bundle $L^*_{u_i}$ corresponds to the divisor defined by the equation $w_i = 0$. In [10] we described the cohomology ring of $X(\nu)$ in terms of the subtorus K as follows.

Theorem 2.2. *Let $\Phi \colon \mathbf{Z}[u_1, \ldots, u_N] \to H^*(X(\nu); \mathbf{Z})$ be a ring homomorphism defined by $\Phi(u_i) = c_1(L_{u_i})$. Then the following holds:*
(1) *The map Φ is surjective. Therefore we have an isomorphism as a ring:*

$$H^*(X(\nu); \mathbf{Z}) \cong \mathbf{Z}[u_1, \ldots, u_N]/\ker\Phi.$$

(2) $\ker\Phi$ *is an ideal generated by all*

1. $\sum_{i=1}^N a_i u_i \in \ker \iota^* \cap (t^N_{\mathbf{Z}})^*$, *and*
2. $\prod_{b_i \neq 0} u_i$ *for* $\sum_{i=1}^N b_i X_i \in k \setminus \{0\}$.

Example. Let $\pi\colon t^5 \to t^3$ be a surjevtive map such that $\pi(X_4) = -\pi(X_1) - \pi(X_2)$ and $\pi(X_5) = -\pi(X_1) - \pi(X_3)$. Then we have a toric hyperKähler manifold $X(\nu)$ for $\nu \in k^* \otimes \mathbf{R}^3$ satisfying the condition mentioned above. Since k is spanned by $\{X_1+X_2+X_4, X_1+X_3+X_5\}$, there are 4 types of elements in k as follows:

$$X_1 + X_2 + X_4, X_1 + X_3 + X_5, X_2 - X_3 + X_4 - X_5,$$
$$\sum_{i=1}^5 a_i X_i \quad \text{where } a_i \neq 0 \text{ for } i = 1, \ldots, 5.$$

Moreover, since $\ker\iota^*$ is spanned by $\{u_2 - u_4, u_3 - u_5, u_1 - u_2 - u_3\}$, Theorem 2.2 implies that in this case $\ker\Phi$ is generated by

$$\{u_2 - u_4, u_3 - u_5, u_1 - u_2 - u_3, u_1u_2u_4, u_1u_3u_5, u_2u_3u_4u_5, u_1u_2u_3u_4u_5\}.$$

§3. The associated polyhedral complex

In this section we associate a polyhedral complex $\mathcal{C}(X(\nu))$ to a toric hyperKähler manifold $X(\nu)$ with $\nu = (\nu_1, 0, 0) \in k^* \otimes \mathbf{R}^3$. We also discuss the relation between the topology of $X(\nu)$ and the associated polyhedral complex. Throughout this section, we assume that $\nu = (\nu_1, 0, 0) \in k^* \otimes \mathbf{R}^3$. We also fix an element $h \in (t^N)^*$ such that $\iota^* h = \nu_1$.

First, let us recall the notion of a polyhedral complex. A polyhedral complex $\mathcal{C}$ is by definition a family of polyhedra in the fixed $\mathbf{R}^n$ satisfying the following conditions:

1. If σ is an element of $\mathcal{C}$, then every face of σ belongs to $\mathcal{C}$.
2. If σ and τ are elements of $\mathcal{C}$ and the intersection $\sigma \cap \tau$ is not empty, then $\sigma \cap \tau$ is a face of both σ and τ.

We define the support of $\mathcal{C}$ by $|\mathcal{C}| = \bigcup_{\sigma \in \mathcal{C}} \sigma$.

Now we associate a polyhedral complex $\mathcal{C}(X(\nu))$ to a toric hyperKähler manifold $X(\nu)$ with $\nu = (\nu_1, 0, 0) \in k^* \otimes \mathbf{R}^3$. Recall that we fixed $h \in (t^N)^*$ such that $\iota^* h = \nu_1$. We define hyperplanes F_i in $(t^n)^*$ by

$$F_i = \{p \in (t^n)^* \mid \langle \pi^* p + h, X_i \rangle = 0\} \quad \text{for } i = 1, \ldots, N.$$

Then these hyperplanes devide $(t^n)^*$ into a finite number of closed convex polyhedra $\{\Delta_\epsilon | \epsilon \in \Theta\}$, where Θ is the set consisting of all maps from

