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The Gaussian Image of Mean Curvature One Surfaces in $\mathbb{H}^3$ of Finite Total Curvature

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Abstract.

The hyperbolic Gauss map G of a complete constant mean curvature one surface M in hyperbolic 3-space, is a holomorphic map from M to the Riemann sphere. When M has finite total curvature, we prove G can miss at most three points unless G is constant. We also prove that if M is a properly embedded mean curvature one surface of finite topology, then G is surjective unless M is a horosphere or catenoid cousin.

We consider complete surfaces M in hyperbolic 3-space $\mathbb{H}^3$ with mean curvature one and of finite total curvature. For a point $q \in M$, the Gauss map G sends q to the point at infinity obtained as the positive limit of the geodesic of $\mathbb{H}^3$ starting at q and having $\overrightarrow{H}(q)$ (the mean curvature vector of M at q) as its tangent at q. Bryant has shown that G is meromorphic on M and M admits a parametrization by meromorphic data analogous to the Weierstrass representation of minimal surfaces in Euclidean 3-space $\mathbb{R}^3$ [1], [4].

Yu [6] has shown that G can omit at most 4 points of the sphere at infinity S_∞, unless M is a horosphere and G is constant. For complete minimal surfaces in $\mathbb{R}^3$ of finite total curvature, Osserman had shown that the Gauss map omits at most 3 points of the sphere, unless M is a plane. In this paper we establish a result of this type in $\mathbb{H}^3$.

The conformal type of a complete surface of mean curvature one with finite total curvature in $\mathbb{H}^3$ is finite, i.e., M is conformally a compact Riemann surface $\overline{M}$ with a finite number of points removed (called the punctures), but G does not necessarily extend meromorphically to the punctures. M is called regular when G does extend meromorphically to the punctures.

Our first result is then:

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Theorem 1. *Let M be a complete surface immersed in $\mathbb{H}^3$ with mean curvature one and of finite total curvature. Then G can omit at most 3 points unless G is constant and M is a horosphere.*

Proof. If G is not regular, then G has an essential singularity at a puncture p_0. By Picard's theorem, G can omit at most two values in a neighborhood of this puncture. Thus in the following we can assume that G is meromorphic on $\overline{M}$, i.e., M is regular.

Let (g,ω) be local Weierstrass data of the minimal cousin of M in $\mathbb{R}^3$ (cf. [1], [4] for the details). The induced metric on M is given by $ds = (1+|g|^2)|\omega|$, and the holomorphic quadratic differential

$$Q = \omega dg$$

is globally defined on M and meromorphic at each puncture of M, with a pole at each puncture which is at worst of order 2. Since dG is meromorphic on $\overline{M}$ (the conformal compactification of M), the 1-form $\omega^{\#} = -Q/dG$ is meromorphic on $\overline{M}$; in a local conformal coordinate, $\omega^{\#} = -(g'(z)/G'(z))\ \omega(z)$.

The Schwarzian quadratic differentials of g, G and Q are related on $\overline{M}$ ([1], [4]):

$$S(g) - S(G) = 2Q, \tag{1}$$

where $S(g)(z) = \big((g''/g')' - (1/2)(g''/g')^2\big)dz^2$. Writing $g(z) = a_0 + z^k(a_1 + a_2 z + \dots)$, a calculation shows that $S(g)$ has at worst a pole of order 2 at z and the coefficient of dz^2/z^2 is $(1-k^2)/2$.

Since Q is holomorphic on M, it follows from (1) that the branch points and non-simple poles of g and G on M coincide with each other and each of them has the same multiplicity (the branching order of g at z is defined to be $k-1 = b_g(z)$). In particular, $\omega^{\#}$ has no poles on M.

We next observe that the zeros of $\omega^{\#}$ on M are the poles of G on M, and a pole of G of order k is a zero of $\omega^{\#}$ of order $2k$. First, suppose that $z \in M$ is a pole of G of order k. Then $k \geq 1$ and z may, or may not, be a pole of g. If it is a pole of g, then z is a pole of g of order k (by the Schwarzian derivative relation) and then is a zero of ω of order $2k$. Hence the order of a zero of $\omega^{\#}$ is of twice the order of the pole of G. If z is not a pole of g, then it is not a zero of ω but a zero of g' of order $k-1$ and a pole of G' of order $k+1$. Consequently $\omega^{\#}$ also has a zero whose order is twice the order of the pole of G. An analogous computation, in the case that G has no poles, implies that $\omega^{\#}$ is holomorphic and not zero.