$\{1,\ldots,N\}$ to $\{1,-1\}$, and $\Delta_\epsilon \subset (t^n)^*$ is defined by

$$\Delta_\epsilon = \{p \in (t^n)^* \mid \epsilon(i)\langle \pi^* p + h, X_i\rangle \geq 0 \quad \text{for any } i = 1,\ldots,N\}.$$

Then the associated polyhedral complex $\mathcal{C}(X(\nu))$ is defined to be a complex consisting of all *compact* faces of all polyhedra Δ_ϵ, where $\epsilon \in \Theta$. It should be remarked that, to define $\mathcal{C}(X(\nu))$, we need $h \in (t^N)^*$ such that $\iota^* h = \nu_1$. However, $\mathcal{C}(X(\nu))$ is determined by ν_1 up to parallel translation. So we use this notation.

For each $\epsilon \in \Theta$, we define a subspace V_ϵ of $\mathbf{H}^N$ as follows: $(z,w) \in V_\epsilon$ if and only if, for any $i = 1,\ldots,N$, $w_i = 0$ if $\epsilon(i) = 1$, and $z_i = 0$ if $\epsilon(i) = -1$. It is easy to see that if we set $M_\epsilon = \mu_{T^n}^{-1}(\Delta_\epsilon, 0, 0)$, then we have

$$M_\epsilon = \{V_\epsilon \cap \mu_{K,1}^{-1}(\nu_1)\}/K.$$

Since $V_\epsilon \cong \mathbf{C}^N$, M_ϵ is an ordinary toric manifold.

Let us recall the fundamental property of $X(\nu)$, which is proved in [10].

Lemma 3.1. (1) $\mu_{T^n}^{-1}((t^n)^*, 0, 0) = \bigcup_{\epsilon\in\Theta} M_\epsilon$.
(2) *Suppose that $\Delta_\epsilon \cap F_i$ is a face of Δ_ϵ with codimension one. Then the homology class represented by $\mu_{T^n}^{-1}(\Delta_\epsilon \cap F_i, 0, 0)$ is the Poincaré dual of $\epsilon(i)c_1(L_{u_i})$ in M_ϵ.*

Then we have the following fact, which was due to [5] in special cases and due to [2] for general toric hyperKähler manifolds.

Fact 3.2. *Let $X(\nu)$ be a toric hyperKähler manifold with $\nu = (\nu_1, 0, 0)$ and $\mathcal{C} = \mathcal{C}(X(\nu))$ the associated polyhedral complex. Then the following holds*:
(1) *For each $\tau \in \mathcal{C}(X(\nu))$, $N_\tau = \mu_{T^n}^{-1}(\tau, 0, 0)$ is a projective toric submanifold of $X(\nu)$.*
(2) *$\bigcup_{\tau\in\mathcal{C}} N_\tau = \mu_{T^n}^{-1}(|\mathcal{C}|, 0, 0)$ is a T^n-equivariant deformation retract of $X(\nu)$.*
(3) *The homeomorphism type of $\bigcup_{\tau\in\mathcal{C}} N_\tau$ is completely determined by the combinatorial structure of the associated polyhedral complex $\mathcal{C}(X(\nu))$.*

Definition. Due to Fact 3.2 we call the union of projective toric manifolds $\bigcup_{\tau\in\mathcal{C}} N_\tau$ *the core of the toric hyperKähler manifold* $X(\nu)$.

Example. Let us consider a toric hyperKähler manifold $X(\nu)$ in Section 2 again. Here we assume $\nu = (\nu_1, 0, 0)$. If we set $v_1 = \iota^* u_4 = \iota^* u_2$ and $v_2 = \iota^* u_5 = \iota^* u_3$, then k^* is devided into six chambers as in Figure 1. Suppose that $\nu_1 \in S_1$. If we define $\epsilon_1, \epsilon_2 \in \Theta$ by

$$\epsilon_1(i) = 1 \quad \text{for } i = 1,2,3,4,5, \qquad \epsilon_2(i) = \begin{cases} 1 & \text{for } i = 1,2,4, \\ -1 & \text{for } i = 3,5, \end{cases}$$

then the associated polyhedral complex $\mathcal{C}(X(\nu))$ consists of all faces of Δ_{ϵ_1} and Δ_{ϵ_2} as in Figure 2, where we take an appropriate coordinate (a_1, a_2, a_3) in $(t^3)^*$ such that $F_i = \{(a_1, a_2, a_3)|a_i = 0\}$ for $i = 1, 2, 3$. We remark that the combinatorial structure of the associated polyhedral complex and the topology of the core depend on the chamber. However, the topology of $X(\nu)$ does not depend on it [10].

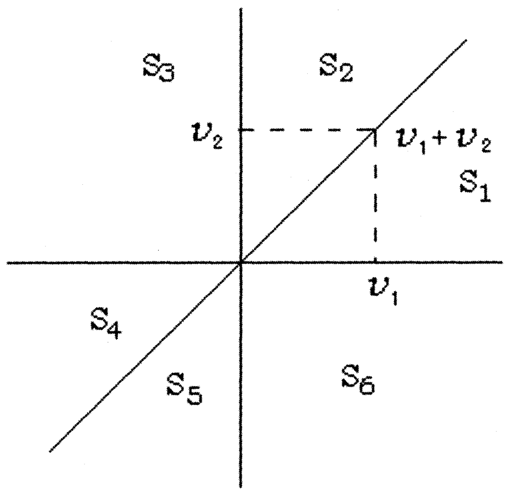

Figure 1.

Thus, to study the cohomology of $X(\nu)$, we have only to study its core $\bigcup_{\tau\in\mathcal{C}} N_\tau$. It is a union of projective toric manifolds, which intersect along toric submanifolds. The topology of projective toric manifolds N_τ is well-known [4]. However, since N_τ's intersect in a comlpicated way, it is not easy to study the topology of the core.

Let us recall the notion of star-collapsibility, which we learned from the earlier version of [2].

Definition . A polyhedral complex $\mathcal{C}$ is *star-collapsible* if there exists a filtration

$$\emptyset = \mathcal{C}_{r+1} \subset \mathcal{C}_r \subset \cdots \subset \mathcal{C}_1 = \mathcal{C}$$

by subcomplexes such that, for $i \leq r$, there exists a vertex $x_i \in \mathcal{C}_i$ and the following conditions are satisfied:

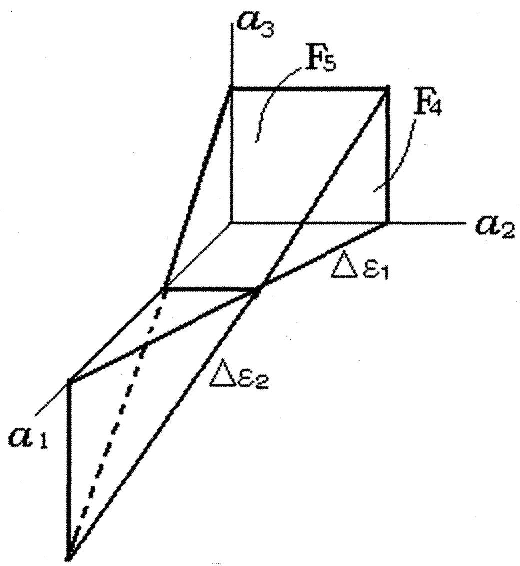

Figure 2.

1. There exists $\sigma_i \in \mathcal{C}_i$ uniquely such that $x_i \in \sigma_i$ and σ_i is a maximal element in $\mathcal{C}_i$.
2. $\mathcal{C}_i \setminus \mathcal{C}_{i+1} = \{\tau \in \mathcal{C}_i \mid x_i \in \tau, \ \ \tau \text{ is a face of } \sigma_i\}$.

Now we show the following lemma. The proof below was suggested by T. Gocho.

Lemma 3.3. *Let $X(\nu)$ be a toric hyperKähler manifold with $\nu = (\nu_1, 0, 0)$. Then the associated polyhedral complex $\mathcal{C}(X(\nu))$ is star-collapsible.*

Proof. Define the S^1-action on $\mathbf{H}^N$ by $(z, w)\beta = (z\beta, w\beta)$ for $\beta \in S^1$. This induces the S^1-action on $X(\nu)$. It is easy to see that this action preserves $\omega_{\nu,1}$, which is the Kähler form with respect to $I_{\nu,1}$. Note that the moment map for this action $\mu_{S^1} \colon X(\nu) \to \mathbf{R}$ is proper and T^n-invariant. If we perturb this function by a small and generic $\xi \in t^n$ as

$$f([z, w]) = \mu_{S^1}([z, w]) + \langle \mu_{T^n,1}([z, w]), \xi \rangle,$$

then f remains proper and the critical point set of f coincides with the fixed point set of T^n, which consists of finite points $\{p_1, \dots, p_r\}$. We may also assume $f(p_1) > f(p_2) > \cdots > f(p_r)$.