Let $p_1, \dots, p_r$ be the punctures, so $\overline{M} = M \cup \{p_1, \dots, p_r\}$. After an isometry of $\mathbb{H}^3$, we can suppose that G has only simple poles on M and has no zeros or poles at the punctures. The metric

$$ds^{\#} = (1 + |G|^2)|\omega^{\#}|$$

is complete on $\overline{M}$, so $\omega^{\#}$ has a pole at each puncture [5]. The order of the pole of $\omega^{\#}$ at p_j is given by

$$P_{p_j}(\omega^{\#}) = \lambda_Q(p_j) + b_G(p_j),$$

where $Q(z) = (\gamma/(z - p_j)^{\lambda_Q(p_j)} + \cdots)dz^2$ is the Laurent expansion of Q at p_j. Then the total order of the poles of $\omega^{\#}$ is

$$P(\omega^{\#}) = \sum_{j=1}^{r} \lambda_Q(p_j) + \sum_{j=1}^{r} b_G(p_j). \tag{2}$$

By Riemann's relation for $\omega^{\#}$ on $\overline{M}$, we have

$$P(\omega^{\#}) - 2N = 2 - 2s, \tag{3}$$

where N is the degree of G (so $2N$ is the order of zeros of $\omega^{\#}$, since G has N simple poles on M) and s is the genus of M.

Let $q_1, ..., q_k$ be the points of S_∞ omitted by G, so that $G^{-1}\{q_1, ..., q_k\} \subset \{p_1, ..., p_r\}$ (we write G also for the meromorphic extension of G to $\overline{M}$). Then we have

$$kN \le \sum_{j=1}^{r} (1 + b_G(p_j)) \le r + b, \tag{4}$$

where b is the total branching order of G. Here $1 + b_G(p_j)$ is the total number of times that G takes its value at p_j, counted with multiplicity.

Riemann's relation applied to the 1-form dG on $\overline{M}$ yields:

$$2N - b = 2 - 2s. \tag{5}$$

Now by Lemma 3 of [5], we have at each puncture p_j:

$$\lambda_Q(p_j) + b_G(p_j) \ge 2.$$

Then equation (2) gives:

$$P(\omega^{\#}) \ge 2r. \tag{6}$$

This last inequality together with the equations (3) and (5) yields:

$$P(\omega^{\#}) = 4N - b \ge 2r.$$

Then the equation (4) implies:

$$4N - kN \geq r \geq 1, \tag{7}$$

and k is at most 3. □

Theorem 2. *Let M be a properly embedded surface in $\mathbb{H}^3$ with mean curvature one and of finite topology. If M is not a horosphere nor a catenoid cousin, then the Gauss map G of M is surjective.*

Proof. We know that M has finite total curvature and each end of M is regular [2]; also each end is asymptotic to an end of a horosphere or an end of a catenoid cousin. We also proved in [2] that the asymptotic boundary of an end is precisely the limiting value of G at the puncture. We can suppose M has at least two ends, since if M had only one end, the asymptotic boundary of M would be one point and M would be a horosphere [2].

We claim that each end of M is asymptotic to a catenoid cousin end. Suppose this were not true. Let E be an end of M asymptotic to a horosphere end. We work in the upper half-space model of $\mathbb{H}^3$, $\{x_3 > 0\}$, and assume E is asymptotic to a horosphere $x_3 = c > 0$. In particular, the mean curvature vector of E points up outside of some compact set of E. There are no ends of M above E. Indeed, their mean curvature vector would also point up (each such end is asymptotic to a horizontal horosphere or a catenoid cousin end whose limiting normal points vertically up) and M separates $\mathbb{H}^3$ into two connected components, so no such end is above E.

Then for $\varepsilon > 0$, the part A of M above $c + \varepsilon$ is compact. At the highest point of A (if A were not empty) the mean curvature vector of M points down. But this highest point can be joined by an arc in $\mathbb{H}^3 \setminus M$, to a point of E where the mean curvature vector points up. Thus M is completely below $x_3 = c$.

Let $\varepsilon > 0$, and let C be a small circle in the plane $x_3 = c - \varepsilon$ so that C is above M. Just as in the proof of the half-space theorem for properly immersed minimal surfaces in $\mathbb{R}^3$ [3], one can take a family of catenoid cousin ends $C(\lambda)$, $\partial C(1) = C$ with $C(1)$ above M, and $C(\lambda)$ converges to the plane $x_3 = c - \varepsilon$ as $\lambda \to 0$. Then some $C(\lambda)$ would touch M at a point $q \in M$, and the maximum principle would yield M equals this catenoid cousin. Thus each end of M is asymptotic to a catenoid cousin.

Next, observe that G is injective on the set of punctures; two distinct ends can not be asymptotic to the same point at infinity. This follows from the fact that each end is asymptotic to a catenoid cousin end and we know the direction of the mean curvature vector along the end. When M is embedded, M separates $\mathbb{H}^3$ and the mean curvature vector points into one of the components of the complement. Thus two ends can not be asymptotic to the same point at infinity.

Now, suppose that G is not surjective and omits a point q. Then there is exactly one catenoid cousin type end E of M asymptotic to q. Let $p \in \overline{M}$ be the puncture of E such that $G(p) = q$. We know G has local degree one at p. There is no other point $p' \in \overline{M}$ sent to q by G. For p' can not be a puncture of M, since G is injective on the punctures, and p' can not be a point of M because q is a value omitted. Hence the degree N of G on $\overline{M}$ is one.

We use the same notation as in Theorem 1. At each puncture p_j of M, $\omega^{\#}$ has a pole exactly of order 2. So, by equation (3), we have

$$2r - 2 = 2 - 2s \text{ and } r + s = 2.$$

Then M is the catenoid cousin ($r = 2$) and Theorem 2 is proved. □

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