Moreover the gradient flow of f is described by the action of 1-parameter subgroup of the complexification of $S^1 \times T^n$. Therefore the gradient flow preserves N_τ for every $\tau \in \mathcal{C}(X(\nu))$.

Note that $f|_{\mu_{T^n}^{-1}((t^n)^*,0,0)}$ desends to the function $\bar{f}$ on $(t^n)^*$. Since $\bar{f}$ is also proper and bounded below, it is easy to see that, for every $x_i = \mu_{T^n,1}(p_i)$, there exists a unique maximal $\sigma_i \in \mathcal{C}(X(\nu))$ such that $x_i \in \sigma_i$ and $\bar{f}|_{\sigma_i}$ has the maximum at x_i. Thus x_i's and σ_i's define a desired filtration on $\mathcal{C}(X(\nu))$. □

Now we discuss the relation between the topology of $X(\nu)$ and the combinatorial property of $\mathcal{C}(X(\nu))$.

Theorem 3.4. *Let $X(\nu)$ be a toric hyperKähler manifold with $\nu = (\nu_1, 0, 0)$ with the associated polyhedral complex $\mathcal{C} = \mathcal{C}(X(\nu))$. Let $\emptyset = \mathcal{C}_{r+1} \subset \mathcal{C}_r \subset \cdots \subset \mathcal{C}_1 = \mathcal{C}$, $x_i \in \mathcal{C}_i$ and $\sigma_i \in \mathcal{C}_i$ be a filtration, vertices and faces concerned with star-collapsibility, respectively. We set $N_i = \mu_{T^n}^{-1}(\sigma_i, 0, 0)$ for $i = 1, \dots, r$. We denote the embedding of N_i into $X(\nu)$ by $\psi_i \colon N_i \to X(\nu)$. Then we have*

$$\ker \Phi = \bigcap_{i=1}^{r} \ker(\psi_i^* \circ \Phi).$$

Proof. Since $\ker \Phi \subset \bigcap_{i=1}^{r} \ker(\psi_i^* \circ \Phi)$ is trivial, we have only to show that $\ker \Phi \supset \bigcap_{i=1}^{r} \ker(\psi_i^* \circ \Phi)$. To prove this, it is sufficient to show that the map

$$\Psi = \bigoplus_{i=1}^{r} \psi_i^* \colon H^*(X(\nu); \mathbf{Z}) \to \bigoplus_{i=1}^{r} H^*(N_i; \mathbf{Z})$$

is injective.

We set $E_i = \mu_{T^n}^{-1}(|\mathcal{C}_i|, 0, 0)$. Since $|\mathcal{C}_i| = |\mathcal{C}_{i+1}| \cup \sigma_i$, we have $E_i = E_{i+1} \cup N_i$. Moreover we prove the following claim.

Claim. *The natural map $H^*(E_i; \mathbf{Z}) \to H^*(E_{i+1}; \mathbf{Z}) \oplus H^*(N_i; \mathbf{Z})$ is injective for $i = 1, \dots, r$.*

Proof of Claim. Since N_i is a projective toric manifold, $H^{\mathrm{odd}}(N_i; \mathbf{Z}) = 0$. Moreover, since $N_i \setminus (E_{i+1} \cap N_i)$ is the biggest cell in N_i, we also have $H^{\mathrm{odd}}(E_{i+1} \cap N_i; \mathbf{Z}) = 0$. To show that $H^{\mathrm{odd}}(E_i; \mathbf{Z}) = 0$, we consider the cohomology exact sequence (This argument is due to Bielawski and Dancer):

$$\to H^{\mathrm{odd}}(E_i, E_{i+1}; \mathbf{Z}) \to H^{\mathrm{odd}}(E_i; \mathbf{Z}) \to H^{\mathrm{odd}}(E_{i+1}; \mathbf{Z}) \to .$$

Since $H^{\mathrm{odd}}(E_i, E_{i+1}; \mathbf{Z}) \cong H^{\mathrm{odd}}(N_i, N_i \cap E_{i+1}; \mathbf{Z}) \cong H^{\mathrm{odd}}(D, \partial D; \mathbf{Z}) \cong 0$, where D is the unit disk in $\mathbf{C}^{\dim \sigma_i}$, we see that $H^{\mathrm{odd}}(E_{i+1}; \mathbf{Z}) \cong 0$ implies $H^{\mathrm{odd}}(E_i; \mathbf{Z}) \cong 0$. Since $H^{\mathrm{odd}}(E_r; \mathbf{Z}) \cong 0$, by the inductive argument we have $H^{\mathrm{odd}}(E_i; \mathbf{Z}) = 0$.

Hence, by applying the standard Mayer-Vietoris argument to $E_i = E_{i+1} \cup N_i$, we can show the claim. □

By the above claim we can conclude that the map

$$H^*(X(\nu); \mathbf{Z}) \cong H^*(E_1; \mathbf{Z}) \to H^*(E_2; \mathbf{Z}) \oplus H^*(N_1; \mathbf{Z})$$

is injective. By using this argument repeatedly, we finish the proof of Theorem 3.4. □

§4. Chern classes

In this section we compute the total Chern class of a toric hyperKähler manifold as a simple application of Theorem 3.4.

Theorem 4.1. *Let $X(\nu)$ be a toric hyperKähler manifold. Let*

$$c(X(\nu)) = 1 + c_1(X(\nu)) + c_2(X(\nu)) + \cdots \in H^*(X(\nu); \mathbf{Z})$$

be the total Chern class of the holomorphic tangent bundle of $X(\nu)$ with respect to the complex structure $I_{\nu,1}$. Then we have

$$c(X(\nu)) = \Phi\left(\prod_{i=1}^{N}(1 - u_i^2)\right) \in H^*(X(\nu); \mathbf{Z}).$$

To prove Theorem 4.1, we need the following lemma, which is a simple generalization of the argument due to Bielawski and Dancer [2]. They showed it in the case $\epsilon_0 \in \Theta$ such that $\epsilon_0(i) = 1$ for all $i = 1, \ldots, N$.

Lemma 4.2. *Let $X(\nu)$ be a toric hyperKähler manifold with $\nu = (\nu_1, 0, 0)$. If M_ϵ is not empty, then its holomorphic cotangent bundle T^*M_ϵ is contained in $X(\nu)$ as an open subset.*

Proof. We first recall the notation in Section 3. Fix $\epsilon \in \Theta$. For $i = 1, \ldots, N$, we define $(q_i^\epsilon, p_i^\epsilon)$ by

$$(q_i^\epsilon, p_i^\epsilon) = \begin{cases} (z_i, w_i) & \text{if } \epsilon(i) = 1, \\ (w_i, -z_i) & \text{if } \epsilon(i) = -1. \end{cases}$$

Then $q^\epsilon = (q_1^\epsilon, \ldots, q_N^\epsilon)$ is a point in the vector space V_ϵ, and $p^\epsilon = (p_1^\epsilon, \ldots, p_N^\epsilon)$ is a point in the dual space V_ϵ^*. In other words, we identify the cotangent bundle T^*V_ϵ with $\mathbf{H}^N$ as above.

Let us recall that we have a holomorphic description of M_ϵ as follows:

$$M_\epsilon = U_\epsilon / K^{\mathbf{C}},$$

where $K^{\mathbf{C}}$ is the complexification of K, and U_ϵ is an open subset of V_ϵ. By the argument in [6], $q^\epsilon \in U_\epsilon$ if and only if the functional l_{q^ϵ} on k defined by

$$l_{q^\epsilon}(Y) = \langle \nu_1, Y\rangle + \frac{1}{4}\sum_{i=1}^N |q_i^\epsilon|^2 e^{-\epsilon(i)4\pi\langle u_i Y\rangle} \quad \text{for } Y \in k$$

has the minimum. Moreover, we have a holomorphic (with respect to the complex structure $I_{\nu,1}$) description of $X(\nu)$ as follows:

$$X(\nu) = W/K^{\mathbf{C}},$$

where W is a subset of $T^*V_\epsilon = \mathbf{H}^N$. Similarly, $(q^\epsilon, p^\epsilon) \in W$ if and only if $(\mu_{K,2} + \sqrt{-1}\mu_{K,3})(q^\epsilon, p^\epsilon) = 0$ and the functional $l_{q^\epsilon, p^\epsilon}$ on k defined by

$$l_{q^\epsilon,p^\epsilon}(Y) = \langle \nu_1, Y\rangle + \frac{1}{4}\sum_{i=1}^N |q_i^\epsilon|^2 e^{-\epsilon(i)4\pi\langle u_i Y\rangle} + \frac{1}{4}\sum_{i=1}^N |p_i^\epsilon|^2 e^{\epsilon(i)4\pi\langle u_i Y\rangle}$$

has the minimum.

Suppose that $q^\epsilon \in U_\epsilon \subset V_\epsilon$ and that $p^\epsilon \in V_\epsilon^*$ defines a cotangent vector of M_ϵ at $[q^\epsilon]$, that is,

$$\langle Y^*_{\text{at } q^\epsilon}, p^\epsilon\rangle = 0 \quad \text{for any } Y = \sum_{i=1}^N a_i X_i \in k,$$

where Y^* is a vector field on V_ϵ generated by Y. If we note

$$Y^*_{\text{at } q^\epsilon} = (2\pi\sqrt{-1}\epsilon(1)a_1 q_1^\epsilon, \ldots, 2\pi\sqrt{-1}\epsilon(N)a_N q_N^\epsilon),$$

then we have

$$\langle Y^*_{\text{at } q^\epsilon}, p^\epsilon\rangle = 2\pi\sqrt{-1}\sum_{i=1}^N a_i\epsilon(i) q_i^\epsilon p_i^\epsilon = 2\pi\sqrt{-1}\left\langle Y, \sum_{i=1}^N z_i w_i u_i\right\rangle.$$

Therefore, $p^\epsilon \in V_\epsilon^*$ defines a cotangent vector of M_ϵ at $[q^\epsilon]$ if and only if $(\mu_{K,2} + \sqrt{-1}\mu_{K,3})(q^\epsilon, p^\epsilon) = 0$. Moreover, if l_{q^ϵ} has the minimum, then it is easy to see that $l_{q^\epsilon,p^\epsilon}$ has also the minimum. Thus we have $(q^\epsilon, p^\epsilon) \in W$, which implies $T^*M_\epsilon \subset X(\nu)$. □

Proof of Theorem 4.1. We may assume $\nu=(\nu_1,0,0)$. Let $i_\epsilon\colon M_\epsilon \to X(\nu)$ be the embedding. By Lemma 4.2, we have

$$i_\epsilon^* TX(\nu) \cong TM_\epsilon \oplus T^*M_\epsilon.$$

By the same argument in [4] and Lemma 3.1, we have

$$c(TM_\epsilon) = \Phi_\epsilon\left(\prod_{i=1}^N (1+\epsilon(i)u_i)\right), \quad c(T^*M_\epsilon) = \Phi_\epsilon\left(\prod_{i=1}^N (1-\epsilon(i)u_i)\right),$$

where $\Phi_\epsilon\colon \mathbf{Z}[u_1,\dots,u_N] \to H^*(M_\epsilon;\mathbf{Z})$ is a ring homomorphism defined by $\Phi_\epsilon(u_i) = c_1(i_\epsilon^* L_{u_i})$. Therefore we have

$$i_\epsilon^* c(X(\nu)) = c(TM_\epsilon)c(T^*M_\epsilon) = i_\epsilon^*\Phi\left(\prod_{i=1}^N (1-u_i^2)\right).$$

On the other hand, by Theorem 2.2, there exsists $f \in \mathbf{Z}[u_1,\dots,u_N]$ such that $\Phi(f) = c(X(\nu))$. Therefore we have

$$i_\epsilon^*\Phi\left(f - \prod_{i=1}^N (1-u_i^2)\right) = 0 \quad \text{for any } \epsilon \in \Theta.$$

Recall now Lemma 3.4. Since any $\tau \in \mathcal{C}(X(\nu))$ is a face of Δ_ϵ for some $\epsilon \in \Theta$, we have

$$f - \prod_{i=1}^N (1-u_i^2) \in \bigcap_{i=1}^r \ker(\psi_i^* \circ \Phi) = \ker\Phi.$$

This implies Theorem 4.1. □

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